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WEAK TYPE ESTIMATES FOR SPHERICAL MULTIPLIERS ON NONCOMPACT SYMMETRIC SPACES

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Abstract. In this paper we prove sharp weak type 1 estimates for spherical Fourier multipliers on symmetric spaces of the noncompact type. This complements earlier results of J.-Ph. Anker and A.D. Ionescu.

1. Introduction

A celebrated result of L. Hörmander [17] states that if B is a translation invariant bounded operator on $L^2(\mathbb{R}^n)$ and the Fourier transform m_B of its convolution kernel satisfies the Mihlin type conditions

$$\left|D^I m_B(\xi)\right| \leq C\, |\xi|^{-|I|} \qquad \forall \xi \in \mathbb{R}^n \setminus \{0\} \tag{1.1}$$

for all multiindices I of length $\leq [\![n/2]\!]+1$, then B extends to an operator bounded on $L^p(\mathbb{R}^n)$ for all p in $(1,\infty)$, and of weak type 1. The operator B is usually referred to as the *Fourier multiplier operator* associated to the *multiplier* m_B.

J.-Ph. Anker [1, 2] and A.D. Ionescu [19, 20] proved analogues of this result for spherical multiplier operators in the setting of Riemannian symmetric spaces of noncompact type. The starting point of our investigation is the observation that these results do not apply to some important operators like the resolvent of the Laplace–Beltrami operator on the symmetric space and, in the higher rank case, its purely imaginary powers. We shall prove sharp weak type 1 estimates for a comparatively wide class of spherical Fourier multiplier operators that include the aforementioned operators. Our main result, Theorem 3.10 below, is formulated in terms of a new condition, which allows the multiplier to be unbounded.

Denote by G a noncompact semisimple Lie group with finite centre, by K a maximal compact subgroup of G, and by X the symmetric space of the noncompact type G/K. We denote by n and ℓ the dimension and the rank of X, respectively. Denote by θ a Cartan involution of the Lie algebra $\mathfrak{g}$ of G, and write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and denote by $\mathfrak{a}^*$ its dual space and by $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of $\mathfrak{a}^*$. Denote by Σ the set of (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$; a choice for the set of positive roots is written Σ^+, and $\mathfrak{a}^+$ denotes the corresponding Weyl chamber. The vector ρ denotes $(1/2)\sum_{\alpha\in\Sigma^+} m_\alpha\, \alpha$, where m_α is the multiplicity of α. We denote by Σ_s the set of simple roots in Σ^+ and by Σ_0^+ the set of indivisible positive roots.

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Denote by W the Weyl group of (G,K) and by $\mathbf{W}$ the interior of the convex hull of the points $\{w\cdot\rho : w\in W\}$. Clearly $\mathbf{W}$ is an open convex polyhedron in $\mathfrak{a}^*$. Recall that the Killing form $B(\cdot\,,\cdot)$ is a nondegenerate bilinear form on $\mathfrak{g}$ that is positive definite when restricted to $\mathfrak{a}$. This induces an inner product on $\mathfrak{a}^*$, and we denote by $|\cdot|$ the associated norm. Sometimes we shall use co-ordinates on $\mathfrak{a}^*$. When we do, we always refer to the co-ordinates associated to the orthonormal basis $\varepsilon_1,\ldots,\varepsilon_{\ell-1},\rho/|\rho|$, where $\varepsilon_1,\ldots,\varepsilon_{\ell-1}$ is any orthonormal basis of $\rho^\perp$. In particular, for each multiindex $I=(i_1,\ldots,i_\ell)$, we denote by D^I the partial derivative $\partial^{|I|}/\partial_1^{i_1}\cdots\partial_\ell^{i_\ell}$ with respect to these co-ordinates.

It is well known that (G,K) is a Gelfand pair, i.e. the convolution algebra $L^1(K\backslash G/K)$ of all K–bi-invariant functions in $L^1(G)$ is commutative. The spectrum of $L^1(K\backslash G/K)$ is the closure $\overline{T}_{\mathbf{W}}$ in $\mathfrak{a}^*_{\mathbb{C}}$ of the tube $T_{\mathbf{W}}=\mathfrak{a}^*+i\mathbf{W}$. Denote by $\widetilde{f}$ the Gelfand transform (also referred to as the spherical Fourier transform, or the Harish-Chandra transform in this setting) of the function f in $L^1(K\backslash G/K)$. It is known that $\widetilde{f}$ is a bounded continuous function on $\overline{T}_{\mathbf{W}}$, holomorphic in $T_{\mathbf{W}}$ and invariant under the Weyl group W. The Gelfand transform extends to K–bi-invariant tempered distributions on G (see, for instance, [11, Ch. 6.1]).

For each q in $[1,\infty)$, denote by ${}^G\mathcal{B}^q(X)$ the Banach algebra of all G invariant bounded linear operators on $L^q(X)$, endowed with the operator norm. It is well known that B is in ${}^G\mathcal{B}^2(X)$ if and only if there exists a K–bi-invariant tempered distribution k_B on G such that $\widetilde{k}_B$ is a bounded Weyl invariant function on $\mathfrak{a}^*$ and

$$Bf=f*k_B\qquad\forall f\in L^2(X)$$

(see [11, Prop. 1.7.1 and Ch. 6.1] for details). We call k_B the *kernel* of B. We denote its spherical Fourier transform $\widetilde{k}_B$ by m_B and call it the *spherical multiplier* associated to B.

For the rest of the Introduction we assume that B is in ${}^G\mathcal{B}^2(X)$. Clearly if B is in ${}^G\mathcal{B}^2(X)$ and is of weak type 1, then B is in ${}^G\mathcal{B}^q(X)$ for all q in $(1,\infty)$. As a consequence of a well known result of J.L. Clerc and E.M. Stein [7], m_B is a Weyl invariant holomorphic function in $T_{\mathbf{W}}$, bounded on closed subtubes thereof. Note, however, that m_B may be unbounded on $T_{\mathbf{W}}$. Indeed, denote by $\mathcal{L}$ the Laplace–Beltrami operator on X, and, for each complex number α such that $0\le\operatorname{Re}\alpha\le 2$, consider the operator $\mathcal{L}^{-\alpha/2}$, spectrally defined. Then $\mathcal{L}^{-\alpha/2}$ is of weak type 1 [2, 3], and

$$m_{\mathcal{L}^{-\alpha/2}}(\zeta)=Q(\zeta)^{-\alpha/2}\qquad\forall\zeta\in T_{\mathbf{W}}$$

is unbounded near the vertices of the polygon $0+i\mathbf{W}$, in particular near $i\rho$. Here Q denotes the Gelfand transform of $\mathcal{L}$ (see (2.12) and (2.13) below). We emphasise that a similar phenomenon cannot occur in the Euclidean case. Indeed, if B is a translation invariant bounded operator on $L^2(\mathbb{R}^n)$, then the associated Fourier multiplier m_B is necessarily bounded on $\mathbb{R}^n$, i.e., on the spectrum of the convolution algebra $L^1(\mathbb{R}^n)$.

Given a multiindex (I',i_ℓ) in $\mathbb{N}^\ell$, where I' is in $\mathbb{N}^{\ell-1}$ and i_ℓ is in $\mathbb{N}$, denote by $|I'|$ the length of I'. For each κ in $[0,\infty)$ consider the following anisotropic condition on the multiplier m_B:

$$\text{(1.2)}\qquad \big|D^{(I',i_\ell)}m_B(\zeta)\big|\le\frac{C}{\min\big(|Q(\zeta)|^{\kappa+i_\ell+|I'|/2},|Q(\zeta)|^{(i_\ell+|I'|)/2}\big)}\qquad\forall\zeta\in T_{\mathbf{W}^+},$$

for all (I', i_ℓ) with $|I'| + i_\ell \leq [\![n/2]\!] + 1$ ($[\![\cdot]\!]$ denotes the integer part function). The set $T_{\mathbf{W}^+}$ is defined in Section 2. Recall that the derivatives $D^{(I', i_\ell)} m_B$ are taken with respect to special co-ordinates on $\mathfrak{a}^*$.

Our main result, Theorem 3.10, states that if m_B satisfies (1.2) and either κ is in $[0,1)$, or κ is 1 and B is a spectral multiplier of $\mathcal{L}$, then B is of weak type 1, and it is bounded on $L^p(X)$ for all p in $(1,\infty)$.

One of the points of strength of Theorem 3.10 is that it allows m_B to be unbounded on $T_{\mathbf{W}}$. Indeed, if m_B satisfies (1.2) and κ is in $(0,1]$, then m_B may be unbounded in each neighbourhood of every point where Q vanishes, in particular in every neighbourhood of $i\rho$. To the best of our knowledge this is the first time that such singular behaviour of the multiplier appears in the literature. Theorem 3.10 (ii) is sharp, and it is strong enough to give the weak type 1 boundedness of $\mathcal{L}^{-\alpha/2}$ for all complex numbers α with $0 \leq \operatorname{Re}\alpha \leq 2$. Note also that the weak type 1 estimate for $\mathcal{L}^{-\alpha/2}$ is derived in [2, 3] from sharp estimates for the heat kernel. It is unlikely that a similar strategy applies to more general multipliers.

To prove Theorem 3.10, we observe that the kernel k_B may be written as the sum of a local part k_B^0, which has compact support near the origin and satisfies Hörmander type integral conditions, and a part at infinity k_B^∞, which is bounded. A standard procedure reduces the problem of proving weak type 1 estimates for the convolution operator $f \mapsto f * k_B^0$ to a similar problem where f is an $L^1(X)$ function supported near the origin. Since k_B^0 satisfies a Hörmander type integral condition, the weak type 1 estimate for $f \mapsto f * k_B^0$ follows from the general theory of singular integrals on spaces of homogeneous type in the sense of Coifman and Weiss [8, 22]. To treat the convolution operator $f \mapsto f * k_B^\infty$, we first obtain pointwise estimates of k_B^∞ and then use a celebrated result of J.-O. Strömberg [23] to show that the operator $f \mapsto f * k_B^\infty$ is indeed of weak type 1. To obtain pointwise estimates for k_B^∞ we use delicate asymptotic expansions for the spherical functions, due to P.C. Trombi and V.S. Varadarajan [25], which have recently been used in [3] to obtain sharp estimates for the heat kernel on X. An interesting consequence of the anisotropy of (1.2) is that the pointwise estimates for k_B^∞ are expressed in terms of an anisotropic homogeneous norm on $\mathfrak{a}$.

It is worth observing that the proof of Theorem 3.10 does not use the cancellations that k_B may have at infinity. This may appear surprising, but it is, in fact, a natural consequence of Strömberg's result, which has no Euclidean analogue and which roughly says that if a bounded K–bi-invariant convolution kernel has a certain prescribed decay at infinity, then the corresponding convolution operator is of weak type 1. This is in striking contrast with the problem of $L^p(X)$ multipliers considered by Ionescu [18, 19], where cancellations of k_B at infinity play a key role. See below for more on this.

We now compare (1.2) with other conditions in the literature. Anker [1], following up earlier results of M. Taylor [24] and J. Cheeger, M. Gromov and Taylor [6] for manifolds with bounded geometry, proved that if m_B satisfies Mihlin type conditions of the form

$$\left|D^I m_B(\zeta)\right| \leq C \left(1 + |\zeta|\right)^{-|I|} \qquad \forall \zeta \in T_{\mathbf{W}} \tag{1.3}$$

for every multiindex I such that $|I| \leq [\![n/2]\!] + 1$, then the operator B is of weak type 1. This extends previous results concerning special classes of symmetric spaces [7, 21, 4].

Anker's result was complemented by A. Carbonaro, G. Mauceri and Meda [5], who showed that if m_B satisfies (1.3), then B is bounded from the Hardy space $H^1(X)$ to $L^1(X)$ and from $L^\infty(X)$ to the space $BMO(X)$ of functions of bounded mean oscillation on X (see [5] for the definition of these spaces). The space $BMO(X)$ has already been defined in the rank one case in [18], where an interesting application to oscillatory multipliers is given.

Observe that $|Q(\zeta)|^{1/2}$ is equivalent to $|\zeta|$ as ζ tends to infinity within the tube $T_{\mathbf{W}}$. Therefore (1.2) and (1.3) are equivalent conditions at infinity. However, they are clearly nonequivalent in each neighbourhood of $i\rho$ for every κ in $[0,1]$. This is reflected in the diverse behaviour of the corresponding kernels at infinity. In fact, if m_B satisfies (1.3), then k_B^∞ is in $L^1(X)$, whereas this may fail if m_B satisfies (1.2). Thus, compared to Theorem 3.10, these results are of a somewhat "local" nature, because the problem of proving weak type estimates for B is reduced to the analogous problem for the operator $f \mapsto f * k_B^0$.

In this paper we prove another multiplier result, which may be formulated in terms of a variant of a condition recently introduced by Ionescu [19, 20] to study $L^p(X)$ boundedness of spherical Fourier multipliers. For p in $(1,2)$, define $\mathbf{W}_p$ and $T_{\mathbf{W}_p}$ by $\mathbf{W}_p = (2/p-1)\,\mathbf{W}$ and $T_{\mathbf{W}_p} = \mathfrak{a}^* + i\mathbf{W}_p$, and the function $d_p : \overline{T}_{\mathbf{W}_p} \to [0,\infty)$ by

$$d_p(\xi+i\eta) = \big[|\xi|^2 + \operatorname{dist}(\eta, \mathbf{W}_p^c)^2\big]^{1/2} \qquad \forall \xi \in \mathfrak{a}^* \quad \forall \eta \in \overline{\mathbf{W}}_p.$$

Ionescu proved that if p is in $(1,2)$ and m_B satisfies the conditions

$$\big|D^I m_B(\zeta)\big| \leq C\, d_p(\zeta)^{-|I|} \qquad \forall \zeta \in T_{\mathbf{W}_p} \tag{1.4}$$

for every multiindex I such that $|I| \leq N$, where N is a sufficiently large integer, then B is bounded on $L^q(X)$ for all q in $[p,p']$. Observe that the derivatives of m_B may be unbounded in every neighbourhood of each point in $0 + i\,\partial\mathbf{W}_p$. Note also that, by the Clerc and Stein condition, if B is bounded on $L^p(X)$, then m_B must be bounded on $T_{\mathbf{W}_p}$.

The proof of this result hinges on an elegant transference result proved by Ionescu in [19, 20], which is a sharp variant of Herz's *principe de majoration* and reduces the problem of finding estimates for the operator B on the symmetric space to a similar problem for a related operator on a Euclidean space. To prove that this transferred operator is bounded on L^p, Ionescu shows that its kernel satisfies Calderón–Zygmund type estimates. This forces the kernel of the original operator B on the symmetric space to have cancellations at infinity.

In Section 3, we consider an extension of Ionescu's condition (1.4). Define the function $d : \overline{T}_{\mathbf{W}} \to [0,\infty)$ by

$$d(\xi+i\eta) = \big[|\xi|^2 + \operatorname{dist}(\eta, \mathbf{W}^c)^2\big]^{1/2} \qquad \forall \xi \in \mathfrak{a}^* \quad \forall \eta \in \overline{\mathbf{W}}. \tag{1.5}$$

We prove that if κ is in $[0,1)$, N is a sufficiently large integer, and m_B satisfies

$$\big|D^I m_B(\zeta)\big| \leq \frac{C}{\min\big(d(\zeta)^{\kappa+|I|}, d(\zeta)^{|I|}\big)} \qquad \forall \zeta \in T_{\mathbf{W}} \tag{1.6}$$

for every multiindex I such that $|I| \leq N$, then the operator B is of weak type 1. Note that in the case where $\kappa = 0$, condition (1.6) reduces to (1.4) with $T_{\mathbf{W}}$ in place of $T_{\mathbf{W}_p}$.

The proof of this result follows the strategy of the proof of Therem 3.10. We just observe that k_B^∞ is not necessarily in $L^1(X)$. Thus, to prove weak type 1 estimates for the convolution operator $f \mapsto f * k_B^\infty$, we first obtain pointwise estimates of k_B^∞, by routine adaptations of techniques developed by Ionescu in [20], and then use [23] to show that the operator $f \mapsto f * k_B^\infty$ is indeed of weak type 1. It is worth pointing out that our proof uses neither the aforementioned transference principle of Ionescu nor cancellations of the kernel k_B at infinity.

Observe that conditions (1.6) and (1.2) are equivalent at infinity, because both $d(\zeta)$ and $|Q(\zeta)|^{1/2}$ are equivalent to $|\zeta|$ as ζ tends to infinity within the tube $T_{\mathbf{W}}$. If $\ell = 1$, then conditions (1.6) and (1.2) are equivalent. However, if $\ell \geq 2$, then they are nonequivalent near $0 + i\partial\mathbf{W}$, and condition (1.6) is not satisfied by the multiplier associated to $\mathcal{L}^{-\alpha/2}$ for all complex α with $0 \leq \operatorname{Re}\alpha \leq 2$ (see Remark 3.3 for details).

Conditions similar to (1.2) on tubes $T_{\mathbf{W}_p}$ may be considered, and corresponding weak or strong type p estimates for spherical multipliers may be proved. To keep the length of this paper reasonable we shall postpone the detailed study of operators satisfying these conditions to a forthcoming paper.

Our paper is organised as follows. Section 2 contains some notation and terminology, and also some preliminary results. In Section 3 we define certain function spaces that appear in the statement of our main result and state Theorem 3.10. Sections 4 and 5 are quite technical. In Section 4 we discuss weak type 1 estimates for two families of convolution operators (see formula (4.1)), which are relevant in the proof of Theorem 3.10. Section 5 is devoted to estimating the kernel k_B when m_B satisfies (1.2). The proof of Theorem 3.10 hinges on the results of Sections 4 and 5, and is given in Section 6.

We will use the "variable constant convention", and denote by C, possibly with sub- or superscripts, a constant that may vary from place to place and may depend on any factor quantified (implicitly or explicitly) before its occurrence, but not on factors quantified afterwards.

2. Notation and background material

We use the standard notation of the theory of Lie groups and symmetric spaces, as in the books of Helgason [14, 15]. We shall also refer to the book [11] and to the paper [3].

In addition to the notation above, denote by $\mathfrak{n}$ the subalgebra $\sum_{\alpha\in\Sigma^+}\mathfrak{g}_\alpha$ of $\mathfrak{g}$. By N, $\overline{N}$, A, and K we denote the subgroups of G corresponding to $\mathfrak{n}$, $\theta\mathfrak{n}$, $\mathfrak{a}$, and $\mathfrak{k}$, respectively, and we write $G = KAN$ and $G = \overline{N}AK$ for the associated Iwasawa decompositions. Given λ in $\mathfrak{a}^*$, define H_λ to be the unique element in $\mathfrak{a}$ such that

$$B(H_\lambda, H) = \lambda(H) \qquad \forall H \in \mathfrak{a},$$

and then an inner product $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{a}^*$ by the rule

$$\langle\lambda, \lambda'\rangle = B(H_\lambda, H_{\lambda'}) \qquad \forall \lambda, \lambda' \in \mathfrak{a}^*.$$

We abuse the notation and denote by $|\cdot|$ both the norms associated to the inner products $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}^*$ and $B(\cdot,\cdot)$ on $\mathfrak{a}$. The inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{a}^*$ extends to a bilinear form, also denoted $\langle\cdot\,,\cdot\rangle$, on $\mathfrak{a}^*_{\mathbb{C}}$. For any R in $\mathbb{R}^+$ define

$$\mathbf{B}_R = \{\lambda \in \mathfrak{a}^* : |\lambda| < R\}. \tag{2.1}$$

The ball $\mathbf{B}_{|\rho|}$ will occur frequently in the analysis of functions of the Laplace–Beltrami operator. For notational convenience, we shall write $\mathbf{B}$ instead of $\mathbf{B}_{|\rho|}$.

If H is in $\mathfrak{a}$, we write $(H_1, \ldots, H_\ell)$ for the vector of its co-ordinates with respect to the dual basis of the basis $\varepsilon_1, \ldots, \varepsilon_{\ell-1}, \rho/|\rho|$ of $\mathfrak{a}^*$ defined in the Introduction. Observe that the last vector of this dual basis is $H_\rho/|H_\rho|$. Sometimes we shall write H' instead of $(H_1, \ldots, H_{\ell-1})$. Define $\mathcal{N} : \mathfrak{a} \to \mathbb{R}$ by

$$\mathcal{N}(H', H_\ell) = \big(|H'|^4 + H_\ell^2\big)^{1/4}. \tag{2.2}$$

Note that $\mathcal{N}$ is homogeneous with respect to the dilations $(H', H_\ell) \mapsto (\varepsilon H', \varepsilon^2 H_\ell)$ and that the homogeneous dimension of $\mathfrak{a}$, endowed with the (quasi) metric induced by $\mathcal{N}$, is $\ell + 1$. Suppose that R is in $\mathbb{R}^+$. Define

$$\mathbf{b}_R = \{H \in \mathfrak{a} : \mathcal{N}(H', H_\ell) < R\}. \tag{2.3}$$

Define the parabolic region $\mathbf{p}$ in $\mathfrak{a}$ by

$$\mathbf{p} = \{H \in \mathfrak{a} : |H'| < H_\ell^{1/2}\}. \tag{2.4}$$

Define the functions $\omega : \mathfrak{a} \to \mathbb{R}$ and $\omega^* : \mathfrak{a}^* \to \mathbb{R}$ by

$$\omega(H) = \min_{\alpha \in \Sigma_s} \alpha(H) \qquad \forall H \in \mathfrak{a} \qquad \text{and} \qquad \omega^*(\lambda) = \min_{\alpha \in \Sigma_s} \langle \alpha, \lambda \rangle \qquad \forall \lambda \in \mathfrak{a}^*. \tag{2.5}$$

Furthermore for each c in $\mathbb{R}^+$, define the subset $\mathbf{s}_c$ of $\overline{\mathfrak{a}^+}$ by

$$\mathbf{s}_c = \{H \in \mathfrak{a} : 0 \le \omega(H) \le c\}. \tag{2.6}$$

Denote by $\big(\mathfrak{a}^*\big)^+$ the interior of the fundamental domain of the action of the Weyl group W that contains ρ. For any subset $\mathbf{E}$ of $\mathfrak{a}^*$ denote by $T_{\mathbf{E}}$ the tube over $\mathbf{E}$, i.e., the set $\mathfrak{a}^* + i\mathbf{E}$ in the complexified space $\mathfrak{a}^*_{\mathbb{C}}$, and by $\overline{T}_{\mathbf{E}}$ its closure in $\mathfrak{a}^*_{\mathbb{C}}$. For each t in $\mathbb{R}$ we denote by $\mathbf{E}^t$ the set

$$\mathbf{E}^t = \{\lambda \in \mathbf{E} : \ \omega^*(\lambda) > t\}. \tag{2.7}$$

Note that if E is open, then $\mathbf{E}^0$ is the interior of $\overline{(\mathfrak{a}^*)^+} \cap \mathbf{E}$. For simplicity, we shall write $\mathbf{E}^+$ instead of $\overline{(\mathfrak{a}^*)^+} \cap \mathbf{E}$. Notice that $\mathbf{W}^+$ is neither open nor closed in $\mathfrak{a}^*$, whereas for each t in $\mathbb{R}^-$ the set $\mathbf{W}^t$ is an open neighbourhood of $\mathbf{W}^+$ that contains the origin. Thus, $T_{\mathbf{W}^t}$ is a neighbourhood of $T_{\mathbf{W}^+}$ in $\mathfrak{a}^*_{\mathbb{C}}$ that contains $\mathfrak{a}^* + i0$.

We write $\mathrm{d}x$ for a Haar measure on G, and we let $\mathrm{d}k$ be the Haar measure on K of total mass one. We identify functions on the symmetric space X with right–K–invariant functions on G, in the usual way. If $E(G)$ denotes a space of functions on G, we define $E(K\backslash X)$ and $E(X)$ to be the closed subspaces of $E(G)$ of the K–bi-invariant and the right–K–invariant functions, respectively. The Haar measure of G induces a G–invariant measure $\mathrm{d}\dot{x}$ on X for which

$$\int_X f(\dot{x}) \, \mathrm{d}\dot{x} = \int_G f(x) \, \mathrm{d}x \qquad \forall f \in C_c(X),$$

where $\dot{x} = xK$. We recall that

$$\int_G f(x)\,\mathrm{d}x = \int_K \int_{\mathfrak{a}^+} \int_K f\big(k_1 (\exp H) k_2\big)\, \delta(H)\,\mathrm{d}k_1\,\mathrm{d}H\,\mathrm{d}k_2,$$

where $\mathrm{d}H$ denotes a suitable nonzero multiple of the Lebesgue measure on $\mathfrak{a}$, and

$$\delta(H) = \prod_{\alpha \in \Sigma^+} \sinh^{m_\alpha}\big(\alpha(H)\big) \le C\,\mathrm{e}^{2\rho(H)} \qquad \forall H \in \mathfrak{a}^+.$$

For any a in A we denote by $\log a$ the element H in $\mathfrak{a}$ such that $\exp H = a$. For any x in G, we denote by $H(x)$ the unique element of $\mathfrak{a}$ such that x is in $K \exp H(x) N$.

Thus, $H(kan) = \log a$. For any λ in $\mathfrak{a}^*_{\mathbb{C}}$, the elementary spherical function φ_λ is defined by the rule

$$\varphi_\lambda(x) = \int_K \exp[(i\lambda - \rho)H(xk)]\,\mathrm{d}k \qquad \forall x \in G.$$

The spherical transform $\widetilde{f}$, also denoted by $\mathcal{H}f$, of an $L^1(G)$-function f is defined by

$$\widetilde{f}(\lambda) = \int_G f(x)\,\varphi_{-\lambda}(x)\,\mathrm{d}x \qquad \forall \lambda \in \mathfrak{a}^*.$$

Harish-Chandra's inversion formula and Plancherel's formula state that

$$f(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\,\varphi_\lambda(x)\;\mathrm{d}\mu(\lambda) \qquad \forall x \in G$$

for "nice" K-bi-invariant functions f on G, and

$$\|f\|_2 = \left[\int_{\mathfrak{a}^*} \left|\widetilde{f}(\lambda)\right|^2\,\mathrm{d}\mu(\lambda)\right]^{1/2} \qquad \forall f \in L^2(K\backslash G/K),$$

where $\mathrm{d}\mu(\lambda) = c_{{}_G}\,|\mathbf{c}(\lambda)|^{-2}\,\mathrm{d}\lambda$ and $\mathbf{c}$ denotes the Harish-Chandra $\mathbf{c}$-function. For the details, see, for instance, [14, IV.7]. Sometimes we shall write $\mathcal{H}^{-1}$ for the inverse Fourier transform. The Harish-Chandra $\mathbf{c}$-function is given by

$$\mathbf{c}(\lambda) = \prod_{\alpha\in\Sigma_0^+} c_\alpha(\langle\alpha,\lambda\rangle),$$

where each Plancherel factor c_α is given by an explicit formula involving several Γ-functions [14, Thm 6.14]. It is well known that

$$|\mathbf{c}(\lambda)|^{-2} \le C\left(1+|\lambda|\right)^{\sum_{\alpha\in\Sigma_0^+} d_\alpha} \le C\left(1+|\lambda|\right)^{n-\ell}, \tag{2.8}$$

where $d_\alpha = \dim\mathfrak{g}_\alpha + \dim\mathfrak{g}_{2\alpha}$. We denote by $\check{\mathbf{c}}$ the function $\check{\mathbf{c}}(\lambda) = \mathbf{c}(-\lambda)$ which is holomorphic in $T_{\mathbf{W}^t}$ for some negative t and satisfies the following estimate:

$$\left|(\check{\mathbf{c}})^{-1}(\zeta)\right| \le C \prod_{\alpha\in\Sigma_0^+} \left(1+|\zeta|\right)^{\sum_{\alpha\in\Sigma_0^+} d_\alpha/2} \le C\left(1+|\zeta|\right)^{(n-\ell)/2} \qquad \forall \zeta \in T_{\mathbf{W}^t}.$$

This, the analyticity of $(\check{\mathbf{c}})^{-1}$ on $T_{\mathbf{W}^t}$, and Cauchy's integral formula imply that for every multiindex I

$$\left|D^I(\check{\mathbf{c}})^{-1}(\zeta)\right| \le C\left(1+|\zeta|\right)^{(n-\ell)/2} \qquad \forall \zeta \in T_{\mathbf{W}^t}. \tag{2.9}$$

Now, we describe the various faces of $\overline{\mathfrak{a}^+}$ which are in one-to-one correspondence with the nontrivial subsets F of Σ_s. We denote by $(\Sigma_F)^+$ the positive root subsystem generated by F and by $(\Sigma_F)_0^+$ the positive indivisible roots in $(\Sigma_F)^+$. Then we may write

$$\mathfrak{a} = \mathfrak{a}_F \oplus \mathfrak{a}^F, \qquad \mathfrak{a}^* = \mathfrak{a}^*_F \oplus (\mathfrak{a}^*)^F, \qquad \mathfrak{n} = \mathfrak{n}_F \oplus \mathfrak{n}^F \qquad \text{and} \qquad N = N_F N^F,$$

where $\mathfrak{a}_F$ is the subspace generated by the vectors $\{H_\alpha : \alpha \in F\}$, $\mathfrak{a}^F$ denotes its orthogonal complement in $\mathfrak{a}$, $\mathfrak{a}^*_F$ is the subspace of $\mathfrak{a}^*$ generated by F, $(\mathfrak{a}^*)^F$ denotes its orthogonal complement in $\mathfrak{a}^*$, $\mathfrak{n}_F = \bigoplus_{\alpha\in(\Sigma_F)^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^F = \bigoplus_{\alpha\in\Sigma^+\setminus(\Sigma_F)^+} \mathfrak{g}_\alpha$. The face $(\mathfrak{a}^F)^+$ of $\overline{\mathfrak{a}^+}$ attached to F is

$$(\mathfrak{a}^F)^+ = \{H \in \mathfrak{a}^F : \alpha(H) > 0 \quad \forall \alpha \in \Sigma_s \setminus F\}.$$

We shall write $H = H_F + H^F$ and $\lambda = \lambda_F + \lambda^F$ according to the decompositions $\mathfrak{a} = \mathfrak{a}_F \oplus \mathfrak{a}^F$ and $\mathfrak{a}^* = \mathfrak{a}_F^* \oplus (\mathfrak{a}^*)^F$, respectively. In particular, $\rho = \rho_F + \rho^F$. Observe that $\ell = \ell_F + \ell^F$, where ℓ_F and ℓ^F denote the dimensions of $\mathfrak{a}_F$ and $\mathfrak{a}^F$, respectively.

We denote by Λ the lattice $\sum_{\alpha\in\Sigma_s} \mathbb{N}\alpha$. Observe that $\Lambda = \Lambda_F + \Lambda^F$, where $\Lambda_F = \sum_{\alpha\in F} \mathbb{N}\alpha$ and $\Lambda^F = \sum_{\alpha\in\Sigma_s\setminus F} \mathbb{N}\alpha$, and

$$\mathbf{c} = \mathbf{c}_F\ \mathbf{c}^F,$$

where

$$\mathbf{c}_F(\lambda) = \prod_{\alpha\in(\Sigma_F)_0^+} c_\alpha(\langle\alpha,\lambda\rangle) \qquad \text{and} \qquad \mathbf{c}^F(\lambda) = \prod_{\alpha\in\Sigma_0^+\setminus(\Sigma_F)_0^+} c_\alpha(\langle\alpha,\lambda\rangle).$$

We shall often use the following estimates:

$$\begin{aligned} |\mathbf{c}_F(\lambda)|^{-2} &\leq C\,\big(1+|\lambda|\big)^{\sum_{\alpha\in(\Sigma_F)_0^+} d_\alpha}, \\ \big|D_\lambda^I(\check{\mathbf{c}}^F)^{-1}(\lambda)\big| &\leq C\,\big(1+|\lambda|\big)^{\sum_{\alpha\in\Sigma_0^+\setminus(\Sigma_F)_0^+} d_\alpha/2}, \end{aligned} \tag{2.10}$$

and for every multiindex I

$$|\mathbf{c}_F(\lambda)|^{-1}\,\big|D_\lambda^I(\check{\mathbf{c}}^F)^{-1}(\lambda)\big| \leq C\,\big(1+|\lambda|\big)^{n-\ell}. \tag{2.11}$$

We denote by P_F the normalizer of N^F in G; it has the Langlands decomposition $P_F = M_F(\exp\mathfrak{a}^F)N^F$, where M_F and $M^F = M_F(\exp\mathfrak{a}^F)$ are closed subgroups of G. We denote by ω^{*F} and ω_F^* the functions defined by

$$\omega^{*F}(\lambda) = \min_{\alpha\in\Sigma_s\setminus F}\langle\alpha,\lambda\rangle \qquad \text{and} \qquad \omega_F^*(\lambda) = \min_{\alpha\in F}\langle\alpha,\lambda\rangle \qquad \forall\lambda\in\mathfrak{a}^*.$$

The height of an element $q = \sum_{\alpha\in\Sigma_s} n_\alpha\alpha$ in Λ is defined by $|q| = \sum_{\alpha\in\Sigma_s} n_\alpha$. The asymptotic expansion of the spherical functions along the walls of the Weyl chamber is due to Trombi and Varadarajan [25, Thm 2.11.2] (see also [11, Thm 5.9.4]). For the reader's convenience we state the following variant of [25, Thm 2.11.2], due to Anker and Ji [3, Theorem 2.2.8].

Theorem 2.1. *Suppose that X is a symmetric space of the noncompact type. Suppose that F is a nontrivial subset of Σ_s, that λ is regular and that H is in $\overline{\mathfrak{a}^+}$ with $\omega^F(H) > 0$. We have an asymptotic expansion*

$$\varphi_\lambda(\exp H) \sim \mathrm{e}^{-\rho^F(H)} \sum_{q\in\Lambda^F}\ \sum_{w\in W_F\setminus W} \mathbf{c}^F(w\cdot\lambda)\,\varphi^F_{w\cdot\lambda,q}(\exp H),$$

where

(i) *$\varphi^F_{\lambda,0}$ is the spherical function of index λ on $M^F = M_F \exp\mathfrak{a}^F$ and*

$$\varphi^F_{\lambda,0}(x) = \varphi^F_{\lambda_F}(y)\,\mathrm{e}^{i\lambda^F(H)} \qquad \forall x = y\exp H \in M_F\exp\mathfrak{a}^F;$$

(ii) *$\varphi^F_{\lambda,q}$ are bi-K_F-invariant C^∞ functions in the variable $x \in M^F$ and W_F-invariant holomorphic functions in the variable λ in the region*

$$\{\lambda = \lambda_F + \lambda^F \in \mathfrak{a}^*_{\mathbb{C}} : |\mathrm{Im}\,\lambda_F| < c,\ \omega^{*F}(\mathrm{Im}\,\lambda^F) > -c\},$$

for some small positive c; moreover,

$$\varphi^F_{\lambda,q}(x) = \varphi^F_{\lambda,q}(y)\,\mathrm{e}^{(i\lambda-q)(H)} \qquad \forall x = y\exp H \in M_F\exp\mathfrak{a}^F;$$

(iii) *for every q in Λ^F there exists a constant $d \geq 0$, and for every positive c there exists a constant $C \geq 0$ such that*
$$\left|\varphi^F_{\lambda,q}(\exp H)\right| \leq C\,\mathrm{e}^{c\,|H_F|}\,(1+|\lambda|)^d\,\mathrm{e}^{-[\mathrm{Im}(\lambda)+\rho_F+q](H)}$$
for all λ in $\mathfrak{a}^* + i\overline{((\mathfrak{a}^*)^F)^+}$ and for all H in $\overline{\mathfrak{a}^+}$;

(iv) *for every positive integer N there exists a constant $d \geq 0$, and for every positive c there exists a constant $C \geq 0$ such that*
$$\Big|\varphi_\lambda(\exp H) - \mathrm{e}^{-\rho^F(H)} \sum_{q\in\Lambda^F,\,|q|<N}\ \sum_{w\in W_F\backslash W} \mathbf{c}^F(w\cdot\lambda)\,\varphi^F_{w\cdot\lambda,q}(\exp H)\Big|$$
$$\leq C\,(1+|\lambda|)^d\,(1+|H|)^d\,\mathrm{e}^{-\rho(H)-N\omega^F(H)}$$
for $\omega^F(H) > c$.

Denote by $\mathfrak{z}(F)$ the centraliser of $\mathfrak{a}^F$ in $\mathfrak{g}$, by $\mathfrak{g}_F$ its derived algebra $[\mathfrak{z}(F),\mathfrak{z}(F)]$, and by $\mathfrak{k}_F$ the algebra $\mathfrak{k}\cap\mathfrak{g}_F$. Then $\mathfrak{g}_F$ is semisimple. Denote by G_F and K_F the connected Lie groups corresponding to the algebras $\mathfrak{g}_F$ and $\mathfrak{k}_F$. Then K_F is a maximal compact subgroup of G_F, and G_F/K_F is a symmetric space of the noncompact type and real rank $|F|$, also denoted by X_F (see [12, pp. 16-20] and the references therein). The restriction to $\exp\mathfrak{a}_F$ of the elementary spherical function $\varphi^F_{\lambda_F}$ coincides with the restriction to $\exp\mathfrak{a}_F$ of the elementary spherical function on the symmetric space X_F associated to λ_F.

Denote by $\mathcal{L}_0$ minus the Laplace–Beltrami operator on X associated to the metric given by the Killing form on $\mathfrak{g}$: $\mathcal{L}_0$ is a symmetric operator on $C^\infty_c(X)$ (the space of smooth complex-valued functions on X with compact support). Its closure is a self-adjoint operator on $L^2(X)$ that we denote by $\mathcal{L}$. It is known that the bottom of the $L^2(X)$ spectrum of $\mathcal{L}$ is $\langle\rho,\rho\rangle$. Note that

$$\mathcal{L}\varphi_\lambda = Q(\lambda)\,\varphi_\lambda \qquad \forall\lambda\in\mathfrak{a}^*_{\mathbb{C}}, \tag{2.12}$$

where Q is the quadratic function on $\mathfrak{a}^*_{\mathbb{C}}$ defined by

$$Q(\zeta) = \langle\zeta,\zeta\rangle + \langle\rho,\rho\rangle \qquad \forall\zeta\in\mathfrak{a}^*_{\mathbb{C}}. \tag{2.13}$$

The operator $\mathcal{L}$ generates a symmetric diffusion semigroup $\{\mathcal{H}^t\}_{t>0}$ on X. For t in $\mathbb{R}^+$, denote by h_t the heat kernel at time t, i.e.,

$$h_t(x) = \int_{\mathfrak{a}^*} e^{-t\,Q(\lambda)}\,\varphi_\lambda(x)\,\mathrm{d}\mu(\lambda) \qquad \forall x\in G. \tag{2.14}$$

3. Statement of the main result

In this section we define some Banach spaces of holomorphic functions that are relevant for our analysis of spherical multipliers and study their relationships. Then we state our main result.

The following definition is motivated by the main result in [19, 20].

Definition 3.1. Suppose that J is a nonnegative integer and that κ is in $[0,\infty)$. We denote by $\widetilde{H}(T_{\mathbf{W}};J,\kappa)$ the space of all holomorphic functions m in $T_{\mathbf{W}}$ such that $\|m\|_{\widetilde{H}(T_{\mathbf{W}};J,\kappa)} < \infty$, where $\|m\|_{\widetilde{H}(T_{\mathbf{W}};J,\kappa)}$ is the infimum of all constants C such that

$$\left|D^I m(\zeta)\right| \leq \frac{C}{\min\big(d(\zeta)^{\kappa+|I|},\, d(\zeta)^{|I|}\big)} \qquad \forall\zeta\in T_{\mathbf{W}} \quad \forall I : |I|\leq J, \tag{3.1}$$

and d is defined in (1.5).

The following result complements the work of Ionescu [19, 20]. Recall that n and ℓ denote the dimension and the rank of X, respectively.

Theorem 3.2. *Assume that κ is in $[0,1)$. Suppose that B is an operator in ${}^G\mathcal{B}^2(X)$ and that m_B is in $\widetilde{H}\big(T_{\mathbf{W}}; [\![n/2]\!] + \ell/2+1, \kappa\big)$. Then B extends to an operator of weak type* 1.

Proof. The proof of this theorem is rather long and technical. The estimates of the kernel k_B may be obtained by following the lines of the proof of the main result in [20]. The result will then follow from Strömberg's theorem. We omit the details. □

Remark 3.3. Note (see [19]) that if $\ell = 1$ and $\kappa = 0$, then Theorem 3.2 applies to the multiplier $m_{\mathcal{L}^{iu}}$, when u is real. However, if $\ell \geq 2$, then the multiplier $m_{\mathcal{L}^{iu}}$ does not belong to $\widetilde{H}(T_{\mathbf{W}}; J, \kappa)$ for any κ in $[0,1]$. We prove this in the case where $\kappa = 0$.

Indeed, suppose that $\mathrm{Re}(\zeta)$ is small. A straightforward computation shows that

$$d(\zeta)\ |\partial_{\zeta_\ell} m_{\mathcal{L}^{iu}}(\zeta)| = 2\ |u|\ d(\zeta)\, \frac{|\zeta_\ell|}{|Q(\zeta)|}\, \mathrm{e}^{-u \arg Q(\zeta)} \qquad \forall \zeta \in T_{\mathbf{W}}. \tag{3.2}$$

Here $\zeta = (\zeta_1, \ldots, \zeta_\ell)$, and $\zeta_1, \ldots, \zeta_\ell$ are the co-ordinates described in the Introduction. We show that if $\ell \geq 2$, then the right hand side cannot possibly stay bounded when ζ tends to $i\rho$ in $\overline{T}_{\mathbf{W}}$. Write $\zeta = \xi + i\eta$, where ξ is in $\mathfrak{a}^*$ and η is in $\mathbf{W}$. Suppose that $\xi \neq 0$, and let η tend to ρ within $\mathbf{W}$. By continuity, the right hand side of (3.2) tends to

$$2\ |u|\ d\big(\xi + i\,\rho\big)\, \frac{|\xi_\ell + i\,\rho|}{\big|Q\big(\xi + i\,\rho\big)\big|}\, \mathrm{e}^{-u \arg Q(\xi + i\eta)}.$$

Now, $d(\xi + i\,\rho) = |\xi|$ and $Q\big(\xi + i\,\rho\big) = |\xi|^2 + 2i\ \langle \xi, \rho\rangle$. Therefore, if ξ is orthogonal to ρ, then the right hand side of (3.2) becomes $2\ |u|\ |\xi|\ |\rho|/|\xi|^2$, which tends to infinity when ξ tends to 0, as required.

Motivated by this observation we first look for Mihlin type conditions satisfied by multipliers of the form $M \circ Q$, where M is a complex-valued function defined on a suitable parabolic region in the complex plane (see Proposition 3.6 below). Multipliers of this form correspond to functions of the Laplace–Beltrami operator, in the sense that $M \circ Q$ is the spherical Fourier multiplier associated to the operator $M(\mathcal{L})$. Note that these multipliers are holomorphic in the tube $T_{\mathbf{B}}$. In the higher rank case most spherical multipliers are not of the form $M \circ Q$, and, in general, do not extend to holomorphic functions in a region larger than $T_{\mathbf{W}}$. We then introduce new Mihlin type conditions for general spherical multipliers holomorphic in $T_{\mathbf{W}}$ (see Definition 3.7 below). Finally we state our main result, Theorem 3.10.

Denote by $\mathbf{P}$ the parabolic region in the plane defined by

$$\mathbf{P} = \{(x,y) \in \mathbb{R}^2 : y^2 < 4\,\langle \rho, \rho\rangle\ x\}.$$

Note that $\mathbf{P}$ is the image of $T_{\mathbf{W}}$ under Q. If $M(\mathcal{L})$ is in ${}^G\mathcal{B}^q(X)$ for all q in $(1,\infty)$, then its spherical multiplier $M \circ Q$ is holomorphic in $T_{\mathbf{W}}$ by the Clerc–Stein condition, and M is holomorphic in $\mathbf{P}$. This partially motivates the definition below.

Definition 3.4. Suppose that J is a nonnegative integer and that κ is in $[0,\infty)$. Denote by $\mathfrak{H}(\mathbf{P};J,\kappa)$ the space of all holomorphic functions M in $\mathbf{P}$ such that $\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}<\infty$, where $\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}$ is the infimum of all constants C such that

$$\left|M^{(j)}(z)\right| \leq \frac{C}{\min\left(|z|^{\kappa+j},|z|^{j}\right)} \qquad \forall z\in\mathbf{P}\quad \forall j\in\{0,1,\ldots,J\}.$$

Clearly for each β such that $\operatorname{Re}\beta\geq 0$ the function $z\mapsto z^{\beta}$ is in $\mathfrak{H}(\mathbf{P};J,\operatorname{Re}\beta)$ for all $J\geq 0$. Note that if M is holomorphic in $\mathbf{P}$, then $M\circ Q$ is, in fact, Weyl invariant and holomorphic in $T_{\mathbf{B}}$. In Proposition 3.6 below we prove that if M is in $\mathfrak{H}(\mathbf{P};J,\kappa)$, then $M\circ Q$ is in the space $H(T_{\mathbf{B}};J,\kappa)$, which we now define.

Definition 3.5. Suppose that J is a positive integer, κ is in $[0,\infty)$, and assume that $\mathbf{E}$ is a convex neighbourhood of the origin in $\mathfrak{a}^*$. Denote by $H(T_{\mathbf{E}};J,\kappa)$ the space of all holomorphic functions m in $T_{\mathbf{E}}$ for which $\|m\|_{H(T_{\mathbf{E}};J,\kappa)}<\infty$, where $\|m\|_{H(T_{\mathbf{E}};J,\kappa)}$ is the infimum of all constants C such that

$$\left|D^{I}m(\zeta)\right| \leq \frac{C}{\min\left(|Q(\zeta)|^{\kappa+|I|},|Q(\zeta)|^{|I|/2}\right)} \qquad \forall \zeta\in T_{\mathbf{E}^+}\quad \forall I:|I|\leq J.$$

See Section 2 for the definition of $\mathbf{E}^+$.

In the rest of the paper we shall consider spaces $H(T_{\mathbf{E}};J,\kappa)$ when $\mathbf{E}$ is either $\mathbf{B}$ or $\mathbf{B}^t$ for some t in $\mathbb{R}^-$.

Proposition 3.6. *Suppose that J is a nonnegative integer and that κ is in $[0,\infty)$. Then there exists a constant C such that*

$$\|M\circ Q\|_{H(T_{\mathbf{B}};J,\kappa)} \leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)} \qquad \forall M\in\mathfrak{H}(\mathbf{P};J,\kappa).$$

Proof. Suppose that I is a multiindex. A straightforward induction argument shows that there exist constants c_P such that

$$D^{I}(M\circ Q)(\zeta)=\sum_{0\leq P\leq I/2} c_P\,\zeta^{I-2P}\,M^{(|I|-|P|)}\big(Q(\zeta)\big) \qquad \forall\zeta\in T_{\mathbf{B}^+}. \tag{3.3}$$

Observe that if ζ is bounded, then so is $|Q(\zeta)|$. Since M is in $\mathfrak{H}(\mathbf{P};J,\kappa)$,

$$\begin{aligned}\left|D^{I}(M\circ Q)(\zeta)\right| &\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}\sum_{0\leq P\leq I/2}|\zeta|^{|I|-2|P|}\,\big|Q(\zeta)\big|^{-\kappa-|I|+|P|}\\ &\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}\sum_{0\leq P\leq I/2}\big|Q(\zeta)\big|^{-\kappa-|I|+|P|}\\ &\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}\big|Q(\zeta)\big|^{-\kappa-|I|} \qquad \forall\zeta\in\mathbf{B}+i\mathbf{B}^+.\end{aligned}$$

If, instead, $\zeta\in\big(\mathfrak{a}^*\setminus\mathbf{B}\big)+i\mathbf{B}^+$, then $|Q(\zeta)|\geq C\,|\zeta|^2$ for some positive constant C. Hence

$$\begin{aligned}\left|D^{I}(M\circ Q)(\zeta)\right| &\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}\sum_{0\leq P\leq I/2}|\zeta|^{|I|-2|P|}\,\big|Q(\zeta)\big|^{-|I|+|P|}\\ &\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}\big|Q(\zeta)\big|^{-|I|/2} \qquad \forall\zeta\in\big(\mathfrak{a}^*\setminus\mathbf{B}\big)+i\mathbf{B}^+.\end{aligned}$$

Thus, $M\circ Q$ is in $H(T_{\mathbf{B}};J,\kappa)$ and $\|M\circ Q\|_{H(T_{\mathbf{B}};J,\kappa)}\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}$, as required. $\square$

As mentioned before, in the higher rank case most spherical multipliers are not of the form $M \circ Q$ with M holomorphic in $\mathbf{P}$, and, in general do not extend to holomorphic functions in a region larger than $T_{\mathbf{W}}$. We would like to prove a result which applies, for instance, to multipliers of the form $m\,(M \circ Q)$, where M is in $\mathfrak{H}(\mathbf{P}; J, \kappa)$, and m is holomorphic and bounded in $T_{\mathbf{W}}$ and satisfies estimates (1.3). To introduce the appropriate function space we need more notation. For every multiindex $I = (i_1, i_2, \dots, i_\ell)$ we shall denote by D^I the differential operator $\partial^{i_1}_{\zeta_1} \partial^{i_2}_{\zeta_2} \cdots \partial^{i_\ell}_{\zeta_\ell}$, where $\zeta = \xi + i\eta$, ξ and η are in $\mathfrak{a}^*$, $\zeta_j = \xi_j + i\eta_j$, and $(\xi_1, \dots, \xi_\ell)$ and $(\eta_1, \dots, \eta_\ell)$ are the co-ordinates of ξ and η with respect to the basis $\varepsilon_1, \dots, \varepsilon_{\ell-1}, \rho/|\rho|$, defined in the Introduction.

Definition 3.7. Suppose that J is a positive integer and that κ is in $[0, \infty)$, and assume that $\mathbf{E}$ is a convex neighbourhood of the origin in $\mathfrak{a}^*$. Denote by $H'(T_{\mathbf{E}}; J, \kappa)$ the space of all holomorphic functions m in $T_{\mathbf{E}}$ for which $\|m\|_{H'(T_{\mathbf{E}}; J, \kappa)} < \infty$, where $\|m\|_{H'(T_{\mathbf{E}}; J, \kappa)}$ is the infimum of all constants C such that

$$\left|D^{(I', i_\ell)} m(\zeta)\right| \le \frac{C}{\min\left(|Q(\zeta)|^{\kappa + i_\ell + |I'|/2}, |Q(\zeta)|^{(|I'| + i_\ell)/2}\right)} \qquad \forall \zeta \in T_{\mathbf{E}^+}$$

for all multiindices (I', i_ℓ) for which $|I'| + i_\ell \le J$.

In the rest of the paper we shall consider spaces $H'(T_{\mathbf{E}}; J, \kappa)$, where $\mathbf{E}$ is either $\mathbf{W}$ or $\mathbf{W}^t$ for some t in $\mathbb{R}^-$. Observe that the functions in $H'(T_{\mathbf{W}^t}; J, \kappa)$ satisfy on $T_{\mathbf{W}^+}$ the same estimates that the functions in $H'(T_{\mathbf{W}}; J, \kappa)$ satisfy, but they need not be holomorphic in the whole tube $T_{\mathbf{W}}$. A similar observation applies to functions in the spaces $H(T_{\mathbf{B}}; J, \kappa)$ and $H(T_{\mathbf{B}^t}; J, \kappa)$ defined above.

Remark 3.8. Suppose that m is in $H'(T_{\mathbf{W}}; J, \kappa)$ and that the function $\xi \mapsto m(\xi + i\rho)$ is smooth on $\mathfrak{a}^* \setminus \{0\}$. By a continuity argument for each multiindex (I', i_ℓ) with $|I'| + i_\ell \le J$ the function m satisfies

$$\left|D^{(I', i_\ell)} m(\xi + i\rho)\right| \le \frac{\|m\|_{H'(T_{\mathbf{W}}; J, \kappa)}}{\min\left(|Q(\xi + i\rho)|^{\kappa + i_\ell + |I'|/2}, |Q(\xi + i\rho)|^{(i_\ell + |I'|)/2}\right)} \tag{3.4}$$

for all ξ in $\mathfrak{a}^* \setminus \{0\}$. Note that $\min\left(|Q(\zeta)|^{\kappa + i_\ell + |I'|/2}, |Q(\zeta)|^{(i_\ell + |I'|)/2}\right)$ is equal to $|Q(\zeta)|^{\kappa + i_\ell + |I'|/2}$ if $|\zeta|$ is small and to $|Q(\zeta)|^{(i_\ell + |I'|)/2}$ if $|\zeta|$ is large. Furthermore,

$$|Q(\xi + i\rho)| = \left||\xi|^2 + 2i\,\langle \xi, \rho\rangle\right|.$$

Thus,

$$|Q(\xi + i\rho)| \asymp \begin{cases} |\xi|^2 & \text{if } \xi \text{ is either large, or small and } \xi \perp \rho, \\ |\xi| & \text{if } \xi = c\,\rho \text{ for } c \in \mathbb{R}^+ \text{ small.} \end{cases}$$

Then, from (3.4) we deduce that

$$\left|D^{(I', i_\ell)} m(\xi + i\rho)\right| \le \begin{cases} \|m\|_{H'(T_{\mathbf{W}}; J, \kappa)}\ |\xi|^{-|I'| - i_\ell} & \text{if } \xi \text{ is large,} \\ \|m\|_{H'(T_{\mathbf{W}}; J, \kappa)}\ |\xi|^{-(\kappa + i_\ell + |I'|/2)} & \text{if } \xi = c\,\rho \text{ for } c \in \mathbb{R}^+ \text{ small,} \\ \|m\|_{H'(T_{\mathbf{W}}; J, \kappa)}\ |\xi|^{-(2\kappa + 2i_\ell + |I'|)} & \text{if } \xi \text{ is small and } \xi \perp \rho. \end{cases}$$

In particular, if $\kappa = 0$, then the function $m(\cdot + i\rho)$ satisfies a standard Mihlin–Hörmander condition of order J at infinity on $\mathfrak{a}^*$ and an anisotropic Mihlin–Hörmander condition of order J near the origin. A similar anisotropy was noticed in [9, Thm 1 (vii) and (ix)] in connection with the kernel of the (modified) Poisson semigroup.

In the next proposition we prove that if $M \in \mathfrak{H}(\mathbf{P}; J, \kappa)$, then the restriction of $M \circ Q$ to $T_{\mathbf{W}}$ belongs to $H'(T_{\mathbf{W}}; J, \kappa)$. A straightforward calculation then implies that if m is holomorphic and bounded in $T_{\mathbf{W}}$ and satisfies estimates (1.3), then the product $m\,(M \circ Q)$ is in $H'(T_{\mathbf{W}}; J, \kappa)$.

We need the following notation. For each c in $\mathbb{R}^+$ define the cone $\mathbf{\Gamma}_c$ by

$$\mathbf{\Gamma}_c = \{H \in \mathfrak{a} : |H'| < c\, H_\ell\}. \tag{3.5}$$

Since H_ρ is in $\mathfrak{a}^+$, there exists c_0 such that $\mathbf{\Gamma}_{c_0} \subset \mathfrak{a}^+$. It is well known (see [13, Lemma 34] or [15, Ch. VII, Lemma 2.20 (iv)]) that the dual Weyl chamber ${}^+\mathfrak{a}$ contains $\mathfrak{a}^+$. Then the dual cone $\mathbf{\Gamma}_{1/c_0}$ contains ${}^+\mathfrak{a}$. Choose $c_1 > 1/c_0$: note that

$$\mathbf{\Gamma}_{c_0} \subset \mathfrak{a}^+ \subset {}^+\mathfrak{a} \subset \mathbf{\Gamma}_{1/c_0} \subset \mathbf{\Gamma}_{c_1}. \tag{3.6}$$

Proposition 3.9. *Suppose that J is a nonnegative integer, and that κ is in $[0, \infty)$. Then there exists a constant C such that*

$$\|M \circ Q\|_{H'(T_{\mathbf{W}}; J, \kappa)} \leq C\, \|M\|_{\mathfrak{H}(\mathbf{P}; J, \kappa)} \qquad \forall M \in \mathfrak{H}(\mathbf{P}; J, \kappa).$$

Proof. By arguing as in the proof of Proposition 3.6, we see that there exists a constant C such that

$$\left|D^{(I', i_\ell)}(M \circ Q)(\zeta)\right| \leq C\, \|M\|_{\mathfrak{H}(\mathbf{P}; J, \kappa)} \left|Q(\zeta)\right|^{-(i_\ell + |I'|)/2} \tag{3.7}$$

for all ζ in $\big(\mathfrak{a}^* \setminus \mathbf{B}\big) + i\mathbf{W}^+$. We claim that there exists a constant C such that

$$|\zeta'| \leq C\, \left|Q(\zeta)\right|^{1/2} \qquad \forall \zeta \in \mathbf{B} + i\mathbf{W}^+. \tag{3.8}$$

Given the claim, we indicate how to conclude the proof of the proposition. Write I for the multiindex (I', i_ℓ). Note that (3.3), the assumption $M \in \mathfrak{H}(\mathbf{P}; J, \kappa)$ and (3.8) imply that there exists a constant C such that

$$\begin{aligned}
&\left|D^{(I', i_\ell)}(M \circ Q)(\zeta)\right| \\
&\leq C\, \|M\|_{\mathfrak{H}(\mathbf{P}; J, \kappa)} \sum_{0 \leq P \leq I/2} |\zeta_1|^{(i_\ell - 2p_\ell)}\, |\zeta'|^{|I'| - 2|P'|}\, \left|Q(\zeta)\right|^{-\kappa - i_\ell - |I'| + |P|} \\
&\leq C\, \|M\|_{\mathfrak{H}(\mathbf{P}; J, \kappa)} \sum_{0 \leq P \leq I/2} \left|Q(\zeta)\right|^{|I'|/2 - |P'|} \left|Q(\zeta)\right|^{-\kappa - i_\ell - |I'| + |P|} \\
&\leq C\, \|M\|_{\mathfrak{H}(\mathbf{P}; J, \kappa)} \left|Q(\zeta)\right|^{-\kappa - i_\ell - |I'|/2} \qquad \forall \zeta \in \mathbf{B} + i\mathbf{W}^+.
\end{aligned}$$

The required conclusion follows directly from this estimate and (3.7).

It remains to prove the claim. We abuse the notation and denote by $\mathbf{\Gamma}_{c_1}$ the cone $\{(\lambda', \lambda_\ell) \in \mathfrak{a}^* : |\lambda'| < c_1\, \lambda_\ell\}$. By [14, Lemma 8.3] $\mathbf{W}^+ = (\mathfrak{a}^*)^+ \cap \big(\rho - {}^+(\mathfrak{a}^*)\big)$, where ${}^+(\mathfrak{a}^*)$ denotes the dual cone of $(\mathfrak{a}^*)^+$. Recall that ${}^+(\mathfrak{a}^*) \subset \mathbf{\Gamma}_{c_1}$ (see (3.6)), so that $\mathbf{W}^+ \subset (\mathfrak{a}^*)^+ \cap \big(\rho - \mathbf{\Gamma}_{c_1}\big)$. Suppose that c is a number such that $(c_1^2 - 1)/(c_1^2 + 1) < c < 1$. Set $\mathbf{V} = \{(\eta', \eta_\ell) : c\, |\rho| < \eta_\ell < |\rho|\, , |\eta'| < c_1\, (|\rho| - \eta_1)\}$. If c is sufficiently close to 1, then $\mathbf{V} \subset (\mathfrak{a}^*)^+$.

Observe that (3.8) is obvious when ζ is in $\mathbf{B} + i(\mathbf{W}^+ \setminus \mathbf{V})$. Indeed, both sides of (3.8) are continuous functions of ζ, and ζ stays at a positive distance from $i\rho$, which is the unique point in $\overline{T}_{\mathbf{W}^+}$ where Q vanishes.

Now suppose that ζ is in $\mathbf{B}+i(\mathbf{W}^+\cap\mathbf{V})$, and write $\zeta=\xi+i\eta$. Note that

$$\begin{aligned}|Q(\zeta)|^2 &= (|\xi|^2+|\rho|^2-|\eta|^2)^2+4\,|\langle\xi,\eta\rangle|^2\\ &\geq (|\xi|^2+|\rho|^2-|\eta|^2)^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}|\rho|^2-|\eta|^2 &= |\rho|^2-\eta_1^2-|\eta'|^2\\ &\geq |\rho|^2-\eta_1^2-c_1^2(|\rho|-\eta_1)^2\\ &= \big(|\rho|-\eta_1\big)\big(|\rho|+\eta_1-c_1^2\,|\rho|+c_1^2\eta_1\big)\\ &\geq |\rho|\,\big(|\rho|-\eta_1\big)\big[1-c_1^2+c\,(1+c_1^2)\big].\end{aligned}$$

Since c has been chosen so that $1-c_1^2+c\,(1+c_1^2)>1$,

$$|\rho|^2-|\eta|^2\geq(|\rho|-\eta_1)\,|\rho|\geq\frac{|\rho|}{c_1}\,|\eta'|\,.$$

Therefore,

$$|Q(\zeta)|^2\geq C\,(|\xi|^2+|\eta'|)^2\geq C\,(|\xi'|^4+|\eta'|^2)\geq C\,|\zeta'|^4\,.$$

This completes the proof of the claim (3.8) and of the proposition. □

Now we state our main result. Its proof is deferred to Section 6. Given B in ${}^G\mathcal{B}^2(X)$, we denote by $|||B|||_{1;1,\infty}$ the quasi-norm of B *qua* operator from $L^1(X)$ to $L^{1,\infty}(X)$.

Theorem 3.10. *Denote by J the integer $[\![n/2]\!]+1$. The following hold:*

(i) *if κ is in $[0,1)$, then there exists a constant C such that for all B in ${}^G\mathcal{B}^2(X)$ for which m_B is in $H'(T_{\mathbf{W}};J,\kappa)$,*

$$|||B|||_{1;1,\infty}\leq C\,\|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)};$$

(ii) *there exists a constant C such that*

$$|||M(\mathcal{L})|||_{1;1,\infty}\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,1)}\qquad\forall M\in\mathfrak{H}(\mathbf{P};J,1).$$

Remark 3.11. The proof of Theorem 3.10 will show that in the case where $\ell>1$ the anisotropic behaviour of the multiplier m_B near the point $i\rho$ (see Remark 3.8 above) implies an anisotropic behaviour of the kernel k_B at infinity. In fact, the bounds of k_B we shall obtain are expressed, in Cartan co-ordinates, in terms of an anisotropic homogeneous "norm" on $\mathfrak{a}$.

Remark 3.12. Observe that Theorem 3.10 (ii) applies to $\mathcal{L}^{-\alpha/2}$ when $0\leq\operatorname{Re}\alpha\leq 2$ (hence we re-obtain Anker's result [2]) and that it is sharp, in the sense that for each $\kappa>1$ the function $z\mapsto z^{-k}$ is in $\mathfrak{H}(\mathbf{P};J,\kappa)$ but $\mathcal{L}^{-\kappa}$ is not of weak type 1. We also remark that if M is in $\mathfrak{H}(\mathbf{P};J,\kappa)$ for some κ in $[0,1)$, then, *a fortiori*, M is in $\mathfrak{H}(\mathbf{P};J,1)$, hence (ii) applies to M.

Remark 3.13. We do not know whether (i) holds with $\kappa=1$. Moreover, if M is in $\mathfrak{H}(\mathbf{P};J,1)$, then $m_{M(\mathcal{L})}$ is in $H'(T_{\mathbf{W}};J,1)$ by Proposition 3.9. Thus, for functions of the Laplace–Beltrami operator $\mathcal{L}$, condition (i) is weaker than (ii).

4. Weak type estimates for certain convolution operators

Suppose that $\varepsilon \in \mathbb{R}$ and consider the K–bi-invariant functions τ_1^ε and τ_2^ε on G, defined by

$$\begin{aligned} \tau_1^\varepsilon\big(\exp H\big) &= \mathrm{e}^{-\rho(H)-|\rho|\,|H|}\,\big(1+\rho(H)\big)^{(1-\ell)/2-\varepsilon} \\ \tau_2^\varepsilon\big(\exp H\big) &= \mathrm{e}^{-2\rho(H)}\,\big(1+\mathcal{N}(H)\big)^{1-\ell-\varepsilon} \end{aligned} \qquad \forall H \in \mathfrak{a}^+. \tag{4.1}$$

The homogeneous norm $\mathcal{N}$ is defined in (2.2). We denote by T_1^ε and T_2^ε the convolution operators $f \mapsto f * \tau_1^\varepsilon$ and $f \mapsto f * \tau_2^\varepsilon$, respectively. In this section we study the weak type 1 boundedness of the operators T_1^ε and T_2^ε.

Proposition 4.1. *Suppose that ε is in $\mathbb{R}$. The following hold:*
- (i) *the operator T_1^ε is of weak type 1 if and only if $\varepsilon \geq 0$;*
- (ii) *if $\ell > 1$, then the operator T_2^ε is of weak type 1 if and only if $\varepsilon > 0$. If $\ell = 1$, then T_2^ε is of weak type 1 if and only if $\varepsilon \geq 0$.*

Proof. It is straightforward to check that the result stated in [23, Remark 2, p. 125] applies to τ_1^ε when $\varepsilon \geq 0$, and to τ_2^ε when either $\varepsilon \geq 0$ and $\ell = 1$ or $\varepsilon > 0$ and $\ell \geq 2$. This proves the weak type 1 estimate for T_1^ε when $\varepsilon \geq 0$, and for T_2^ε when either $\varepsilon \geq 0$ and $\ell = 1$, or $\varepsilon > 0$ and $\ell \geq 2$. A self-contained proof of these results that does not use [23, Remark 2, p. 125] (Strömberg does not provide a proof for his statement) may be found in a preliminary version of this paper on the arXiv.

To conclude the proof of (i) it remains to show that T_1^ε is not of weak type 1 when $\varepsilon < 0$. By a standard argument, it suffices to prove that the corresponding kernel τ_1^ε is not in $L^{1,\infty}(X)$. We give the details in the case where $\ell \geq 2$. Those in the case where $\ell = 1$ are easier and are omitted. Observe that

$$\tau_1^\varepsilon(\exp H) = \mathrm{e}^{-2\rho(H)}\,U(H) \qquad \forall H \in \mathfrak{a}^+,$$

where $U(H) = \mathrm{e}^{\rho(H)-|\rho|\,|H|}\,\big(1+\rho(H)\big)^{(1-\ell)/2-\varepsilon}$. Write $H = (H', H_\ell)$ and recall that $\rho(H) = |\rho|\,H_\ell$. A straightforward computation shows that

$$\begin{aligned} \rho(H) - |\rho|\;|H| &= -\,|\rho|\;\frac{|H'|^2}{H_\ell + \sqrt{H_\ell^2 + |H'|^2}} \\ &\geq -\,|\rho|\;\frac{|H'|^2}{H_\ell}. \end{aligned}$$

Now, if H is in $\mathbf{p}$ (see (2.4)), then $|H'|^2/H_\ell \leq 1$, so that there exists a positive constant c such that

$$U(H) \geq c\,\big(1+H_\ell\big)^{(1-\ell)/2-\varepsilon} \qquad \forall H \in \mathbf{p}.$$

For each t in $\mathbb{R}^+$, define $E_t = \{k_1 \exp(H) k_2 \in K \exp(\mathfrak{a}^+) K : \tau_1^\varepsilon(\exp H) > t\}$. Set $h := \inf\{H_\ell \in \mathbb{R}^+ : (H', H_\ell) \in \mathfrak{a}^+ \cap \mathbf{p}\}$, and denote by s_t the unique point in $\mathbb{R}^+$ such that

$$\frac{\mathrm{e}^{2|\rho| s_t}}{(1+s_t)^{(1-\ell)/2-\varepsilon}} = (ct)^{-1}. \tag{4.2}$$

Denote by $|E_t|$ the Haar measure of E_t. Note that

$$\begin{aligned} |E_t| &\geq \Big|\Big\{k_1 \exp(H', H_\ell)\,k_2 \in K \exp(\mathfrak{a}^+ \cap \mathbf{p})\,K : \frac{\mathrm{e}^{-2|\rho|\,H_\ell}}{c\,\big(1+H_\ell\big)^{(\ell-1)/2+\varepsilon}} > t\Big\}\Big| \\ &\geq \Big|\Big\{k_1 \exp(H', H_\ell)\,k_2 \in K \exp(\mathfrak{a}^+ \cap \mathbf{p})\,K : h < H_\ell < s_t\Big\}\Big|. \end{aligned}$$

It is straightforward to check that this measure is estimated from below by a constant times $\int_h^{s_t} s^{(\ell-1)/2}\,\mathrm{e}^{2|\rho|s}\,\mathrm{d}s$. By (4.2) s_t tends to ∞ as t tends to 0^+. Integration by parts shows that the integral above is comparable to $s_t^{(\ell-1)/2}\,\mathrm{e}^{2|\rho|s_t}$ as t tends to 0^+. Thus, there exists a positive constant C such that

$$|E_t| \geq C\, s_t^{(\ell-1)/2}\,\mathrm{e}^{2|\rho|s_t} \geq C\,\frac{s_t^{-\varepsilon}}{t}.$$

Hence $\sup_{t>0}\big(t\,|E_t|\big) = \infty$, so that $\tau_1^\varepsilon \notin L^{1,\infty}(X)$ if $\varepsilon < 0$. The proof of (i) is complete.

Next we prove the negative results in (ii). Suppose first that $\ell = 1$. Then

$$\begin{aligned} \tau_1^\varepsilon(\exp H) &= \mathrm{e}^{-2\rho(H)}\,(1+|\rho|\ H)^{-\varepsilon} \\ \tau_2^\varepsilon(\exp H) &= \mathrm{e}^{-2\rho(H)}\,(1+\sqrt{H})^{-\varepsilon} \end{aligned} \qquad \forall H \in \mathfrak{a}^+.$$

It is straightforward to check that there exist positive constants C_1 and C_2 such that

$$C_1\,\tau_1^{\varepsilon/2} \leq \tau_2^\varepsilon \leq C_2\,\tau_1^{\varepsilon/2}.$$

By (i), $T_1^{\varepsilon/2}$ is not of weak type 1 when $\varepsilon < 0$. Hence the same holds for T_2^ε, as required.

Now suppose that $\ell \geq 2$ and that $\varepsilon \leq 0$. To conclude the proof of (ii), it remains to show that T_2^0 is not of weak type 1. It suffices to prove that τ_2^0 is not in $L^{1,\infty}(X)$. Denote by τ' the K–bi-invariant function on G defined by

$$\tau'\big(k_1\exp(H',H_\ell)k_2\big) = (1+|H'|)^{1-\ell}\,\mathrm{e}^{-2|\rho|\,H_\ell}\,\mathbf{1}_{\mathbf{p}^c\cap\mathbf{\Gamma}_{c_0}}(H',H_\ell)$$

for all H in $\mathfrak{a}^+$ and for all k_1,k_2 in K. Note that $\tau_2^0\big(\exp H\big) \geq C\,\tau'(\exp H)$ for all H in $\mathfrak{a}^+$. Indeed, $H_\ell \leq |H'|^2$, because H is in $\mathbf{p}^c$. Hence $1+\mathcal{N}(H) \leq 1+2^{1/4}\ |H'|$, from which the inequality above follows directly.

We show that τ' is not in $L^{1,\infty}(X)$. Clearly this implies that τ_2^0 is not in $L^{1,\infty}(X)$ either, as required. For each t in $(0,\mathrm{e}^{-2|\rho|}\,2^{1-\ell}]$ define

$$\Omega_t = \big\{k_1\exp(H)\,k_2 \in K\exp(\mathfrak{a}^+)\,K : \tau'(k_1\exp(H)k_2) > t\big\},$$

and the function $b_t : \mathbb{R} \to \mathbb{R}$ by

$$b_t(s) = (t\,\mathrm{e}^{2|\rho|s})^{-1/(\ell-1)} - 1 \qquad \forall s \in \mathbb{R}.$$

Denote by u_t and v_t the unique solutions to the equations $s = b_t(s)$ and $s^{1/2} = b_t(s)$. It is straightforward to check that $1 < u_t < v_t$ for all t in $(0,\mathrm{e}^{-2|\rho|}\,2^{1-\ell}]$ and that $s^{1/2} < b_t(s) < s$ for all s in (u_t,v_t). Note also that $\tau'(\exp H) > t$ if and only if H is in $\mathbf{p}^c\cap\mathbf{\Gamma}_{c_0}$ and $|H'| < b_t(H_\ell)$. Therefore

$$\Omega_t \supset \big\{k_1\exp(H',H_\ell)\,k_2 \in K\exp(\mathfrak{a}^+)\,K : u_t < H_\ell < v_t,\ H_\ell^{1/2} < |H'| < b_t(H_\ell)\big\}$$

and

$$|\Omega_t| \geq \int_{u_t}^{v_t} \mathrm{e}^{2|\rho|s}\,\lambda_{\ell-1}(A_s)\,\mathrm{d}s,$$

where A_s denotes the annulus $\{H' \in \rho^\perp : s^{1/2} < |H'| < b_t(s)\}$. Therefore

$$\lambda_{\ell-1}(A_s) = c\,b_t(s) - c\,s^{(\ell-1)/2},$$

where c is the volume of the unit ball in $\mathbb{R}^{\ell-1}$ with respect to the Lebesgue measure. Observe that u_t tends to ∞ as t tends to 0^+. Hence s is large in the formula above. Now, there exists a positive constant C such that if s is large, then

$$\lambda_{\ell-1}(A_s) \geq \frac{C}{t}\, \mathrm{e}^{-2|\rho| s},$$

so that

$$|\Omega_t| \geq C\, \frac{v_t - u_t}{t} \qquad \forall t \in (0, \mathrm{e}^{-2|\rho|}\, 2^{1-\ell}].$$

To conclude the proof, it suffices to show that $v_t - u_t$ does not stay bounded as t tends to 0^+. From the definition of u_t and v_t we deduce that

$$\mathrm{e}^{2|\rho|(v_t - u_t)} = \Big(\frac{1+v_t}{1+u_t^{1/2}}\Big)^{\ell-1}.$$

Now, if $v_t - u_t$ stays bounded, then so does the right hand side in the formula above. Hence there exists a constant C such that $1 + v_t \leq C\,(1+u_t^{1/2})$, but this is impossible, because $v_t > u_t$ and u_t tends to ∞ as t tends to 0^+.

This proves that T_2^0 is not of weak type 1, as required to conclude the proof of (ii) and of the proposition. $\square$

5. Kernel estimates

In this section we prove some technical lemmata which will be used in the proof of Theorem 3.10. The ball $\mathbf{B}$ is defined just below formula (2.1).

Lemma 5.1. *Suppose that γ is in $\mathbb{R}^+$. Then there exists a constant C such that for every η in $(\mathfrak{a}^*)^+$ with $|\eta| = |\rho|$ and for every ε in $(0, 1/4)$,*

$$\int_{\mathbf{B}} \big|Q\big(\lambda + i(1-\varepsilon)\eta\big)\big|^{-\gamma}\,\mathrm{d}\lambda \leq \begin{cases} C\,\big(1 + \varepsilon^{(\ell+1)/2-\gamma}\big) & \text{if } \gamma \neq (\ell+1)/2, \\ C\,\log(1/\varepsilon) & \text{if } \gamma = (\ell+1)/2. \end{cases} \tag{5.1}$$

Proof. Given η in $(\mathfrak{a}^*)^+$ such that $|\eta| = |\rho|$, we choose an orthonormal basis of $\mathfrak{a}^*$ whose last vector is $\eta/|\eta|$. For any λ in $\mathfrak{a}^*$ we write $\lambda = (\lambda'_\eta, \lambda_\eta)$, where $\lambda'_\eta \in \mathbb{R}^{\ell-1}$ and $\lambda_\eta \in \mathbb{R}$ for the co-ordinates of λ with respect to this orthonormal basis. Notice that

$$\big|Q\big(\lambda + i(1-\varepsilon)\eta\big)\big|^2 = \big[\lambda_\eta^2 + |\lambda'_\eta|^2 + (2\varepsilon - \varepsilon^2)\,|\rho|^2\big]^2 + 4\,(1-\varepsilon)^2\,|\rho|^2\,\lambda_\eta^2.$$

Then there exists a constant C such that

$$\big|Q\big(\lambda + i(1-\varepsilon)\eta\big)\big|^2 \geq C\,\big[(|\lambda'_\eta|^2 + \varepsilon)^2 + \lambda_\eta^2\big] \qquad \forall \lambda \in \mathbf{B}.$$

Therefore,

$$\int_{\mathbf{B}} \big|Q\big(\lambda + i(1-\varepsilon)\eta\big)\big|^{-\gamma}\,\mathrm{d}\lambda \leq C \int_{\mathbf{B}} \frac{1}{\big[(|\lambda'_\eta|^2 + \varepsilon)^2 + \lambda_\eta^2\big]^{\gamma/2}}\,\mathrm{d}\lambda_\eta\,\mathrm{d}\lambda'_\eta. \tag{5.2}$$

If $\ell + 1 > 2\gamma$, then the integral on the right hand side of (5.2) is estimated from above by

$$\int_{\mathbf{B}} \big[|\lambda'_\eta|^4 + \lambda_\eta^2\big]^{-\gamma/2}\,\mathrm{d}\lambda_\eta\,\mathrm{d}\lambda'_\eta,$$

which is finite, so that (5.1) is proved in this case.

Now suppose that $\ell + 1 \leq 2\gamma$. We abuse the notation and denote by $\mathbf{b}_R$ the set of all $(\lambda'_\eta, \lambda_\eta)$ in $\mathbb{R}^{\ell-1} \times \mathbb{R}$ such that $|\lambda'_\eta|^4 + \lambda_\eta^2 < R^4$. Observe that $\mathbf{B} \subset \mathbf{b}_{2|\rho|}$.

Indeed, if $(\lambda_\eta', \lambda_\eta)$ is in $\mathbf{B}$, then $\left|\lambda_\eta'\right|^2 + \lambda_\eta^2 < |\rho|^2$. In particular $\left|\lambda_\eta'\right| < |\rho|$ and $|\lambda_\eta| < |\rho|$, whence

$$\left|\lambda_\eta'\right|^4 + \lambda_\eta^2 < |\rho|^2 \left|\lambda_\eta'\right|^2 + \lambda_\eta^2 < \max(1,|\rho|^2)\,(\left|\lambda_\eta'\right|^2 + \lambda_\eta^2) < \max(1,|\rho|^4) \leq (2\,|\rho|)^4,$$

because $|\rho|$ is always at least $1/2$. We majorise the integral on the right hand side of (5.2) by integrating on $\mathbf{b}_{2|\rho|}$ instead of on $\mathbf{B}$. Then, changing variables $(\lambda_\eta', \lambda_\eta) = (\varepsilon^{1/2}\, v', \varepsilon\, v_\ell)$, we see that

$$\int_{\mathbf{B}} \left|Q\big(\lambda + i(1-\varepsilon)\eta\big)\right|^{-\gamma} \mathrm{d}\lambda \leq C \int_{\mathbf{b}_{4|\rho|/\sqrt{\varepsilon}}} \frac{\varepsilon^{(\ell+1)/2-\gamma}}{\big[(|v'|^2+1)^2 + v_\ell^2\big]^{\gamma/2}}\, \mathrm{d}v'\, \mathrm{d}v_\ell.$$

If $\ell+1 = 2\gamma$, then (5.2) is bounded by $C\,\log(1/\varepsilon)$, as required.

If $\ell+1 < 2\gamma$, then (5.2) is bounded by

$$\varepsilon^{(\ell+1)/2-\gamma}\Big[\int_{\mathbf{b}_1} \frac{\mathrm{d}v'\,\mathrm{d}v_\ell}{\big[(|v'|^2+1)^2 + v_\ell^2\big]^{\gamma/2}} + \int_{\mathbf{b}_1{}^c} \frac{\mathrm{d}v'\,\mathrm{d}v_\ell}{(|v'|^4 + v_\ell^2)^{\gamma/2}}\Big] \leq C\,\varepsilon^{(\ell+1)/2-\gamma},$$

as required. □

Lemma 5.2 below will be used in Step II of the proof of Theorem 3.10 to control the kernel k_B away from the walls of $\mathfrak{a}^+$, whereas Lemma 5.6 below is needed in Step III of the same proof to control the size of k_B near the walls of $\mathfrak{a}^+$.

Lemma 5.2. *Suppose that κ is in $[0,1]$. Set $J := \ell+1$ and denote by L the least integer $\geq (\ell+1)/2$. For any function m, which is holomorphic in $T_{\mathbf{W}^t}$ for some t in $\mathbb{R}^-$ and such that $m\,\mathrm{e}^{-Q/2}$ is in $L^1(\mathfrak{a}^*)$ (with respect to the Lebesgue measure), define $k_1 : \mathfrak{a}^+ \to \mathbb{C}$ by*

$$k_1(H) = \int_{\mathfrak{a}^*} m(\lambda)\,\mathrm{e}^{-Q(\lambda)/2}\,\mathrm{e}^{i\lambda(H)}\,\mathrm{d}\lambda \qquad \forall H \in \mathfrak{a}^+.$$

The following hold:

(i) *there exists a constant C such that for all m in $H'(T_{\mathbf{W}^t}; J, \kappa)$ and for all H in $\mathfrak{a}^+$,*

$$|k_1(H)| \leq \begin{cases} C\,\|m\|_{H'(T_{\mathbf{W}^t};J,\kappa)}\,\mathrm{e}^{-\rho(H)}\big[1+\mathcal{N}(H)\big]^{-\ell-1+2\kappa} & \text{if } 0<\kappa\leq 1,\\ C\,\|m\|_{H'(T_{\mathbf{W}^t};J,0)}\,\mathrm{e}^{-\rho(H)}\,\big[1+\mathcal{N}(H)\big]^{-\ell-1}\,\log\big[2+\mathcal{N}(H)\big] & \text{if } \kappa = 0;\end{cases}$$

(ii) *if either $0<\kappa\leq 1$ or $\kappa=0$ and ℓ is even, then there exists a constant C such that for all m in $H'(T_{\mathbf{B}^t}; J, \kappa)$,*

$$|k_1(H)| \leq C\,\|m\|_{H(T_{\mathbf{B}^t};L,\kappa)}\,\mathrm{e}^{-|\rho|\,|H|}\big[1+\rho(H)\big]^{-(\ell+1)/2+\kappa} \qquad \forall H \in \mathfrak{a}^+.$$

Similarly, if $\kappa = 0$ and ℓ is odd, then there exists a constant C such that for all m in $H'(T_{\mathbf{B}^t}; J, \kappa)$,

$$|k_1(H)| \leq C\,\|m\|_{H(T_{\mathbf{B}^t};L,0)}\,\mathrm{e}^{-|\rho|\,|H|}\,\big[1+\rho(H)\big]^{-(\ell+1)/2}\,\log\big[2+\rho(H)\big] \qquad \forall H \in \mathfrak{a}^+.$$

Proof. We denote by m_1 the function defined by

$$m_1(\zeta) = m(\zeta)\,\mathrm{e}^{-Q(\zeta)/2} \qquad \forall \zeta \in T_{\mathbf{W}^t}.$$

Observe that k_1 (which is the inverse Fourier transform of m_1) is bounded, because m_1 is in $L^1(\mathfrak{a}^*)$. Therefore all the estimates in (i) and (ii) hold trivially for H in $\mathfrak{a}^+ \cap \mathbf{b}_2$, and we may assume that H is in $\mathfrak{a}^+ \cap \mathbf{b}_2{}^c$.

First we prove (i). For the duration of the proof of (i) we write ρ_ε instead of $(1-\varepsilon)\rho$. An application of Leibniz's rûle shows that there exists a constant C such that for every multiindex (I', i_ℓ) such that $|I'| + i_\ell \leq J$, for every ε in $(0, 1/4)$, and for every m in $H'(T_{\mathbf{W}^t}; J, \kappa)$,

$$\begin{aligned} &\left| D^{(I', i_\ell)} m_1(\lambda + i\rho_\varepsilon) \right| \\ &\leq \begin{cases} C \, \|m\| \, \mathrm{e}^{-\operatorname{Re} Q(\lambda + i\rho_\varepsilon)/4} \left| Q\big(\lambda + i\rho_\varepsilon\big) \right|^{-(i_\ell + |I'|)/2} & \forall \lambda \in \mathfrak{a}^* \setminus \mathbf{B}, \\ C \, \|m\| \, \left| Q\big(\lambda + i\rho_\varepsilon\big) \right|^{-\kappa - i_\ell - |I'|/2} & \forall \lambda \in \mathbf{B}, \end{cases} \end{aligned} \tag{5.3}$$

where here we write $\|m\|$ instead of $\|m\|_{H'(T_{\mathbf{W}^t}; J, \kappa)}$. Assume that ε is in the interval $\big(0, C/\rho(H)\big)$ for some fixed constant C. Since m_1 is holomorphic in $T_{\mathbf{W}^t}$, we may move the contour of integration to the space $\mathfrak{a}^* + i\rho_\varepsilon$ and obtain

$$\begin{aligned} |k_1(H)| &= \mathrm{e}^{-(1-\varepsilon)\rho(H)} \left| \int_{\mathfrak{a}^*} m_1\big(\lambda + i\rho_\varepsilon\big) \, \mathrm{e}^{i\lambda(H)} \, \mathrm{d}\lambda \right| \\ &\leq C \, \mathrm{e}^{-\rho(H)} \left| \int_{\mathfrak{a}^*} m_1\big(\lambda + i\rho_\varepsilon\big) \, \mathrm{e}^{i\lambda(H)} \, \mathrm{d}\lambda \right| \qquad \forall H \in \mathfrak{a}^+ \cap {\mathbf{b}_2}^c. \end{aligned}$$

We shall treat the cases where H is in $\mathfrak{a}^+ \cap {\mathbf{b}_2}^c \cap \mathbf{p}$ and H is in $\mathfrak{a}^+ \cap {\mathbf{b}_2}^c \cap \mathbf{p}^c$ separately (the region $\mathbf{p}$ is defined in (2.4)).

First suppose that H is in $\mathfrak{a}^+ \cap {\mathbf{b}_2}^c \cap \mathbf{p}$ and choose $\varepsilon = 1/\rho(H)$. By integrating by parts J times with respect to the variable λ_ℓ, we see that

$$\begin{aligned} |k_1(H)| &\leq C \, \mathrm{e}^{-\rho(H)} \left| i^{-J} \, H_\ell^{-J} \int_{\mathfrak{a}^*} m_1(\lambda + i\rho_\varepsilon) \; \partial_\ell^J \mathrm{e}^{i\lambda(H)} \, \mathrm{d}\lambda \right| \\ &= C \, \mathrm{e}^{-\rho(H)} \left| (-i)^{-J} \, H_\ell^{-J} \int_{\mathfrak{a}^*} \partial_\ell^J m_1(\lambda + i\rho_\varepsilon) \, \mathrm{e}^{i\lambda(H)} \, \mathrm{d}\lambda \right| \\ &\leq \frac{C}{H_\ell^J} \, \mathrm{e}^{-\rho(H)} \Big[\int_{\mathfrak{a}^* \setminus \mathbf{B}} \left| \partial_\ell^J m_1(\lambda + i\rho_\varepsilon) \right| \, \mathrm{d}\lambda + \int_{\mathbf{B}} \left| \partial_\ell^J m_1(\lambda + i\rho_\varepsilon) \right| \, \mathrm{d}\lambda \Big]. \end{aligned}$$

We use estimates (5.3) with $I' = 0'$ and $i_\ell = J$, and obtain

$$\begin{aligned} |k_1(H)| \leq C \, \|m\|_{H'(T_{\mathbf{W}^t}; J, \kappa)} \, \frac{\mathrm{e}^{-\rho(H)}}{H_\ell^J} \Big[&\int_{\mathfrak{a}^* \setminus \mathbf{B}} \frac{\mathrm{e}^{-\operatorname{Re} Q(\lambda + i\rho_\varepsilon)/4}}{\left| Q\big(\lambda + i\rho_\varepsilon\big) \right|^{i_\ell/2}} \, \mathrm{d}\lambda \\ &+ \int_{\mathbf{B}} \left| Q(\lambda + i\rho_\varepsilon) \right|^{-\kappa - J} \, \mathrm{d}\lambda \Big]. \end{aligned}$$

It is straightforward to check that $\operatorname{Re} Q(\lambda + i\rho_\varepsilon) \geq |\lambda|^2$ for all λ in $\mathfrak{a}^*$. Hence the first integral is majorised by $\int_{\mathfrak{a}^* \setminus \mathbf{B}} \exp(-|\lambda|^2/4) \, |\lambda|^{-i_\ell} \, \mathrm{d}\lambda$, which is clearly convergent and independent of ε. To estimate the second integral we observe that $\kappa + J > (\ell + 1)/2$ for every κ in $[0, 1]$. Then Lemma 5.1 (with $\gamma = \kappa + J$) implies that

$$\int_{\mathbf{B}} |Q(\lambda + i\rho_\varepsilon)|^{-\kappa - J} \, \mathrm{d}\lambda \leq C \, (1 + \varepsilon^{(\ell+1)/2 - J - \kappa}) \qquad \forall \varepsilon \in (0, 1/4).$$

Recall that $\varepsilon = 1/\rho(H)$ and that H is in $\mathfrak{a}^+ \cap {\mathbf{b}_2}^c \cap \mathbf{p}$, so that H_ℓ is (positive and) bounded away from 0. Therefore,

$$\begin{aligned} |k_1(H)| &\leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,\kappa)} \frac{\mathrm{e}^{-\rho(H)}}{H_\ell^J} \big[1 + H_\ell^{J+\kappa-(\ell+1)/2}\big] \\ &\leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,\kappa)}\, \mathrm{e}^{-\rho(H)}\, H_\ell^{\kappa-(\ell+1)/2} \\ &\leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,\kappa)}\, \mathrm{e}^{-\rho(H)} \big[1+\mathcal{N}(H)\big]^{2\kappa-(\ell+1)} \qquad \forall H \in \mathfrak{a}^+ \cap {\mathbf{b}_2}^c \cap \mathbf{p}, \end{aligned}$$

as required.

Next suppose that H is in $\mathfrak{a}^+ \cap {\mathbf{b}_2}^c \cap \mathbf{p}^c$ and choose $\varepsilon = 1/|H'|^2$. Note that $\varepsilon \leq C/\rho(H)$, where C does not depend on H. Suppose that $H = (H', H_\ell)$ is given. Denote by ∂' the directional derivative on $\mathfrak{a}^*$ in the direction of H'. By integrating by parts, we see that

$$\begin{aligned} |k_1(H)| &\leq C\, \mathrm{e}^{-\rho(H)} \Big| i^{-J}\ |H'|^{-J} \int_{\mathfrak{a}^*} m_1\big(\lambda + i\rho_\varepsilon\big)\, (\partial')^J \mathrm{e}^{i\lambda(H)}\, \mathrm{d}\lambda \Big| \\ &= C\, \mathrm{e}^{-\rho(H)} \Big| |H'|^{-J} \int_{\mathfrak{a}^*} (\partial')^J m_1\big(\lambda + i\rho_\varepsilon\big)\, \mathrm{e}^{i\lambda(H)}\, \mathrm{d}\lambda \Big|. \end{aligned} \tag{5.4}$$

By arguing much as above (we use (5.3) with $|I'| = J$ and $i_\ell = 0$), we see that if $\kappa > 0$, then

$$\begin{aligned} |k_1(H)| &\leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,\kappa)} \frac{\mathrm{e}^{-\rho(H)}}{|H'|^J}\ \big[1 + \varepsilon^{(\ell+1-J)/2-\kappa}\big] \\ &\leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,\kappa)} \frac{\mathrm{e}^{-\rho(H)}}{|H'|^J}\ \big[1 + |H'|^{J+2\kappa-(\ell+1)}\big] \\ &\leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,\kappa)}\, \mathrm{e}^{-\rho(H)} \big[1+\mathcal{N}(H)\big]^{2\kappa-(\ell+1)} \qquad \forall H \in \mathfrak{a}^+ \cap {\mathbf{b}_2}^c \cap \mathbf{p}^c, \end{aligned}$$

as required to conclude the proof of (i) in the case $\kappa > 0$. If, instead, $\kappa = 0$, then by arguing much as above we see that

$$\begin{aligned} |k_1(H)| \leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,0)} \frac{\mathrm{e}^{-\rho(H)}}{|H'|^J} \Big[\int_{\mathfrak{a}^*\setminus\mathbf{B}} \frac{\mathrm{e}^{-\operatorname{Re} Q(\lambda+i\rho_\varepsilon)/4}}{\big|Q\big(\lambda+i\rho_\varepsilon\big)\big|^J}\, \mathrm{d}\lambda \\ + \int_{\mathbf{B}} \big|Q(\lambda+i\rho_\varepsilon)\big|^{-J}\, \mathrm{d}\lambda \Big]. \end{aligned}$$

By Lemma 5.1 the last integral is estimated by $C\,\log(1/\varepsilon)$, so that

$$\begin{aligned} |k_1(H)| &\leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,0)} \frac{\mathrm{e}^{-\rho(H)}}{|H'|^J}\ \log|H'| \\ &\leq C\, \|m\|_{H'(T_{\mathbf{W}^t};J,0)}\, \mathrm{e}^{-\rho(H)} \big[1+\mathcal{N}(H)\big]^{-\ell-1}\ \log\big[2+\mathcal{N}(H)\big], \end{aligned}$$

where we have used the fact that there exists a positive constant c such that

$$c\,\mathcal{N}(H', H_\ell) \leq |H'| \leq \mathcal{N}(H', H_\ell) \qquad \forall (H', H_\ell) \in \mathfrak{a}^+ \cap {\mathbf{b}_2}^c \cap \mathbf{p}^c.$$

The proof of (i) is complete.

Next we prove (ii). Observe that for any vector η in $\partial\mathbf{B}^+$ and any positive integer $j \leq L$ the derivative $\partial_\eta^j m$ of order j in the direction of η may be written as a linear combination of the derivatives $D^I m$ with $|I| = j$. Therefore,

$$\big|\partial_\eta^j m(\zeta)\big| \leq C\, \|m\|_{H(T_{\mathbf{B}^t};J,\kappa)}\ |Q(\zeta)|^{-\kappa-j} \qquad \forall \zeta \in \mathbf{B} + i\mathbf{B}^+. \tag{5.5}$$

By the Leibniz rûle, m_1 satisfies a similar estimate. Given H in $\mathfrak{a}^+ \cap {\mathbf{b}_2}^c$, define ε and η by $\varepsilon = 1/(|\rho|\ |H|)$ and $\eta = (|\rho|/|H|)\, H$. For the duration of the proof of (ii) we write η_ε instead of $(1-\varepsilon)\eta$. By shifting the integration to the space $\mathfrak{a}^* + i\eta_\varepsilon$ and integrating by parts L times, we see that

$$\begin{aligned} k_1(H) &= \mathrm{e}^{-(1-\varepsilon)|\rho||H|} \int_{\mathfrak{a}^*} m_1\big(\lambda + i\eta_\varepsilon\big)\, \mathrm{e}^{i\lambda(H)}\, \mathrm{d}\lambda \\ &= \frac{\mathrm{e}^{-(1-\varepsilon)|\rho||H|}}{(i\,\eta(H))^L} \int_{\mathfrak{a}^*} m_1\big(\lambda + i\eta_\varepsilon\big)\, \partial_\eta^L \mathrm{e}^{i\lambda(H)}\, \mathrm{d}\lambda \\ &= \frac{\mathrm{e}^{-(1-\varepsilon)|\rho||H|}}{(-i\ |\rho|\ |H|)^L} \int_{\mathfrak{a}^*} \partial_\eta^L m_1\big(\lambda + i\eta_\varepsilon\big)\, \mathrm{e}^{i\lambda(H)}\, \mathrm{d}\lambda. \end{aligned}$$

By arguing as in the proof of (i) we see that there exists a constant C such that for every ε in $(0, 1/4)$,

$$\int_{\mathfrak{a}^* \setminus \mathbf{B}} \big|\partial_\eta^L m_1(\lambda + i\eta_\varepsilon)\big|\ \mathrm{d}\lambda \le C\, \|m\|_{H(T_{\mathbf{B}^t}; L,\kappa)} \qquad \forall m \in H(T_{\mathbf{B}^t}; L, \kappa). \tag{5.6}$$

This and (5.5) imply that

$$|k_1(H)| \le C\, \|m\|_{H(T_{\mathbf{B}^t}; L,\kappa)}\, \frac{\mathrm{e}^{-|\rho||H|}}{|H|^L} \Big[1 + \int_{\mathbf{B}} \big|Q\big(\lambda + i\eta_\varepsilon\big)\big|^{-\kappa-L}\, \mathrm{d}\lambda\Big].$$

We use Lemma 5.1 to estimate the last integral. If $\kappa = 0$ and ℓ is odd, then $L = (\ell+1)/2$. Therefore the last integral is majorised by $C\, \log(1/\varepsilon)$. Thus,

$$\begin{aligned} |k_1(H)| &\le C\, \|m\|_{H(T_{\mathbf{B}^t}; L,0)}\, \frac{\mathrm{e}^{-|\rho||H|}}{|H|^L}\, \log(1/\varepsilon) \\ &\le C\, \|m\|_{H(T_{\mathbf{B}^t}; L,0)}\, \mathrm{e}^{-|\rho|\,|H|} \big[1 + \rho(H)\big]^{-(\ell+1)/2} \log\big[2 + \rho(H)\big], \end{aligned}$$

where we have used the fact that if H is in $\mathfrak{a}^+$, then $\rho(H) = |\rho|\, H_1 \le |\rho|\, |H|$. If, instead, either ℓ is even or $\kappa > 0$, then $L + \kappa > (\ell+1)/2$, so that by Lemma 5.1

$$\begin{aligned} |k_1(H)| &\le C\, \|m\|_{H(T_{\mathbf{B}^t}; L,\kappa)}\, \frac{\mathrm{e}^{-|\rho||H|}}{|H|^L} \big[1 + |H|^{L+\kappa-(\ell+1)/2}\big] \\ &\le C\, \|m\|_{H(T_{\mathbf{B}^t}; L,\kappa)}\, \mathrm{e}^{-|\rho||H|} \big[1 + |H|\big]^{\kappa-(\ell+1)/2} \\ &\le C\, \|m\|_{H(T_{\mathbf{B}^t}; L,\kappa)}\, \mathrm{e}^{-|\rho||H|} \big[1 + \rho(H)\big]^{\kappa-(\ell+1)/2} \qquad \forall H \in \mathfrak{a}^+ \cap {\mathbf{b}_2}^c. \end{aligned}$$

The proof of (ii) is complete. □

Definition 5.3. For any s in $[0, \infty)$ define the function Υ^s and the measure μ^s by

$$\Upsilon^s(\lambda) = (1 + |\lambda|)^s \qquad \text{and} \qquad \mathrm{d}\mu^s(\lambda) = \Upsilon^s(\lambda)\ \mathrm{d}\lambda \qquad \forall \lambda \in \mathfrak{a}^*.$$

Suppose that $\mathbf{E}$ is a Weyl invariant subset of $\mathbf{W}$ and that J is a nonnegative integer. Denote by $Y(\mathbf{E}, J)$ the vector space of all Weyl invariant holomorphic functions m in $T_{\mathbf{E}}$ such that $S^s_{\mathbf{E},J}(m) < \infty$ for all s in $[0, \infty)$, where

$$S^s_{\mathbf{E},J}(m) = \max_{|I| \le J} \sup_{\eta \in \mathbf{E}} \int_{\mathfrak{a}^*} \big|D^I m(\lambda + i\eta)\big|\ \mathrm{d}\mu^s(\lambda).$$

We endow $Y(\mathbf{E}, J)$ with the locally convex topology induced by the family of seminorms $\{S^s_{\mathbf{E},J} :\ s \in [0, \infty)\}$. With this topology $Y(\mathbf{E}, J)$ becomes a Fréchet space.

Remark 5.4. Observe that for every s in $[0,\infty)$ there exists a constant C such that

$$\Upsilon^s(\lambda) \leq (1+|\lambda+i\eta|)^s \leq C\,\Upsilon^s(\lambda) \qquad \forall \lambda \in \mathfrak{a}^* \quad \forall \eta \in \mathbf{W}.$$

Consequently,

$$\begin{aligned} S^s_{\mathbf{E},J}(m) &\leq \max_{|I|\leq J} \sup_{\eta\in\mathbf{E}} \int_{\mathfrak{a}^*} \left|D^I m(\lambda+i\eta)\right| (1+|\lambda+i\eta|)^s \,\mathrm{d}\lambda \\ &\leq C\, S^s_{\mathbf{E},J}(m) \qquad \forall m \in Y(\mathbf{E},J). \end{aligned}$$

We shall use this observation without any further comment.

For any nontrivial subset F of Σ_s and $0<\delta\leq\varepsilon<\infty$, define the region $\mathfrak{w}(F;\delta,\varepsilon)$ by

(5.7)
$$\mathfrak{w}(F;\delta,\varepsilon) = \{H\in\mathbf{s}_2 : \alpha(H)\leq\delta\,|H| \;\; \forall\alpha\in F, \text{ and } \alpha(H)\geq\varepsilon\,|H| \;\; \forall\alpha\in\Sigma_s\setminus F\}.$$

In the following proposition we put together some useful facts concerning the sets $\mathfrak{w}(F;\delta,\varepsilon)$ that will be used below. For any c in $\mathbb{R}^+$, define $(\mathbf{s}_F)_c$ and $(\mathbf{s}^F)_c$ by

$$(\mathbf{s}_F)_c = \{H_F \in \overline{\mathfrak{a}_F} : 0 \leq \omega_F(H_F) \leq c\}$$

and

$$(\mathbf{s}^F)_c = \{H^F \in \overline{\mathfrak{a}^F} : 0 \leq \omega^F(H^F) \leq c\}.$$

Lemma 5.5. *Suppose that F is a nontrivial subset of Σ_s. The following hold:*

(i) *if H is in $\mathfrak{w}(F;\delta,\varepsilon)\cap{\mathbf{b}_1}^c$, then H_F is in $\overline{(\mathfrak{a}_F)^+}$, H^F is in $(\mathfrak{a}^F)^+$,*

$$\omega^F(H^F)\geq\omega^F(H)\geq\varepsilon\,|H|\geq\varepsilon \qquad \text{and} \qquad |H_F|\leq\gamma\,\delta\,|H|, \tag{5.8}$$

where γ is a positive constant which depends on the root system Σ;

(ii) *if H is in $\mathfrak{w}(F;\delta,\varepsilon)$, then H_F is in $(\mathbf{s}_F)_2$.*

Proof. For the proof of (i) see [3, 3.16.2-3.16.4].

To prove (ii) suppose that H is in $\mathfrak{w}(F;\delta,\varepsilon)$ and that $\omega(H)=\alpha(H)$ for some α in Σ_s. If α is in F, then $\alpha(H_F)=\alpha(H)\leq 2$. If, instead, α is in $\Sigma_s\setminus F$, then

$$\alpha(H)\geq\varepsilon\;|H|\geq\frac{\varepsilon}{\delta}\,\beta(H) \qquad \forall\beta\in F.$$

Hence

$$\omega_F(H_F)\leq\frac{\delta}{\varepsilon}\,\alpha(H)\leq 2\,\frac{\delta}{\varepsilon}\leq 2,$$

so that H_F is in $(\mathbf{s}_F)_2$, as required. □

Define σ by

$$\sigma = \min\{|\rho_F| : \emptyset\subset F\subseteq\Sigma_s\}, \tag{5.9}$$

and denote by $\mathbf{E}_\sigma$ the Weyl invariant subset of $\mathbf{W}$ defined by

$$\mathbf{E}_\sigma = \{\eta\in\mathbf{W} : |\eta - w\cdot\rho|\geq\sigma \text{ for all } w\in W\}.$$

Set $\mathrm{Cosh}_{2\rho}(H) := \sum_{w\in W} \mathrm{e}^{2w\cdot\rho(H)}$ for all H in $\mathfrak{a}$ and denote by $\mathcal{M}_{2\rho}$ the multiplication operator acting on K–bi-invariant functions f on G by

$$\big(\mathcal{M}_{2\rho}f\big)(\exp H) = \mathrm{Cosh}_{2\rho}(H)\, f(\exp H) \qquad \forall H\in\mathfrak{a}.$$

Note that there exist positive constants C_1 and C_2 such that

$$C_1\,\mathrm{e}^{2\rho(H)} \leq \mathrm{Cosh}_{2\rho}(H) \leq C_2\,\mathrm{e}^{2\rho(H)} \qquad \forall H\in\overline{\mathfrak{a}^+}. \tag{5.10}$$

The proof of the following lemma is reminiscent of the proof of [3, Thm 3.7] and of the main result in [11, Section 7.10]. All these proofs use the Trombi–Varadarajan expansion of spherical functions and an induction argument.

Lemma 5.6. *The following hold:*

(i) *the map $\mathcal{M}_{2\rho} \circ \mathcal{H}^{-1}$ is bounded from $Y(\mathbf{E}_\sigma, 0)$ to $L^\infty(\mathbf{s}_2)$;*
(ii) *if $J \geq \ell + 1$, then the map $\mathcal{M}_{2\rho} \circ \mathcal{H}^{-1}$ is bounded from $Y(\mathbf{E}_\sigma, J)$ to $L^1(\mathbf{s}_2)$ (with respect to the Lebesgue measure).*

Proof. Suppose that m is in $Y(\mathbf{E}_\sigma, O)$, and denote by k its inverse spherical Fourier transform

$$k(\exp H) = \int_{\mathfrak{a}^*} \varphi_\lambda(\exp H)\, m(\lambda)\, \mathrm{d}\mu(\lambda) \qquad \forall H \in \mathfrak{a}.$$

It is straightforward to check that this integral is absolutely convergent.

First suppose that $\ell = 1$. Then $\mathbf{s}_2$ is the interval $\{H \in \overline{\mathfrak{a}^+} : 0 \leq \alpha(H) \leq 2\}$, where α denotes the unique simple positive root. In particular, $\mathbf{s}_2$ is a bounded subset of $\overline{\mathfrak{a}^+}$, and the function $H \mapsto \mathrm{e}^{2\rho(H)}$ is bounded on $\mathbf{s}_2$. Furthermore, $\sigma = |\rho|$, so that $\mathbf{E}_\sigma = \{0\}$. Now, (2.8) and the fact that $\|\varphi_\lambda\|_\infty = 1$ for any λ in $\mathfrak{a}^*$ imply that

$$\begin{aligned} \left|\mathrm{e}^{2\rho(H)} k(\exp H)\right| &\leq C \int_{\mathfrak{a}^*} |m(\lambda)|\, \big(1 + |\lambda|\big)^{n-\ell}\, \mathrm{d}\lambda \\ &= C\, S^{n-\ell}_{\mathbf{E}_\sigma, 0}(m) \qquad \forall H \in \mathbf{s}_2, \end{aligned}$$

where C does not depend on m in $Y(\mathbf{E}_\sigma, O)$. Therefore, by (5.10),

$$\|\mathcal{M}_{2\rho} k\|_{L^\infty(\mathbf{s}_2)} \leq C\, S^{n-\ell}_{\mathbf{E}_\sigma, 0}(m),$$

whence

$$\|\mathcal{M}_{2\rho} k\|_{L^1(\mathbf{s}_2)} \leq C\, S^{n-\ell}_{\mathbf{E}_\sigma, 0}(m),$$

because $\mathbf{s}_2$ has finite measure. This proves both (i) and (ii) in the case where $\ell = 1$.

Now suppose that $\ell \geq 2$ and that m is in $Y(\mathbf{E}_\sigma, O)$. We observe preliminarily that, arguing as we did above in the case where $\ell = 1$, we may show that

$$\|\mathcal{M}_{2\rho} k\|_{L^\infty(\mathbf{s}_2 \cap \mathbf{b}_1)} \leq C\, S^{n-\ell}_{\mathbf{E}_\sigma, 0}(m).$$

Since $\mathbf{s}_2 \cap \mathbf{b}_1$ has finite measure,

$$\|\mathcal{M}_{2\rho} k\|_{L^1(\mathbf{s}_2 \cap \mathbf{b}_1)} \leq C\, S^{n-\ell}_{\mathbf{E}_\sigma, 0}(m). \tag{5.11}$$

Thus, in the rest of the proof we may assume that $H \in \mathbf{s}_2 \setminus \mathbf{b}_1$.

A consequence of [3, Lemma 2.1.7] is that $\mathbf{s}_2$ is covered by a finite number of regions $\mathfrak{w}(F; \delta_F, \varepsilon_F)$, where $\emptyset \subset F \subseteq \Sigma_s$, δ_F and ε_F may be chosen so that $0 < \delta_F \leq \varepsilon_F < \infty$, and δ_F is as small as we need. We shall prove that $\mathcal{M}_{2\rho} k$ is either bounded or integrable in $\mathbf{s}_2$ by showing that $\mathcal{M}_{2\rho} k$ is bounded or integrable respectively in $\mathfrak{w}(F; \delta_F, \varepsilon_F)$ for every nontrivial subset F of Σ_s.

Fix $F \subseteq \Sigma_s$, δ_F and ε_F as above. By using the Trombi–Varadarajan asymptotic expansion for the spherical functions and the Weyl invariance of m, for each positive integer N we may write

$$k(\exp H) = \sum_{q \in \Lambda^F,\, |q| < N} h^F_q(H) + r^F_N(H) \qquad \forall H \in \mathfrak{w}(F; \delta_F, \varepsilon_F),$$

where $h_q^F(H)$ is defined, for every H in $\mathfrak{w}(F;\delta_F,\varepsilon_F)$, by

$$h_q^F(H) = |W_F\backslash W|\;\mathrm{e}^{-\rho^F(H)} \int_{\mathfrak{a}^*} |\mathbf{c}_F(\lambda)|^{-2}\;\big[(\check{\mathbf{c}}^F)^{-1} m\big](\lambda)\,\varphi^F_{\lambda,q}(\exp H)\,\mathrm{d}\lambda \tag{5.12}$$

and r_N^F is a remainder term. We extend h_q^F and r_N^F to $\mathbf{s}_2$ by setting them equal to 0 outside $\mathfrak{w}(F;\delta_F,\varepsilon_F)$.

First we prove (i). We argue by induction on the rank ℓ of the symmetric space. We have already proved (i) in the case where $\ell = 1$. Suppose that (i) holds for all symmetric spaces of the noncompact type and rank $\leq \ell-1$, and consider a symmetric space X of the noncompact type and rank ℓ.

Consider the remainder term r_N^F. By Theorem 2.1 (iv) and (5.8) there exist positive constants C and d such that

$$\begin{aligned}\big|r_N^F(H)\big| &\leq C\,\mathrm{e}^{-\rho(H)-N\omega^F(H)}\,\big(1+|H|\big)^d \int_{\mathfrak{a}^*} |m(\lambda)|\,\big(1+|\lambda|\big)^d\,\mathrm{d}\lambda \\ &\leq C\,\mathrm{e}^{-2\rho(H)}\,\mathrm{e}^{|\rho||H|-N\varepsilon_F|H|}\,\big(1+|H|\big)^d\,S^d_{\mathbf{E}_\sigma,0}(m) \qquad \forall H\in\mathfrak{w}(F;\delta_F,\varepsilon_F).\end{aligned} \tag{5.13}$$

Choose $N > |\rho|/\varepsilon_F$. Then

$$\|\mathcal{M}_{2\rho} r_N^F\|_{L^\infty(\mathbf{s}_2)} \leq C\,S^d_{\mathbf{E}_\sigma,0}(m). \tag{5.14}$$

Next, suppose that q is in $\Lambda^F\setminus\{0\}$ with $|q|<N$. We may write the integral in (5.12) as an iterated integral, where the outer integral is on $(\mathfrak{a}^*)_F$ and the inner integral on $(\mathfrak{a}^*)^F$.

For the rest of the proof, for each $v\in(0,1)$ we shall write ρ_v^F instead of $(1-v)\,\rho^F$.

Since m is holomorphic in $T_{\mathbf{W}}$, $\varphi^F_{\lambda,q}$ and $(\check{\mathbf{c}}^F)^{-1}$ are holomorphic in a neighbourhood of $T_{((\mathfrak{a}^*)^F)^+}$, for each $v\in(0,1)$ we may move the contour of integration in the inner integral to the space $(\mathfrak{a}^*)^F + i\rho_v^F$ and obtain

$$h_q^F(H) = |W_F\backslash W|\;\mathrm{e}^{-\rho^F(H)} \int_{(\mathfrak{a}^*)_F} \big|\mathbf{c}_F(\lambda_F)\big|^{-2}\,m_q(\lambda_F)\,\mathrm{d}\lambda_F \qquad \forall H\in\mathfrak{w}(F;\delta_F,\varepsilon_F),$$

where

$$m_q(\lambda_F) = \int_{(\mathfrak{a}^*)^F} \big[(\check{\mathbf{c}}^F)^{-1}\,m\big]\big(\lambda_F+\lambda^F+i\rho_v^F\big)\,\varphi^F_{\lambda+i\rho_v^F,q}(\exp H)\,\mathrm{d}\lambda^F.$$

Set $v = 1/\rho^F(H)$, and note that $\big|\rho-\rho_v^F\big| \geq |\rho_F| \geq \sigma$, so that ρ_v^F is in $\mathbf{E}_\sigma$. By the estimate (2.10) on the Harish-Chandra function,

$$\begin{aligned}|\mathbf{c}_F(\lambda_F)|^{-2} &\leq C\,(1+|\lambda_F|)^{\sum_{\alpha\in(\Sigma_F)_0^+} d_\alpha} \\ &\leq C\,(1+\big|\lambda_F+\lambda^F\big|)^{\sum_{\alpha\in\Sigma_0^+} d_\alpha} \\ &= C\,(1+|\lambda|)^{n-\ell}.\end{aligned}$$

By Theorem 2.1 (iii), (2.11) and (5.8) we have that for all H in $\mathfrak{w}(F;\delta_F,\varepsilon_F)$,

$$\begin{aligned}\big|h_q^F(H)\big| &\leq C\,\mathrm{e}^{-\rho^F(H)+\varepsilon_F|H_F|-\rho_v^F(H)-\rho_F(H)-q(H)} \int_{\mathfrak{a}^*} \big|m\big(\lambda+i\rho_v^F\big)\big|\,\mathrm{d}\mu^{n-\ell+d}(\lambda) \\ &\leq C\,\mathrm{e}^{-2\rho(H)+(\varepsilon_F+|\rho_F|)|H_F|-\varepsilon_F|q||H|}\,S^{n-\ell+d}_{\mathbf{E}_\sigma,0}(m) \\ &\leq C\,\mathrm{e}^{-2\rho(H)+(\varepsilon_F+|\rho_F|)\gamma\delta_F|H|-\varepsilon_F|q||H|/2}\,\mathrm{e}^{-\varepsilon_F|q|/2}\,S^{n-\ell+d}_{\mathbf{E}_\sigma,0}(m).\end{aligned} \tag{5.15}$$

Thus, if $\delta_F \leq \gamma^{-1}\,(\varepsilon_F + |\rho_F|)^{-1}\,\varepsilon_F/2$, then

$$\|\mathcal{M}_{2\rho} h_q^F\|_{L^\infty(\mathbf{s}_2)} \leq C\,\mathrm{e}^{-\varepsilon_F |q|/2}\, S^{n-\ell+d}_{\mathbf{E}_\sigma,0}(m).$$

By summing over all q in Λ^F such that $0 < |q| < N$, we see that

$$\Big\|\mathcal{M}_{2\rho}\Big(\sum_{q\in\Lambda^F,\, 0<|q|<N} h_q^F\Big)\Big\|_{L^\infty(\mathbf{s}_2)} \leq C\, S^{n-\ell+d}_{\mathbf{E}_\sigma,0}(m). \tag{5.16}$$

Finally, we consider h_0^F. By arguing much as above, we move the contour of integration to the space $\mathfrak{a}^* + i\rho_v^F$ with $v = 1/\rho^F(H)$. Then (5.12) and the formula for $\varphi^F_{\lambda,0}$ given in Theorem 2.1 (i) imply that for all H in $\mathfrak{w}(F;\delta_F,\varepsilon_F)$,

(5.17)

$$h_0^F(H) = |W_F\backslash W|\,\mathrm{e}^{(v-2)\rho^F(H)} \int_{\mathfrak{a}_F^*} \varphi^F_{\lambda_F}(\exp H_F)\, m_0(\lambda_F;H^F)\ |\mathbf{c}_F(\lambda_F)|^{-2}\,\mathrm{d}\lambda_F,$$

where

$$m_0(\lambda_F;H^F) = \int_{(\mathfrak{a}^*)^F} \big[(\check{\mathbf{c}}^F)^{-1} m\big](\lambda_F + \lambda^F + i\rho_v^F)\,\mathrm{e}^{i\lambda^F(H^F)}\,\mathrm{d}\lambda^F. \tag{5.18}$$

Define σ_F by

$$\sigma_F = \min\{|\rho_{F'}| : \emptyset \subset F' \subseteq F\}.$$

Clearly $\sigma_F \geq \sigma$. Denote by $(\mathbf{E}_F)_{\sigma_F}$ the W_F invariant subset of $\mathbf{W}_F$ defined by

$$(\mathbf{E}_F)_{\sigma_F} = \{\eta_F \in \mathbf{W}_F : |\eta_F - w\cdot\rho_F| \geq \sigma_F \ \text{ for all } w \in W_F\}.$$

Observe that if η_F is in $(\mathbf{E}_F)_{\sigma_F}$, then $\eta = \eta_F + \rho_v^F$ is in $\mathbf{E}_\sigma$. Indeed,

$$\big|\eta_F + \rho_v^F - \rho\big| = \big|\eta_F - \rho_F - v\,\rho^F\big| \geq \big|\eta_F - \rho_F\big| \geq \sigma_F \geq \sigma. \tag{5.19}$$

Now we prove that $m_0(\cdot;H^F)$ is in $Y((\mathbf{E}_F)_{\sigma_F},0)$, uniformly with respect to H^F. Indeed, for any r in $[0,\infty)$,

$$S^r_{(\mathbf{E}_F)_{\sigma_F},0}(m_0(\cdot;H^F)) = \sup_{\eta_F\in(\mathbf{E}_F)_{\sigma_F}} \int_{(\mathfrak{a}^*)_F} \big|m_0(\lambda_F + i\eta_F;H^F)\big|\,\Upsilon^r(\lambda_F)\,\mathrm{d}\lambda_F.$$

By (2.10)

$$\big|m_0(\lambda_F + i\eta_F;H^F)\big| \leq C\int_{(\mathfrak{a}^*)^F} \big|m(\lambda_F + \lambda^F + i\eta_F + i\rho_v^F)\big|\ \Upsilon^{n-\ell}(\lambda_F + \lambda^F)\,\mathrm{d}\lambda^F.$$

Hence, by Tonelli's Theorem and the fact that

$$\Upsilon^r(\lambda_F)\,\Upsilon^{n-\ell}(\lambda_F + \lambda^F) \leq \Upsilon^{n-\ell+r}(\lambda_F + \lambda^F),$$

we have that

$$\begin{aligned} S^r_{(\mathbf{E}_F)_{\sigma_F},0}(m_0(\cdot;H^F)) &\leq \sup_{\eta_F\in(\mathbf{E}_F)_{\sigma_F}} \int_{\mathfrak{a}^*} \big|m(\lambda + i\eta_F + i\rho_v^F)\big|\ \mathrm{d}\mu^{n-\ell+r}(\lambda) \\ &\leq C\, S^{n-\ell+r}_{\mathbf{E}_\sigma,0}(m), \end{aligned}$$

where C is independent of H^F.

Note that the restriction of $\varphi^F_{\lambda_F}$ to $\exp(\mathfrak{a}_F)$ may be interpreted as the restriction to $\exp(\mathfrak{a}_F)$ of an elementary spherical function on X_F, which is a symmetric space of the noncompact type and rank $|F|$ (see Section 2). By Lemma 5.5 (ii) if H is in $\mathfrak{w}(F;\delta_F,\varepsilon_F)$, then H_F is in $(\mathbf{s}_F)_2$. By induction, there exists s in $[0,\infty)$ such that

$$\sup_{H_F\in(\mathbf{s}_F)_2}\Big|\mathrm{e}^{2\rho_F(H_F)}\int_{\mathfrak{a}_F^*}\varphi^F_{\lambda_F}(\exp H_F)\, m_0(\lambda_F;H^F)\;|\mathbf{c}_F(\lambda_F)|^{-2}\,\mathrm{d}\lambda_F\Big| \\ \leq C\, S^s_{(\mathbf{E}_F)_{\sigma_F},0}\big(m_0(\cdot;H^F)\big).$$

Hence

$$\tag{5.20} \begin{aligned}\big|h^F_0(H)\big| &\leq C\,\mathrm{e}^{-2\rho^F(H)}\,\mathrm{e}^{-2\rho_F(H)}\,S^s_{(\mathbf{E}_F)_{\sigma_F},0}\big(m_0(\cdot;H^F)\big)\\ &\leq C\,\mathrm{e}^{-2\rho(H)}\,S^{n-\ell+s}_{\mathbf{E}_\sigma,0}(m) \qquad \forall H\in\mathfrak{w}(F;\delta_F,\varepsilon_F).\end{aligned}$$

From (5.14), (5.16) and (5.20) we deduce that

$$\big\|\mathcal{M}_{2\rho}\,\mathcal{H}^{-1}m\big\|_{L^\infty(\mathbf{s}_2)} \leq C\,S^{s'}_{\mathbf{E}_\sigma,0}(m) \qquad \forall m\in Y(\mathbf{E}_\sigma,0),$$

where $s'=\max\{n-\ell+d,n-\ell+s\}$, and (i) is proved.

Now we prove (ii). Suppose that m is in $Y(\mathbf{E}_\sigma,J)$ with $J\geq\ell+1$. By arguing as in the proof of (i), we may write

$$k(\exp H)=\sum_{q\in\Lambda^F,\,|q|<N} h^F_q(H)+r^F_N(H) \qquad \forall H\in\mathbf{s}_2.$$

Observe that if $N>|\rho|/\varepsilon_F$, from the pointwise estimate (5.13) we deduce that

$$\tag{5.21} \begin{aligned}\int_{\mathfrak{w}(F;\delta_F,\varepsilon_F)}\big|r^F_N(H)\big|\,\mathrm{e}^{2\rho(H)}\,\mathrm{d}H &\leq C\,S^d_{\mathbf{E}_\sigma,0}(m)\int_{\mathfrak{a}^+}\mathrm{e}^{|\rho||H|-N\varepsilon_F|H|}\,\big(1+|H|\big)^d\,\mathrm{d}H\\ &\leq C\,S^d_{\mathbf{E}_\sigma,0}(m).\end{aligned}$$

Similarly, if $\delta_F<\gamma^{-1}\,(\varepsilon_F+|\rho_F|)^{-1}\,\varepsilon_F/2$, then the pointwise estimate (5.15) implies that

$$\begin{aligned}&\int_{\mathfrak{w}(F;\delta_F,\varepsilon_F)}\big|h^F_q(H)\big|\,\mathrm{e}^{2\rho(H)}\,\mathrm{d}H\\ &\leq C\,\mathrm{e}^{-\varepsilon_F|q|/2}\,S^{n-\ell+d}_{\mathbf{E}_\sigma,0}(m)\int_{\mathfrak{a}^+}\mathrm{e}^{(\varepsilon_F+|\rho_F|)\gamma\delta_F|H|}\,\mathrm{e}^{-\varepsilon_F|q||H|/2}\,\mathrm{d}H\\ &\leq C\,\mathrm{e}^{-\varepsilon_F|q|/2}\,S^{n-\ell+d}_{\mathbf{E}_\sigma,0}(m).\end{aligned}$$

By summing over all q in Λ^F such that $0<|q|<N$, we see that

$$\tag{5.22} \int_{\mathfrak{w}(F;\delta_F,\varepsilon_F)}\Big|\sum_{q\in\Lambda^F,\,0<|q|<N} h^F_q(H)\Big|\,\mathrm{e}^{2\rho(H)}\,\mathrm{d}H \leq C\,S^{n-\ell+d}_{\mathbf{E}_\sigma,0}(m).$$

It remains to estimate $\int_{\mathbf{s}_2}\big|h^F_0(H)\big|\,\mathrm{e}^{2\rho(H)}\,\mathrm{d}H$. By arguing as in the proof of (i), we may write

$$h^F_0(H)=|W_F\backslash W|\;\mathrm{e}^{(v-2)\rho^F(H)}\int_{\mathfrak{a}_F^*}\varphi^F_{\lambda_F}(\exp H_F)\,m_0(\lambda_F;H^F)\;|\mathbf{c}_F(\lambda_F)|^{-2}\,\mathrm{d}\lambda_F,$$

where m_0 is defined in (5.18). By integrating by parts $\ell+1$ times with respect to the variable λ^F in the integral in (5.18), we see that

$$m_0(\lambda_F;H)=\frac{1}{\big[i\,B(H_{\rho^F},H^F)\big]^{\ell+1}}\;m_{\ell+1}(\lambda_F;H^F),$$

where

$$m_{\ell+1}(\lambda_F; H^F) = \int_{(\mathfrak{a}^*)^F} \partial_{\rho^F}^{\ell+1}\big[(\check{\mathbf{c}}^F)^{-1} m\big](\lambda_F + \lambda^F + i\rho_v^F)\, \mathrm{e}^{i\lambda^F(H^F)}\, \mathrm{d}\lambda^F.$$

We claim that $m_{\ell+1}(\cdot; H^F)$ is in $Y((\mathbf{E}_F)_{\sigma_F}, 0)$, uniformly with respect to H^F.

Indeed, by Leibniz's rûle $m_{\ell+1}$ may be written as a linear combination of terms of the form

$$\int_{(\mathfrak{a}^*)^F} \big[\partial_{\rho^F}^{\ell+1-j}\big((\check{\mathbf{c}}^F)^{-1}\big)\,\big(\partial_{\rho^F}^{j} m\big)\big](\lambda_F + \lambda^F + i\rho_v^F)\, \mathrm{e}^{i\lambda^F(H)}\, \mathrm{d}\lambda^F,$$

where $0 \leq j \leq \ell + 1$. Therefore (2.10) implies that for any η_F in $(\mathbf{E}_F)_{\sigma_F}$,

$$\begin{aligned}
&\big|m_{\ell+1}(\lambda_F + i\eta_F; H^F)\big| \\
&\leq C \sum_{j=0}^{\ell+1} \int_{(\mathfrak{a}^*)^F} \Big|\partial_{\rho^F}^{j} m(\lambda_F + \lambda^F + i\eta_F + i\rho_v^F)\Big|\, \Upsilon^{n-\ell}(\lambda_F + \lambda^F)\, \mathrm{d}\lambda^F.
\end{aligned}$$

Hence, for any r in $[0, \infty)$,

$$\begin{aligned}
S^r_{(\mathbf{E}_F)_{\sigma_F},0}\big(m_{\ell+1}(\cdot; H^F)\big) &= \sup_{\eta_F \in (\mathbf{E}_F)_{\sigma_F}} \int_{(\mathfrak{a}^*)_F} \big|m_{\ell+1}(\lambda_F + i\eta_F; H^F)\big|\, \Upsilon^r(\lambda_F)\, \mathrm{d}\lambda_F \\
&\leq C \sum_{j=0}^{\ell+1} \sup_{\eta_F \in (\mathbf{E}_F)_{\sigma_F}} \int_{\mathfrak{a}^*} \Big|\partial_{\rho^F}^{j} m(\lambda + i\eta_F + i\rho_v^F)\Big|\, \mathrm{d}\mu^{n-\ell+r}(\lambda) \\
&\leq C\, S^{n-\ell+r}_{\mathbf{E}_\sigma,\ell+1}(m),
\end{aligned}$$

thereby proving the claim. In the last inequality we have used the fact proved above (see (5.19)) that if η_F is in $(\mathbf{E}_F)_{\sigma_F}$, then $\eta_F + \rho_v^F$ is in $\mathbf{E}_\sigma$.

By (i) there exists s in $[0, \infty)$ such that for all H_F in $(\mathbf{s}_F)_2$,

$$\begin{aligned}
&\Big|\mathrm{e}^{2\rho_F(H_F)} \int_{\mathfrak{a}_F^*} \varphi^F_{\lambda_F}(\exp H_F)\, m_{\ell+1}(\lambda_F; H^F)\; |\mathbf{c}_F(\lambda_F)|^{-2}\, \mathrm{d}\lambda_F\Big| \\
&\leq C\, S^s_{(\mathbf{E}_F)_{\sigma_F},0}\big(m_{\ell+1}(\cdot; H^F)\big).
\end{aligned}$$

Hence,

$$\begin{aligned}
\big|h_0^F(H)\big| &\leq C\, \frac{\mathrm{e}^{-2\rho^F(H) - 2\rho_F(H)}}{|H^F|^{\ell+1}}\, S^s_{(\mathbf{E}_F)_{\sigma_F},0}\big(m_{\ell+1}(\cdot; H^F)\big) \\
&\leq C\, \frac{\mathrm{e}^{-2\rho(H)}}{|H^F|^{\ell+1}}\, S^{n-\ell+s}_{\mathbf{E}_\sigma,\ell+1}(m).
\end{aligned}$$

Observe that, by (5.8),

$$|H|^2 = |H_F|^2 + \big|H^F\big|^2 \leq \gamma^2\, \delta_F^2\; |H|^2 + \big|H^F\big|^2 \qquad \forall H \in \mathfrak{w}(F; \delta_F, \varepsilon_F) \cap {\mathbf{b}_1}^c.$$

Hence, if $\delta_F < 1/\gamma$, then

$$\big|H^F\big|^2 \geq (1 - \gamma^2\, \delta_F^2)\; |H|^2 \qquad \forall H \in \mathfrak{w}(F; \delta_F, \varepsilon_F) \cap {\mathbf{b}_1}^c.$$

Therefore,

$$\begin{aligned}
\int_{\mathfrak{w}(F;\delta_F,\varepsilon_F)\cap {\mathbf{b}_1}^c} \big|h_0^F(H)\big|\, \mathrm{e}^{2\rho(H)}\, \mathrm{d}H &\leq C\, S^{n-\ell+s}_{\mathbf{E}_\sigma,\ell+1}(m) \int_{\mathfrak{w}(F;\delta_F,\varepsilon_F)\cap {\mathbf{b}_1}^c} \big|H\big|^{-(\ell+1)}\, \mathrm{d}H \\
&\leq C\, S^{n-\ell+s}_{\mathbf{E}_\sigma,\ell+1}(m) \qquad \forall m \in Y(\mathbf{E}_\sigma, J).
\end{aligned}$$

This, (5.21), (5.22) and (5.11) imply that

$$\big\|\mathcal{M}_{2\rho}\,\mathcal{H}^{-1}m\big\|_{L^1(\mathbf{s}_2)} \leq C\, S^{s'}_{\mathbf{E}_\sigma,\ell+1}(m) \qquad \forall m \in Y(\mathbf{E}_\sigma, J),$$

where $s' = \max\{n-\ell+d, n-\ell+s\}$.

This concludes the proof of (ii) and of the lemma. □

6. Proof of the main result

In the proof of Theorem 3.10 we use Harish-Chandra's expansion of spherical functions away from the walls of the Weyl chamber. Denote by Λ the positive lattice generated by the simple roots in Σ^+. For all H in $\mathfrak{a}^+$ and λ in $\mathfrak{a}^*$,

$$|\mathbf{c}(\lambda)|^{-2}\,\varphi_\lambda(\exp H) = \mathrm{e}^{-\rho(H)} \sum_{q\in\Lambda} \mathrm{e}^{-q(H)} \sum_{w\in W} \mathbf{c}(-w\cdot\lambda)^{-1}\,\Gamma_q(w\cdot\lambda)\,\mathrm{e}^{i(w\cdot\lambda)(H)}. \tag{6.1}$$

The coefficient Γ_0 is equal to 1; the other coefficients Γ_q are rational functions, holomorphic in $T_{\mathbf{W}^t}$ for some t in $\mathbb{R}^-$ (see (2.7) for the definition of $T_{\mathbf{W}^t}$). Moreover, there exists a constant d, and, for each positive integer N, another constant C such that

$$\big|D^I\Gamma_q(\zeta)\big| \leq C\,(1+|q|)^d \qquad \forall \zeta \in T_{\mathbf{W}^t} \qquad \forall I : |I| \leq N. \tag{6.2}$$

Note that the estimate for the derivatives is a consequence of Gangolli's estimate for Γ_q [10] and Cauchy's integral formula. The Harish-Chandra expansion is pointwise convergent in $\mathfrak{a}^+$ and uniformly convergent in $\mathfrak{a}^+ \setminus \mathbf{s}_c$ for every $c > 0$.

Remark 6.1. Suppose that L is a positive integer. There exists a constant C such that

$$\big\|(\check{\mathbf{c}})^{-1}\,\Gamma_q\, m\,\big[1-(1-\mathrm{e}^{-Q})^L\big]\,\mathrm{e}^{Q/2}\big\|_{H'(T_{\mathbf{W}^t};J,\kappa)} \leq C\,(1+|q|)^d\,\|m\|_{H'(T_{\mathbf{W}};J,\kappa)}$$

for all m in $H'(T_{\mathbf{W}};J,\kappa)$ and for all q in Λ. Similarly, there exists a constant C such that

$$\big\|(\check{\mathbf{c}})^{-1}\,\Gamma_q\,(M\circ Q)\,\big[1-(1-\mathrm{e}^{-Q})^L\big]\,\mathrm{e}^{Q/2}\big\|_{H(T_{\mathbf{B}^t};J,\kappa)} \leq C\,(1+|q|)^d\,\|M\|_{\mathfrak{H}(\mathbf{P};J,\kappa)}$$

for all M in $\mathfrak{H}(\mathbf{P};J,\kappa)$ and for all q in Λ.

To prove the first estimate we compute the derivatives of order at most J of

$$(\check{\mathbf{c}})^{-1}\,\Gamma_q\, m_B\,\big[1-(1-\mathrm{e}^{-Q})^L\big]\,\mathrm{e}^{Q/2}$$

by using Leibnitz's rûle. To estimate each of the summands, we use (6.2) and the fact that for some t in $\mathbb{R}^-$ the function $(\check{\mathbf{c}})^{-1}$ is holomorphic in $T_{\mathbf{W}^t}$ and $(\check{\mathbf{c}})^{-1}$ and its derivatives grow at most polynomially at infinity in $T_{\mathbf{W}^t}$ (see (2.9)).

The proof of the second estimate is similar and is omitted.

Remark 6.2. Observe that if $\kappa < 1$, then for every c in $\mathbb{R}^+$,

$$\int_{\mathfrak{a}^+\setminus\mathbf{s}_c} \frac{\mathrm{e}^{-\omega(H)}}{\big[1+\mathcal{N}(H)\big]^{\ell+1-2\kappa}}\,\mathrm{d}H < \infty \qquad \text{and} \qquad \int_{\mathfrak{a}^+\setminus\mathbf{s}_c} \frac{\mathrm{e}^{\rho(H)-|\rho|\,|H|-\omega(H)}}{\big[1+\rho(H)\big]^{(\ell-1)/2}}\,\mathrm{d}H < \infty.$$

We prove that the first integral above is convergent. The proof that the second is convergent is easier and is omitted.

Observe that there exists ε in $\mathbb{R}^+$ such that $\omega(H) \geq \varepsilon\,|H|$ for all H in $\mathbf{\Gamma}_{c_0} \setminus \mathbf{s}_c$. Therefore,

$$\int_{\mathbf{\Gamma}_{c_0}\setminus\mathbf{s}_c} \frac{\mathrm{e}^{-\omega(H)}}{\big[1+\mathcal{N}(H)\big]^{\ell+1-2\kappa}}\,\mathrm{d}H \leq \int_{\mathbf{\Gamma}_{c_0}\setminus\mathbf{s}_c} \mathrm{e}^{-\varepsilon\,|H|}\,\mathrm{d}H < \infty.$$

Moreover, there exists a constant C such that $\mathcal{N}(H) \geq C\,|H'| \geq C\,\rho(H)$ for every H in $\mathfrak{a}^+ \setminus (\mathbf{s}_c \cup \mathbf{\Gamma}_{c_0})$. Hence,

$$\int_{\mathfrak{a}^+\setminus(\mathbf{s}_c\cup\mathbf{\Gamma}_{c_0})} \frac{\mathrm{e}^{-\omega(H)}}{\big[1+\mathcal{N}(H)\big]^{\ell+1-2\kappa}}\,\mathrm{d}H \leq C \int_{\mathfrak{a}^+} \frac{\mathrm{e}^{-\omega(H)}}{\big[1+\rho(H)\big]^{\ell+1-2\kappa}}\,\mathrm{d}H,$$

which is easily seen to be convergent [20, Lemma 3.5].

Now we prove our main result, which we restate for the reader's convenience.

Theorem (2.10). *Denote by J the integer $[\![n/2]\!]+1$. The following hold:*

(i) *if κ is in $[0,1)$, then there exists a constant C such that for all B in ${}^G\mathcal{B}^2(X)$ for which m_B is in $H'(T_{\mathbf{W}};J,\kappa)$,*

$$|\!|\!| B |\!|\!|_{1;1,\infty} \leq C\,\|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)};$$

(ii) *there exists a constant C such that*

$$|\!|\!| M(\mathcal{L}) |\!|\!|_{1;1,\infty} \leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,1)} \qquad \forall M \in \mathfrak{H}(\mathbf{P};J,1).$$

Proof. First we prove (i). Suppose that L is a positive integer $> \kappa + J$. We denote by B_1 and B_2 the operators defined by

$$B_1 = B\left(1-\mathrm{e}^{-\mathcal{L}}\right)^L \qquad \text{and} \qquad B_2 = B\left[1-\left(1-\mathrm{e}^{-\mathcal{L}}\right)^L\right].$$

Thus, $B = B_1 + B_2$. Denote by h_1 the heat kernel at time 1 (see (2.14)). The spherical multipliers associated to B_1 and B_2 are the functions m_{B_1} and m_{B_2} on $T_{\mathbf{W}}$ defined by

$$m_{B_1} = m_B\left(1-\widetilde{h}_1\right)^L \qquad \text{and} \qquad m_{B_2} = m_B\left[1-\left(1-\widetilde{h}_1\right)^L\right].$$

Denote by ψ a smooth K–bi-invariant function such that $\psi(\exp H) = 0$ for H in $\mathbf{s}_1 \cap {\mathbf{b}_2}^c$ and $\psi(\exp H) = 1$ for H in $\mathbf{s}_2^c \cup \mathbf{b}_1$. We decompose k_{B_2} as follows:

$$k_{B_2} = (1-\psi)\,k_{B_2} + \psi\,k_{B_2}.$$

Step I. B_1 is of weak type 1. Since $L > \kappa + J$, the function m_{B_1} and its derivatives up to the order J are bounded on $T_{\mathbf{W}}$. This is due to the fact that $(1-\widetilde{h}_1)^L$ vanishes at the point $i\rho$, together with all its derivatives up to the order $L-1$, and this compensates for the fact that m_{B_1} and its derivatives may be unbounded near $i\rho$. A straightforward computation shows that m_{B_1} satisfies the hypotheses of [2, Corollary 17]. Therefore B_1 is of weak type 1, and $|\!|\!| B_1 |\!|\!|_{1;1,\infty} \leq C\,\|m\|_{H'(T_{\mathbf{W}};J,\kappa)}$.

Step II. Estimates away from the walls. We claim that the function $\psi\,k_{B_2}$ may be written as the sum of two K–bi-invariant functions $k_{B_2}^{(0)}$ and $k_{B_2}^{(1)}$, where $k_{B_2}^{(1)}$ is in $L^1(K\backslash G/K)$ and $k_{B_2}^{(0)}$ satisfies the following estimates in Cartan co-ordinates: there exists a constant C such that for all H in $\mathfrak{a}^+ \setminus \mathbf{s}_1$,

$$
\begin{aligned}
&\left|k_{B_2}^{(0)}(\exp H)\right| \\
&\leq \begin{cases} C\,\|m_B\|\,\mathrm{e}^{-2\rho(H)}\,\big[1+\mathcal{N}(H)\big]^{-\ell-1}\,\log\big(2+\mathcal{N}(H)\big) & \text{if } \kappa=0, \\ C\,\|m_B\|\,\mathrm{e}^{-2\rho(H)}\,\big[1+\mathcal{N}(H)\big]^{2\kappa-\ell-1} & \text{if } 0<\kappa\leq 1 \end{cases}
\end{aligned} \tag{6.3}
$$

(see (2.2) for the definition of $\mathcal{N}$), where we wrote $\|m_B\|$ instead of $\|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)}$.

To prove this, we observe preliminarily that if H is in $\mathfrak{a}^+\setminus\mathbf{s}_1$ and $q=\sum_{\alpha\in\Sigma_s} n_\alpha\,\alpha$, then

$$
q(H)=\sum_{\alpha\in\Sigma_s} n_\alpha\,\alpha(H)\geq\omega(H)\sum_{\alpha\in\Sigma_s} n_\alpha=\omega(H)\,|q|\,, \tag{6.4}
$$

so that

$$
\sum_{q\in\Lambda\setminus\{0\}} \mathrm{e}^{-q(H)}\,(1+|q|)^d\leq \mathrm{e}^{-\omega(H)}\sum_{q\in\Lambda\setminus\{0\}} \mathrm{e}^{1-|q|}\,(1+|q|)^d\leq C\,\mathrm{e}^{-\omega(H)}. \tag{6.5}
$$

This, (6.2) and (2.9) (with $I=0$) imply that

$$
\begin{aligned}
&\sum_{q\in\Lambda\setminus\{0\}} \mathrm{e}^{-q(H)}\int_{\mathfrak{a}^*}\left|m_{B_2}(\lambda)\,\mathbf{c}(-\lambda)^{-1}\,\Gamma_q(\lambda)\,\mathrm{e}^{i\lambda(H)}\right|\,\mathrm{d}\lambda \\
&\leq C\,\|m_B\|_{L^\infty(\mathfrak{a}^*)}\int_{\mathfrak{a}^*}\mathrm{e}^{-Q(\lambda)/2}\,\mathrm{d}\lambda \\
&\leq C\,\|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)}.
\end{aligned} \tag{6.6}
$$

Now, we substitute Harish-Chandra expansion (6.1) in the inversion formula

$$
k_B(\exp H)=c_G\int_{\mathfrak{a}^*} m_B(\lambda)\,\varphi_\lambda(\exp H)\,\,\mathrm{d}\mu(\lambda)\qquad\forall H\in\mathfrak{a}^+,
$$

use the fact that the integrand is Weyl invariant, and obtain

$$
\begin{aligned}
&\psi(H)\,k_{B_2}(\exp H) \\
&=c_G\,\,|W|\,\,\psi(H)\,\mathrm{e}^{-\rho(H)}\sum_{q\in\Lambda}\mathrm{e}^{-q(H)}\int_{\mathfrak{a}^*} m_{B_2}(\lambda)\,\mathbf{c}(-\lambda)^{-1}\,\Gamma_q(\lambda)\,\mathrm{e}^{i\lambda(H)}\,\mathrm{d}\lambda,
\end{aligned}
$$

where $|W|$ denotes the cardinality of the Weyl group and the term by term integration is justified by (6.6). Write $\psi\,k_{B_2}=k_{B_2}^{(0)}+k_{B_2}^{(1)}$, where

$$
\begin{aligned}
k_{B_2}^{(0)}(\exp H)&=c_G\,\,|W|\,\,\psi(H)\,\mathrm{e}^{-\rho(H)}\int_{\mathfrak{a}^*} m_{B_2}(\lambda)\,\mathbf{c}(-\lambda)^{-1}\,\mathrm{e}^{i\lambda(H)}\,\mathrm{d}\lambda, \\
k_{B_2}^{(1)}(\exp H)&=c_G\,\,|W|\,\,\psi(H)\,\mathrm{e}^{-\rho(H)}\sum_{q\in\Lambda\setminus\{0\}}\mathrm{e}^{-q(H)}\int_{\mathfrak{a}^*} m_{B_2}(\lambda)\,\mathbf{c}(-\lambda)^{-1}\,\Gamma_q(\lambda)\,\mathrm{e}^{i\lambda(H)}\,\mathrm{d}\lambda.
\end{aligned}
$$

To prove estimate (6.3) above for $k_{B_2}^{(0)}$ in the case where $0<\kappa\leq 1$, we apply Lemma 5.2 (i) (with $(\check{\mathbf{c}})^{-1}\,m_{B_2}\,\mathrm{e}^{Q/2}$ in place of m), and then Remark 6.1 (with

$q = 0$), and we then obtain that

$$\begin{aligned}\left|k_{B_2}^{(0)}(\exp H)\right| &\leq C \ \left\|(\check{\mathbf{c}})^{-1}\, m_{B_2}\, \mathrm{e}^{Q/2}\right\|_{H'(T_{\mathbf{W}^+};J,\kappa)} \ \frac{\mathrm{e}^{-2\rho(H)}}{\left[1+\mathcal{N}(H)\right]^{\ell+1-2\kappa}} \\ &\leq C\, \|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)}\, \frac{\mathrm{e}^{-2\rho(H)}}{\left[1+\mathcal{N}(H)\right]^{\ell+1-2\kappa}} \qquad \forall H \in \mathfrak{a}^+,\end{aligned}$$

as required. The required estimate for $\kappa = 0$ is proved similarly.

It remains to show that $k_{B_2}^{(1)}$ is in $L^1(K\backslash G/K)$ for all κ in $[0, 1]$. We give the details when $0 < \kappa \leq 1$. Those in the case where $\kappa = 0$ are similar and are omitted. We apply Lemma 5.2 (i) (with the function $(\check{\mathbf{c}})^{-1}\, \Gamma_q\, m_{B_2}\, \mathrm{e}^{Q/2}$ in place of m) to each summand of the series that appears in the definition of $k_{B_2}^{(1)}$, and obtain that

$$\begin{aligned}\left|k_{B_2}^{(1)}(\exp H)\right| &\leq C\, \frac{\mathrm{e}^{-2\rho(H)}}{\left[1+\mathcal{N}(H)\right]^{\ell+1-2\kappa}} \sum_{q\in\Lambda\setminus\{0\}} \mathrm{e}^{-q(H)} \ \left\|(\check{\mathbf{c}})^{-1}\, \Gamma_q\, m_{B_2}\, \mathrm{e}^{Q/2}\right\|_{H'(T_{\mathbf{W}^+},J,\kappa)} \\ &\leq C\, \|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)}\, \frac{\mathrm{e}^{-2\rho(H)-\omega(H)}}{\left[1+\mathcal{N}(H)\right]^{\ell+1-2\kappa}} \qquad \forall H \in \mathfrak{a}^+ \setminus \mathbf{s}_1,\end{aligned}$$

where we have used Remark 6.1, (6.4) and (6.5). Therefore,

$$\begin{aligned}\|k_{B_2}^{(1)}\|_{L^1(G)} &\leq C\, \|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)} \int_{\mathfrak{a}^+\setminus\mathbf{s}_1} \frac{\mathrm{e}^{-\omega(H)}}{\left[1+\mathcal{N}(H)\right]^{\ell+1-2\kappa}}\, \mathrm{d}H \\ &\leq C\, \|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)},\end{aligned}$$

where we have used Remark 6.2. This concludes the proof of Step II.

Step III. Estimates near the walls. We shall prove that the function $(1-\psi)\, k_{B_2}$ is integrable. By Lemma 5.6 (ii) there exists an integer s such that

$$\begin{aligned}\|(1-\psi)\, k_{B_2}\|_{L^1(X)} &\leq C\, \|\mathcal{M}_{2\rho} k_{B_2}\|_{L^1(\mathbf{s}_2)} \\ &\leq C\, S^s_{\mathbf{E}_\sigma,\ell+1}(m_{B_2}).\end{aligned}$$

To conclude the proof of Step III it suffices to show that there exists a constant C such that

$$S^s_{\mathbf{E}_\sigma,\ell+1}(m_{B_2}) \leq C\ \|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)}. \tag{6.7}$$

Indeed, by Leibniz's rûle there exists a constant C such that for every multiindex (I', i_ℓ) with $|I'| + i_\ell \leq \ell + 1$ and for every ζ in $T_{\mathbf{W}^+}$,

$$\begin{aligned}&\left|D^I m_{B_2}(\zeta)\right| \\ &\leq C\ \|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)}\, \mathrm{e}^{-\operatorname{Re} Q(\zeta)/2}\, \max\left[|Q(\zeta)|^{-\kappa-i_\ell-|I'|/2}, |Q(\zeta)|^{-(i_\ell+|I'|)/2}\right].\end{aligned}$$

Then for every η in $\mathbf{E}_\sigma$ and for every multiindex $I = (I', i_\ell)$ with $|I| = |I'| + i_\ell \leq \ell + 1$,

$$\begin{aligned}&\int_{\mathfrak{a}^*} \left|D^{(I',i_\ell)} m_{B_2}(\lambda + i\eta)\right|\, \mathrm{d}\mu^s(\lambda) \\ &\leq C\, \|m_B\|_{H'(T_{\mathbf{W}};J,\kappa)} \Big[\int_{\mathfrak{a}^*\setminus\mathbf{B}_R} (1+|\lambda|)^{s-|I|/2}\, \mathrm{e}^{-\operatorname{Re} Q(\lambda+i\eta)/2}\, \mathrm{d}\lambda \\ &\qquad + \int_{\mathbf{B}_R} |Q(\lambda+i\eta)|^{-\kappa-i_\ell-\left|I'\right|/2}\, \mathrm{d}\lambda\Big],\end{aligned}$$

where R is large enough. Observe that the first integral on the right hand side is dominated by $C\int_{\mathfrak{a}^+}\mathrm{e}^{-|\lambda|^2/3}\,\mathrm{d}\lambda$, where C is a constant depending on s but not on η. Furthermore, since $|Q(\lambda+i\eta)|$ is continuous and does not vanish when η is in $\mathbf{E}_\sigma$ and λ stays in a compact neighbourhood of the origin, we may conclude that it is bounded away from 0. Thus, the second integral on the right hand side in the formula above is finite, and (6.7) is proved.

Step IV. Conclusion. Recall that $k_{B_2}=k_{B_2}^{(0)}+k_{B_2}^{(1)}+(1-\psi)\,k_{B_2}$ and that $k_{B_2}^{(1)}$ and $(1-\psi)\,k_{B_2}$ are in $L^1(K\backslash G/K)$. Thus, the operators $f\mapsto f*k_{B_2}^{(1)}$ and $f\mapsto f*\big[(1-\psi)\,k_{B_2}\big]$ are bounded on $L^1(X)$, hence, *a fortiori*, of weak type 1. The estimates proved in Step II imply that the convolution operator $f\mapsto f*k_{B_2}^{(0)}$ is of weak type 1 by Proposition 4.1. Therefore B_2 is of weak type 1. Since B_1 is of weak type 1 (see Step I), we may conclude that B is of weak type 1, as required to conclude the proof of (i).

The proof of (ii) is similar to the proof of (i). We briefly indicate the changes needed. We decompose $M(\mathcal{L})$ as the sum $M_1(\mathcal{L})+M_2(\mathcal{L})$, where M_1 and M_2 are the functions defined by

$$M_1(z)=M(z)\big(1-\mathrm{e}^{-z}\big)^L \qquad\text{and}\qquad M_2(z)=M(z)\big[1-\big(1-\mathrm{e}^{-z}\big)^L\big].$$

We denote by $m_{M_1(\mathcal{L})}$ and $m_{M_2(\mathcal{L})}$ the spherical multipliers associated to $M_1(\mathcal{L})$ and to $M_2(\mathcal{L})$, respectively. We write $k_{M_2(\mathcal{L})}=(1-\psi)\,k_{M_2(\mathcal{L})}+\psi\,k_{M_2(\mathcal{L})}$, where ψ is defined at the beginning of the proof of (i). By arguing as in Step I above, we see that $M_1(\mathcal{L})$ is of weak type 1 and that $|\!|\!|M_1(\mathcal{L})|\!|\!|_{1;1,\infty}\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,1)}$.

We claim that the function $\psi\,k_{M_2(\mathcal{L})}$ may be written as the sum of two K–bi-invariant functions $k_{M_2(\mathcal{L})}^{(0)}$ and $k_{M_2(\mathcal{L})}^{(1)}$, where $k_{M_2(\mathcal{L})}^{(1)}$ is in $L^1(K\backslash G/K)$ and $k_{M_2(\mathcal{L})}^{(0)}$ satisfies the following estimates in Cartan co-ordinates:

(6.8)

$$\big|k_{M_2(\mathcal{L})}^{(0)}(\exp H)\big|\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,1)}\,\mathrm{e}^{-\rho(H)-|\rho|\,|H|}\,\big[1+\rho(H)\big]^{(1-\ell)/2}\qquad\forall H\in\mathfrak{a}^+\setminus\mathbf{s}_1.$$

Indeed, since M is in $\mathfrak{H}(\mathbf{P};J,1)$, $M\circ Q$ is in $H(T_{\mathbf{B}};J,1)$ by Proposition 3.6. Then we may apply Lemma 5.2 (ii) (with $(\check{\mathbf{c}})^{-1}\,(M_2\circ Q)\,\mathrm{e}^{Q/2}$ in place of m) and obtain that

$$\begin{aligned}
&\big|k_{M_2(\mathcal{L})}^{(0)}(\exp H)\big|\\
&\leq C\,\,\big\|(\check{\mathbf{c}})^{-1}\,(M_2\circ Q)\,\mathrm{e}^{Q/2}\big\|_{H(T_{\mathbf{B}};J,1)}\,\,\mathrm{e}^{-\rho(H)-|\rho|\,|H|}\,\big[1+\rho(H)\big]^{(1-\ell)/2}\\
&\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,1)}\mathrm{e}^{-\rho(H)-|\rho|\,|H|}\,\big[1+\rho(H)\big]^{(\ell-1)/2}\qquad\forall H\in\mathfrak{a}^+,
\end{aligned}$$

thereby proving (6.8). Notice that we have used Remark 6.1 in the last inequality.

It remains to show that $k_{M_2(\mathcal{L})}^{(1)}$ is in $L^1(K\backslash G/K)$. By arguing as in Step II above, we see that $k_{M_2(\mathcal{L})}^{(1)}$ satisfies the following estimate:

$$\big|k_{M_2(\mathcal{L})}^{(1)}(\exp H)\big|\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,1)}\,\mathrm{e}^{-\rho(H)-|\rho|\,|H|-\omega(H)}\,\big[1+\rho(H)\big]^{(1-\ell)/2}.$$

We now use Remark 6.2, and obtain that

$$\begin{aligned}
\|k_{M_2(\mathcal{L})}^{(1)}\|_{L^1(G)}&\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,1)}\int_{\mathfrak{a}^+\setminus\mathbf{s}_1}\frac{\mathrm{e}^{\rho(H)-|\rho|\,|H|-\omega(H)}}{\big[1+\rho(H)\big]^{(\ell-1)/2}}\,\mathrm{d}H\\
&\leq C\,\|M\|_{\mathfrak{H}(\mathbf{P};J,1)}.
\end{aligned}$$

The proof that the function $(1-\psi)\, k_{M_2(\mathcal{L})}$ is integrable with

$$\|(1-\psi)\, k_{M_2(\mathcal{L})}\|_{L^1(K\backslash G/K)} \leq C\, \|M\|_{\mathfrak{H}(\mathbf{P};J,1)}$$

is almost *verbatim* the same as the proof of the corresponding statement in case (i) (see Step III) and is omitted. The required conclusion follows as in Step IV in case (i).

The proof of (ii), and of the theorem, is complete. □

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