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A Three-Stage Game Model of the Supply Chain in Disaster Relief Operations

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Abstract

We propose a three-stage non-cooperative game theoretic model to describe the distribution of relief supplies in response to slow-onset disasters. The first-stage game is modeled as a finite standard game, where each humanitarian organization (HO) has to decide whether to form a coalition with other HOs to negotiate framework agreements with the carriers before the disaster occurs. The second-stage game models the actual negotiation between HOs and carriers: it is a generalized Nash game between carriers, the HOs outside of the coalition, and the coalition formed in the first stage. After the disaster occurs, the third-stage game models the competition between HOs which have to purchase and distribute the relief items: it is formulated as a generalized Nash game where the HOs which had taken part in the coalition in the first stage share some constraints. First, we prove that the second-stage and third-stage games have a unique variational equilibrium. Next, under suitable assumption on the parameters, we prove that the variational equilibrium of the second-stage game can be written in closed form and the grand coalition formed by all the HOs in the first stage is the most efficient in terms of the social welfare computed at the third-stage equilibrium. Moreover, we provide sufficient conditions for the grand coalition to be a Nash equilibrium of the first-stage game. Finally, some preliminary numerical experiments are shown.

Keywords Generalized Nash equilibrium problem · Variational inequality · Humanitarian logistics

Mathematics Subject Classification 90C33 · 91A80 · 90B06 · 65K15

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1 Introduction

The role played by operations research to improve the supply chain to provide efficient food delivery to crisis areas has acquired a growing importance in both the academic and the non academic environment in the last decade. For instance, The United Nations World Food Programme (WFP) [27] was named the winner of the 2021 Franz Edelman Award, for its success in developing mathematical models, and related analytic tools which enabled the WFP in carrying out extremely complex humanitarian operations in countries such as Yemen, Syria and South-Sudan.

The literature about mathematical models of relief activities after the so called slow-onset disasters is, however, quite scarce, although this term has been used for decades by specialists in the United Nations system who help people prepare for and deal with catastrophic events, such as droughts, that can be reasonably predicted as the ancient and familiar patterns begin to set in, see e.g. [4, 12, 25]. In contrast to sudden-onset disasters, like for example the recent earthquake in Turkey, phenomena like famine, floods and the like have the distinctive feature that they can be geographically and, within a certain approximation, temporally delimited, before they happen. Humanitarian organizations (HOs) thus aim at establishing framework agreements with potential carriers of relief items well before the crises actually unfolds. Indeed, transportation is a critical aspect in the supply chain relief operations, and the possibility, for HOs, of allocating a predetermined budget before the emergency is beneficial. Indeed, by making agreements with the carriers one or two years before the actual transportation takes place, HOs will presumably avoid the rapid growth of fares which follows the onset of a disaster. Moreover, the simultaneous presence of a (small) number of HOs in the field may trigger competition effects that could yield to a suboptimal allocation of relief items. Thus, the development of new models to make the relief activities as efficient as possible is a well justified need. The relationships within service providers and HOs has been investigated in [3], and [19], while for a survey on the supply chain in humanitarian logistics, the reader can refer to [14] and to [10] for a game theory approach.

Our contribution is inspired by the recent paper [9] where the authors put forward a game-theoretical model for the negotiation of framework agreements between HOs and carriers and the subsequent purchase and distribution of relief items where needed, and also brief the reader on the recent related literature. With respect to the above-mentioned paper, we still consider a two-phase model (before and after the humanitarian crisis), but we differ in the following aspects. In [9], the first phase is split into two different subgames: one between HOs (where the prices chosen by carriers are fixed) and one between carriers (where the volumes projected for transport chosen by HOs are fixed). The two games are then connected through an *ad hoc* economic argument to define an equilibrium as the solution of the sum of two suitable variational inequalities. On the other hand, we directly formulate the interaction between HOs and carriers through a single game. Moreover, we allow that some of the HOs form a coalition to have a stronger bargaining power. In our game, the players are the carriers, the HOs outside of the coalition, and the coalition considered as a single player. Since all the HOs shared carrier capacity constraints, we formulate the problem as a generalized Nash equilibrium problem (GNEP). It is interesting noticing that, in the particular case of no coalition, the variational equilibrium of our single game coincides with the equilibrium defined in [9]. We compute the variational equilibrium of the game for all the possible scenarios of HOs forming one coalition. However, the analysis above does not investigate the problem of the coalition formation, i.e., whether an HO should find convenient to be in the coalition or not. Since HOs are supposed to act selfishly, we model the first phase of the problem as a two-stage non-cooperative game: the first-stage game is a finite Nash game between HOs, where each HO controls a binary variable describing its membership in the coalition, while the second-stage game is the negotiation game between HOs and carriers. We prove that the variational equilibrium of the negotiation game is unique and, under suitable assumptions on the model parameters, can be written in closed form.

The second phase is the actual purchase and distribution of relief items and is modeled as a non-cooperative game played between the HOs, also considering the possibility that the HOs which had cooperated in the negotiation game join again. However, now the HOs compete to obtain financial support, thus we consider all of them as individual players, but the same HOs that had taken part in a coalition in the first-stage game share some budget constraints also in the purchase and distribution game. This is another aspect that differs our model from that proposed in [9]. Due to the presence of shared constraints between some HOs, this game is formulated as a GNEP as well. Since the variational equilibrium of the negotiation game is used as input data of the purchase and distribution game, we consider the latter game as the third-stage game of our model. We prove that the purchase and distribution game, for any coalition formed by HOs, has a unique variational equilibrium that coincides with the maximizer of the social welfare function. Moreover, the grand coalition formed by all the HOs in the first-stage game is proved to be the most efficient in terms of social welfare under suitable assumptions on parameters. Finally, we show that the grand coalition may be not a Nash equilibrium of the first-stage game, but we provide sufficient conditions on the parameters for it to be so. We mention that the effect of cooperation between HOs has been studied in the very recent paper [20], with a system optimization approach.

The contribution of the paper is therefore twofold. From a modeling viewpoint, we propose a three-stage game model: the first-stage game is a finite standard game that models the membership problem between HOs, the second-stage game is a GNEP that models the negotiation of framework agreements between HOs and carriers, while the third-stage game is a GNEP modeling the competition between HOs in the purchase and distribution of relief items. Thus, our model generalizes the approach proposed in [9], where HOs are not allowed to form a coalition when negotiating with carriers. From a methodological viewpoint, we prove several theoretical results on the proposed game model: the grand coalition, under suitable assumptions, is a Nash equilibrium of the first-stage game; the second-stage game, for any coalition formed by HOs, has a unique variational equilibrium that can be written in closed form; the third-stage game, for any coalition formed by HOs, has a unique variational equilibrium that coincides with the maximizer of the social welfare function. Moreover, the grand coalition, under suitable assumptions, is the most efficient in terms of the social welfare at equilibrium.

We anticipate here that our methodology requires to solve the problem backward. We first solve the second stage and use its solution as an input of the third stage, then

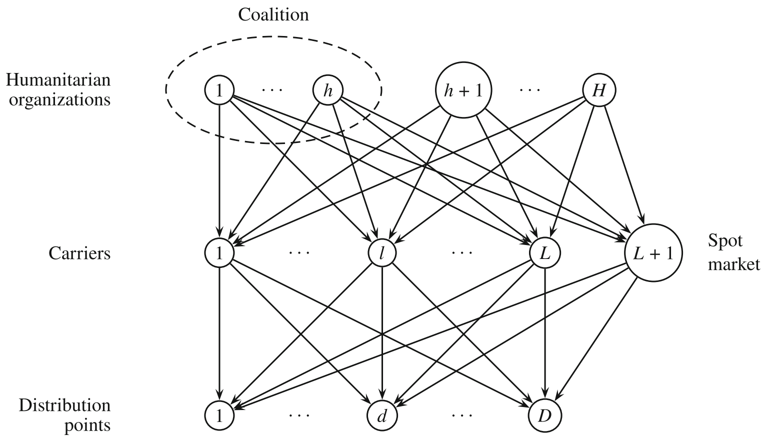


Fig. 1 Network structure of the model

we combine the solutions of second and third stages and pass them on as input to the first stage, coalition formation. This approach is common to several economic models (see, e.g., [6, 11, 15, 16]) and differs from a multilevel hierarchical approach, as there is no leader-follower structure but some players (HOs) appear in each stage of the game.

The paper is organized as follows. In Sect. 2 we describe the three-stage game model: the first-stage coalition formation game is outlined in Sect. 2.1, the second-stage negotiation game is described in Sect. 2.2 and the third-stage purchase and distribution game is described in Sect. 2.3. Section 3 provides some theoretical results on the three game models described above: Sect. 3.1 is devoted to the second-stage game, Sect. 3.2 to the third-stage game, while Sect. 3.3 to the first-stage game. In Sect. 4 we illustrate our model with some numerical experiments: Example 4.1 shows that the grand coalition is the most efficient in terms of both social welfare and total volume transported, however it is not stable because it is not a Nash equilibrium of the first-stage game, while Example 4.2 shows that the grand coalition is both a Nash equilibrium and the most efficient in terms of social welfare at equilibrium. In the last concluding section we summarize our findings and outline future research perspectives.

2 Model Formulation

We illustrate the network structure of the model with the help of Fig. 1.

The network consists of three tiers. The H nodes of the first tier represent humanitarian organizations and we denote with h the generic node of this tier. Some HOs can form a coalition to have a stronger bargaining power in the negotiation phase with the carriers (in the figure, for simplicity, the coalition is made up with the first h HOs). In the second tier, we find the L carriers with whom HOs establish framework agreements, to be possibly exploited in the distribution process. Framework agreements are usually negotiated well before that a humanitarian crisis actually starts. In the same tier we also put the spot market, where HOs can buy carrier services at the current price (i.e., without negotiating), at the onset of the crises; the generic carrier is denoted with index l. The last tier consists of D distribution points where the relief items are needed, and we denote with d the generic distribution point.

We distinguish two phases in our model: the negotiation phase (before the humanitarian crisis) and the actual purchase/distribution phase (after the humanitarian crisis). However, as we allow HOs to form a coalition in the negotiation process with the carriers, each HO has to make two decisions in the first phase: first, it must decide whether to join the coalition or not, and then, once the coalition is formed, it has to establish an agreement with the carriers in order to guarantee a minimum volume to be transported to the distribution points. Finally, after the humanitarian crisis, the HOs have to buy and distribute the relief items and can either exploit the framework agreements previously established with carriers, or find more convenient to pay the fares offered by the spot market.

Since the utility of each HO in the last phase depends on the choices made by other HOs, the framework agreements with the carriers during the negotiation phase and the coalition formed by the HOs, we formulate this problem as a three-stage non-cooperative game. Specifically, the first-stage game (or coalition formation game) is a standard non-cooperative game between HOs, where each HO h controls a binary variable s_h that indicates whether it takes part in the coalition ($s_h = 1$) or not ($s_h = 0$). Once the coalition given by the vector s is fixed, the second-stage game (or negotiation game) is a generalized Nash equilibrium problem (GNEP) where the players are the carriers, the HOs outside of the coalition and the coalition considered as a single player. The third-stage game (or purchase and distribution game) is a further GNEP where only the HOs are the players. We assume that the HOs that had formed the coalition in the first stage jointly satisfy a common budget constraint and a common upper bound on the volumes to be distributed. However, this time also the HOs in the coalition act selfishly and wish to maximize their individual utility. The structure of the three-stage model is illustrated in Fig. 2. The detailed model of each stage (with variables, objective functions and constraints) is described in the following subsections.

2.1 First Stage: The Coalition Formation Game

The first decision each HO must make is whether or not to be part of the coalition that will negotiate with the carriers. It is natural to expect that smaller HOs will have an interest in joining the coalition in order to have more bargaining power with carriers, while for larger HOs this advantage may not be so obvious. The choice of coalition among HOs affects both the negotiation with carriers (second-stage game) and the purchase and distribution of goods (third-stage game). To measure whether it is convenient for each HO to join the coalition or not, we use the value of the HO utility function calculated at the variational equilibrium in the last stage game.

Formally, the first-stage game (or coalition formation game) is a standard H-person finite game, where each HO h controls a binary variable

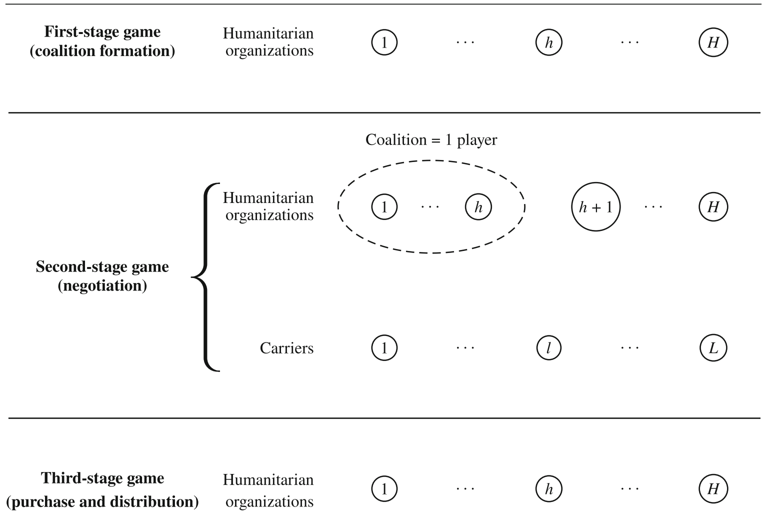


Fig. 2 The three-stage game model

$$s_h = \begin{cases} 1 & \text{if HO } h \text{ takes part in the coalition,} \\ 0 & \text{otherwise,} \end{cases}$$

and aims to solve the problem

$$\max_{s_h} T_h(s) \tag{1a}$$

$$\text{subject to } s_h \in \{0, 1\}. \tag{1b}$$

The utility function T_h is defined as

$$T_h(s) := V_h(Y^*(s)),$$

where V_h and $Y^*(s)$ are the utility function of HO h and the (unique) variational equilibrium in the third-stage game (see Sect. 2.3), respectively. Notice that the utility function T_h is well defined since we will prove in Theorem 3.3 a) that the third-stage game has a unique variational equilibrium $Y^*(s)$ for each coalition s.

Hence, s^* is a Nash equilibrium of the first-stage game if every HO that is part of the coalition has no incentive to leave it and every HO that is outside it has no incentive to enter in.

Definition 2.1 s^* is a Nash equilibrium of the first-stage game iff

$$T_h(s_h^*, s_{-h}^*) \geq T_h(1 - s_h^*, s_{-h}^*)$$

holds for each $h = 1, \ldots, H$.

Notice that a trivial Nash equilibrium of the first-stage game is $s^* = (0, \dots, 0)$, that is, the case where no HO takes part in the coalition. In Sect. 3, we will provide sufficient conditions for the *grand coalition* $s^* = (1, \dots, 1)$ to be a Nash equilibrium as well.

We remark that the combinatorial nature of the first-stage game allows to treat the case of 10–20 HOs. However, this number is often met in several real-world scenarios (see [10, 14] and references therein). For instance, the paper [5] deals with a descriptive analysis of agency partnerships involved in post-disaster operations relief in Haiti, where the number of HOs involved was 18, but only 7 were involved in all stages of relief operations.

2.2 Second Stage: The Negotiation Game

Once the coalition of HOs has been formed in the first stage, the second-stage game models the negotiation between the carriers and the HOs.

Each HO h controls the variables x_{hld}, for any $l = 1, \dots, L$ and $d = 1, \dots, D$, which represent the volume *projected* by HO h for transport with carrier l to distribution point d. X_h denotes the vector of all the variables controlled by HO h, while X the vector of the variables of all HOs.

On the other hand, each carrier l controls the variables p_{hld}, for any $h = 1, \dots, H$ and $d = 1, \dots, D$, which represent the unit price paid by HO h to transport relief items to distribution point d with carrier l. P^l denotes the vector of all the variables controlled by carrier l, while P the vector of the variables of all carriers.

To denote the coalition defined in the first stage, we introduce the index set $\mathcal{O} := \{h : s_h = 1\}$. Each HO h which does not take part in the coalition $\mathcal{O}$ establishes individual agreements with each carrier and aims to solve the following problem:

$$\max_{X_h} \; U_h(X, P) = -\sum_{l=1}^{L}\sum_{d=1}^{D} p_{hld}\, x_{hld} - \omega_h^R \sum_{l=1}^{L}\sum_{d=1}^{D} r_{hl}\, x_{hld}^2 \tag{2a}$$

$$\text{subject to} \quad \sum_{l=1}^{L} x_{hld} \ge M_{hd}, \qquad d = 1, \dots, D, \tag{2b}$$

$$\sum_{i=1}^{H}\sum_{d=1}^{D} x_{ild} \le G_l, \qquad l = 1, \dots, L, \tag{2c}$$

$$x_{hld} \ge 0, \qquad l = 1, \dots, L, \; d = 1, \dots, D. \tag{2d}$$

HO h wishes at the same time to minimize the transportation cost and avoid allocating a large fraction of the (expected) volume of goods to a particular carrier. Hence, the utility function (2a) is the opposite of the weighted sum of two terms: the first term is the total transportation cost, while the second one is a dependency risk function, where r_{hl} is the relative risk that HO h assigns to carrier l and ω_h^R is a weight which describes the relative importance of risk compared to cost for HO h. Constraints (2b) guarantees that a minimum volume M_{hd} is transported by all the carriers to each destination point d. Constraints (2c) are shared by all HOs and ensure that the total projected volume assigned to carrier l does not exceed its maximum capacity G_l. The associated Lagrange multiplier, say γ_{il}, represents the marginal value (of each HO i outside the coalition, and of the coalition) of increasing the carrier's projected capacity. In economic terms, γ_{il} measures how much the utility of the humanitarian organizations outside the coalition, and the coalition, would improve if the carrier had one additional unit of projected capacity available. A positive γ_{il} indicates that the capacity constraint is binding, meaning that the carrier's limit is restricting the optimal allocation. If all HOs outside the coalition, and the coalition, share the same Lagrange multiplier γ_l for all l, then the marginal utility of increasing projected carrier capacity is the same across different humanitarian organizations. This can also be interpreted as equity in allocation: if the Lagrange multipliers were different, it would suggest that some HOs are more constrained than others. The equality suggests a balanced or regulated negotiation process where capacity is fairly allocated. This would be the case, for instance, when different agencies of the UN act under a General Director.

The HOs that signed an agreement to join the coalition $\mathcal{O}$, aim at transporting to each point d a projected volume of at least $\sum_{i\in\mathcal{O}} M_{id}$, thus allowing more flexibility of the signatory players, with respect to the initial quantity M_{hd}. All HOs that are part of the coalition bargain with each carrier jointly to obtain the same price, thus they act collectively as one player to solve the following problem:

$$\max_{(X_i)_{i\in\mathcal{O}}} \sum_{i\in\mathcal{O}} U_i(X,P) = \sum_{i\in\mathcal{O}} \left[-\sum_{l=1}^{L}\sum_{d=1}^{D} p_{ild}\, x_{ild} - \omega_i^R \sum_{l=1}^{L}\sum_{d=1}^{D} r_{il}\, x_{ild}^2 \right] \tag{3a}$$

$$\text{subject to } \sum_{i\in\mathcal{O}}\sum_{l=1}^{L} x_{ild} \geq \sum_{i\in\mathcal{O}} M_{id}, \qquad d=1,\ldots,D, \tag{3b}$$

$$\sum_{i=1}^{H}\sum_{d=1}^{D} x_{ild} \leq G_l, \qquad l=1,\ldots,L, \tag{3c}$$

$$x_{ild} \geq 0, \qquad i\in\mathcal{O},\ l=1,\ldots,L,\ d=1,\ldots,D. \tag{3d}$$

Let us notice that constraints (3b) involve only the variables of the HOs in the coalition, while (3c) are shared by all the HOs.

On the other hand, each carrier l, which controls the price variables p_{hld} for any $h=1,\ldots,H$ and $d=1,\ldots,D$, wishes to solve the problem:

$$\max_{P^l} U^l(X,P) = \sum_{h=1}^{H}\sum_{d=1}^{D}(p_{hld}-c_d^t)x_{hld} + \omega_l^S \sum_{h\notin\mathcal{O}}\sum_{d=1}^{D} M_{hd}\left[1-\left(\frac{p_{hld}}{p_{hd}^{\max}}\right)^2\right]$$
$$+\,\omega_l^S \sum_{h\in\mathcal{O}}\sum_{d=1}^{D} M_{hd}\left[1-\left(\frac{p_{hld}}{p_d^{\max}(\mathcal{O})}\right)^2\right] \tag{4a}$$

$$
\begin{aligned}
\text{subject to}\quad & c_d^t \le p_{hld} \le p_{hd}^{\max}, && h \notin \mathcal{O},\ d = 1,\dots,D, && \text{(4b)}\\
& c_d^t \le p_{hld} \le p_d^{\max}(\mathcal{O}), && h \in \mathcal{O},\ d = 1,\dots,D, && \text{(4c)}\\
& p_{hld} = p_{ild}, && h,i \in \mathcal{O},\ h \ne i,\ d = 1,\dots,D, && \text{(4d)}
\end{aligned}
$$

where c_d^t is the unit transportation cost faced by any carrier for transport to point d; $p_{hd}^{\max}$ is the maximum price that HO h is willing to pay to any carrier for transportation to point d, while $p_d^{\max}(\mathcal{O})$ is the maximum price that each HO in the coalition $\mathcal{O}$ is willing to pay to any carrier for transportation to point d, that is defined as $p_d^{\max}(\mathcal{O}) = \min_{h\in\mathcal{O}} p_{hd}^{\max}$.

The utility function (4a) is the weighted sum of the expected profit (first term) and the customer satisfaction of the HOs not participating in the coalition (second term) and of those participating (third term), where ω_l^S is a weight which describes the relative importance of customer satisfaction with respect to profit for carrier l. Constraints (4b)–(4c) guarantee that the prices set by carrier l are not lower than the transportation cost and not higher than the price each HO is willing to pay, while constraints (4d) imply that the price fixed by carrier l must be the same for all the HOs in the coalition.

Since the HOs' utility functions (2a) and (3a) depend on the carrier variables, the carrier utility function (4a) depends on the HO variables, and constraints (2c) and (3c) are shared by all the HOs, the second-stage game model is a GNEP with shared constraints. This class of GNEPs was introduced in the seminal paper by Rosen [24] and reformulated many years later within the framework of variational and quasi-variational inequalities in finite dimension [7, 18] and in infinite dimension [8, 17].

The feasible region of the second-stage game is denoted by

$$
K_2 = \Big\{(X,P) \in \mathbb{R}_+^{2HLD} :\ \text{constraints } (2b)\ \forall h \notin \mathcal{O}, (3b), (3c), (4b), (4c) \text{ and } (4d)\ \forall l = 1,\dots,L \text{ hold}\Big\}.
$$

Definition 2.2 Given any coalition $\mathcal{O}$ of HOs, defined in the first-stage game, $(\bar{X}, \bar{P})$ is a generalized Nash equilibrium of the negotiation game iff:

$$
\begin{aligned}
& U_h(\bar{X}_h, \bar{X}_{-h}, \bar{P}) \ge U_h(X_h, \bar{X}_{-h}, \bar{P}), && \forall h \notin \mathcal{O},\ \forall X_h : (X_h, \bar{X}_{-h}, \bar{P}) \in K_2,\\
& \sum_{i\in\mathcal{O}} U_i(\bar{X}_{\mathcal{O}}, \bar{X}_{-\mathcal{O}}, \bar{P}) \ge \sum_{i\in\mathcal{O}} U_i(X_{\mathcal{O}}, \bar{X}_{-\mathcal{O}}, \bar{P}), && \forall X_{\mathcal{O}} : (X_{\mathcal{O}}, \bar{X}_{-\mathcal{O}}, \bar{P}) \in K_2,\\
& U^l(\bar{X}, \bar{P}^l, \bar{P}^{-l}) \ge U^l(\bar{X}, P^l, \bar{P}^{-l}), && \forall l = 1,\dots,L,\ \forall P^l : (\bar{X}, P^l, \bar{P}^{-l}) \in K_2,
\end{aligned}
$$

where $X_{-h} = (X_i)_{i\ne h}$, $X_{\mathcal{O}} = (X_i)_{i\in\mathcal{O}}$, $X_{-\mathcal{O}} = (X_i)_{i\notin\mathcal{O}}$ and $P^{-l} = (P^i)_{i\ne l}$.

It is well known that GNEPs may have infinite solutions, among which the so called *variational solutions* are of particular interest for their economic interpretation (see, e.g., [7]), and can be computed solving a suitable variational inequality. Specifically, Lagrange multipliers corresponding to the shared constraints are the same for each player. *Variational solutions* of the above GNEP can be obtained by solving the following variational inequality $VI(F_2, K_2)$: find $(X^*(s), P^*(s)) \in K_2$ such that

$$F_2(X^*(s), P^*(s))^\top [(X, P) - (X^*(s), P^*(s))] \geq 0, \qquad \forall (X, P) \in K_2, \tag{5}$$

where the map $F_2 : \mathbb{R}^{2HLD} \to \mathbb{R}^{2HLD}$ is defined by:

$$F_2(X, P) = -\Big(\nabla_{X_\mathcal{O}} \sum_{i \in \mathcal{O}} U_i(X, P),\ \nabla_{X_{-\mathcal{O}}} U_{-\mathcal{O}}(X, P), \nabla_{P^1} U^1(X, P), \ldots, \nabla_{P^L} U^L(X, P)\Big). \tag{6}$$

The notation $\nabla_{X_\mathcal{O}}$ means that the gradient is with respect to the group of variables of the HOs in the coalition, while with the notation $\nabla_{X_{-\mathcal{O}}} U_{-\mathcal{O}}(X, P)$ we denote the vector made up of the partial gradients of the utility functions of the HOs outside of the coalition.

In Sect. 3 it will be proved that the second-stage game has a unique variational equilibrium $(X^*(s), P^*(s))$ for any coalition chosen by HOs in the first stage and represented by the vector s.

Remark 2.1 The model described in this subsection differs from the negotiation model proposed in [9] in several respects. First, in [9] a purely non-cooperative approach is considered, in which no coalition between HOs is allowed. Furthermore, the equilibrium pricing mechanism is also different between the two approaches. In fact, we formulate the negotiation phase as a single GNEP between HOs and carriers, whereas in [9] the model is split into two different subgames: a GNEP only between HOs (where the prices p^*_{hld} chosen by carriers are fixed) and a Nash game only between carriers (where the quantities x^*_{hld} chosen by HOs are fixed). Subsequently, the authors in [9] consider two VIs: the first provides the variational equilibrium of the GNEP between HOs, and the second the Nash equilibrium of the game between carriers. Finally, they define the equilibrium (X^*, P^*) of the negotiation phase as the solution of the sum of the two previous VIs. However, we note that, in the case where no coalition is formed, the latter equilibrium coincides with the solution of $VI(F_2, K_2)$, i.e. the variational equilibrium of our GNEP model.

2.3 Third Stage: The Purchase and Distribution Game

In the third stage, after the humanitarian crisis occurs, HOs know the coalition $\mathcal{O}$ formed in the first stage, defined by the vector s, and the variational equilibrium $(X^*(s), P^*(s))$ of the second-stage game which represents the volume projected by HOs for transport and the prices chosen by carriers at equilibrium. In this phase, the HOs have to buy and distribute the relief items and can either exploit the framework agreements previously established with carriers, or find more convenient to pay the fares offered by the spot market.

Each HO h controls the variables y_{hld}, for any $l = 1, \ldots, L+1$ and $d = 1, \ldots, D$, which represent the quantity to be transported with carrier l to distribution point d. Y_h

denotes the vector of all the variables controlled by HO h, while Y the vector of the variables of all HOs.

Each HO h which does not take part in the coalition $\mathcal{O}$ aims to solve the following problem:

$$\max_{Y_h} \; V_h(Y) = \sum_{d=1}^{D} u_d \left[\sum_{l=1}^{L+1} y_{hld} - \frac{\alpha_h}{2} \sum_{l=1}^{L+1} y_{hld}^2 \right] + \omega_h^A \sum_{d=1}^{D} \sum_{l=1}^{L+1} i_{hd} \, y_{hld} \tag{7a}$$

$$\text{subject to} \; \sum_{l=1}^{L+1} \sum_{d=1}^{D} \left[c_h^p + p_{hld}^*(s) \right] y_{hld} \le B_h, \tag{7b}$$

$$y_{hld} \le x_{hld}^*(s), \qquad l = 1, \ldots, L, \; d = 1, \ldots, D, \tag{7c}$$

$$\sum_{i=1}^{H} y_{ild} \le K_{ld}, \qquad l = 1, \ldots, L+1, \; d = 1, \ldots, D, \tag{7d}$$

$$\sum_{i=1}^{H} \sum_{l=1}^{L+1} y_{ild} \le n_d, \qquad d = 1, \ldots, D, \tag{7e}$$

$$y_{hld} \ge 0, \qquad l = 1, \ldots, L+1, \; d = 1, \ldots, D. \tag{7f}$$

The utility function V_h is the weighted sum of its impact (first term) and its activity signal (second term), where the parameter ω_h^A describes the relative importance of the activity signal with respect to the impact. The impact is assessed according to the needs n_d of the recipients at the distribution point d and on the volume of items distributed by all the HOs, where u_d weights the urgency of the distribution point d and α_h is a saturation parameter. The activity signal term models the fact that, by its distribution campaign, an HO acquires visibility which will influence future donations, where i_{hd} quantifies the relative importance of point d for HO h.

Constraint (7b) guarantees that the budget B_h is met, where c_h^p is the unit purchase price, $p_{hld}^*(s)$, for $l = 1, \ldots, L$, are the equilibrium fares obtained during the negotiation phase, and $p_{h(L+1)d}^*$ are the fares fixed by the spot market. Constraints (7c) impose the quantities y_{hld} to be transported with carriers $l = 1, \ldots, L$ can not be greater than the equilibrium volume $x_{hld}^*(s)$ obtained during the negotiation phase. Constraints (7d), shared by all HOs (in the coalition and outside the coalition), ensure that the total quantity transported using carrier l to distribution point d does not exceed the carrier's capacity K_{ld}. The associated Lagrange multiplier, say λ_{hld}, represents the marginal value of increasing the carrier's capacity. In economic terms, λ_{hld} measures how much the objective function (the utility of the HOs) would improve if the carrier had one additional unit of capacity available. A positive λ_{hld} indicates that the capacity constraint is binding, meaning that the carrier's limited capacity is restricting the optimal allocation. If all HOs share the same Lagrange multiplier λ_{ld} for all l, d, it implies that the capacity constraints affect all HOs equally, meaning that the marginal utility of increasing carrier capacity is the same across different HOs. On the other hand, if the Lagrange multipliers were different, it would suggest that some HOs are more constrained than others. The equality thus describes a balanced or regulated transport market where capacity is fairly allocated.

Finally, constraints (7e), shared by all the HOs, ensure that the total amount of goods delivered to distribution point d does not exceed the needs n_d of the recipients. For each HO h, the associated Lagrange multiplier, say μ_{hd}, represents the marginal value of increasing the needs of recipients. In economic terms, μ_{hd} measures how much the utility of the HOs would improve if the demand at a distribution point increased by one unit. A positive μ_{hd} means that the constraint is binding, indicating that all the demand is met and that increasing the available supply would be beneficial. Consider now the case where all μ_{hd} are equal to some μ_d for all HOs. If all HOs share the same Lagrange multiplier μ_d for all d, this can be interpreted as a *uniform demand pressure*: the demand constraints affect all HOs equally, suggesting a balanced distribution system where all organizations face the same limitations in meeting demand. Equal multipliers thus suggest a regulated distribution effort. Furthermore, equal μ_d values indicate that no single HO has an advantage in meeting recipient needs, ensuring that relief efforts are evenly distributed.

On the other hand, each HO h which takes part in the coalition $\mathcal{O}$ aims to solve the following problem:

$$\max_{Y_h} \; V_h(Y) = \sum_{d=1}^{D} u_d \left[\sum_{l=1}^{L+1} y_{hld} - \frac{\alpha_h}{2} \sum_{l=1}^{L+1} y_{hld}^2 \right] + \omega_h^A \sum_{d=1}^{D} \sum_{l=1}^{L+1} i_{hd}\, y_{hld} \tag{8a}$$

$$\text{subject to} \sum_{i\in\mathcal{O}} \sum_{l=1}^{L+1} \sum_{d=1}^{D} [c_i^p + p_{ild}^*(s)]\, y_{ild} \le \sum_{i\in\mathcal{O}} B_i, \tag{8b}$$

$$\sum_{i\in\mathcal{O}} y_{ild} \le \sum_{i\in\mathcal{O}} x_{ild}^*(s), \qquad l = 1, \ldots, L, \; d = 1, \ldots, D, \tag{8c}$$

$$\sum_{i=1}^{H} y_{ild} \le K_{ld}, \qquad l = 1, \ldots, L+1, \; d = 1, \ldots, D, \tag{8d}$$

$$\sum_{i=1}^{H} \sum_{l=1}^{L+1} y_{ild} \le n_d, \qquad d = 1, \ldots, D, \tag{8e}$$

$$y_{hld} \ge 0, \qquad l = 1, \ldots, L+1, \; d = 1, \ldots, D. \tag{8f}$$

The utility function (8a) has the same form as (7a). Constraints (8b)–(8c) are similar to (7b)–(7c) but are shared by all the HOs in the coalition since they negotiated collectively with carriers, while (8d)–(8e) are shared by all the HOs. Specifically, constraints (8b) ensure that the total spending of all HOs in the coalition does not exceed their total budget. The associated Lagrange multiplier, say ν_i, represents the marginal value of increasing the budget for HO i within the coalition. A positive ν_i means that the budget constraint is binding for HO i. If all HOs in the coalition have the same ν_i, it follows that each HO in the coalition experiences the same marginal impact of budget increases and thus the coalition operates efficiently under a shared financial structure. Constraints (8c) ensure that the total quantity transported by all HOs in the coalition does not exceed the total projected volume from the second-stage game. The associated Lagrange multiplier, say τ_{hld}, represents the marginal value of increasing the transport volume available to members of the coalition. If all HOs in the coalition share the same τ_{hld}, it implies that all members experience the same marginal limitation on transport volume and no single HO disproportionately dominates the coalition's transport capacity. We can thus conclude that the coalition efficiently manages its shared transport constraints.

The third-stage game model is therefore a GNEP with shared constraint. Notice that this model is well-defined since the variational equilibrium $(X^*(s), P^*(s))$ of the second-stage game is unique. We denote the feasible region of this GNEP as

$$K_3 = \left\{ Y \in \mathbb{R}_+^{H(L+1)D} : (7b) \text{ and } (7c)\ \forall h \notin \mathcal{O}, (8b), (8c), (8d), (8e) \text{ hold} \right\}.$$

Definition 2.3 $\bar{Y}$ is a generalized Nash equilibrium of the purchase and distribution game iff:

$$V_h(\bar{Y}_h, \bar{Y}_{-h}) \geq V_h(Y_h, \bar{Y}_{-h}), \qquad \begin{array}{l} \forall h = 1, \ldots, H, \\ \forall Y_h \text{ such that } (Y_h, \bar{Y}_{-h}) \in K_3. \end{array}$$

Similar to the second stage, we can now search for variational solutions of the GNEP introduced above. Specifically, a variational solution of the purchase/distribution game is a solution of the following variational inequality $VI(F_3, K_3)$: find $Y^*(s) \in K_3$ such that

$$F_3(Y^*(s))^\top (Y - Y^*(s)) \geq 0, \qquad \forall Y \in K_3, \tag{9}$$

where $F_3 : \mathbb{R}^{H(L+1)D} \to \mathbb{R}^{H(L+1)D}$ is the operator defined by:

$$F_3(Y) = -\left(\nabla_{Y_1} V_1(Y), \ldots, \nabla_{Y_H} V_H(Y)\right).$$

Since each utility function V_h only depends on the variables Y_h, the GNEP admits as potential function the sum of the utility functions:

$$W(Y) := \sum_{h=1}^{H} V_h(Y),$$

that can be interpreted as the total welfare of all the HOs.

In Sect. 3 it will be proved that the third-stage game has a unique variational equilibrium $Y^*(s)$ for any coalition chosen by HOs in the first stage.

For the reader's convenience we summarize the main features of the three-stage model in Table 1, the variables in Table 2 and the parameters in Table 3.

Table 1 Main features of the three-stage model

	Game	Players	Optimization problems
First-stage	Finite	HOs	(1)
Second-stage	GNEP	HOs	(2) if $s_h = 0$; (3) if $s_h = 1$
		Carriers	(4)
Third-stage	Potential GNEP	HOs	(7) if $s_h = 0$; (8) if $s_h = 1$

Table 2 Variables of the three-stage model

Variables		
First stage	s_h	1 if HO h take part in the coalition, 0 otherwise
	$s \in \{0, 1\}^H$	Vector of all s_h
Second stage	x_{hld}	Volume *projected* by HO h for transport with carrier l to d
	$X_h \in \mathbb{R}_+^{LD}$	Vector of all x_{hld} for HO h
	$X \in \mathbb{R}_+^{HLD}$	Vector of all x_{hld}
	p_{hld}	Rate for transportation from HO h to d with carrier l
	$P^l \in \mathbb{R}_+^{HD}$	Vector of all p_{hld} for carrier l
	$P \in \mathbb{R}_+^{HLD}$	Vector of all p_{hld}
Third stage	y_{hld}	Volume *transported* by carrier l to d on behalf of HO h
	$Y_h \in \mathbb{R}_+^{(L+1)D}$	Vector of all y_{hld} for HO h
	$Y \in \mathbb{R}_+^{H(L+1)D}$	Vector of all y_{hld}

3 Theoretical Results

In this section we collect the theoretical results on the games of each model stage. Since the utility functions of HOs in the first-stage game are defined as a function of the (variational) equilibrium of the third-stage game, which in turn depends on the (variational) equilibrium of the second-stage game, we begin the analysis of equilibria from the second-stage game, then move on to the third-stage game and finally analyze the first-stage game.

3.1 Second-Stage Game

Since the set K_2 is compact and the map F_2 is continuous, $VI(F_2, K_2)$ admits at least one solution (see, e.g., [13]). A well-known sufficient condition for the uniqueness of the solution is the strict monotonicity property of the operator entering the variational inequality.

Definition 3.1 An operator $F : \mathbb{R}^m \to \mathbb{R}^m$ is said to be monotone iff

$$[F(x) - F(y)]^\top (x - y) \geq 0, \qquad \forall x, y \in \mathbb{R}^m.$$

Table 3 Parameters of the three-stage model

Parameters	
r_{hl}	Relative risk that HO h assigns to carrier l
ω_h^R	Relative importance of risk compared to cost for HO h
M_{hd}	HO h's volume of framework agreement for transport to d
G_l	Maximum transportation volume for carrier l
c_d^t	Unit transportation cost of each carrier for transport to d
ω_l^S	Relative importance of customer satisfaction w.r.t. profit for carrier l
$p_{hd}^{\max}$	Max price HO h is willing to pay for transport to d
$p_d^{\max}(\mathcal{O})$	Max price each HO in coalition $\mathcal{O}$ is willing to pay for transport to d
u_d	Urgency of distribution point d
α_h	Saturation parameter
i_{hd}	Relative importance of point d for HO h
ω_h^A	Relative importance of the activity signal w.r.t. the impact for HO h
B_h	Budget of HO h
c_h^p	Unit purchase cost of HO h
K_{ld}	Capacity of carrier l for transportation to d
n_d	Needs of beneficiaries at d
$p^*_{h(L+1)d}$	Price HO h has to pay on spot market for transport to d

F is said to be strictly monotone if in the definition above the equality holds only for $x = y$.

Theorem 3.1 *For any vector $s \in \{0,1\}^H$ defined in the first-stage game, the map F_2 defined in (6) is linear and strictly monotone on $\mathbb{R}^{2HLD}$. Therefore, the second-stage game has a unique variational equilibrium $(X^*(s), P^*(s))$ for any $s \in \{0,1\}^H$.*

Proof The first block of F_2 related to HOs is

$$
\begin{aligned}
&-\Big(\nabla_{X_{\mathcal{O}}} \sum_{i\in\mathcal{O}} U_i(X,P),\ \nabla_{X_{-\mathcal{O}}} U_{-\mathcal{O}}(X,P)\Big) \\
&\quad = -\Big(\nabla_{X_1} U_1(X,P), \ldots, \nabla_{X_H} U_H(X,P)\Big) \\
&\quad = \Big(p_{hld} + 2\,\omega_h^R\, r_{hl}\, x_{hld}\Big)_{hld},
\end{aligned}
$$

while the second block related to carriers is

$$
\begin{aligned}
&- \left(\nabla_{P^1} U^1(X, P), \ldots, \nabla_{P^L} U^L(X, P)\right) \\
&= \begin{cases} \left(-x_{hld} + \dfrac{2\,\omega_l^S\, M_{hd}}{(p_{hd}^{\max})^2}\, p_{hld}\right)_{hld} & \text{if } s_h = 0, \\ \left(-x_{hld} + \dfrac{2\,\omega_l^S\, M_{hd}}{(p_d^{\max}(\mathcal{O}))^2}\, p_{hld}\right)_{hld} & \text{if } s_h = 1. \end{cases}
\end{aligned}
$$

Hence, F_2 is a linear operator, which can be written as

$$
F_2(X, P) = A \begin{pmatrix} X \\ P \end{pmatrix},
$$

where A is a $2HLD \times 2HLD$ matrix. Since the terms $p_{hld}\, x_{hld}$ appear with opposite signs in the utility functions of HOs and carriers, the symmetric part of A results to be a diagonal matrix with positive entries, thus it is positive definite. Therefore, also the matrix A is positive definite, the operator F_2 is strictly monotone on $\mathbb{R}^{2HLD}$ and $VI(F_2, K_2)$ has a unique solution. □

We now show that, under suitable assumptions on the model parameters, it is possible to write the variational equilibrium $(X^*(s), P^*(s))$ in closed form. We start with a preliminary result about the best-reply strategies of carriers.

Lemma 3.1 *For any vector $s \in \{0, 1\}^H$ defined in the first-stage game, the best-reply strategy of each carrier $l = 1, \ldots, L$ to the strategy X of HOs, i.e., the optimal solution of problem (4), is*

$$
p_{hld}^* = \begin{cases} \min\left\{ p_{hd}^{\max},\ \max\left\{ c_d^t,\ \dfrac{(p_{hd}^{\max})^2\, x_{hld}}{2\omega_l^S\, M_{hd}} \right\}\right\} & \textit{if } s_h = 0, \\ \min\left\{ p_d^{\max}(\mathcal{O}),\ \max\left\{ c_d^t,\ \dfrac{(p_d^{\max}(\mathcal{O}))^2 \sum_{i\in\mathcal{O}} x_{ild}}{2\omega_l^S \sum_{i\in\mathcal{O}} M_{id}} \right\}\right\} & \textit{if } s_h = 1, \end{cases} \tag{10}
$$

for any $d = 1, \ldots, D$.

Proof It is easy to check that problem (4) is separable with respect to indices h and d. Hence, for any $d = 1, \ldots, D$, carrier l must solve the problem

$$
\max \quad -\frac{\omega_l^S M_{hd}}{(p_{hd}^{\max})^2}(p_{hld})^2 + p_{hld}\, x_{hld} + \omega_l^S\, M_{hd} - c_d^t\, x_{hld} \tag{11a}
$$

$$
\text{subject to } \; c_d^t \le p_{hld} \le p_{hd}^{\max}, \tag{11b}
$$

if $s_h = 0$, and the problem

$$
\max \quad -\frac{\omega_l^S \sum_{i\in\mathcal{O}} M_{id}}{(p_d^{\max}(\mathcal{O}))^2}(p_{hld})^2 + p_{hld} \sum_{i\in\mathcal{O}} x_{ild} + \omega_l^S \sum_{i\in\mathcal{O}} M_{id} - c_d^t \sum_{i\in\mathcal{O}} x_{ild} \tag{12a}
$$

$$\text{subject to } \; c_d^t \leq p_{hld} \leq p_d^{\max}(\mathcal{O}), \tag{12b}$$

if $s_h = 1$. Since (11) and (12) are quadratic optimization problems of one variable defined on a closed interval, it is easy to check that their optimal solution is given by (10). □

Let us now consider the best-reply strategies of HOs.

Lemma 3.2 *Suppose that parameters G_l are sufficiently large for any carrier $l = 1, \ldots, L$. Then, for any vector $s \in \{0, 1\}^H$ defined in the first-stage game, problem (2) that has to be solved by each HO h with $s_h = 0$ can be decomposed into the following D optimization problems, one for each $d = 1, \ldots, D$:*

$$\max \; \sum_{l=1}^{L} \left[-\omega_h^R \, r_{hl} \, x_{hld}^2 - p_{hld} \, x_{hld} \right] \tag{13a}$$

$$\text{subject to } \; \sum_{l=1}^{L} x_{hld} = M_{hd}, \tag{13b}$$

$$x_{hld} \geq 0, \qquad l = 1, \ldots, L, \tag{13c}$$

and problem (3) that has to be collectively solved by HOs in the coalition can be decomposed into the following D optimization problems, one for each $d = 1, \ldots, D$:

$$\max \; \sum_{i \in \mathcal{O}} \sum_{l=1}^{L} \left[-\omega_i^R \, r_{il} \, x_{ild}^2 - p_{ild} \, x_{ild} \right] \tag{14a}$$

$$\text{subject to} \sum_{i \in \mathcal{O}} \sum_{l=1}^{L} x_{ild} = \sum_{i \in \mathcal{O}} M_{id}, \tag{14b}$$

$$x_{ild} \geq 0, \qquad i \in \mathcal{O}, \; l = 1, \ldots, L. \tag{14c}$$

Proof If the capacities G_l are large enough, then constraints (2c) can be deleted from problem (2), that can thus be decomposed into the following D optimization problems:

$$\max \; \sum_{l=1}^{L} \left[-\omega_h^R \, r_{hl} \, x_{hld}^2 - p_{hld} \, x_{hld} \right] \tag{15a}$$

$$\text{subject to } \; \sum_{l=1}^{L} x_{hld} \geq M_{hd}, \tag{15b}$$

$$x_{hld} \geq 0, \qquad l = 1, \ldots, L. \tag{15c}$$

It is easy to check that constraint (15b) has to be satisfied as equality at the optimal solution, thus the thesis is proved. A similar argument can be applied to decompose problem (3). □

Notice that (13) and (14) have the same special structure, i.e., they are quadratic optimization problems where the objective function is the sum of quadratic functions of one variable and the feasible region is a simplex. The following result provides a simple procedure to find their optimal solution.

Lemma 3.3 *Consider the following quadratic optimization problem*

$$\min \sum_{i=1}^{n} \left(\frac{1}{2} a_i z_i^2 + b_i z_i \right) \tag{16a}$$

$$\text{subject to} \sum_{i=1}^{n} z_i = c, \tag{16b}$$

$$z_i \geq 0, \qquad i = 1, \ldots, n, \tag{16c}$$

where $c > 0$, $a_i > 0$ for any $i = 1, \ldots, n$, and b_i are supposed to be in non-decreasing order, i.e., $b_1 \leq b_2 \leq \cdots \leq b_n$. Let $b_{n+1} = +\infty$. Then, the optimal solution of problem (16) is

$$z_i^* = \begin{cases} \dfrac{\mu - b_i}{a_i} & \forall i = 1, \ldots, k, \\ 0 & \forall i = k+1, \ldots, n, \end{cases} \tag{17}$$

where $k \in \{1, \ldots, n\}$ is the index such that

$$\mu := \frac{c + \sum_{i=1}^{k} \frac{b_i}{a_i}}{\sum_{i=1}^{k} \frac{1}{a_i}} \in (b_k, b_{k+1}].$$

Proof Problem (16) is convex and admits a unique optimal solution z^* since the objective function is strictly convex. Hence, z^* is the unique solution of the corresponding Karush-Kuhn-Tucker system:

$$\begin{aligned} & a_i z_i + b_i - \mu - \lambda_i = 0, && i = 1, \ldots, n, \\ & \sum_{i=1}^{n} z_i = c, && \\ & \lambda_i z_i = 0, \quad z_i \geq 0, \quad \lambda_i \geq 0, && i = 1, \ldots, n, \end{aligned}$$

where μ and λ_i are the multiplier associated to constraints (16b) and (16c), respectively. Let $I_+ := \{i : z_i^* > 0\}$ and $I_0 = \{i : z_i^* = 0\}$. Notice that constraint (16b) implies $I_+ \neq \emptyset$. If $i \in I_+$, then $\lambda_i = 0$ and $0 < z_i^* = (\mu - b_i)/a_i$. Hence, $\mu > b_i$. Moreover, we have

$$c = \sum_{i=1}^{n} z_i^* = \sum_{i \in I_+} z_i^* = \sum_{i \in I_+} \frac{\mu - b_i}{a_i},$$

thus

$$\mu = \frac{c + \sum_{i \in I_+} \frac{b_i}{a_i}}{\sum_{i \in I_+} \frac{1}{a_i}}.$$

On the other hand, if $i \in I_0$, then $0 \leq \lambda_i = b_i - \mu$, hence $\mu \leq b_i$.

Since coefficients b_i are in non-decreasing order, we have $I_+ = \{1, \ldots, k\}$ and $I_0 = \{k+1, \ldots, n\}$, where k is the index such that $\mu \in (b_k, b_{k+1}]$. □

Lemma 3.3 guarantees that problem (16) can be solved by the following polynomial time algorithm.

Algorithm for solving problem (16).

Step 0. Set $k = 1$.
Step 1. Compute

$$\mu = \frac{c + \sum_{i=1}^{k} \frac{b_i}{a_i}}{\sum_{i=1}^{k} \frac{1}{a_i}}.$$

Step 2. If $\mu \in (b_k, b_{k+1}]$, then STOP (the optimal solution is given by (17)) else set $k = k + 1$ and go to step 1.

The following special case follows directly from Lemma 3.3.

Corollary 3.1 *If problem (16) is defined with $a_i = a$ and $b_i = b$ for any $i = 1, \ldots, n$, then the optimal solution is $z_i^* = c/n$ for any $i = 1, \ldots, n$.*

The next result shows that, under suitable assumptions on the parameters, the unique variational equilibrium of the second-stage game can be written in closed form.

Theorem 3.2 *Suppose that the model parameters satisfy the following conditions:*

a) G_l are sufficiently large for any carrier $l = 1, \ldots, L$,
b) $\omega_h^R = \omega^R$ for any $h = 1, \ldots, H$,
c) $r_{hl} = r$ for any $h = 1, \ldots, H$ and $l = 1, \ldots, L$,
d) $\omega_l^S = \omega^S$ for any $l = 1, \ldots, L$.

Then, for any vector $s \in \{0, 1\}^H$ defined in the first-stage game, the unique variational equilibrium $(X^*(s), P^*(s))$ of the second-stage game is

$$x_{hld}^*(s) = \begin{cases} \dfrac{M_{hd}}{L} & \text{if } s_h = 0, \\ \dfrac{\sum_{i \in \mathcal{O}} M_{id}}{L|\mathcal{O}|} & \text{if } s_h = 1, \end{cases} \tag{18}$$

$$p_{hld}^*(s) = \begin{cases} \min\left\{ p_{hd}^{\max},\ \max\left\{ c_d^t,\ \dfrac{(p_{hd}^{\max})^2}{2\,L\,\omega^S} \right\} \right\} & \text{if } s_h = 0, \\ \min\left\{ p_d^{\max}(\mathcal{O}),\ \max\left\{ c_d^t,\ \dfrac{(p_d^{\max}(\mathcal{O}))^2}{2\,L\,\omega^S} \right\} \right\} & \text{if } s_h = 1, \end{cases} \tag{19}$$

for any $h = 1, \ldots, H,\ l = 1, \ldots, L,\ d = 1, \ldots, D.$

Proof First, we remark that Lemma 3.2 implies that the second-stage game is a standard NEP since there are no shared constraints between HOs. In order to prove the thesis, we show that the HOs strategy $X^*(s)$ defined in (18) is the best-reply to the carriers strategy $P^*(s)$ defined in (19), and vice versa.

Lemma 3.1 and assumption d) guarantee that $P^*(s)$ defined in (19) is the best-reply strategy of each carrier to the HOs strategy $X^*(s)$ defined in (18).

On the other hand, Lemma 3.2 and assumptions b), c) guarantee that each HO h that does not take part in the coalition has to solve the following problems

$$\max \sum_{l=1}^{L} \left[-\omega^R\, r\, x_{hld}^2 - p_{hld}^*(s)\, x_{hld} \right] \tag{20a}$$

$$\text{subject to} \sum_{l=1}^{L} x_{hld} = M_{hd}, \tag{20b}$$

$$x_{hld} \geq 0, \qquad l = 1, \ldots, L, \tag{20c}$$

for any $d = 1, \ldots, D$. Since $p_{hld}^*(s)$ defined in (19) does not depend on the carrier l, problems (20) are of the same form as (16) in the special case considered in Corollary 3.1. Therefore, the optimal solution of (20) is given by (18). Moreover, Lemma 3.2 and assumptions b), c) guarantee that the HOs in the coalition have to solve collectively the following problems

$$\max \sum_{i \in \mathcal{O}} \sum_{l=1}^{L} \left[-\omega^R\, r\, x_{ild}^2 - p_{ild}^*(s)\, x_{ild} \right] \tag{21a}$$

$$\text{subject to} \sum_{i \in \mathcal{O}} \sum_{l=1}^{L} x_{ild} = \sum_{i \in \mathcal{O}} M_{id}, \tag{21b}$$

$$x_{ild} \geq 0, \qquad i \in \mathcal{O},\ l = 1, \ldots, L, \tag{21c}$$

for any $d = 1, \ldots, D$. Notice that the price $p_{ild}^*(s)$ defined in (19), for HOs in the coalition, does not depend on the HO i or the carrier l, hence also problems (21) are of the same form as (16) in the special case considered in Corollary 3.1. Thus, the optimal solution of (21) is given by (18).

Therefore, the strategy $X^*(s)$ defined in (18) is also the best-reply to the strategy $P^*(s)$ defined in (19), thus the unique equilibrium of the second-stage game is given by (18)–(19) □

The assumptions of Theorem 3.2 correspond to an ideal case, but allow the derivation of a closed-form solution, which can be useful to visually assess the sensitivity of the solution with respect to changing the parameters.

3.2 Third-Stage Game

In the next result we prove that the third-stage game has a unique variational equilibrium for any coalition chosen by HOs in the first stage and it coincides with the maximizer of the potential function. Moreover, we show that the total welfare computed at the variational equilibrium of the third-stage game obtains its maximum value when all HOs take part in the coalition at the first stage.

Theorem 3.3 *The following statements hold:*

a) For any vector $s \in \{0, 1\}^H$ defined in the first-stage game, the third-stage game has a unique variational equilibrium $Y^*(s)$ that coincides with the optimal solution of the problem

$$\max \; W(Y) \tag{22a}$$
$$\text{subject to } Y \in K_3. \tag{22b}$$

b) If the model parameters satisfy the assumptions of Theorem 3.2, then

$$W(Y^*(1, \dots, 1)) \geq W(Y^*(s)), \qquad \forall s \in \{0, 1\}^H. \tag{23}$$

Proof a) Since each utility function V_h only depends on the variable Y_h, we have $\nabla_{Y_h} W(Y) = \nabla_{Y_h} V_h(Y_h)$ for any $h = 1, \dots, H$. Hence, the map $F_3(Y) = -\nabla W(Y)$. Moreover, each V_h is a strictly concave quadratic function, thus W is strictly concave as well and the variational equilibrium coincides with the unique maximizer of W over the set K_3.

b) Let $\mathbf{1} = (1, \dots, 1)$ be the vector representing the grand coalition $\mathcal{H} = \{1, \dots, H\}$ of all the HOs, $s \in \{0, 1\}^H$ any strategy vector chosen in the first-stage game and $\mathcal{O} = \{h : s_h = 1\}$ the corresponding coalition. It follows from a) that $Y^*(\mathbf{1})$ is the optimal solution of the problem

$$\max_Y \; W(Y) \tag{24a}$$
$$\text{s.t.} \; \sum_{h=1}^{H} \sum_{l=1}^{L+1} \sum_{d=1}^{D} [c_h^p + p_{hld}^*(\mathbf{1})] \, y_{hld} \leq \sum_{h=1}^{H} B_h, \tag{24b}$$
$$\sum_{h=1}^{H} y_{hld} \leq \sum_{h=1}^{H} x_{hld}^*(\mathbf{1}), \qquad l = 1, \dots, L, \; d = 1, \dots, D, \tag{24c}$$
$$\sum_{h=1}^{H} y_{hld} \leq K_{ld}, \qquad l = 1, \dots, L+1, \; d = 1, \dots, D, \tag{24d}$$
$$\sum_{h=1}^{H} \sum_{l=1}^{L+1} y_{hld} \leq n_d, \qquad d = 1, \dots, D, \tag{24e}$$

$$y_{hld} \geq 0, \qquad \begin{array}{r} h = 1, \ldots, H, \\ l = 1, \ldots, L+1, \\ d = 1, \ldots, D, \end{array} \tag{24f}$$

while $Y^*(s)$ is the optimal solution of

$$\max_{Y} \; W(Y) \tag{25a}$$

$$\text{s.t.} \; \sum_{l=1}^{L+1} \sum_{d=1}^{D} \left[c_h^p + p_{hld}^*(s)\right] y_{hld} \leq B_h, \qquad h \notin \mathcal{O}, \tag{25b}$$

$$y_{hld} \leq x_{hld}^*(s), \qquad \begin{array}{r} h \notin \mathcal{O}, \\ l = 1, \ldots, L, \\ d = 1, \ldots, D, \end{array} \tag{25c}$$

$$\sum_{h \in \mathcal{O}} \sum_{l=1}^{L+1} \sum_{d=1}^{D} [c_h^p + p_{hld}^*(s)] y_{hld} \leq \sum_{h \in \mathcal{O}} B_h, \tag{25d}$$

$$\sum_{h \in \mathcal{O}} y_{hld} \leq \sum_{h \in \mathcal{O}} x_{hld}^*(s), \qquad \begin{array}{r} l = 1, \ldots, L, \\ d = 1, \ldots, D, \end{array} \tag{25e}$$

$$\sum_{h=1}^{H} y_{hld} \leq K_{ld}, \qquad \begin{array}{r} l = 1, \ldots, L+1, \\ d = 1, \ldots, D, \end{array} \tag{25f}$$

$$\sum_{h=1}^{H} \sum_{l=1}^{L+1} y_{hld} \leq n_d, \qquad d = 1, \ldots, D, \tag{25g}$$

$$y_{hld} \geq 0, \qquad \begin{array}{r} h = 1, \ldots, H \\ l = 1, \ldots, L+1, \\ d = 1, \ldots, D. \end{array} \tag{25h}$$

Theorem 3.2 guarantees that, for any $h = 1, \ldots, H$, $l = 1, \ldots, L+1$ and $d = 1, \ldots, D$, we have

$$x_{hld}^*(\mathbf{1}) = \frac{\sum_{i=1}^{H} M_{id}}{H\,L} \quad \text{and} \quad x_{hld}^*(s) = \begin{cases} \dfrac{M_{hd}}{L} & \text{if } h \notin \mathcal{O}, \\[2ex] \dfrac{\sum_{i \in \mathcal{O}} M_{id}}{L|\mathcal{O}|} & \text{if } h \in \mathcal{O}. \end{cases} \tag{26}$$

Moreover, for any $h \in \mathcal{O}$ the inequality

$$\begin{aligned} p_{hld}^*(\mathbf{1}) &= \min\left\{p_d^{\max}(\mathcal{H}),\; \max\left\{c_d^t,\; \frac{(p_d^{\max}(\mathcal{H}))^2}{2\,L\,\omega^S}\right\}\right\} \\ &\leq \min\left\{p_d^{\max}(\mathcal{O}),\; \max\left\{c_d^t,\; \frac{(p_d^{\max}(\mathcal{O}))^2}{2\,L\,\omega^S}\right\}\right\} = p_{hld}^*(s) \end{aligned} \tag{27}$$

holds for any $l = 1, \ldots, L+1$ and $d = 1, \ldots, D$, while the inequality

$$p^*_{hld}(\mathbf{1}) = \min\left\{p^{\max}_d(\mathcal{H}),\ \max\left\{c^t_d,\ \frac{(p^{\max}_d(\mathcal{H}))^2}{2\,L\,\omega^S}\right\}\right\} \\ \leq \min\left\{p^{\max}_{hd},\ \max\left\{c^t_d,\ \frac{(p^{\max}_{hd})^2}{2\,L\,\omega^S}\right\}\right\} = p^*_{hld}(s) \tag{28}$$

holds for any $h \notin \mathcal{O}$, $l = 1, \ldots, L+1$ and $d = 1, \ldots, D$.
We denote

$$K_{\mathbf{1}} := \left\{Y \in \mathbb{R}^{H(L+1)D} :\ \text{constraints (24}b\text{)} - -\text{(24}f\text{) hold}\right\}, \\ K_s := \left\{Y \in \mathbb{R}^{H(L+1)D} :\ \text{constraints (25}b\text{)} - -\text{(25}h\text{) hold}\right\}$$

the feasible regions of problems (24) and (25), respectively, and we prove that $K_s \subseteq K_{\mathbf{1}}$. Let $Y \in K_s$ be arbitrary. Then, (25d) and (27) imply that

$$\sum_{h\in\mathcal{O}}\sum_{l=1}^{L+1}\sum_{d=1}^{D}[c^p_h + p^*_{hld}(\mathbf{1})]y_{hld} \\ \leq \sum_{h\in\mathcal{O}}\sum_{l=1}^{L+1}\sum_{d=1}^{D}[c^p_h + p^*_{hld}(s)]y_{hld} \leq \sum_{h\in\mathcal{O}} B_h. \tag{29}$$

Moreover, (25b) and (28) imply that

$$\sum_{h\notin\mathcal{O}}\sum_{l=1}^{L+1}\sum_{d=1}^{D}\left[c^p_h + p^*_{hld}(\mathbf{1})\right]\, y_{hld} \\ \leq \sum_{h\notin\mathcal{O}}\sum_{l=1}^{L+1}\sum_{d=1}^{D}\left[c^p_h + p^*_{hld}(s)\right]\, y_{hld} \leq \sum_{h\notin\mathcal{O}} B_h. \tag{30}$$

If we sum inequalities (29) and (30), we get Y satisfies (24b). If we sum inequalities (25c), for $h \notin \mathcal{O}$, and (25e), we get from (26) the following chain of equalities and inequalities:

$$\sum_{h=1}^{H} y_{hld} \leq \sum_{h\in\mathcal{O}} x^*_{hld}(s) + \sum_{h\notin\mathcal{O}} x^*_{hld}(s) = \frac{\sum_{i\in\mathcal{O}} M_{id}}{L} + \frac{\sum_{i\notin\mathcal{O}} M_{id}}{L} \\ = \frac{\sum_{i=1}^{H} M_{id}}{L} = \sum_{i=1}^{H} x^*_{hld}(\mathbf{1}),$$

thus Y satisfies (24c) for any $l = 1, \ldots, L$ and $d = 1, \ldots, D$. Since (24d)–(24f) coincide with (25f)–(25h), we get $Y \in K_{\mathbf{1}}$, hence $K_s \subseteq K_{\mathbf{1}}$ and inequality (23) is proved.

□

3.3 First-Stage Game

Theorem 3.3 shows that, under suitable assumptions on the parameters, the grand coalition formed by all the HOs in the first stage is efficient from a social viewpoint because it maximizes the total welfare at the variational equilibrium of the third-stage game. On the other hand, the grand coalition might be unstable, that is the vector $(1, \ldots, 1)$ might be not a Nash equilibrium of the first-stage game (see Example 4.1) since some HOs might prefer to exit the grand coalition to increase their utility function. However, if the model parameters do not depend on the specific HO, then it is possible to prove that the grand coalition is a Nash equilibrium, as the following result shows.

Theorem 3.4 *Suppose that the model parameters satisfy the assumptions of Theorem 3.2 and the following conditions:*

a) $\alpha_h = \alpha$, $\omega_h^A = \omega^A$, $B_h = B$ *and* $c_h^p = c^p$ *for any* $h = 1, \ldots, H$,
b) $i_{hd} = i_d$, $M_{hd} = M_d$ *and* $p_{hd}^{\max} = p_d^{\max}$ *for any* $h = 1, \ldots, H$, $d = 1, \ldots, D$.

Then, $\mathbf{1} = (1, \ldots, 1)$ *is a Nash equilibrium of the first-stage game.*

Proof Let i be an arbitrary HO and $\tilde{s}$ the vector defined as

$$\tilde{s}_h = \begin{cases} 1 & \text{if } h \neq i, \\ 0 & \text{if } h = i, \end{cases}$$

that represent the coalition $\mathcal{O}$ formed by all HOs except i.

The assumptions and Theorem 3.3 imply that the variational equilibrium $Y^*(\mathbf{1})$ is the optimal solution of the following problem:

$$\max_{Y} \sum_{h=1}^{H} \sum_{l=1}^{L+1} \sum_{d=1}^{D} \left[-\frac{\alpha}{2}\, u_d\, y_{hld}^2 + u_d\, y_{hld} + \omega^A\, i_d\, y_{hld} \right] \tag{31a}$$

$$\text{subject to} \quad \sum_{h=1}^{H}\sum_{l=1}^{L+1}\sum_{d=1}^{D}[c^p + p^*_{hld}(\mathbf{1})]\, y_{hld} \le H\, B, \tag{31b}$$

$$\sum_{h=1}^{H} y_{hld} \le \sum_{h=1}^{H} x^*_{hld}(\mathbf{1}), \qquad \begin{array}{r} l = 1, \dots, L, \\ d = 1, \dots, D, \end{array} \tag{31c}$$

$$\sum_{h=1}^{H} y_{hld} \le K_{ld}, \qquad \begin{array}{r} l = 1, \dots, L+1, \\ d = 1, \dots, D, \end{array} \tag{31d}$$

$$\sum_{h=1}^{H}\sum_{l=1}^{L+1} y_{hld} \le n_d, \qquad d = 1, \dots, D, \tag{31e}$$

$$y_{hld} \ge 0, \qquad \begin{array}{r} h = 1, \dots, H, \\ l = 1, \dots, L+1, \\ d = 1, \dots, D, \end{array} \tag{31f}$$

where

$$x^*_{hld}(\mathbf{1}) = \left(\sum_{i=1}^{H} M_{id}\right) / (H\, L) = M_d / L \tag{32}$$

and

$$p^*_{hld}(\mathbf{1}) = \min\left\{ p_d^{\max},\ \max\left\{ c_d^t,\ \frac{(p_d^{\max})^2}{2\, L\, \omega^S} \right\} \right\} \tag{33}$$

hold for any $h = 1, \dots, H$, $l = 1, \dots, L+1$ and $d = 1, \dots, D$. Notice that the coefficients of variables y_{hld}, both in the objective function (31a) and constraints (31b)–(31f), do not depend on the indices h and l, thus $Y^*(\mathbf{1})$ satisfy the following condition

$$y^*_{hld}(\mathbf{1}) = y^*_{ild}(\mathbf{1})$$

for any $h = 1, \dots, H$, $l = 1, \dots, L+1$ and $d = 1, \dots, D$. Hence, we have

$$\sum_{h=1}^{H} y^*_{hld}(\mathbf{1}) = H\, y^*_{ild}(\mathbf{1}), \qquad \forall l = 1, \dots, L+1,\ \forall d = 1, \dots, D.$$

Therefore, constraint (31b) implies

$$\sum_{l=1}^{L+1}\sum_{d=1}^{D}[c^p + p^*_{hld}(\mathbf{1})]\, y^*_{hld}(\mathbf{1}) \le B, \qquad \forall h = 1, \dots, H, \tag{34}$$

and constraint (31c) implies

$$y^*_{hld}(\mathbf{1}) \le x^*_{hld}(\mathbf{1}), \qquad \begin{array}{l} \forall h = 1, \dots, H, \\ \forall l = 1, \dots, L+1, \\ \forall d = 1, \dots, D. \end{array} \tag{35}$$

Theorem 3.2 guarantees that

$$x^*_{hld}(\tilde{s}) = \begin{cases} \dfrac{M_d}{L} & \text{if } h = i, \\ \dfrac{\sum\limits_{i \in \mathcal{O}} M_{id}}{L|\mathcal{O}|} = \dfrac{M_d}{L} & \text{if } h \ne i, \end{cases}$$

$$p^*_{hld}(\tilde{s}) = \begin{cases} \min\left\{p_d^{\max},\ \max\left\{c_d^t,\ \dfrac{(p_d^{\max})^2}{2\,L\,\omega^S}\right\}\right\} & \text{if } h = i, \\ \min\left\{p_d^{\max}(\mathcal{O}),\ \max\left\{c_d^t,\ \dfrac{(p_d^{\max}(\mathcal{O}))^2}{2\,L\,\omega^S}\right\}\right\} & \text{if } h \ne i. \end{cases}$$

Since $p_d^{\max}(\mathcal{O}) = p_d^{\max}$, we get

$$x^*_{hld}(\tilde{s}) = x^*_{hld}(\mathbf{1}) \qquad \text{and} \qquad p^*_{hld}(\tilde{s}) = p^*_{hld}(\mathbf{1}) \tag{36}$$

hold for any $h = 1, \dots, H$, $l = 1, \dots, L+1$ and $d = 1, \dots, D$. Therefore, (34)–(36) guarantee that $Y^*(\mathbf{1})$ is a feasible solution of the problem

$$\max_{Y} \sum_{h=1}^{H} \sum_{l=1}^{L+1} \sum_{d=1}^{D} \left[-\frac{\alpha}{2}\, u_d\, y^2_{hld} + u_d\, y_{hld} + \omega^A\, i_d\, y_{hld} \right] \tag{37a}$$

$$\text{s.t.} \sum_{h \in \mathcal{O}} \sum_{l=1}^{L+1} \sum_{d=1}^{D} [c^p + p^*_{hld}(\tilde{s})]\, y_{hld} \le (H-1)\, B, \tag{37b}$$

$$\sum_{l=1}^{L+1} \sum_{d=1}^{D} [c^p + p^*_{ild}(\tilde{s})]\, y_{ild} \le B, \tag{37c}$$

$$\sum_{h \in \mathcal{O}} y_{hld} \le \sum_{h \in \mathcal{O}} x^*_{hld}(\tilde{s}), \qquad \begin{array}{r} l = 1, \dots, L, \\ d = 1, \dots, D, \end{array} \tag{37d}$$

$$y_{ild} \le x^*_{ild}(\tilde{s}), \qquad \begin{array}{r} l = 1, \dots, L, \\ d = 1, \dots, D, \end{array} \tag{37e}$$

$$\sum_{h=1}^{H} y_{hld} \le K_{ld}, \qquad \begin{array}{r} l = 1, \dots, L+1, \\ d = 1, \dots, D, \end{array} \tag{37f}$$

$$\sum_{h=1}^{H} \sum_{l=1}^{L+1} y_{hld} \le n_d, \qquad d = 1, \dots, D, \tag{37g}$$

$$y_{hld} \geq 0, \qquad \begin{array}{r} h = 1, \ldots, H, \\ l = 1, \ldots, L+1, \\ d = 1, \ldots, D. \end{array} \tag{37h}$$

Since problem (31) is a relaxation of problem (37) (see proof of Theorem 3.3), we get $Y^*(\mathbf{1})$ is also optimal for (37), hence $Y^*(\mathbf{1}) = Y^*(\tilde{s})$. Therefore, we have

$$T_i(\tilde{s}) = V_i(Y^*(\tilde{s})) = V_i(Y^*(\mathbf{1})) = T_i(\mathbf{1}),$$

that is HO i has no incentive to leave the grand coalition. Since the index i is arbitrary, we proved that $\mathbf{1}$ is a Nash equilibrium of the first-stage game. □

4 Numerical Experiments

We now illustrate the model described in the previous sections with some numerical experiments. Two examples based on synthetic data are provided. The choice of parameters was partially inspired by [9] concerning the order of magnitude. However, we tried to vary some significant parameters such as HOs' budget and carriers' transportation capacities in order to describe a somewhat realistic situation and differ from our previous closed form results. Example 4.1 shows that the grand coalition is not a Nash equilibrium in the coalition formation game, while it is in Example 4.2.

Example 4.1 We consider three HOs ($H = 3$), two carriers ($L = 2$) and two distribution points ($D = 2$). In the first phase, the three HOs wish to sign framework agreements covering a volume of $M_{1d} = 500$, $M_{2d} = 1000$, $M_{3d} = 2500$ tons per distribution point, respectively. The carriers want to limit the total volume signed to $G_1 = 2000$ and $G_2 = 6000$ tons, respectively. Assume that the transportation cost $c_d^t = 0.2$ k€/ton for each carrier and distribution point, while the reservation price $p_{hd}^{\max} = 0.9$ k€/ton for each HO, carrier and distribution point. Moreover, the relative risk $r_{hl} = 1$/ton for any HO h and carrier l, the relative importance of risk compared to costs $\omega_h^R = 0.2$ k€/ton for any HO h and the relative importance of satisfaction compared to profit $\omega_l^S = 0.4$ k€/ton for any carrier l. In the second phase, the three HOs have a budget $B_1 = 1000$, $B_2 = 2000$, $B_3 = 5000$ k€, respectively. Each HO h has a purchase price $c_h^p = 0.7$ k€/ton, a saturation parameter $\alpha_h = 0.001$/ton and a relative importance of impact with respect to activity signal $\omega_h^A = 1$ k€/ton. The two carriers offer a transportation capacity $K_{1d} = 2000$ and $K_{2d} = 6000$ tons for each distribution point, respectively. The spot market transportation cost $p_{h,L+1,d} = 0.8$ k€/ton for each HO h and distribution point d. Each distribution point d has needs $n_d = 5000$ tons, a relative urgency $u_d = 1$ k€/ton, and a relative importance $i_{hd} = 1$ for any HO h.

For each possible coalition between HOs, given by the binary vector s, we numerically computed the variational equilibrium $(X^*(s), P^*(s))$ of the second-stage game and the variational equilibrium $Y^*(s)$ of the third-stage game. Since variational inequality (5) to be solved has an affine and (strictly) monotone map and a polyhedral feasible region, we reformulated it as an equivalent convex quadratic optimization problem (see [1]). Indeed, given a map $F : \mathbb{R}^n \to \mathbb{R}^n$ and a closed convex set $K \subseteq \mathbb{R}^n$, the well-known gap function approach (see, e.g. [2, 21]) guarantees that a vector $z^* \in K$ solves the variational inequality $VI(F, K)$, i.e.,

$$F(z^*)^\top (y - z^*) \geq 0, \qquad \forall y \in K,$$

if and only if z^* is a minimizer of the gap function

$$g(z) = \max_y \left\{ F(z)^\top (z - y) : \ y \in K \right\}$$

over the set K with $g(z^*) = 0$. If F is affine and monotone, i.e., $F(z) = Pz + r$, where $P \in \mathbb{R}^{n \times n}$ is a positive semidefinite matrix and $r \in \mathbb{R}^n$, and K is a polyhedral set, i.e., $K = \{y \in \mathbb{R}^n : \ Ay \leq b\}$, where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, then the duality theory of linear programming allows us to rewrite the gap function as follows:

$$\begin{aligned} g(z) &= \max_y \left\{ (Pz + r)^\top (z - y) : \ Ay \leq b \right\} \\ &= z^\top (Pz + r) + \max_y \left\{ (-Pz - r)^\top y : \ Ay \leq b \right\} \\ &= \frac{1}{2} z^\top (P + P^\top) z + r^\top z + \min_\lambda \left\{ b^\top \lambda : \ A^\top \lambda = -Pz - r, \ \lambda \geq 0 \right\}. \end{aligned}$$

Therefore, z^* solves $VI(F, K)$ if and only if there exists a vector $\lambda^* \in \mathbb{R}^m$ such that (z^*, λ^*) is an optimal solution of the following convex quadratic optimization problem

$$\begin{aligned} \min_{(z,\lambda)} \quad & \frac{1}{2} z^\top (P + P^\top) z + r^\top z + b^\top \lambda \\ \text{subject to} \quad & Pz + A^\top \lambda + r = 0, \\ & Az \leq b, \\ & \lambda \geq 0, \end{aligned}$$

and its optimal value is equal to zero. The latter optimization problem and problem (22) to be solved to find the variational equilibrium of the third-stage game were solved by means of the MATLAB function `quadprog` from the optimization toolbox. Computations were implemented in MATLAB R2024a and tested on an Apple M1 Max running under macOS 15.0.

Table 4 shows, for any possible coalition between HOs, the values of the social welfare $W(Y^*(s))$ at the variational equilibrium, the total volume transported to all the distribution points and the corresponding average need fulfillment, defined as

$$\frac{\sum_{h=1}^{H} \sum_{l=1}^{L+1} \sum_{d=1}^{D} y^*_{hld}(s)}{\sum_{d=1}^{D} n_d}.$$

Table 4 Social welfare at equilibrium, total volume transported to distribution points and corresponding average need fulfillment for any possible coalition between HOs

Coalition	Social welfare at equilibrium (k€)	Total volume transported to distribution points (tons)	Average need fulfillment
None	10420.89	6030.52	60.31%
{1, 2}	10353.11	6002.06	60.02%
{1, 3}	11112.03	6103.66	61.04%
{2, 3}	10885.54	6097.02	60.97%
{1, 2, 3}	11199.63	6122.01	61.22%

Table 5 Variational equilibria of the second-stage and third-stage games when no coalition between HOs is formed

	Second-stage game							
	x^*_{hld}				p^*_{hld}			
	Carrier 1		Carrier 2		Carrier 1		Carrier 2	
	$d = 1$	$d = 2$	$d = 1$	$d = 2$	$d = 1$	$d = 2$	$d = 1$	$d = 2$
HO 1	0	0	500	500	0.2000	0.2000	0.9000	0.9000
HO 2	125	125	875	875	0.2000	0.2000	0.8859	0.8859
HO 3	875	875	1625	1625	0.3544	0.3544	0.6581	0.6581

	Third-stage game					
	y^*_{hld}					
	Carrier 1		Carrier 2		Spot market	
	$d = 1$	$d = 2$	$d = 1$	$d = 2$	$d = 1$	$d = 2$
HO 1	0	0	103.95	103.95	222.45	222.45
HO 2	125	125	241.27	241.27	336.57	336.57
HO 3	875	875	627.17	627.17	483.76	483.76

The grand coalition involving all the HOs turns out to be the one both with the highest value of social welfare at equilibrium and the maximum total volume transported to the distribution points. However, the grand coalition is not stable because it is not a Nash equilibrium of the first-stage game. In fact, the following relations hold:

$$T_1(1, 1, 1) = 3733.21 > 1245.32 = T_1(0, 1, 1),$$
$$T_2(1, 1, 1) = 3733.21 > 2718.58 = T_2(1, 0, 1),$$
$$T_3(1, 1, 1) = 3733.21 < 6543.43 = T_3(1, 1, 0),$$

that is the first two HOs would have no advantage in leaving the grand coalition, while the third (which has a greater budget than the other two) would benefit from leaving.

For a more detailed analysis, Tables 5 and 6 report the variational equilibria of the

Table 6 Variational equilibria of the second-stage and third-stage games when all the HOs are in the coalition

Second-stage game

	x^*_{hld}				p^*_{hld}			
	Carrier 1		Carrier 2		Carrier 1		Carrier 2	
	$d=1$	$d=2$	$d=1$	$d=2$	$d=1$	$d=2$	$d=1$	$d=2$
HO 1	333.33	333.33	1000	1000	0.2531	0.2531	0.7594	0.7594
HO 2	333.33	333.33	1000	1000	0.2531	0.2531	0.7594	0.7594
HO 3	333.33	333.33	1000	1000	0.2531	0.2531	0.7594	0.7594

Third-stage game

	y^*_{hld}					
	Carrier 1		Carrier 2		Spot market	
	$d=1$	$d=2$	$d=1$	$d=2$	$d=1$	$d=2$
HO 1	333.33	333.33	366.24	366.24	320.76	320.76
HO 2	333.33	333.33	366.24	366.24	320.76	320.76
HO 3	333.33	333.33	366.24	366.24	320.76	320.76

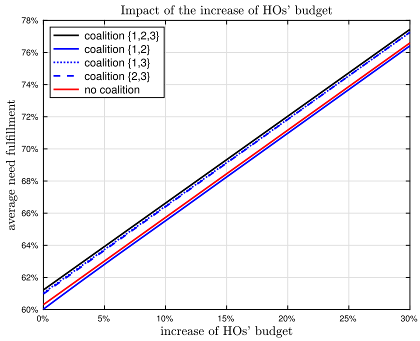
second-stage and third-stage games in the case of empty coalition or grand coalition, respectively.

In the case of an empty coalition (Table 5), in the first phase any HO negotiates with each carrier independently: HO 3 (the largest one) is able to get good prices from both carriers; HO 2 (the intermediate one) signs a framework agreements with carrier 1 at the minimum rate for a small target volume while the price set by carrier 2 is close to its maximum price; HO 1 (the smallest one) is not able to get a framework agreement with carrier 1 and it is forced to pay the highest possible price to the carrier 2. In the second phase, HO 2 and 3 ship with carrier 1 all the quantity previously decided in the framework agreement; HO 1 and 2 prefer to use spot market over carrier 2 because of the lower price, while HO 3 prefer carrier 2 to the spot market. The total volume transported by the HOs to all the distribution points is about 6030 tons.

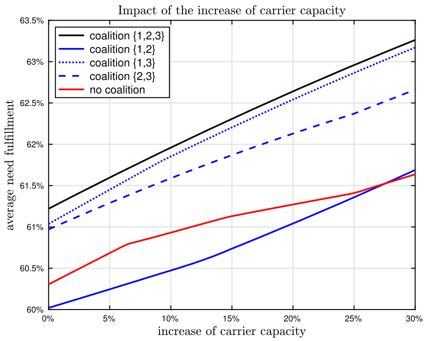
Instead, in the case of the grand coalition (Table 6), HOs collectively negotiate with each carrier in the first phase: they get a very good price from carrier 1 (the smallest one) and a price lower than the spot market price from carrier 2. In the second phase, they take full advantage of the framework agreement with carrier 1 and ship a larger volume with carrier 2 than the spot market. Overall, the total volume transported by the HOs to all the distribution points is about 6122 tons.

In order to analyze which model parameters are most important and how to improve the impact of disaster relief, we performed some numerical tests by varying the values of the transportation cost c^t_d, purchase price c^p_h, spot market transportation cost $p_{h,L+1,d}$, budget of HOs B_h and transportation capacities G_l and K_{ld} of carriers.

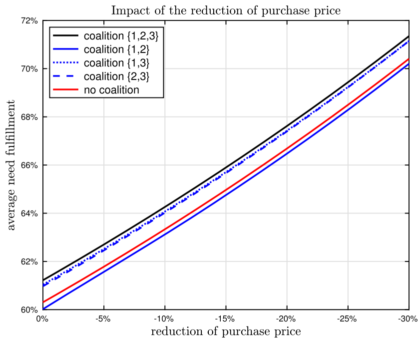
Figure 3 shows how the reduction or increasing of the above parameters has an impact on the average need fulfillment at equilibrium, for any possible coalition between HOs. Results suggest that the reduction of transportation cost has no significant impact, the reduction of the spot market price and the increase of capacity of carrier has a limited impact, while the reduction of purchase price and the increase



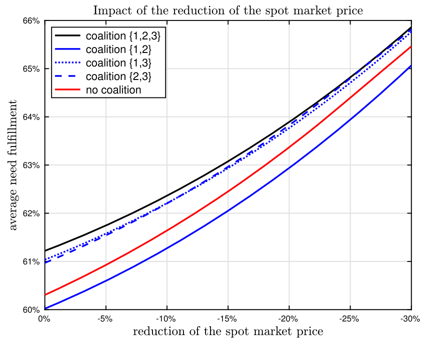
(a)



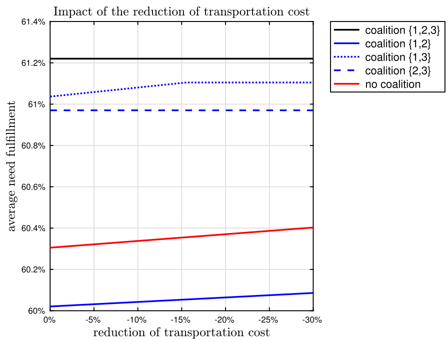
(b)



(c)



(d)



(e)

Fig. 3 Impact of the increase of HOs' budget (**a**), increase of carrier capacity (**b**), reduction of purchase price (**c**), reduction of spot market price (**d**), reduction of transportation cost (**e**) on the average need fulfillment

Table 7 Social welfare at equilibrium, total volume transported to distribution points and corresponding average need fulfillment for any possible coalition between HOs for Example 4.2

Coalition	Social welfare at equilibrium (k€)	Total volume transported to distribution points (tons)	Average need fulfillment
none	4381.11	2352.16	23.52%
$\{1, 2\}$	4294.46	2293.82	22.94%
$\{1, 3\}$	5021.58	2716.34	27.16%
$\{2, 3\}$	4775.41	2667.86	26.68%
$\{1, 2, 3\}$	5126.50	2743.56	27.44%

of budget of HOs have a major impact to improve the need fulfillment of distribution points. Finally, notice that the coalition involving all HOs provides an average need fulfillment always greater than the other coalitions.

Example 4.2 We consider the same situation described in Example 4.1 except each HO h has a budget $B_h = 1000$ k€. Again, the grand coalition is the most efficient in terms of social welfare and total volume transported, as shown in Table 7. Moreover, contrary to Example 4.1, the grand coalition is also a Nash equilibrium of the first-stage game. Indeed, we have

$$\begin{aligned} T_1(1, 1, 1) &= 1708.83 > 1245.32 = T_1(0, 1, 1), \\ T_2(1, 1, 1) &= 1708.83 > 1534.10 = T_2(1, 0, 1), \\ T_3(1, 1, 1) &= 1708.83 > 1651.81 = T_3(1, 1, 0), \end{aligned}$$

i.e., no HO would have any advantage in leaving the grand coalition.

5 Conclusions

In this paper we considered a three-stage game theory model of supply chain in humanitarian operations. In the first stage the humanitarian organizations (HOs) must decide which coalition to form among themselves to negotiate framework agreements with the carriers. This is a finite game where each HO has only two strategies: to enter into a coalition with other HOs or not. In the second stage the actual negotiation takes place: carriers and HOs are the players of a generalized Nash game whose solution are the projected volumes of relief items and the associated transportation rates. Since the kind of crises considered can be approximately foreseen, the negotiation stage takes place a few years early. The variational equilibrium of the second-stage game is then used as input data for the third-stage game, which describes the actual purchase and distribution of relief items. This latter game is potential GNEP with a unique variational equilibrium. We proved that, under suitable assumption on the model parameters, it is possible to write the variational equilibrium of the second-stage game in closed form and the grand coalition between HOs formed in the first stage results to be the most efficient in terms of social welfare computed at the equilibrium of the third stage.

Moreover, under further assumptions on parameters, we proved the grand coalition is stable, i.e., is a Nash equilibrium of the first-stage game. Some preliminary numerical experiments confirm our theoretical results. Humanitarian logistic models under uncertainty have been described in [26]. Since the second stage is based on an estimate of the needs that will be required when the crises actually unfolds, it could be interesting, to further refine the model, to introduce some data via their probability distributions, and solve the model within the theory of stochastic variational inequalities (see, e.g. [22, 23]). Another possible future development may be the modeling of the negotiation phase between HOs and carriers as a bilevel game in order to analyze the advantages and disadvantages compared to a one-level game model.

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Data Availability The datasets generated and/or analyzed during the current study are available from the corresponding author on reasonable request.

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