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3d mirror duality: From eight to zero supercharges

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Declaration

This dissertation is a result of my own efforts. The work to which it refers is based on my PhD research projects:

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- S. Benvenuti, R. Comi and S. Pasquetti. *Mirror dualities with four supercharges.* JHEP, 10:234, 2024.
- S. Benvenuti, R. Comi and S. Pasquetti. *Star-triangle dualities and supersymmetric improved bifundamentals.* JHEP, 09:171, 2025.
- S. Benvenuti, R. Comi, S. Pasquetti, G. Pedde Ungureanu, S. Rota and A. Shri. *Planar Abelian Mirror Duals of $\mathcal{N} = 2$ $SQCD_3$.* Accepted for publication on Phys. Rev. D.
- S. Benvenuti, R. Comi, S. Pasquetti, G. Pedde-Ungureanu, S. Rota and A. Shri. *A Chiral-Planar dualization algorithm for 3d $\mathcal{N} = 2$ Chern-Simons-matter theories.* JHEP, 10:211 , 2025.
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I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university.

Riccardo Comi
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Abstract

The low-energy dynamics of strongly coupled Quantum Field Theories displays a rich variety of interesting phenomena. These phenomena are notoriously difficult to study, but the introduction of Supersymmetry and/or Conformal symmetry offers valuable tools to probe them. Among non-perturbative phenomena, a central role is played by Infra-Red dualities where two distinct Ultra-Violet QFTs flow to the same interacting CFT in the IR. The first example of an IR duality is Seiberg duality, which relates two 4d $\mathcal{N} = 1$ theories and was formulated exploiting crucial properties of supersymmetric QFTs. Over time, many more IR dualities have been proposed, motivating the search for organizing principles.

In this thesis we focus on mirror duality, which is an IR duality that is typical of 3d $\mathcal{N} = 4$ theories. Mirror duality exchanges the Higgs and the Coulomb branches of the moduli space of a 3d $\mathcal{N} = 4$ theory and, as a consequence, it maps local operators constructed from elementary fields into local disorder operators. Mirror duality for 3d $\mathcal{N} = 4$ theories was understood as being induced from $\mathcal{S}$-duality acting on the Type IIB brane setups engineering these theories. It was also later understood that mirror duality for Abelian theories can be derived in a field-theoretical framework as a consequence of an elementary Abelian mirror duality, relating an SQED with one flavor to a free hypermultiplet.

A first goal of this thesis is to extend this field-theoretical approach to the case of non-Abelian $\mathcal{N} = 4$ mirror dualities. This is achieved by developing an algorithmic strategy in which non-Abelian $\mathcal{N} = 4$ theories are first chopped into QFT blocks, that are collections of hypermultiplets in one-to-one correspondence with five-branes. Each block is then dualized using a set of basic dualities that mimic the action of $SL(2, \mathbb{Z})$ on five-branes, and the resulting components are recombined to obtain the mirror dual of the original theory.

A second objective of this thesis is the generalization of mirror dualities to $\mathcal{N} = 2$ theories. This problem is addressed along two complementary directions. First, the mirror dualization algorithm is extended introducing QFT blocks composing $\mathcal{N} = 2$ theories and their corresponding basic duality moves. Starting from the example of the $\mathcal{N} = 2$ adjoint SQCD, we discover that its mirror dual is a quasi-Lagrangian quiver, in the sense that it is obtained from the gauging of emergent symmetries of strongly coupled SCFTs. This feature is general to an infinite class of $\mathcal{N} = 2$ dualities, which we interpret as the field-theoretical realization of $\mathcal{S}$-duality acting on Type IIB brane setups preserving four supercharges. In this context, we associate to an NS brane a strongly-coupled theory instead of a collection of free hypermultiplets, as in the case of eight supercharges.
A second strategy to derive $\mathcal{N} = 2$ mirror dualities is to perform SUSY-breaking deformations of known $\mathcal{N} = 4$ dualities. This is a non-trivial task, as the mapping between $\mathcal{N} = 2$ vacua of the mirror theories is hard to achieve. We show that the result of this deformation is a novel type of mirror duality relating a $\mathcal{N} = 2$ Chern-Simons SQCD to a purely Abelian planar quiver theory. Finally, inspired by this $\mathcal{N} = 2$ mirror dualities, we propose novel $\mathcal{N} = 0$ mirror dualities for non-Abelian gauge theories coupled to scalars and fermions. These dualities are tested through the mapping of global symmetries, the spectrum of the lightest operators and massive phases.

Contents

Chapter 1

Introduction

Quantum Field Theory (QFT) is one of the greatest achievements of theoretical physics of the last century. Its influence permeates a vast number of topics in modern physics, having obtained an astonishing number of experimental validations, for example, in the description of the fundamental interactions, up to all experimentally accessible energies, as well as in the description of condensed matter systems and even with applications to cosmology.
Despite these remarkable successes, QFT still faces many problems and remains far from being fully understood. One of the most persistent challenges is to understand theories beyond the perturbative regime. This is unsatisfactory since many interesting and physically relevant phenomena arise in the strong coupling regime, as for example color confinement in QCD. A possible solution to this problem consists of extending the symmetries of the systems in order to gain greater control over them. This is the case, for instance, in Conformal Field Theories (CFTs) where the Poincaré group is extended to include conformal transformations, that is the symmetry group of most, if not all, physical systems at the critical point. The space-time symmetry can also be extended by introducing supersymmetry transformations, leading to Supersymmetric QFTs (SQFTs) or to Supersymmetric CFTs (SCFTs). Although some of these models may not be realized in nature, they provide a valuable theoretical laboratory for investigating strongly coupled systems and their possible non-perturbative phenomena. Insights obtained from these theories can then be exploited to explore systems with reduced or even no supersymmetry at all.

A characteristic phenomenon of strongly coupled theories, indeed one that will play a crucial role in this thesis, is *Infra-Red* (IR) *dualities*. We say that we have an IR duality whenever two different Ultra-Violet (UV) QFTs flow to the same IR fixed point at the end of the Renormalization Group (RG) flow, where they are hence described by the same CFT. In some cases a given quantity may be easier to compute in a particular UV description and having access to many dual descriptions provides an extremely powerful tool. The first example of an IR duality among 4-dimensional SQFTs was provided by Seiberg [1][1] exploiting crucial properties of Supersymmetric theories, such as holomorphy and non-renormalization theorems. Seiberg duality relates an $\mathcal{N}=1$ $SU(N)$ gauge theory, coupled to F fundamental and anti-fundamental chiral multiplets, to an $\mathcal{N}=1$ $SU(F-N)$ gauge theory coupled to F fundamentals and anti-fundamentals chiral which interact with a matrix of gauge singlets via a cubic superpotential. The duality is useful, for example, to understand in which range of the parameters the 4d

[1] The first examples of dualities among QFTs were actually much older and can be traced back to the duality among the 2d Sine-Gordon $\leftrightarrow$ Thirring models [2, 3].

$\mathcal{N} = 1$ $SU(N)$ SQCD should flow to an interacting SCFT in the IR. In fact, from the analysis of the β-function one expects that an interacting fixed point may exist only in the range $\frac{3}{2}N < F < 3N$, but this can not be confirmed directly as perturbatively only a Banks-Zaks fixed point near the upper bound $F \sim 3N$ is found. Studying the dual description, one finds that it admits a non-trivial fixed point in the same range of the N, F parameters, however in the dual the Banks-Zaks fixed point is found close to the lower bound $F \sim \frac{3}{2}N$. This observation suggests that an interacting SCFT exists throughout the whole range $\frac{3}{2}N < F < 3N$. This prediction is an example of the power of an IR duality.

Over the years many other dualities have been proposed for theories in various dimensions and with a variety of possible gauge groups and matter contents. It is therefore quite natural to seek an organizing principle among dualities to restrict the landscape of the independent ones.
String theory provides a natural framework for this purpose. In fact, certain SCFTs can be engineered as the low-energy theories living on a stack of branes, possibly interacting with other branes. In such cases, the existence of a duality of the brane setup implies an IR duality for the engineered SCFT. For example, Seiberg duality can be interpreted in this way using the construction of [4], inspired by the pioneering work of [5]. The 4d $\mathcal{N} = 1$ $SU(N)$ SQCD can be engineered in Type IIA as the theory living on a stack of N D4-branes suspended between two NS-branes with F D6-branes inserted, where the branes are oriented so that the setup preserves four supercharges. By performing a sequence of moves that rearranges the NS and D6-branes, the setup can engineer a different SCFT, that is the Seiberg dual of the 4d $\mathcal{N} = 1$ SCFT. Since these moves do not change the physics of the theory on the D4-branes, we conclude that the two SQFTs are IR dual to each other.
Dualities can also be obtained through across-dimensional RG flows (see [6] and references therein), which is the idea that starting from an SCFT in d-dimensions one can compactify it on a $(d - d')$-dimensional manifold to obtain a new d'-dimensional SCFT. A duality for the final theory may arise as a consequence of a duality of the mother theory or also due to symmetries of the manifold on which the theory is compactified. An example of the former is given in [7, 8], in which Seiberg duality is compactified on a circle to produce a similar duality among 3d $\mathcal{N} = 2$ theories. Instead, an example of the latter is the construction of dualities among 4d $\mathcal{N} = 2$ theories via the compactification of 6d $\mathcal{N} = (2, 0)$ theories [9]. For instance, compactifying the 6d $\mathcal{N} = (2, 0)$ A_{N-1} theory on a torus yields the 4d $\mathcal{N} = 4$ $SU(N)$ Yang-Mills theory which inherits the $SL(2, \mathbb{Z})$ duality group of the torus. Dualities can arise also from RG flows within a fixed dimension, such as those resulting from superpotential or mass deformation of other dualities. A notable example is the construction of dualities for 3d $\mathcal{N} = 2$ theories with Chern-Simons interactions as real mass deformations of Aharony duality [10, 11, 12].
There is also the program of *dualities from dualities* in which the goal is to show that a given duality is a consequence of a more fundamental one. An example of this approach is the deconfinement technique [13, 14, 15, 16] in which matter in a tensor representation of the gauge group is "deconfined", that is, it is replaced by a gauge theory using a more fundamental IR duality that does not involve tensors. One can then apply dualities to the resulting quiver theory to derive a new duality for the original theory. In recent years the idea of dualities from dualities has been expanded in many directions. For instance part of this thesis is devoted to showing that 3d mirror dualities are consequence of much more fundamental Seiberg-like dualities.

It is important to keep in mind that IR dualities are conjectures and therefore

should be tested to validate them. The typical tests performed involve the matching of RG invariants, which are quantities of the SCFT that can be computed from the UV description as they do not change along the flow. The same quantities can then be computed in both the UV descriptions and the duality predicts that they should match. One of the most powerful matchings usually performed is that of the partition function on various compact manifolds, which can be computed exactly in supersymmetric theories using localization techniques (see [17] for a review and references therein). In particular, the role of the partition function of 3d $\mathcal{N} \geqslant 2$ theories on the squashed three-sphere, denoted by $\mathbf{S}_b^3$, will be central in this thesis [18, 19, 20]. But one can also match many other quantities in order to better understand the features of a duality, such as anomalies and the spectrum of gauge invariant operators.
It is also possible to match the global symmetries of the SCFT; however, this matching should be treated with care, as global symmetries are not protected along the RG flow. It is, in fact, an interesting question to relate the global symmetry of the UV theory, $\mathcal{G}_{\text{UV}}$, to that of the IR theory, $\mathcal{G}_{\text{IR}}$.
One possibility is that $\mathcal{G}_{\text{IR}} \subset \mathcal{G}_{\text{UV}}$. This occurs whenever a UV global symmetry does not act on the IR spectrum of the theory, which can happen, for instance, simply because the states transforming non-trivially under the symmetry become massive along the RG flow. But it could also be that $\mathcal{G}_{\text{IR}} \supset \mathcal{G}_{\text{UV}}$. In this case, we say that the global symmetry of the theory *enhances* in the IR. Intuitively, this could happen if the states or interactions restricting $\mathcal{G}_{\text{UV}}$ become respectively massive or irrelevant along the RG flow, so that $\mathcal{G}_{\text{IR}}$ is larger than $\mathcal{G}_{\text{UV}}$. Indeed, IR dualities play a prominent role in this story as they might relate two theories with manifestly different UV global symmetry groups. Therefore, in order for the IR symmetries to match, there must be an enhancement or trivialization of the UV global symmetries.
An example of symmetry enhancement is provided by the 3d $\mathcal{N} = 4$ theory with $U(1)$ gauge group and two hypermultiplets of charge 1. The $U(1)$ topological symmetry enhances to $SU(2)$ in the IR, which can be observed as certain monopole operators become conserved currents, thus giving rise to a larger global symmetry. Indeed, this enhancement can be also observed from a very special IR duality that relates the 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with two hypermultiplets to itself. Under this duality, local gauge invariant operators constructed from elementary fields map to monopole operators and vice versa. Thus the $SU(2)$ flavor symmetry rotating the hypermultiplets is expected to match the topological symmetry, justifying the enhancement.
Indeed, also supersymmetry can be enhanced. For example, the $\mathcal{N} = 4$ SQED with two flavors has a second IR dual that is a $\mathcal{N} = 2$ SQCD with gauge group $SU(2)$, Chern-Simons level 1 and four fundamental chiral multiplets Q_i [21, 22]. This theory has also a superpotential that couples cubically three gauge singlets to the gauge invariant operators constructed out of $Q_{1,2}$. The presence of this superpotential breaks the global symmetry from an $SU(4)$ rotating the four flavors, to two $SU(2)$ rotating each a pair of flavors $Q_{1,2}$ and $Q_{3,4}$. This second dual frame for the $\mathcal{N} = 4$ SQED with two flavors has thus the interesting feature of having manifest the $SU(2) \times SU(2)$ global symmetry in the UV, however it exibit only $\mathcal{N} = 2$ supersymmetry which must be then enhanced to $\mathcal{N} = 4$ in the IR.

The self-mirror duality of the $\mathcal{N} = 4$ SQED with two flavors plays a central role in this thesis, as it is a prototypical example of a *mirror duality.* Mirror duality is an IR duality that typically appears in 3d $\mathcal{N} = 4$ theories, but it can be generalized to theories with less amount of supersymmetry and to theories with no supersymmetry at all, as we will discuss in depth in this thesis. The history of mirror duality is long, as it has been studied extensively over the past three decades. Its original formulation, due to Intriligator and Seiberg [23], originates from a simple observation. In 3d $\mathcal{N} = 4$ theories the moduli space of vacua consist of two distinct branches: The Higgs branch, parameterized by the Vacuum Expectation Values (VEVs) of the complex scalar fields in hypermultiplets, modulo gauge transformations; and the Coulomb branch, parameterized by the VEVs of the real scalar fields in $\mathcal{N} = 4$ vector multiplets and of monopole operators. Both branches are hyper-Kähler varieties and, due to relations imposed by the F- and D-terms, these two cones intersect only at the origin, where the SCFT lives[2]. One may then wonder whether there exists a duality relating different 3d $\mathcal{N} = 4$ theories with exchanged Higgs and Coulomb branches. This is indeed the defining feature of mirror duality[3].

It was later understood that certain 3d $\mathcal{N} = 4$ theories can be engineered in string theory, and mirror duality is induced from dualities of the brane setup. The first two results in this direction are the M-theory construction of [27] and the Type IIB construction of [5]. In particular, in the latter, the main idea is that certain 3d $\mathcal{N} = 4$ theories can be engineered via brane setups composed of NS, D5 and D3-branes. Mirror duality is then induced by $\mathcal{S}$-duality which transforms an NS into a D5-brane and vice versa. Later, it was understood by Gaiotto and Witten [28] that $\mathcal{S}$-duality can be performed locally on the brane setup. One can apply $\mathcal{S}$-duality to a spacetime region containing only a single five-brane transforming it (an NS into a D5 and viceversa) and producing domain walls: An $\mathcal{S}$-wall on its left and an $\mathcal{S}^{-1}$-wall on its right. The theory living on a stack of D3 branes intersecting an $\mathcal{S}$-wall was identified as a special 3d $\mathcal{N} = 4$ SCFT called $T[SU(N)]$[4]. As the name suggests, the fusion of an $\mathcal{S}$ and $\mathcal{S}^{-1}$-walls yields a trivial domain wall; therefore if the local dualization is applied on each five-brane in the setup, the global effect of transforming all the five-branes is recovered and no domain walls are left. Indeed one can also consider more general $SL(2, \mathbb{Z})$ transformations, that is the full duality group acting on the five-branes. These transformations give rise to more IR dualities for the 3d $\mathcal{N} = 4$ SCFTs, which are commonly reffered to as generalized mirror dualities.

Another interesting intuition was proposed by Kapustin and Strassler [31]. In this work, the authors observed that Abelian mirror dualities can be understood using a purely field-theoretical approach[5]. This approach has been reformulated in modern language in several recent works, including some of the content of this thesis. In this formulation, a theory is first decomposed into its elementary hypermultiplets, that are either bifundamentals charged under a pair of $U(1)$ gauge groups, or flavors charged under a single $U(1)$ gauge group. This decomposition is achieved by freezing gauge symmetries that are restored later. At this stage, each hypermultiplet is mirror-dualized using a basic duality that relates a free hypermultiplet to a gauged one, that is an SQED with one flavor. This duality is established as a path integral identity showing that

[2]This statement is not true in general, as certain 3d $\mathcal{N} = 4$ theories exhibit a more complicated moduli space structure that is not simply the intersection of two cones, which is the case in the so-called *bad* theories. In what follows, however, we restrict our attention to *good* theories.

[3]It is worth noting that in the last decade mirror duality for 3d $\mathcal{N} = 4$ theories has proven useful in probing the Higgs branches of theories in $d > 3$-dimensions. Initially this strategy was employed for 4d $\mathcal{N} = 2$ theories, by studying the 3d $\mathcal{N} = 4$ mirror of their circle compactification (see [24] and following references). Later, a similar approach was developed to study 5d theories [25] and 6d theories [26].

[4]This statement was later supported from the point of view of AGT correspondence [29] by the fact that the partition function of the $T[SU(N)]$ theory coincides with the $\mathcal{S}$-duality kernel for the Liouville conformal block on the torus with one puncture [30].

[5]An interesting observation made in [31], worth mentioning but not discussed here, is that mirorr duality appears to be a symmetry valid at all scales, and not only as a relation between the IR fixed point of the two theories.

the gauged fermionic superdeterminant coincides with the free one. It was shown also that the scaling dimension of monopole operators in the SQED with one flavor is $\frac{1}{2}$, as expected for free hypermultiplets [32]. Finally, the results of all the dualizations are gauged back together by restoring the previously frozen gauge symmetries. The result of this procedure is a new theory that precisely coincides with the mirror dual of the original theory.
This approach resambles the logic of Gaiotto and Witten [28], where each hypermultiplet composing the theory can be associated to a single five-brane. The basic duality relating each hypermultiplet to a gauged one can then be interpreted as a local dualization, where the gauging corresponds to the action of the $\mathcal{S}$ and $\mathcal{S}^{-1}$-walls.
Moreover, this approach shows that Abelian mirror dualities are not fundamental dualities, in the sense that they can be derived assuming a basic Abelian duality relating the SQED with one flavor to a free hypermultiplet. An immediate question then arises: is it possible to generalize the Kapustin-Strassler approach (and all its consequences) to non-Abelian theories? The answer appears to be positive and part of this thesis will be devoted to this problem.

Another question that we will address in this thesis is whether mirror dualities can be generalized to theories with a lower amount of supersymmetry, in particular to $\mathcal{N}=2$ theories. This is a question that has already been addressed in the literature, notably in [33, 34, 35], where the first examples of $\mathcal{N}=2$ mirror dualities were proposed[6]. Moreover, the Kapustin–Strassler approach can be naturally extended also to the case of $\mathcal{N}=2$ Abelian theories [37]. Together, these observations set the ground for our investigation. The construction of $\mathcal{N}=2$ mirror dualities requires as a first step a careful definition, because lowering supersymmetry causes many of the desired properties of the moduli space to be lost. Two approaches are possible. On one hand, one can consider Type IIB brane setups preserving four supercharges, thus engineering $\mathcal{N}=2$ theories on a stack of D3-branes. These theories are still expected to enjoy IR dualities induced by $\mathcal{S}$-duality, which can be naturally interpreted as mirror dualities. However, it turns out that identifying the precise $\mathcal{N}=2$ theory engineered by a generic Type IIB brane setup preserving four supercharges is challenging, making the string-theoretical approach quite difficult.
One the other hand, one can define $\mathcal{N}=2$ mirror dualities as the result of a SUSY-breaking deformation of $\mathcal{N}=4$ ones. This second approach also proves to be non-trivial as it is difficult to determine the mapping between $\mathcal{N}=2$ vacua of the original $\mathcal{N}=4$ theories.
In this thesis we address both these challenges. Although these two approaches are very different in their nature, one being more string theory inspired and the other more field theory inspired, we will show that the novel $\mathcal{N}=2$ mirror dualities preserve the key property that mesonic operators are exchanged with monopoles, and flavor symmetries with topological symmetries.

Dualities that exchange operators constructed from elementary fields and disorder operators are also well known in non-supersymmetric theories. The most notable example is particle-vortex duality, which relates a critical complex scalar to an Abelian Higgs model [38]. This suggests that, in a certain sense, mirror duality can be extended also to $\mathcal{N}=0$ theories, which presents another intriguing question driving this thesis.

With the problem thus established, we now outline the content of this thesis.

Chapter 2 provides a brief review of preliminary facts regarding 3d $\mathcal{N}=4$ and $\mathcal{N}=2$ theories and mirror duality [39, 34]. We also review the Hanany–Witten Type

[6]Examples with even lower supersymmetry, namely $\mathcal{N}=1$, have also been proposed [36].

IIB construction [5, 28] and the results of Kapustin and Strassler on Abelian mirror dualities [31].

In Chapter 3 we extend the Kapustin–Strassler approach to non-Abelian mirror dualities and generalized $SL(2,\mathbb{Z})$ dualities among 3d $\mathcal{N}=4$ linear quiver theories. The content of this chapter is based on [40].

We begin by defining the elementary QFT blocks that compose these theories, that are in one-to-one correspondence with five-branes that can be inserted in Hanany–Witten brane setups. We then define the field-theoretical realization of $SL(2,\mathbb{Z})$ operators and the notion of product, to verify that these operators satisfy the expected $SL(2,\mathbb{Z})$ relations. With all these ingredients at our disposal, we construct basic duality moves, which correspond to the action of a $SL(2,\mathbb{Z})$ operator on a QFT block, producing a new block as a result. We argue that all these dualities can be shown to follow from the more fundamental Aharony duality, which then induces generalized $SL(2,\mathbb{Z})$ dualities, including mirror duality. Finally, we present an example of a duality web obtained by acting with a sequence of $SL(2,\mathbb{Z})$ operators on a theory.

Starting from Chapter 4 we initiate the program of generalizing mirror dualities to theories with fewer than eight supercharges, starting from $\mathcal{N}=2$ theories. The content of this chapter is based on [41], with connections to [42].

To achieve this result, we generalize the mirror dualization algorithm of the previous chapter to include $\mathcal{N}=2$ QFT blocks. In particular we introduce $\mathcal{N}=2$ flavor blocks and its $\mathcal{S}$-dualization. Differently from the $\mathcal{N}=4$ case, where the $\mathcal{S}$-dualization of a flavor results in a free bifundamental hypermultiplet, in the case of $\mathcal{N}=2$ flavors we obtain a strongly coupled SCFT called $FM[U(N)]$. We argue that this theory can be regarded as an *improved bifundamental*, since it shares key properties with a standard bifundamental hypermultiplet. For instance, it contains a bifundamental operator in it spectrum of gauge invariant operators, while improving a standard bifundamental hypermultiplet by supporting a larger global symmetry. Using this new basic move we construct the mirror dual of the $\mathcal{N}=2$ adjoint $U(N)$ SQCD, which takes the form of a quasi-Lagrangian linear quiver, in the sense that the theory is constructed by gauging diagonal combinations of emergent symmetries of multiple copies of $FM[U(N)]$ theories. The $\mathcal{N}=2$ adjoint SQCD admits a superpotential deformation that recovers the $\mathcal{N}=4$ SQCD, and we show that the dual flows to the correct $\mathcal{N}=4$ mirror theory when we perform the dual superpotential deformation.

We conclude the chapter by arguing that the basic $\mathcal{N}=2$ mirror duality can be interpreted as the field-theoretical realization of the $\mathcal{S}$-dualization of a D5 into an NS-brane when this operation is performed in a Type IIB brane setup preserving four supercharges. This leads to the proposal that, in a setup preserving four supercharges, an NS-brane is not associated to a standard bifundamental hypermultiplet, as in the case with eight supercharges, but instead to an $FM[U(N)]$ *improved bifundamental* theory. We then propose a solution to the long standing problem of how $\mathcal{S}$-duality is realized in $\mathcal{N}=2$ theories engineered by Type IIB brane setups.

In Chapter 5 we continue the program of $\mathcal{N}=2$ mirror dualities but following a different path. The content of this chapter is based on [43, 44].

The strategy is to perform real mass deformation of 3d $\mathcal{N}=4$ mirror dualities with the effect of breaking supersymmetry to $\mathcal{N}=2$. If the deformation is performed correctly, starting from an $\mathcal{N}=4$ mirror dual pair we arrive at an $\mathcal{N}=2$ mirror pair that inherits many properties of the original duality, such as the mapping between mesonic and monopole operators and the exchange of flavor and topological symmetry. We start by focusing on the mirror duality for the $\mathcal{N}=4$ SQCD that we deform by giving a real mass to the adjoint chiral in the $\mathcal{N}=4$ vector multiplet and to the anti-

fundamental chirals in the flavor hypermultiplets. As a result, we obtain an $\mathcal{N}=2$ Chern-Simons SQCD, where the CS level is generated due to fermionic fields acquiring a mass. Mapping this deformation in the mirror dual is a non-trivial task as the same real mass deformation may lead to many inequivalent vacua at different points of the Coulomb branch. Only one of those vacua is expected to be dual to the $\mathcal{N}=2$ CS-SQCD. To individuate this vacuum we analyze the problem using the $\mathbf{S}_b^3$ partition function, discovering that the dual vacuum lies at a point of the Coulomb branch where all the gauge groups are Higgsed to their Cartan subgroup. By carefully analyzing this vacuum, we propose a mirror dual of the $\mathcal{N}=2$ CS-SQCD that is a purely Abelian theory with the shape of a planar quiver, with CS interactions and both a mesonic and monopole superpotentials.
We then study further examples by starting from other $\mathcal{N}=4$ mirror pairs and performing similar deformations. Surprisingly, we discover that all the examples share the same pattern: the $\mathcal{N}=4$ mirror duality becomes a duality relating an $\mathcal{N}=2$ non-Abelian gauge theory to a planar Abelian quiver. To better understand this result, we develop an algorithmic procedure to derive these Abelian planar dualities. This is achieved by starting from the QFT blocks, $SL(2,\mathbb{Z})$ operators and basic duality moves of the $\mathcal{N}=4$ dualization algorithm, and then performing a suitable real mass deformation. We thus obtain an algorithmic strategy analogous to the $\mathcal{N}=4$ one, where $\mathcal{N}=2$ non-Abelian QFT blocks are related to Abelian planar quivers via basic duality moves. The role of the $\mathcal{S}$-wall is taken by a new $\mathcal{N}=2$ theory which we baptize the $G[U(N)]$ theory.

In Chapter 6 we conclude by exploring a further SUSY breaking deformation, starting from $\mathcal{N}=2$ mirror pairs, to obtain $\mathcal{N}=0$ dualities. The content of this chapter is based on [45].
The strategy is to make educated guesses starting from the $\mathcal{N}=2$ mirror dualities that relate non-Abelian gauge theories to planar Abelian quivers. In this way, we propose a new dual for the $U(N)$ CS-QCD with fundamental scalars and fermions, which again takes the form of a planar Abelian quiver, with scalars and fermions charged under the various gauge groups. These new dualities are tested by matching the global symmetries and the spectrum of the lightest gauge invariant operators, for which we compute the spin and the representation under the global symmetry. It is worth mentioning that the proposed dualities can be thought to result from a very mild deformation of the $\mathcal{N}=2$ mirorr duality, where in each chiral multiplet in the two theories, only one among the scalar and the fermion component acquires a mass, appropriately shifting the CS-interactions whenever a fermion becomes massive. These novel dualities provide examples of non-Abelian mirror dualities among $\mathcal{N}=0$ theories, where a key feature is that the lightest operators constructed from elementary fields are mapped to disorder operators, that are dressed monopoles.

Chapter 2

3d mirror duality: a lightning review

In this chapter we review preliminaries about mirror duality for 3d $\mathcal{N}=4$ theories. We review the original formulation of mirror duality [23], and the basics of moduli spaces. We describe the Type IIB construction of 3d $\mathcal{N}=4$ theories of [5] and its relation to mirror duality. Finally we also review the Kapustin-Strassler [31] which provides a field theoretical derivation of Abelian mirror dualities in terms of a fundamental Abelian mirror duality.

2.1 Mirror duality

Mirror duality is a specific type of IR duality that appears (mostly but not only) in 3d $\mathcal{N}=4$ theories. The following review summarizes the original work introducing the idea of mirror duality by Intriligator and Seiberg [23] and [34, 46].

We start from a brief resume of generalities about 3d $\mathcal{N}=4$ theories. The Lagrangian of these theories is constructed from two types of supermultiplets. Vector multiplets provide gauge fields, along with their superpartners. An $\mathcal{N}=4$ vector multiplet contains a real scalar component that transforms in the $(\mathbf{1},\mathbf{3})$ representation of the $SO(4)_R = SU(2)_H \times SU(2)_C$ R-symmetry, thus transforming only under the $SU(2)_C$ subgroup. The matter is provided by hypermultiplets in generic representations of the gauge group. An hypermultiplet contains a complex scalar component transforming in the $(\mathbf{2},\mathbf{1})$ representation of the R-symmetry, thus only under the $SU(2)_H$ subgroup.
We also consider 3d theories with lower amount of SUSY, namely $\mathcal{N}=2$. The R-symmetry of these theories is $U(1)_R$. To construct these theories we consider $\mathcal{N}=2$ vector multiplets, containing a single real scalar component, and chiral multiplets, contaning a complex scalar component. For $\mathcal{N}<4$ it is also possible to consider Chern-Simons interaction satisfying a suitable quantization condition.

In this thesis we will frequently make use of the so-called $\mathcal{N}=2^*$ notation [37], which consist in rewriting $\mathcal{N}=4$ theories in $\mathcal{N}=2$ language. The $SU(2)_H \times SU(2)_R$ R-symmetry decomposes into a $U(1)_R$ $\mathcal{N}=2$ R-symmetry given by the diagonal combination of the two Cartan generators $U(1)_{H+C}$. The anti-diagonal combination $U(1)_{H-C}$ commutes with $U(1)_R$ and therefore provides a global symmetry from the $\mathcal{N}=2^*$ point of view. An $\mathcal{N}=4$ vector multiplet decomposes into a $\mathcal{N}=2$ one, plus a chiral multiplet in the adjoint representation of the gauge group. An hypermultiplet decomposes into a pair of chiral multiplets in conjugated representations. Also, in the $\mathcal{N}=2^*$

language there is a cubic superpotential coupling the two chirals composing an hypermultiplet with the adjoint chiral in the vector multiplet.

The moduli space, i.e. the space of supersymmetric vacua, of 3d $\mathcal{N}=4$ theories consist of two different branches.

- The Higgs branch is an hyper-Kähler variety that is parameterized by VEVs of the scalar fields in hypermultiplets, modded by gauge transformations. By construction, only the $SU(2)_H$ subgroup of the R-symmetry acts non-trivially on this branch. It follows from non-renormalization theorems that the Higgs branch can not be corrected by quantum effects and, therefore, it is said to be classically exact.

- The Coulomb branch is parameterized semi-classically by VEVs of the scalar component of vector multiplets. By construction, only the $SU(2)_C$ subgroup of the R-symmetry acts non-trivially on this branch. Differently from the Higgs branch, it is not protected from quantum corrections that indeed play a crucial role in determining the geometry of this branch, which turns out to be also hyper-Kähler.

Due to its central role, the Coulomb branch deserves a longer exaplanation. In 3d theories in general it is possible to construct a scalar field by Hodge-dualizing the field-strenght tensor:

$$\partial^i \gamma = \epsilon^{ijk} F_{jk} \,. \tag{2.1}$$

If the field-strenght is associated to an Abelian gauge field then it follows from the Bianchi identity that $\epsilon^{ijk}F_{jk}$ is a conserved current. The symmetry associated to this current is called topological and consist of constant shifts for the γ field, which indeed leave the definition (2.1) invariant.
The supersymmetric version of (2.1) is that a vector multiplet can be dualized into an hypermultiplet by pairing together γ with the real scalar component ϕ of the vector multiplet to form a complex field. The exponentiation of this combination defines semi-classically a monopole operator:

$$\mathfrak{M}^{\pm} \sim \mathrm{e}^{\frac{1}{g^2}(\phi \pm i\gamma)} \,. \tag{2.2}$$

The monopoles then transform with charge ± 1 under the $U(1)$ topological symmetry. The insertion of a monopole operator in the path integral can be thought as the insertion of a local state that carries non-zero magnetic flux through an $\mathbf{S}^2$ surface surrounding it. If we use monopole operators to parameterize the Coulomb branch we obtain a variety that looks like an $\mathbb{R}^3 \times \mathbf{S}^1$, where g dictates the radius of $\mathbf{S}^1$. Notice that this metric is singular for $g \to +\infty$, i.e. in the strong coupling regime. As we get closer to this singular point the Coulomb branch receives quantum corrections, so that it is deformed to a cone whose singular point coincides with the origin of the moduli space.

From F and D-term relations it follows that in 3d $\mathcal{N}=4$ theories the Higgs and Coulomb branches can intersct only at the origin of the moduli space which is then given by two hyper-Kähler cones[1]. By observing the peculiar property that the Higgs and Coulomb branch have the same geometrical structure one may wonder if there is the possibility to have a dual description which swaps the two. Mirror duality does

[1]This is actually non true in general. Certain pathological 3d $\mathcal{N}=4$ theories can have a more complicated moduli space which does not look like the intersection of two cones. This is the case of the so-called *bad* theories [28], the precise geometry of the moduli space of these theories is yet not completely understood, even if progresses have been made in this direction [47, 48, 49].

precisely this job and can be formulated as follows [39]. Mirror duality is an IR duality relating two (generically) different 3d $\mathcal{N} = 4$ theories flowing to the same SCFT in the IR. The duality exchanges:

- The $SU(2)_H$ and $SU(2)_C$ subgroups of the R-symmetry.
- The Higgs and the Coulomb branches.
- Mass parameters and Fayet-Iliopoulos parameters.

One of the simplest example of this duality is that of a $U(1)$ gauge theory with two fundamental hypermultiplets, which is self-mirror dual. The Higgs branch of this theory is $\mathbb{C}^2/\mathbb{Z}_2$. Since the theory is Abelian it is possible to construct two independent monopole operators $\mathfrak{M}^{\pm}$ and a third gauge invariant operator which is simply the trace of the scalar field in the vector multiplet. A careful analysis shows that also the Coulomb branch is $\mathbb{C}^2/\mathbb{Z}_2$. Mirror duality swaps the two branches and, since they are identical, the mirror theory is again a $U(1)$ gauge theory with two fundamental hypermultiplets. This is an example of a self-mirror duality.

It was observed shortly later [34] that mirror duality can be generalized to certain 3d $\mathcal{N} = 2$ theories. The idea is that in these theories there is still a notion of Higgs and Coulomb branch, as two branches parameterized either by mesonic or monopole operators. Even though the geometry of the moduli space is not rigidly constrained to be that of two intersecting cones, it is still possible to think of mirror duality as an IR duality that exchanges (some) mesonic and monopole operators of the two theories.

The most notable example is given by a 3d $\mathcal{N} = 2$ $U(1)$ gauge theory with one fundamental hypermultiplet. This theory is dual to the so-called XYZ model, which is a Wess-Zumino model of three chiral multiplets X, Y, Z, with a cubic superpotential $\mathcal{W} = XYZ$, hence the name. Under this duality the two monopoles $\mathfrak{M}^{\pm}$ maps to two chiral fields, say X, Y, while the only mesonic gauge invariant operator that can be costructed maps to the third chiral Z. The moduli space of this theory looks like the intersection of three cones each parameterized by the VEV of a single chiral multiplet. The three cones are hence the same and indeed they can be permuted by an S_3 symmetry. In fact, the gauge theory admits also a self-dual description under which the three operators are mapped non-trivially by swapping them. It follows that under this duality monopole operators may map to mesons and vice-versa. This situation is then a prototyipical example of a $\mathcal{N} = 2$ mirror duality. More examples have been constructed later, for instance in [37] for Abelian quiver theories and, indeed, in the works presented in this thesis.

2.2 Hanany-Witten brane setups and mirror duality

Given the important role in this thesis we now briefly review the Type IIB Hanany-Witten brane construction of 3d $\mathcal{N} = 4$ theories and its connection to mirror duality [5].

We consider setups composed of D3, NS and D5-branes that are oriented in the 10d spacetime as reported in Table 2.1. The orientation is chosen so that any setup involving all three type of branes preserves 8 supercharges.

A linear setup consists of a sequence of five-branes at different x_6 position at which they cross a stack D3 branes, see for example the first row in Figure 2.1. The gauge theory lives on the non-compact x_{012} worldvolume of the D3 branes, and the specific

	0	1	2	3	4	5	6	7	8	9
D3	x	x	x				x			
NS	x	x	x	x	x	x				
D5	x	x	x					x	x	x

Table 2.1: Orientation of the branes.

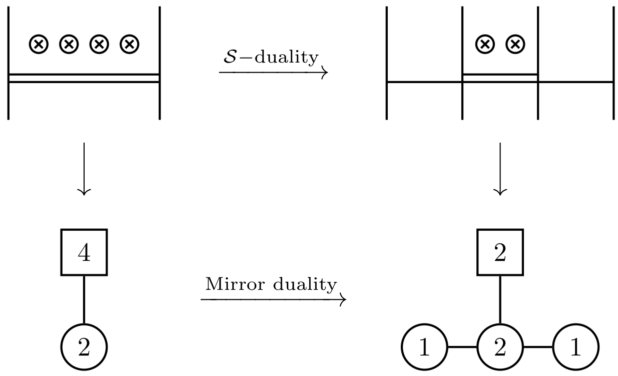


Figure 2.1: Example of $\mathcal{S}$-dual brane setups and their corresponding mirror dual 3d $\mathcal{N}=4$ theories. Vertical line represent NS, crosses represent D5 and horizontal lines are D3-branes. The brane setup on the top-right corner is obtained by performing $\mathcal{S}$-duality on that in the top-left corner followed by HW moves.

Lagrangian described by the setup can be understood by studying the boundary condition imposed by NS and D5-branes. Any interval between two NS branes with N D3-branes in between consist in a $U(N)$ gauge group with an associated $\mathcal{N}=4$ vector multiplet. An NS-brane with N and M D3 branes respectively on its left and on its right, engineers an hypermultiplet in the bifundamental of $U(N)\times U(M)$. Any D5 placed in an interval between two NS branes consists in a flavor for the corresponding gauge group (see Figure 2.1 for an example).

A brane setup constructed as described enjoys a duality called $\mathcal{S}$, which is inherited from its uplift to M-theory. This duality acts on the five-branes by transforming an NS-brane into a D5 and vice-versa. By performing $\mathcal{S}$-duality one produces a second brane setup from which it is possible to read off a second different gauge theory which, by means of the brane duality, is IR dual to the starting theory (see Figure 2.1 for an example).

There is only one last subtlety to consider which is that after performing $\mathcal{S}$-duality on the brane setup one has to reshuffle around the five-branes so that each D5 sees an equal number of D3 branes on both sides. This is necessary to read off the gauge theory from the brane setup using the rules explained above. This can be done using the Hanany-Witten transition which says that whenever a D5 crosses an NS-brane it is created or annihilated a D3-brane stretching between the two.

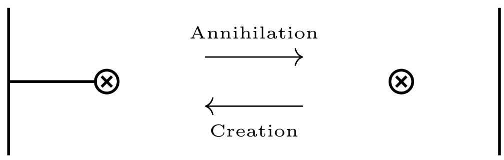


Figure 2.2: Hanany-Witten transition. From left to right a D3-brane is annihilated, similarly from right to left it is created.

This story can be generalized. One can consider the action of the whole $SL(2,\mathbb{Z})$ duality group of the brane setup, that induce so-called generalised mirror dualities for the engineered 3d $\mathcal{N}=4$ theories.
We pick the following choice for the matrix representation of the $\mathcal{S}$ and $\mathcal{T}$ generators of $SL(2,\mathbb{Z})$:

$$\mathcal{S}=\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad \mathcal{T}=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} . \tag{2.3}$$

In this representation the five-branes are pairs of coprime integers (p,q), with the NS and D5-brane being $(1,0)$ and $(0,1)$ respectively. One can produce generic (p,q)-branes by acting with combinations of $\mathcal{S}$ and $\mathcal{T}$ on NS and D5-branes. A (p,q) brane can be always obtained starting from an NS-brane by acting with the operator $\mathcal{T}^{k_1}\mathcal{S}\mathcal{T}^{k_2}\mathcal{S}\dots\mathcal{S}\mathcal{T}^{k_r}$, where the integers k_i are determined from the ratio p/q written as a continued fraction:

$$\frac{p}{q}=\frac{1}{k_1-\frac{1}{k_2-\frac{1}{\dots-\frac{1}{k_r}}}} . \tag{2.4}$$

A (p,q)-brane is another admitted five-brane in a Type IIB setup which consist of a bound state of p NS and q D5 branes and it spans the x_{012} plus the $x_{37}^{[\theta]}x_{48}^{[\theta]}x_{59}^{[\theta]}$ directions, where $x_{ab}^{[\theta]}$ is the direction in the x_{ab} plane with angle θ that is determined as a function of (p,q). A generic brane setup involving also (p,q)-branes preserves 6 supercharges [50].

This picture was later reformulated by Gaiotto and Witten [28]. It was understood that one can perform the action of any $SL(2,\mathbb{Z})$ operator, let us call it $\mathcal{O}$, locally in a spacetime region containing a single five-brane. This has the effect of $\mathcal{O}$-transforming the brane and produces an $\mathcal{O}$-duality wall on its left and a $\mathcal{O}^{-1}$-duality wall on its right, that are domain walls at fixed positions along x_6. It was also identified the field theory that corresponds to a $\mathcal{O}$-duality wall intersecting a stack of D3 branes. In the case of the $\mathcal{S}$-duality wall this theory was identified with the $T[SU(N)]$ SCFT, while in the case of the $\mathcal{T}$-duality wall the theory consists of a CS term at level $+1$.

This whole picture can be generalized in many ways, by introducing for example orientifold planes and to generic d-dimensional field theories with various amount of supersymmetry. As a complete list of citation would be hard to achieve, we refer to [51] for a review of the early works following [5]. In this thesis we will be interested mainly in 3d linear quivers with unitary gauge nodes, even if it is expected that the strategies can be generalized further.

2.3 A field theory "proof" for mirror duality

An interesting question revolving around the topic of mirror duality has been whether it is possible to derive mirror dualities, and possibly, $SL(2,\mathbb{Z})$ dualities, for $\mathcal{N}=4$ theories in a purely field-theoretical framework.

The kick that initiated this program was made by Kapustin and Strassler [31]. It was shown that mirror duality for Abelian $\mathcal{N}=4$ theories can be proved using a purely field theoretical algorithmic strategy. The first step is to take an Abelian $\mathcal{N}=4$ theory and chop it into hypermultiplets that are either bifundamentals, that are charged either under a pair of $U(1)$ gauge symmetries, or flavors that are charged under a single $U(1)$ gauge symmetry. This is achieved by freezing the gauge interactions that are later restored. We then dualize each hypermultiplet using a basic Abelian mirror duality which relates a free hypermultiplet to an $U(1)$ gauge theory with one single flavor. It is crucial to observe that this fundamental duality can be established as a path integral identity, however, for the purpose of this thesis, we can deal just with the $\mathbf{S}_b^3$ partition function [18, 19, 20] identity which reads:

$$\int \mathrm{d}z \; \mathrm{e}^{2\pi i(\eta_1-\eta_2)z} s_b\left(-\frac{iQ}{2}+2m_A\right)\left(\frac{iQ}{2}-m_A\pm z\right) = s_b\left(m_A \pm (\eta_1-\eta_2)\right) . \quad (2.5)$$

See Appendix A for the $\mathbf{S}_b^3$ partition function notation. In the expression the combination $(\eta_1-\eta_2)$ is the FI parameter, defined in this way for convenience, and m_A is the real mass associated to the $U(1)$ flavor subgroup of the $\mathcal{N}=4$ R-symmetry in the $\mathcal{N}=2^*$ language. This identity can be interpreted as the mirror dualization of a flavor hypermultiplet into a bifundamental hypermultiplet. Or in stringy words, a D5 is being dualized into an NS-brane by the action of an $\mathcal{S}$ and $\mathcal{S}^{-1}$-wall. Notice that in the Abelian case the $\mathcal{S}$-duality wall is simply a BF coupling which in partition function terms is $e^{2\pi i XY}$. Similarly also the $\mathcal{S}^{-1}$-wall is a BF coupling but at opposite level. The relation $\mathcal{S}\mathcal{S}^{-1}=1$ is therefore the following:

$$\int \mathrm{d}z \; e^{2\pi i(\eta_1-\eta_2)z} s_b\left(-\frac{iQ}{2}+2m_A\right) = \frac{\delta(\eta_1-\eta_2)}{s_b\left(-\frac{iQ}{2}+2m_A\right)} \quad (2.6)$$

The Dirac-delta function combined with the double-sine function is thus interpreted as the identity operator. Notice that the gauging is performed with an $\mathcal{N}=4$ vector multiplet measure, which includes a double-sine function encoding the adjoint chiral multiplet. The same double-sine function normalizes the identity such that it gives exactly one when is integrated over a measure corresponding to a $\mathcal{N}=4$ Abelian vector multiplet.
The field theory interpretation of this functional identity is that the theory obtained by gluing two Abelian $\mathcal{S}$-walls is a pure $U(1)$ $\mathcal{N}=4$ SYM, which is an example of a bad theory, in the sense of [28]. This theory has a quantum deformed moduli space so that everywhere the topological symmetry is spontaneously broken. This phenomenon is indicated by the Dirac-delta function appearing on the r.h.s. of (2.6) as the attempt of gauging the topological symmetry, which translates to an integration over the FI parameter, would be immediately frozen by the delta. The spectrum of this theory is composed only by the two monopoles whose scaling dimension is 0 when computed from the UV R-symmetry. These operators therefore decouple and one of the two acquires a VEV, which is causing the spontanous symmetry breaking and whose contribution is thus encoded in the delta function, in a spirit similar to that of [52]. The other monopole operator becomes free and its contribution is instead encoded in the double-sine function at the denominator of (2.6).

Similarly one can derive an inverse basic duality relating a free flavor hypermultiplet to a bifundamental with $\mathcal{S}$ and $\mathcal{S}^{-1}$-walls acting on it. This is not an independent identity since it follows from (2.5) by acting with the appropriate walls.

After each basic block is dualized we glue together the result of the dualization and use the $\mathcal{S}\mathcal{S}^{-1}=1$ relation. Finally we obtain a different Lagrangian which is precisely the mirror dual of the starting $\mathcal{N}=4$ theory.

It is important to notice that this strategy proves that Abelian mirror dualities are consequence of the basic duality only, which is then a more fundamental duality. The basic duality and the $\mathcal{S}\mathcal{S}^{-1}=1$ relation can be seen as Aharony duality respectively in the confining and in the quantum deformed moduli space regimes. Moreover, these dualities have been established as a path integral identity [31] and the integral identity as been proved mathematically for the partition function on various 3-manifolds.

In more recent years, in the work of [53, 54], this logic has been partially extended to non-Abelian theories by studying their $\mathbf{S}^3$ partition function. The aim of Chapter 3 is to partially review the work contained in [40][2] in which it was finally established this idea showing that, similarly to the Abelian case, all mirror dualities (for linear unitary quivers) follow from basic dualities that can be derived assuming only Aharony duality.

[2]See also [55, 56] for preliminary works on the subject.

Chapter 3

$\mathcal{N} = 4$: The dualization algorithm

In this chapter we discuss a purely field-theoretical approach to derive generalized mirror dualities for 3d $\mathcal{N} = 4$ linear unitary quivers, that reflect the $SL(2, \mathbb{Z})$ duality group of Type IIB brane setup engineering these theories. This requires us to define the field-theoretic realization of the $SL(2, \mathbb{Z})$ operators and of QFT blocks, that are the objects upon which the operators act.
The content of this chapter is meant to be a partial review of [40] in which the mirror dualization algorithm was finally established building upon results first presented in [55, 56]. More results present in this paper are discussed in the conclusions section.

3.1 Introduction

Mirror duality in 3d $\mathcal{N} = 4$ theories [23] has been studied extensively in the last decades. One of the major breakthroughs was the discovery that mirror duality can be understood as being induced by dualities acting on the string theoretical engineering of these theories. The two earliest and most notable examples are the M-theory origin of [27] or the Type IIB origin of [5], in particular we now focus on the latter which states that for those 3d $\mathcal{N} = 4$ theories that can be engineered in a Type IIB brane setup made of NS, D5 and D3 branes, mirror duality is induced from $\mathcal{S}$-duality which acts by transforming an NS-brane into a D5-brane and vice-versa (see the review in Section 2.2).

Shortly after this discovery, it was noted that mirror duality for Abelian gauge theories can be derived purely from field-theoretical arguments [31]. The idea is that one can think of a (quiver) theory as being constructed from elementary blocks that are hypermultiplets in either the fundamental or bifundamental representation of one or two $U(1)$ gauge groups respectively. One can then cut the theory into these elementary components by freezing gauge interactions and it is possible to dualize each hypermultiplet using a basic mirror duality relating a free hypermultiplet to a $U(1)$ theory with one flavor. Moreover, it was shown by the authors of [31] that this duality can be established as a path integral identity relating a free and a gauged fermionic superdeterminant. After dualizing each elementary block one can reintroduce gauge interactions and, after some manipulations, one obtains the mirror dual theory.

Interestingly, this construction can be connected to an idea that was later introduced by Gaiotto and Witten in [28]. In this work, the authors explain that from the Type IIB point of view it is possible to perform $\mathcal{S}$-duality locally in the space-time region containing a single five-brane, with the effect of $\mathcal{S}$-transforming the brane and producing domain walls on its left and right. These domain walls, or $\mathcal{S}$-wall, when they intersect a stack of D3 branes, can be interpreted as the $T[SU(N)]$ SCFT. In these terms, the idea of Kapustin and Strassler can be formulated as follows. Each elementary block corresponds to the contribution of a single five-brane, and the basic duality move can be interpreted as the $\mathcal{S}$-dualization of a five-brane with the production of $\mathcal{S}$-walls on the two sides.

The aim of this chapter is to generalize the Kapustin-Strassler approach to non-Abelian theories and to any $SL(2,\mathbb{Z})$ generator. This is a task that has been partially achieved in [53, 54], in which the authors studied the S^3 partition function realization of (p,q)-branes and $SL(2,\mathbb{Z})$ operators and relations. The work presented in this chapter is twofold. One one hand we to generalize the picture to the $\mathbf{S}_b^3$ partition function which captures more details w.r.t. the $\mathbf{S}^3$ partition function. For example on $\mathbf{S}^3$ the partition function of the $T[SU(N)]$ theory is extremely similar to that of a BF coupling [53] and the contribution of (p,q)-branes resembles that of a bifundamental hypermultiplet with CS interaction. However, the $T[SU(N)]$ theory is a much more complicated object, being an SCFT, and also the presence of (p,q)-branes is expected to lead to quasi-Lagrangian theories, that are theories constructed from the gauging of symmetries that appear only in the IR due to enhancements. We expect that, differently from the partition function on $\mathbf{S}^3$, that on $\mathbf{S}_b^3$ can capture more of these details. The second aim is also to show that all the basic dualities involved in the dualization of a 3d $\mathcal{N}=4$ theory can be proved to be a consequence of the much more fundamental Aharony duality [7]. Thus motivating the statement that mirror duality is a consequence of Aharony duality.

3.2 The mirror dualization algorithm

In this section, we introduce the 3d $\mathcal{N}=4$ mirror dualization algorithm. In particular, we present all of its basic ingredients that are the $SL(2,\mathbb{Z})$ operators and the QFT blocks on which the operators acts.

3.2.1 $SL(2,\mathbb{Z})$ operators: duality-walls

We begin by introducing the field theory realization of the $SL(2,\mathbb{Z})$ generators that are the $\mathcal{S}$-wall and the $\mathcal{T}$-wall.

$\mathcal{S}$-wall. We associate the $\mathcal{S}$ generator of $SL(2,\mathbb{Z})$ with the $FT[U(N)]$ theory [57] (see Appendix C.1 for a review). This theory is a strongly coupled SCFT which is defined as the $T[SU(N)]$ theory of [28] with an extra BF coupling and a chiral field coupled the moment map of the manifest $U(N)$ symmetry. The $\mathcal{S}$-wall partition function is then defined to be:[1]

$$\mathcal{Z}_{\mathcal{S}}^{(N)}(\vec{X};\vec{Y};m_A) \equiv \mathcal{Z}_{FT[U(N)]}(\vec{X};-\vec{Y};m_A) = \mathcal{Z}_{FT[U(N)]}(-\vec{X};\vec{Y};m_A)\,. \tag{3.1}$$

The convention for the $\mathbf{S}_b^3$ partition function can be found in Appendix A. For the purpose of this chapter is useful to just know that $Z_{FT[U(N)]}(\vec{X},\vec{Y};m_A)$ is the partition function of the $FT[U(N)]$ theory which has two $U(N)$ global symmetries, one emergent and one manifest in the UV. The first entry $\vec{X}$ is the set of real masses associated to the manifest $U(N)_X$ symmetry while the second entry $\vec{Y}$ is that of the emergent $U(N)_Y$. Due to the exact self-mirror property of the $FT[U(N)]$ we are free to swap the emergent

[1] Notice that w.r.t. [55, 56] here we include an extra minus sign in front of either the $\vec{X}$ or $\vec{Y}$ mass parameters.

and manifest symmetry. Notice that the self-duality of the $FT[U(N)]$ theory swaps the two $U(N)$ symmetries and does not act on m_A which is the real mass associated to the flavor subgroup of the $\mathcal{N}=4$ R-symmetry in the $\mathcal{N}=2^*$ langauge, differently w.r.t. the usual self-mirror duality of the $T[SU(N)]$ theory. This is because the self-duality of the $FT[U(N)]$ theory is a combination of mirror duality, which indeed acts non-trivially on m_A, and a second duality called flip-flip [57].

The inverse $\mathcal{S}^{-1}$-wall it is still defined to be the $FT[U(N)]$ with a different choice of BF coupling w.r.t. the $\mathcal{S}$-wall:

$$\mathcal{Z}^{(N)}_{\mathcal{S}^{-1}}(\vec{X};\vec{Y};m_A) = \mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{X};-\vec{Y};m_A) = \mathcal{Z}^{(N)}_{\mathcal{S}}(-\vec{X};\vec{Y};m_A)\,. \tag{3.2}$$

We schematically represent the $3d$ $\mathcal{S}$-wall as in Figure 3.1 to display both of its $U(N)$ global symmetries. Graphycally we distinguish between $\mathcal{S}$ and $\mathcal{S}^{-1}$ by a $(+)$ and $(-)$ label respectively.
The definition of the $\mathcal{S}^{-1}$-wall is motivated by the relation $\mathcal{S}\mathcal{S}^{-1}=1$. We stress that the partition function of the $\mathcal{S}$-wall should be thought of as a matrix with indices given by the real mass parameters for the two $U(N)$ global symmetries. The product operation is then given by the diagonal gauging of two $U(N)$ global symmetries, which first identifies the set of real mass parameters and then we integrate over with a suitable $\mathcal{N}=4$ measure. We will see this concretely momentarely.

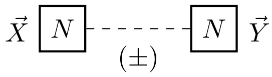


Figure 3.1: Short notation of the $\mathcal{S}^{\pm 1}$-walls. Notice that we label flavor nodes with the real mass parameters.

We also define an asymmetric $\mathcal{S}$-wall, where one of the $U(N)$ symmetries is broken to $U(M)\times U(1)$ with $M<N$. This deformed $\mathcal{S}$-wall coincide with the $FT_{[N-M,1^M]}[U(N)]$ obtained via a nilpotent VEV deformation of the $FT[U(N)]$ theory (see Appendix C.1). We schematically represent the asymmetric $\mathcal{S}$-wall as in Figure 3.2.

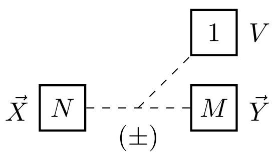


Figure 3.2: Short notation of the asymmetric $\mathcal{S}^{\pm 1}$-wall.

The identity-wall. As a natural check of the definition of the $\mathcal{S}$ and $\mathcal{S}^{-1}$-walls we verify that their product correctly gives the identity element. Moreover, we expect the $\mathcal{S}$ operator to also satisfy the $SL(2,\mathbb{Z})$ relation $\mathcal{S}^2=-1$[2].

These two relations coincide with variations of the gluing to identity property of the $FT[U(N)]$ theory[3]. In particular the $\mathcal{S}\mathcal{S}^{-1}=1$ relation corresponds to the following

[2]These identities were already known to follow from the closed-form expression of the round sphere partition function of the $T[SU(N)]$ found in [53, 58].

[3]This relation was interpreted in [49] from the point of view of bad theories. In short terms the interpretation is that the UV completion of the gluing of two $\mathcal{S}$-walls is a bad theory in the sense that certain monopole operators have scaling dimension 0, thus violating the unitarity bound, when computed with the UV R-symmetry. Some of these monopole operators acquire a VEV along the

$\mathbf{S}_b^3$ partition function identitiy:

$$\int \mathrm{d}\vec{Z}_N\, \Delta_N(\vec{Z};m_A)\mathcal{Z}_{\mathcal{S}}^{(N)}(\vec{Z};\vec{X};m_A)\mathcal{Z}_{\mathcal{S}}^{(N)}(\vec{Z};-\vec{Y};m_A) = {}_{\vec{X}}\mathbb{I}_{\vec{Y}}(m_A) \tag{3.3}$$

while the $\mathcal{S}^2 = -1$ relation corresponds to:

$$\int \mathrm{d}\vec{Z}_N\, \Delta_N(\vec{Z};m_A)\mathcal{Z}_{\mathcal{S}}^{(N)}(\vec{Z};\vec{X};m_A)\mathcal{Z}_{\mathcal{S}}^{(N)}(\vec{Z};\vec{Y};m_A) = {}_{\vec{X}}\mathbb{I}_{-\vec{Y}}(m_A)\,. \tag{3.4}$$

On the r.h.s. of the two relations appears an object that is a collection of delta-functions:

$${}_{\vec{X}}\mathbb{I}_{\vec{Y}}(m_A) = \frac{\sum_{\sigma\in S_N}\prod_{j=1}^N \left(X_j - Y_{\sigma(j)}\right)}{\Delta_N(\vec{X};m_A)} \tag{3.5}$$

with

$$\begin{aligned}
\Delta_N(\vec{X};m_A) &= \Delta_N(\vec{X})\prod_{i,j=1}^N s_b\left(-i\frac{Q}{2} + (X_i - X_j) + 2m_A\right)\,,\\
\Delta_N(\vec{X}) &= \frac{1}{\prod_{i<j}^N s_b\left(i\frac{Q}{2} \pm (X_i - X_j)\right)}\,.
\end{aligned} \tag{3.6}$$

Where in the first line we have the $\mathbf{S}_b^3$ partition function contribution of a $\mathcal{N}=4$ vector that splits into a $\mathcal{N}=2$ vector, whose contribution is encoded in $\Delta_N(\vec{X})$ and an adjoint chiral multiplet that in our convention has R-charge 2 and $U(1)_A$ charge -2.

This object is naturally interpreted with the identity element since it is easy to verify that the product of any partition function with it gives back the original partition function. Notice also that the presence of a minus sign in (3.6), w.r.t. (3.5), is interpreted as a minus sign.
Notice that the product operation is taken to be the diagonal gauging of two $U(N)$ global symmetries with a $\mathcal{N}=4$ measure. In the $\mathcal{N}=2^*$ language this implies the introduction of a chiral multiplet in the adjoint representation of $U(N)$, encoded in $\Delta_N(\vec{Z};m_A)$, which couples to the moment map operators of the two blocks, that are also in the adjoint representation of the gauged $U(N)$ symmetry.

We also define the three-dimensional asymmetric Identity-wall:

$${}_{\vec{X}}\mathbb{I}_{\vec{Y},V}(m_A) = \frac{1}{\Delta_N(\vec{X};m_A)}\sum_{\sigma\in S_N}\prod_{i=1}^N \delta\left(X_i - Y_{\sigma(i)}\right)\Bigg|_{Y_{M+j}=\frac{N-M+1-2j}{2}(iQ-2m_A)+V}\,. \tag{3.7}$$

which in analogy is obtained by multiplying together a standard $\mathcal{S}$-wall and an asymmetric $\mathcal{S}^{-1}$-wall. Both the symmetric and the asymmetric Identity relations are represented in Figure 3.3.

The fusion to identity property of the $FT[U(N)]$ theory can be proved to be a consequence of Aharony duality and the functional identity that relates a pure $\mathcal{N}=4$ $U(1)$ gauge theory to a delta-function [31], that we reviewed in Chpater 2. This is done by iterating Aharony duality on the Lagrangian UV description of the gluing of two

RG flow so that the moduli space is deformed and the naive $U(N)\times U(N)$ global symmetry is not supperted anywhere in it, but only the diagonal $U(N)$ subgroup. The partition function of bad theories was proved to be a distribution [48, 59], the delta function on the r.h.s. of the relation express the presence of monopole operators acquiring a VEV, along the lines of [52], and the sum over $N!$ *frames* is interpreted as equivalent vacua.

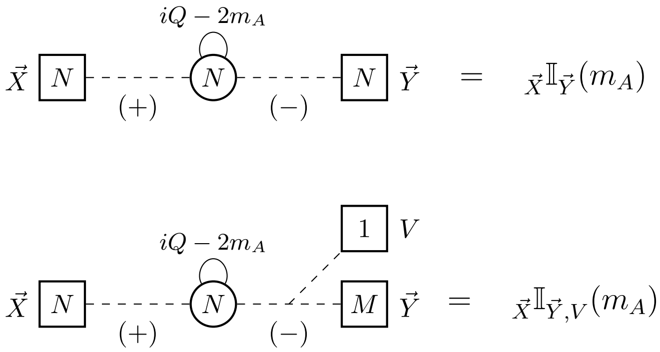


Figure 3.3: Gluing two $\mathcal{S}$-walls gives an Identity-wall, in the symmetric case first line and in the asymmetric case second line. In this figure we also introduce the notation that beside solid lines representing chiral multiplets we give the R-charge r and $U(1)_A$ charge q_A in a form resembling the $\mathbf{S}_b^3$ partition function contribution: $\frac{iQ}{2}r + q_A m_A$.

$FT[U(N)]$ theories, which is a linear quiver of $2N-1$ nodes with rank increasing from 1 to N and then back from N to 1, as decribed in detail in [55].

$\mathcal{T}$-wall. The $\mathcal{T}$ generator of $SL(2,\mathbb{Z})$ is simply associated with the addition of a CS level (see Figure 3.4).

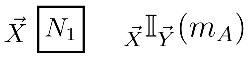


Figure 3.4: The $\mathcal{T}$-wall. The subscript 1 indicates a CS-level of +1.

We then define its contribution to the $\mathbf{S}_b^3$ partition function as:

$$\mathcal{Z}_{\mathcal{T}}^{(N)}(\vec{X};\vec{Y};m_A) = \mathrm{e}^{\frac{i\pi N}{24}(8m_A^2(N-1)-4im_AQ(N-1)+Q^2)}\mathrm{e}^{-i\pi\sum_{i=1}^N X_i^2}\,{}_{\vec{X}}\mathbb{I}_{\vec{Y}}(m_A)\,, \tag{3.8}$$

The first prefactor is included so to simplify the identities for the basic duality moves and to recover the $SL(2,\mathbb{Z})$ relation $(\mathcal{ST})^3 = 1$ as we will see momentarily. Notice also the presence of an identity-wall in the definition which is included so that the $\mathcal{T}$ operator can be thought of as a matrix carrying two indices, that are real parameters associated to two $U(N)$ symmetries.

The $\mathcal{T}^{-1}$-wall is simply defined as the addition of a CS-level with opposite sign. It hence follows trivially that $\mathcal{T}\mathcal{T}^{-1} = 1$. Equivalently, from the relation $(\mathcal{ST})^3 = 1$ one can write the inverse as $\mathcal{T}^{-1} = \mathcal{STSTS}$, which it is possible to verify that coincides with a CS level -1.

$SL(2,\mathbb{Z})$ **relations**. We have already discussed the relation $\mathcal{S}^2 = -1$. The relation $(\mathcal{ST})^3 = 1$ is represented in Figure 3.5.

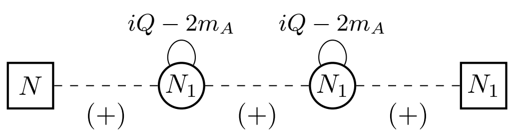


Figure 3.5: The $SL(2,\mathbb{Z})$ relation $(\mathcal{ST})^3 = 1$.

To prove this identity it is enough to simply prove that $\mathcal{STS} = (\mathcal{TST})^{-1}$. The proof of this relation can be found in [55].

$\mathcal{T}^T$**-wall**. Finally, for completeness we also discuss the $\mathcal{T}^T$ operator. This is not an independent operator, since it can be written in terms of the $\mathcal{S}$ and $\mathcal{T}$ generators as $\mathcal{T}^T = -\mathcal{T}\mathcal{S}\mathcal{T}$, which coincides with the transpose of the $\mathcal{T}$ matrix. Accordingly, we associate to it one copy of the $FT[U(N)]$ $\mathcal{S}$-wall theory with background CS level $+1$ for both of its $U(N)$ global symmetries (see Figure 3.6).

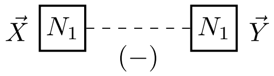


Figure 3.6: The $\mathcal{T}^T$-wall.

We then define its contribution to the $\mathbf{S}_b^3$ partition function as:

$$\begin{aligned}\mathcal{Z}^{(N)}_{\mathcal{T}^T}(\vec{X};\vec{Y};m_A) &= \int \mathrm{d}\vec{Z}^{(1)}_N\,\mathrm{d}\vec{Z}^{(2)}_N\,\Delta^{3d}_N(\vec{Z}^{(1)};m_A)\Delta^{3d}_N(\vec{Z}^{(2)};m_A)\mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{X};\vec{Z}^{(1)};m_A)\\ &\quad\times \mathcal{Z}^{(N)}_{\mathcal{S}^{-1}}(\vec{Z}^{(1)};\vec{Z}^{(2)};m_A)\mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{Z}^{(2)};\vec{Y};m_A)\\ &= \mathrm{e}^{\frac{i\pi N}{12}(8m_A^2(N-1)-4im_AQ(N-1)+Q^2)}\mathrm{e}^{-i\pi(\sum_{i=1}^N X_i^2+Y_i^2)}\mathcal{Z}^{(N)}_{\mathcal{S}^{-1}}(\vec{X};\vec{Y};m_A)\end{aligned} \tag{3.9}$$

If we replace $\mathcal{Z}^{(N)}_{\mathcal{S}^{-1}}(\vec{X};\vec{Y};m_A)$ by

$$\mathcal{Z}^{(N)}_{\mathcal{S}^{-1}}(\vec{X};\vec{Y};m_A) = \int \mathrm{d}\vec{Z}_N\,\Delta^{3d}_N(\vec{Z};m_A)\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{X};\vec{Z};m_A)\,{}_{\vec{Z}}\mathbb{I}^{3d}_{-\vec{Y}}(m_A)\,, \tag{3.10}$$

which corresponds to $\mathcal{S}^{-1} = -\mathcal{S}$, we can see this is exactly the relation $\mathcal{T}^T = -\mathcal{T}\mathcal{S}\mathcal{T}$.

3.2.2 QFT building blocks

The fundamental QFT building blocks in are those naturally associated with the possible types of 5-branes that we can have in the Hanany–Witten brane setup. We focus in particular on NS5, D5 and (1,1)-branes.

The $\mathcal{B}_{10}$ block. To an NS5-brane with N D3-branes ending on its left and M on its right is associated a $U(N)\times U(M)$ bifundamental hypermultiplet, which we also call $\mathcal{B}_{10}$ block (see Figure 3.7).

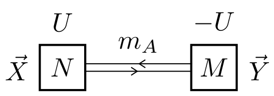


Figure 3.7: The $\mathcal{B}_{10}$ block. Here and in the following figures we label flavor nodes with their associated real mass parameters. Above the nodes we give the background FI parameter. Finally, close the chirals we give the R-charge r and $U(1)_A$ charge q_A in short as a polynomial $\frac{iQ}{2}r + q_A m_A$, in analogy with $\mathbf{S}_b^3$ partition function convention.

Its $\mathbf{S}_b^3$ partition function contribution is given by:

$$\mathcal{Z}^{(N,M)}_{(1,0)}(\vec{X};\vec{Y};U;m_A) = \mathrm{e}^{2\pi iU\left(\sum_{i=1}^N X_i-\sum_{j=1}^M Y_j\right)}\prod_{i=1}^N\prod_{j=1}^M s_b\left(\frac{iQ}{2}-m_A\pm(X_i-Y_j)\right)\,. \tag{3.11}$$

Here U appears as a BF coupling between a background topological symmetry and the $U(N)$ and $U(M)$ global symmetries. Indeed as these symmetries are gauged, U becomes an FI parameter.

We can obtain the block $\mathcal{B}_{-10}$ by acting with the -1 operator on each side of $\mathcal{B}_{10}$, which can be expressed by the following identity:

$$\begin{aligned}\mathcal{Z}^{(N,M)}_{(-1,0)}(\vec{X};\vec{Y};U;m_A) &= \oint \mathrm{d}\vec{Z}_N\,\mathrm{d}\vec{W}_M\;\Delta^{3d}_N(\vec{Z};m_A)\Delta^{3d}_M(\vec{W};m_A)\,{}_{\vec{Z}}\mathbb{I}^{3d}_{-\vec{X}}(m_A)\,{}_{\vec{W}}\mathbb{I}^{3d}_{-\vec{Y}}(m_A)\\ &\times \mathrm{e}^{2\pi i U\left(\sum_{i=1}^N Z_i-\sum_{j=1}^M W_j\right)}\prod_{i=1}^N\prod_{j=1}^M s_b\Big(\frac{iQ}{2}-m_A\pm(Z_i-W_j)\Big)\\ &= \mathrm{e}^{-2\pi i U\left(\sum_{i=1}^N X_i-\sum_{j=1}^M Y_j\right)}\prod_{i=1}^N\prod_{j=1}^M s_b\Big(\frac{iQ}{2}-m_A\pm(X_i-Y_j)\Big)\,,\end{aligned} \tag{3.12}$$

which is simply $\mathcal{Z}^{(N,M)}_{(1,0)}(\vec{X};\vec{Y};-U;m_A)$ where the sign of U is flipped.

The $\mathcal{B}_{01}$ block. To a D5-brane with N D3-branes ending both on its left and on its right we associated a single $U(N)$ fundamental hypermultiplet, which we also call $\mathcal{B}_{01}$ block (see Figure 3.8).

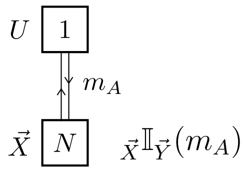


Figure 3.8: The $\mathcal{B}_{01}$ block.

Its $\mathbf{S}^3_b$ partition function contribution is given by

$$\mathcal{Z}^{(N,N)}_{(0,1)}(\vec{X};\vec{Y};U;m_A) = \prod_{j=1}^N s_b\left(\frac{iQ}{2}-m_A\pm(X_i-U)\right)\,{}_{\vec{X}}\mathbb{I}_{\vec{Y}}(m_A)\,. \tag{3.13}$$

We quip this block with an identity operator in order to make it carry two indices and obtain the desired matrix structure.

It is useful to generalize this to a D5 suspended between different numbers of D3-branes, say N D3's on the left and M D3's on the right. The associated partition function is given by

$$\mathcal{Z}^{(N,M)}_{(0,1)}(\vec{X};\vec{Y};U;m_A) = \prod_{j=1}^M s_b\left(\frac{iQ}{2}-m_A(N-M+1)\pm(Y_j-U)\right)\,{}_{\vec{X}}\mathbb{I}_{\vec{Y},U}(m_A)\,. \tag{3.14}$$

We can also introduce the block $\mathcal{B}_{0-1}$ by acting with $-I$ on each side of $\mathcal{B}_{01}$ which results into:

$$\mathcal{Z}^{(N,M)}_{(0,-1)}(\vec{X};\vec{Y};U;m_A) = \mathcal{Z}^{(N,M)}_{(0,1)}(\vec{X};\vec{Y};-U;m_A)\,. \tag{3.15}$$

The $\mathcal{B}_{11}$ block. Finally, the QFT building block associated to a $(1,1)$-brane with N D3 branes ending on its left and M on its right is a $U(N)\times U(M)$ bifundamental hypermultiplet with the addition of background CS levels $+1$ and -1 respectively for

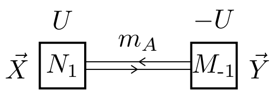


Figure 3.9: The $\mathcal{B}_{11}$ block.

the $U(N)$ and the $U(M)$ global symmetries, which we also call $\mathcal{B}_{11}$ block (see Figure 3.9).
Its $\mathbf{S}_b^3$ partition function contribution is given by

$$\mathcal{Z}_{(1,1)}^{(N,M)}(\vec{X};\vec{Y};U;m_A) = \mathrm{e}^{-i\pi\left(\sum_{i=1}^N X_i^2-\sum_{j=1}^M Y_j^2\right)}\mathrm{e}^{2\pi iU\left(\sum_{i=1}^N X_i-\sum_{j=1}^M Y_j\right)} \times \prod_{i=1}^N\prod_{j=1}^M s_b\left(\frac{iQ}{2}-m_A\pm(X_i-Y_j)\right)\,, \tag{3.16}$$

Which indeed coincides with the action of the T-wall on the $\mathcal{B}_{10}$ block, as we will comment in a moment.

3.2.3 Basic duality moves

Having defined the $3d$ QFT blocks and the $SL(2,\mathbb{Z})$ operators, we are ready to present the basic duality moves that play a central role in the dualization algorithm. These are simply the action of the $SL(2,\mathbb{Z})$ operators on the QFT blocks. For a generic operator $\mathcal{O}$ its action on the $\mathcal{B}_{pq}$ block is given as $\mathcal{O}\mathcal{B}_{pq}\mathcal{O}^{-1}$, which in field theory language corresponds to diagonally gauge the two $U(N)$ global symmetries of the $\mathcal{B}_{pq}$ QFT block, together with the field theory realization of the $\mathcal{O}$ and $\mathcal{O}^{-1}$-walls.

Crucially, all the dualities presented in this section can be proved by iterations of the Aharony duality [7] as demonstrated in [55, 40]. In this terms one can state that mirror duality for unitary linear quivers is proved to be consequence of Aharony duality, which is then a more fundamental duality.

$\mathcal{S}$-dualization

$\boldsymbol{\mathcal{B}_{-10} = \mathcal{S}\mathcal{B}_{01}\mathcal{S}^{-1}}$. The first duality move is the $\mathcal{S}$-dualization of a $(0,1)$-brane into a $(-1,0)$-brane, a $\overline{\mathrm{NS5}}$, which we interpret as the field theory duality in Figure 3.10.

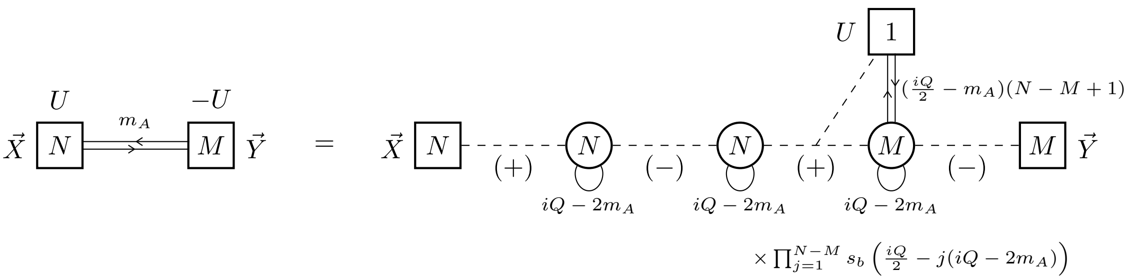


Figure 3.10: The $\mathcal{B}_{-10} = \mathcal{S}\mathcal{B}_{01}\mathcal{S}^{-1}$ duality move. The presence of extra chiral fields in the duality is given below the theory in $\mathbf{S}_b^3$ partition function notation.

This corresponds to the following identity of $\mathbf{S}^3_b$ partition functions:

$$\mathcal{Z}^{(N,M)}_{(-1,0)}(\vec{X};\vec{Y};U;m_A) = \prod_{j=1}^{N-M} s_b\left(i\frac{Q}{2} - j(iQ-2m_A)\right) \int \left(\prod_{k=1}^{2} \mathrm{d}\vec{Z}^{(k)}_M \, \Delta^{3d}_M(\vec{Z}^{(k)};m_A)\right) \\ \times \mathcal{Z}^{(N)}_{\mathcal{S}}\left(\vec{X};\vec{Z}^{(1)}\right) \mathcal{Z}^{(N,M)}_{(0,1)}\left(\vec{Z}^{(1)};\vec{Z}^{(2)};U;i\frac{Q}{2}-m_A\right) \mathcal{Z}^{(M)}_{\mathcal{S}^{-1}}(\vec{Z}^{(2)};\vec{Y};m_A)\,, \tag{3.17}$$

where the partition function for the $\mathcal{B}_{-10}$ block on the left hand side is given by that of $\mathcal{B}_{10}$ with the FI parameter $-U$ as shown in (3.12).

As anticipated, this duality can be proven to be a consequence of Aharony duality by simply iterating the more fundamental duality as described in [55]. This is the case for every basic duality move shown in this section.

$\boldsymbol{\mathcal{B}_{01} = \mathcal{S}\mathcal{B}_{10}\mathcal{S}^{-1}}$. We next consider the QFT analogue of the $\mathcal{S}$-dualization of a $(1,0)$-brane into a $(0,1)$-brane shown in Figure 3.11.

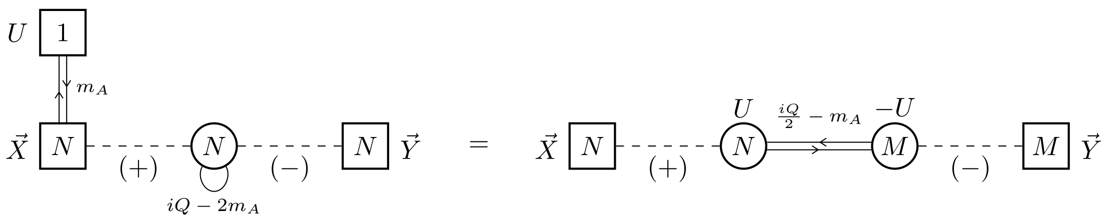


Figure 3.11: The $\mathcal{B}_{01} = \mathcal{S}\mathcal{B}_{10}\mathcal{S}^{-1}$ duality move.

In this case, the associated partition function identity is

$$\mathcal{Z}^{(N)}_{(0,1)}(\vec{X};\vec{Y};U;m_A) = \int \mathrm{d}\vec{Z}^{(1)}_N \,\mathrm{d}\vec{Z}^{(2)}_N \, \Delta^{3d}_N(\vec{Z}^{(1)})\Delta^{3d}_N(\vec{Z}^{(2)}) \mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{X};\vec{Z}^{(1)};m_A) \\ \times \mathcal{Z}^{(N,N)}_{(1,0)}\left(\vec{Z}^{(1)};\vec{Z}^{(2)};U;i\frac{Q}{2}-m_A\right) \mathcal{Z}^{(N)}_{\mathcal{S}^{-1}}(\vec{Z}^{(2)};\vec{Y};m_A)\,. \tag{3.18}$$

As anticipated, this identity can be proved by iterating Aharony duality or simply starting from the duality in 3.17 and multiplying by $\mathcal{S}$ and $\mathcal{S}^{-1}$-walls on the left and right hand side.

$\boldsymbol{\mathcal{B}_{11} = \mathcal{S}\mathcal{B}_{1-1}\mathcal{S}^{-1}}$. The last $\mathcal{S}$-dualization we consider is the one relating a $(1,1)$-brane and a (1,-1)-brane shown in Figure 3.12.

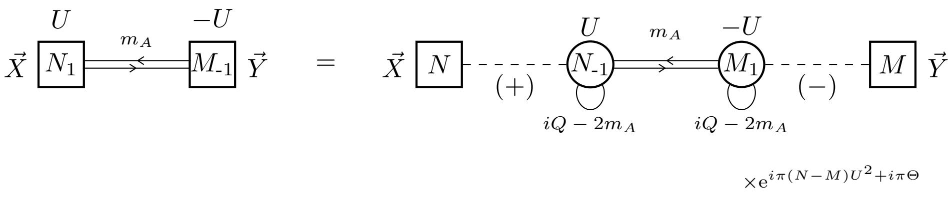


Figure 3.12: The $\mathcal{B}_{11} = \mathcal{S}\mathcal{B}_{1-1}\mathcal{S}^{-1}$ duality move. BF couplings are given below the theory in $\mathbf{S}^3_b$ partition function notation.

At the level of the $\mathbf{S}^3_b$ partition function, this duality move is encoded in the following identity:

$$
\begin{aligned}
\mathcal{Z}^{(N,M)}_{(1,1)}(\vec{X};\vec{Y};U;m_A) = \mathrm{e}^{i\pi(N-M)U^2+i\pi\Theta}\int \mathrm{d}\vec{Z}^{(1)}_N\,\mathrm{d}\vec{Z}^{(2)}_M\,\Delta^{3d}_N(\vec{Z}^{(1)};m_A)\Delta^{3d}_M(\vec{Z}^{(2)};m_A)\\
\times\,\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{X};\vec{Z}^{(1)};m_A)\mathcal{Z}^{(N,M)}_{(1,-1)}(\vec{Z}^{(1)};\vec{Z}^{(2)};U;m_A)\mathcal{Z}^{(M)}_{\mathcal{S}^{-1}}(\vec{Z}^{(2)};\vec{Y};m_A)\,,
\end{aligned}
\tag{3.19}
$$

where

$$
\begin{aligned}
\Theta = \frac{N-M}{24}\Big[&8m_a^2\left((N-M)^2-3(N+M)+2\right)\\
&-4im_AQ\left(2(N-M)^2-3(N+M)+1\right)-Q^2\left(2(N-M)^2+1\right)\Big]\,.
\end{aligned}
\tag{3.20}
$$

We have defined the contribution of the $(1,-1)$-brane, that is the $\mathsf{B}_{1\text{-}1}$ block, as

$$
\begin{aligned}
\mathcal{Z}^{(N,M)}_{(1,\text{-}1)}(\vec{X};\vec{Y};U;m_A) = \mathrm{e}^{i\pi\left(\sum_{i=1}^N X_i^2-\sum_{j=1}^M Y_j^2\right)}\mathrm{e}^{2\pi iU\left(\sum_{i=1}^N X_i-\sum_{j=1}^M Y_j\right)}\\
\times\prod_{i=1}^N\prod_{j=1}^M s_b\frac{iQ}{2}-m_A\pm(X_i-Y_j).
\end{aligned}
\tag{3.21}
$$

Notice that this differs from the contribution of a $(1,1)$ for the fact that the CS levels are inverted.

$\mathcal{T}$-dualization

$\boldsymbol{\mathcal{B}_{01}=\mathcal{T}\mathcal{B}_{01}\mathcal{T}^{-1}}$**.** A D5-brane is invariant under the $\mathcal{T}$-dualization as shown in Figure 3.13.

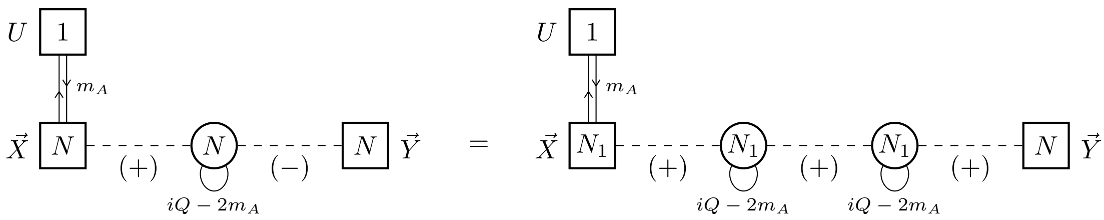


Figure 3.13: The $\mathcal{B}_{01}=\mathcal{T}\mathcal{B}_{01}\mathcal{T}^{-1}$ duality move.

Recalling that the inverse of $\mathcal{T}$ is given by $\mathcal{T}^{-1}=\mathcal{S}\mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S}$, we find that this dualization implies the following identity of $\mathbf{S}^3_b$ partition functions:

$$
\begin{aligned}
\mathcal{Z}^{(N)}_{(0,1)}(\vec{X};\vec{Y};U;m_A) = \int\left(\prod_{k=1}^6\mathrm{d}\vec{Z}^{(k)}_N\,{}^{3d}_N(\vec{Z}^{(k)};m_A)\right)\mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{X};\vec{Z}^{(1)};m_A)\\
\times\,\mathcal{Z}^{(N)}_{(0,1)}(\vec{Z}^{(1)};\vec{Z}^{(2)};U;m_A)\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{Z}^{(2)};\vec{Z}^{(3)};m_A)\mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{Z}^{(3)};\vec{Z}^{(4)};m_A)\\
\times\,\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{Z}^{(4)};\vec{Z}^{(5)};m_A)\mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{Z}^{(5)};\vec{Z}^{(6)};m_A)\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{Z}^{(6)};\vec{Y};m_A)\,.
\end{aligned}
\tag{3.22}
$$

We point out that this identity doesn't have any additional prefactor thanks to the fact that we defined the $\mathcal{T}$ generator in (3.8) with an extra prefactor.

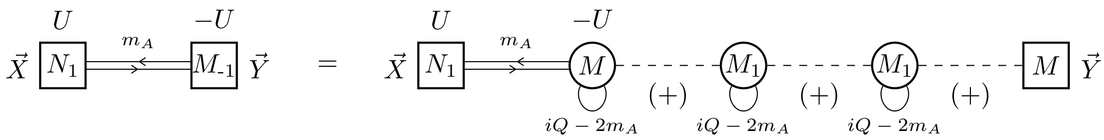


Figure 3.14: The $\mathcal{B}_{11} = \mathcal{T}\mathcal{B}_{10}\mathcal{T}^{-1} = \mathcal{T}\mathcal{B}_{10}\mathcal{S}\mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S}$ duality move.

$\boldsymbol{\mathcal{B}_{11} = \mathcal{T}\mathcal{B}_{10}\mathcal{T}^{-1}}$**.** We now consider the $\mathcal{T}$-dualization of a $(1,0)$-brane into a $(1,1)$-brane. Using again that $\mathcal{T}^{-1} = \mathcal{S}\mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S}$ this corresponds to the duality in Figure 3.14.

At the level of partition functions we have the following identity:

$$\begin{aligned}\mathcal{Z}^{(N,M)}_{(1,1)}(\vec{X};\vec{Y};U;m_A) = &\int \mathrm{d}\vec{Z}^{(1)}_N\, \Delta^{3d}_N(\vec{Z}^{(1)};m_A) \left(\prod_{k=2}^{6} \mathrm{d}\vec{Z}^{(k)}_M\, \Delta^{3d}_M(\vec{Z}^{(k)};m_A)\right) \mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{X};\vec{Z}^{(1)};m_A)\\ &\times \mathcal{Z}^{(N,M)}_{(1,0)}(\vec{Z}^{(1)};\vec{Z}^{(2)};U;m_A)\mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(2)};\vec{Z}^{(3)};m_A)\mathcal{Z}^{(M)}_{\mathcal{T}}(\vec{Z}^{(3)};\vec{Z}^{(4)};m_A)\\ &\times \mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(4)};\vec{Z}^{(5)};m_A)\mathcal{Z}^{(M)}_{\mathcal{T}}(\vec{Z}^{(5)};\vec{Z}^{(6)};m_A)\mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(6)};\vec{Y};m_A)\,,\end{aligned} \tag{3.23}$$

$\boldsymbol{\mathcal{B}_{10} = \mathcal{T}\mathcal{B}_{1-1}\mathcal{T}^{-1}}$**.** Lastly, we consider the $\mathcal{T}$-dualization of a $(1,-1)$-brane into a $(1,0)$-brane, which corresponds to the duality in Figure 3.15.

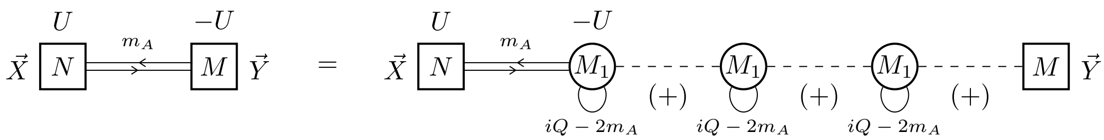


Figure 3.15: The $\mathcal{B}_{10} = \mathcal{T}\mathcal{B}_{1-1}\mathcal{T}^{-1} = \mathcal{T}\mathcal{B}_{1-1}\mathcal{S}\mathcal{T}\mathcal{S}\mathcal{T}\mathcal{S}$ duality move.

This translates into the identity

$$\begin{aligned}\mathcal{Z}^{(N,M)}_{(1,0)}(\vec{X};\vec{Y};U;m_A) = &\int \mathrm{d}\vec{Z}^{(1)}_N\, \Delta^{3d}_N(\vec{Z}^{(1)};m_A) \left(\prod_{k=2}^{6} \mathrm{d}\vec{Z}^{(k)}_M\, \Delta^{3d}_M(\vec{Z}^{(k)};m_A)\right) \mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{X};\vec{Z}^{(1)};m_A)\\ &\times \mathcal{Z}^{(N,M)}_{(1\text{,-}1)}(\vec{Z}^{(1)};\vec{Z}^{(2)};U;m_A)\mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(2)};\vec{Z}^{(3)};m_A)\mathcal{Z}^{(M)}_{\mathcal{T}}(\vec{Z}^{(3)};\vec{Z}^{(4)};m_A)\\ &\times \mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(4)};\vec{Z}^{(5)};m_A)\mathcal{Z}^{(M)}_{\mathcal{T}}(\vec{Z}^{(5)};\vec{Z}^{(6)};m_A)\mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(6)};\vec{Y};m_A)\,,\end{aligned} \tag{3.24}$$

where we recall that the contribution of a $(1,-1)$-brane was defined in (3.21).

$\mathcal{T}^T$-dualization

We also discuss below the basic moves involving the $SL(2,\mathbb{Z})$ element $\mathcal{T}^T = -\mathcal{T}\mathcal{S}\mathcal{T}$ with inverse $\left(\mathcal{T}^T\right)^{-1} = -\mathcal{S}\mathcal{T}\mathcal{S}$.

$\boldsymbol{\mathcal{B}_{11} = \mathcal{T}^T\mathcal{B}_{01}\left(\mathcal{T}^T\right)^{-1}}$**.** The $\mathcal{T}^T$-dualization of a $(0,1)$-brane into a $(1,1)$-brane corresponds to the duality in Figure 3.16.

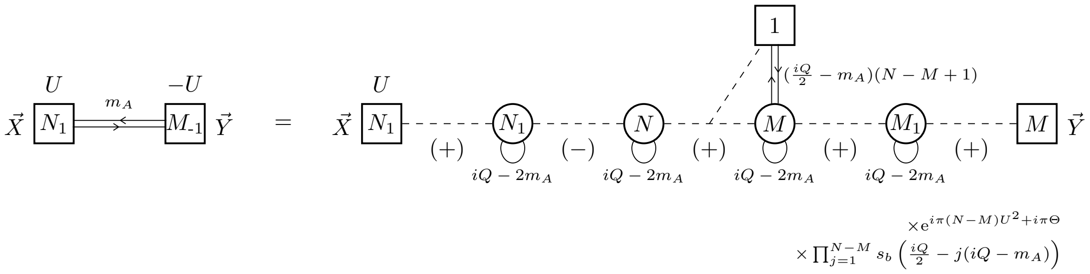


Figure 3.16: The $\mathcal{B}_{11} = \mathcal{T}^T\mathcal{B}_{01}\left(\mathcal{T}^T\right)^{-1} = \mathcal{TST}\mathcal{B}_{01}\mathcal{STS}$ duality move.

We then have the following identity of $\mathbf{S}_b^3$ partition functions:

$$
\begin{aligned}
\mathcal{Z}^{(N,M)}_{(1,1)}(\vec{X};\vec{Y};U;m_A) &= \mathrm{e}^{i\pi(N-M)U^2 - i\pi\Theta}\prod_{j=1}^{N-M} s_b\left(i\frac{Q}{2} - j(iQ-2m_A)\right)\\
&\times \int\left(\prod_{k=1}^{3}\mathrm{d}\vec{Z}^{(k)}_N\,{}^{3d}_{N}(\vec{Z}^{(k)};m_A)\right)\left(\prod_{k=4}^{6}\mathrm{d}\vec{Z}^{(k)}_M\,{}^{3d}_{N}(\vec{Z}^{(M)};m_A)\right)\\
&\times \mathcal{Z}^{(N)}_{\mathcal{T}}(-\vec{X};\vec{Z}^{(1)};m_A)\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{Z}^{(1)};\vec{Z}^{(2)};m_A)\mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{Z}^{(2)};\vec{Z}^{(3)};m_A)\\
&\times \mathcal{Z}^{(N,M)}_{(0,1)}\left(\vec{Z}^{(3)};\vec{Z}^{(4)};U;i\frac{Q}{2}-m_A\right)\mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(4)};\vec{Z}^{(5)};m_A)\mathcal{Z}^{(M)}_{\mathcal{T}}(\vec{Z}^{(5)};\vec{Z}^{(6)};m_A)\\
&\times \mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(6)};-\vec{Y};m_A)\,,
\end{aligned}
\tag{3.25}
$$

where we recall that Θ was defined in (3.20). Note that we have inserted the extra signs of $-\vec{X}$ and $-\vec{Y}$ on the right hand side, which reflect the minus sign of $\mathcal{T}^T = -\mathcal{TST}$ and its inverse.

$\boldsymbol{\mathcal{B}_{10} = \mathcal{T}^T\mathcal{B}_{10}\left(\mathcal{T}^T\right)^{-1}}$**.** A $(1,0)$-brane is transparent under the $\mathcal{T}^T$-dualization, which is translated into the non-trivial duality in Figure 3.17.

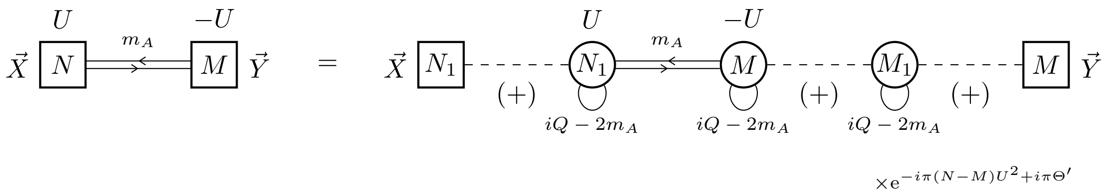


Figure 3.17: The $\mathcal{B}_{10} = \mathcal{T}^T\mathcal{B}_{10}\left(\mathcal{T}^T\right)^{-1} = \mathcal{TST}\mathcal{B}_{10}\mathcal{STS}$ duality move.

The associated $\mathbf{S}_b^3$ partition function identity is

$$
\begin{aligned}
\mathcal{Z}^{(N,M)}_{(1,0)}(\vec{X};\vec{Y};U;m_A) &= \mathrm{e}^{-i\pi(N-M)U^2-i\pi\Theta'} \\
&\times \int \left(\prod_{k=1}^{3} \mathrm{d}\vec{Z}^{(k)}_N \,{}^{3d}_{N}(\vec{Z}^{(k)};m_A)\right)\left(\prod_{k=4}^{6} \mathrm{d}\vec{Z}^{(k)}_M \,{}^{3d}_{N}(\vec{Z}^{(M)};m_A)\right) \\
&\times \mathcal{Z}^{(N)}_{\mathcal{T}}(-\vec{X};\vec{Z}^{(1)};m_A)\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{Z}^{(1)};\vec{Z}^{(2)};m_A)\mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{Z}^{(2)};\vec{Z}^{(3)};m_A) \\
&\times \mathcal{Z}^{(N,M)}_{(1,0)}(\vec{Z}^{(3)};\vec{Z}^{(4)};U;m_A)\mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(4)};\vec{Z}^{(5)};m_A)\mathcal{Z}^{(M)}_{\mathcal{T}}(\vec{Z}^{(5)};\vec{Z}^{(6)};m_A) \\
&\times \mathcal{Z}^{(M)}_{\mathcal{S}}(\vec{Z}^{(6)};-\vec{Y};m_A)\,,
\end{aligned}
\tag{3.26}
$$

where we defined

$$
\Theta' = \frac{N-M}{12}(iQ-2m_A)^2\left[(N-M)^2-1\right]\,. \tag{3.27}
$$

$\boldsymbol{\mathcal{B}_{0-1} = \mathcal{T}^T\mathcal{B}_{1-1}\left(\mathcal{T}^T\right)^{-1}}$**.** Lastly, we consider the $\mathcal{T}^T$-dualization of a $(1,-1)$-brane into a $(0,-1)$-brane, or $\overline{\text{D5}}$, which corresponds to the duality in Figure 3.18

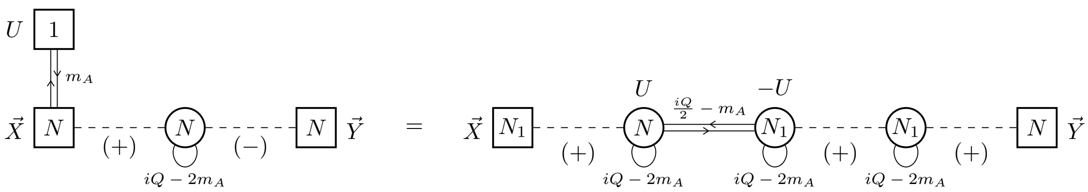


Figure 3.18: The $\mathcal{B}_{0-1} = \mathcal{T}^T\mathcal{B}_{1-1}\left(\mathcal{T}^T\right)^{-1}$ duality move.

At the level of $\mathbf{S}_b^3$ partition function we have the following identity:

$$
\begin{aligned}
\mathcal{Z}^{(N)}_{(0,-1)}(\vec{X};\vec{Y};U;m_A) &= \int \left(\prod_{k=1}^{6} \mathrm{d}\vec{Z}^{(k)}_N \,\Delta^{3d}_N(\vec{Z}^{(k)};m_A)\right)\mathcal{Z}^{(N)}_{\mathcal{T}}(-\vec{X};\vec{Z}^{(1)};m_A) \\
&\times \mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{Z}^{(1)};\vec{Z}^{(2)};m_A)\mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{Z}^{(2)};\vec{Z}^{(3)};m_A) \\
&\times \mathcal{Z}^{(N,N)}_{(1,\text{-}1)}\left(\vec{Z}^{(3)};\vec{Z}^{(4)};U;i\frac{Q}{2}-m_A\right)\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{Z}^{(4)};\vec{Z}^{(5)};m_A) \\
&\times \mathcal{Z}^{(N)}_{\mathcal{T}}(\vec{Z}^{(5)};\vec{Z}^{(6)};m_A)\mathcal{Z}^{(N)}_{\mathcal{S}}(\vec{Z}^{(6)};-\vec{Y};m_A)\,,
\end{aligned}
\tag{3.28}
$$

where $\mathcal{Z}^{(N,M)}_{(0,-1)}(\vec{X};\vec{Y};U;m_A) = \mathcal{Z}^{(N,M)}_{(0,1)}(\vec{X};\vec{Y};-U;m_A)$.

3.2.4 The Hanany–Witten duality move

The algorithm to dualize a quiver into its mirror consist in the following steps:

- Chop the quiver into its basic blocks, that are generic $\mathcal{B}_{pq}$ blocks glued together. This is done by freezing gauge interactions that are later restored.
- Once we choose the $SL(2,\mathbb{Z})$ operator, let us call it $\mathcal{O}$ in general, we dualize each block using the appropriate basic move corresponding to: $\mathcal{B}_{pq} = \mathcal{O}\mathcal{B}_{p'q'}\mathcal{O}^{-1}$.
- We glue back together the result of each dualization by reintroducing the gauge interactions frozen at the first step.

- We implement the property $\mathcal{O}^{-1}\mathcal{O} = 1$ to elimate the duality-walls produced at the second step.

After all these steps are performed we are not completely done, let us consider for simplicity the case where $\mathcal{O} = \mathcal{S}$. In particular we might end up with a theory in which some operator has a non-trivial VEV at the end of the steps described above. This happens every time we start from a quiver with gauge nodes of unequal ranks, and it is the manifestation of the fact that in the brane setup after applying $\mathcal{S}$-duality we have D5-branes with different numbers of D3's on the left and on the right. One then has to reorder the branes using Hanany–Witten (HW) moves until we reach the situation in which all D5-branes have an equal number of D3's on the left and on the right. In such a situation we are able to read off the mirror dual quiver gauge theory. In field theory this operation amounts to studying the aforementioned VEVs. An effective way to do that is is to use a duality that realizes the HW move in the field theory language. The HW duality move is given in Figure 3.19.

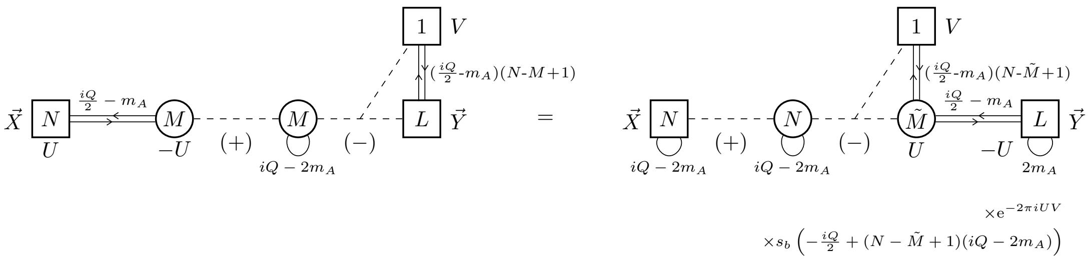


Figure 3.19: The Hanany–Witten duality move.

This duality corresponds to the following $\mathbf{S}_b^3$ partition function identity for $N \geqslant \widetilde{M} \geqslant 0$ as consistent with the S-rule:

$$
\begin{aligned}
&\int \mathrm{d}\vec{Z}_M \, \Delta_M(\vec{Z}) \mathcal{Z}^{(N,M)}_{(1,0)}\left(\vec{X};\vec{Z};U;i\frac{Q}{2}-m_A\right) {}_{\vec{Z}}\mathbb{I}_{\vec{Y},V}(m_A) \\
&\quad \times \prod_{j=1}^{L} s_b \frac{iQ}{2} - \left(\frac{iQ}{2}-m_A\right)(M-L+1) \pm (Y_j - V) \\
&\quad = \mathrm{e}^{-2i\pi UV} s_b\left(-i\frac{Q}{2} + (N-\widetilde{M}+1)(iQ-2m_A)\right) \\
&\quad \times \prod_{i,j=1}^{N} s_b\left(-i\frac{Q}{2} + 2m_A \pm (X_i - X_j)\right) \prod_{i,j=1}^{L} s_b\left(i\frac{Q}{2} - 2m_A \pm (Y_i - Y_j)\right) \\
&\quad \times \int \mathrm{d}\vec{Z}_{\widetilde{M}} \, \Delta_{\widetilde{M}}(\vec{Z}) \prod_{j=1}^{\widetilde{M}} s_b \frac{iQ}{2} - \left(\frac{iQ}{2}-m_A\right)(N-\widetilde{M}+1) \pm (Z_j - V) \\
&\quad \times {}_{\vec{X}}\mathbb{I}_{\vec{Z},V}(m_A) \mathcal{Z}^{(\widetilde{M},L)}_{(1,0)}\left(\vec{Z};\vec{Y};U;i\frac{Q}{2}-m_A\right) .
\end{aligned}
\tag{3.29}
$$

This encodes in field theory the Hanany–Witten move that swaps a $(1,0)$ with a $(0,1)$-brane as shown in Figure 3.20, where before the transition we have N-D3 branes on the left of the $(1,0)$, M D3-branes between the $(1,0)$ and the $(0,1)$-brane, and L D3-branes on the right of the $(0,1)$, while after the transition we have N D3-branes on the left of the $(0,1)$, $\widetilde{M}$ D3-branes between the $(0,1)$ and the $(1,0)$, and L D3-branes

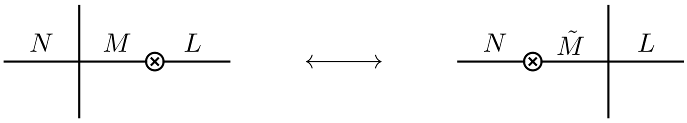


Figure 3.20: The HW brane move swapping NS5 (a solid vertical line) and D5 (a crossed circle), where $\widetilde{M} = N + L - M + 1$.

on the right of the $(1,0)$. More in general one can also consider different HW moves, for example swapping a $(1,0)$ and a $(1,k)$-brane as shown in Figure 3.21. These moves are not strictly necessary, since we can read off a gauge theory from the brane setup even if we have $(1,k)$-branes with a different number of D3's on each side. Nevertheless, this is still interesting since the two equivalent brane configurations before and after the HW move are usually associated with distinct gauge theories, leading to a non-trivial IR duality. In field theory, this amounts to using the Giveon–Kutasov duality [10] to dualize the gauge node associated with the D3-branes suspended between the $(1,0)$ and the $(1,k)$-brane involved in the HW move.

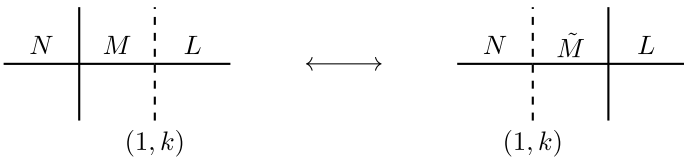


Figure 3.21: The HW brane move swapping NS5 (a solid vertical line) and $(1,k)$ (a dashed vertical line), where $\widetilde{M} = N + L - M + |k|$, which in field theory is realized by a local application of the Giveon–Kutasov duality

The Giveon–Kutasov duality relates a $3d$ $\mathcal{N} = 2$ $U(N_c)_k$ theory with N_f fundamental flavors Q^i, $\tilde{Q}_i$, $i = 1, \cdots N_f$ and no superpotential, $\mathcal{W} = 0$, to a $U(N_f + |k| - N_c)_{-k}$ theory with N_f fundamental flavors q_i, $\tilde{q}^i$, N_f^2 gauge singlets M^i_j and superpotential $\hat{\mathcal{W}} = M^i_j q_i \tilde{q}^j$. The reason why we can apply this duality is because the node of the quiver that we want to dualize has a non-zero CS level due to the presence of the $(1,1)$-brane, which makes the $\mathcal{N} = 2$ adjoint chiral in the $\mathcal{N} = 4$ vector multiplet massive so that at low energies we have an effective $\mathcal{N} = 2$ theory with only fundamental flavors and a quartic superpotential. The identity of the $\mathbf{S}^3_b$ partition functions associated with the Giveon–Kutasov duality is [60, 12]

$$
\begin{aligned}
&\int \mathrm{d}\vec{Z}_{N_c}\, \Delta_{N_c}(\vec{Z}) \mathrm{e}^{-ik\pi \sum_{i=1}^{N_c} Z_i^2 - \pi i \eta \sum_{i=1}^{N} Z_i} \prod_{i=1}^{N_c} \prod_{a=1}^{N_f} s_b(i\frac{Q}{2} \mp Z_i - \mu_a^{\pm})) \\
&= e^{\frac{i\pi}{24}\left(k^2+2\right) - \frac{i\pi}{2}\phi} \prod_{a,b=1}^{N_f} s_b(i\frac{Q}{2} - \mu_a^+ - \mu_b^-) \\
&\quad \times \int \mathrm{d}\vec{Z}_{N_c'}\, \Delta^{3d}_{N_c'}(\vec{Z}) \mathrm{e}^{ik\pi \sum_{i=1}^{N_c'} Z_i^2 + \pi i \eta \sum_{i=1}^{N_c'} Z_i} \prod_{i=1}^{N_c'} \prod_{a=1}^{N_f} s_b \mp Z_i + \mu_a^{\pm}\,.
\end{aligned}
\tag{3.30}
$$

where $k > 0$ and

$$\begin{aligned}\phi = k \Bigg(& \sum_{a=1}^{N_f} \left(\left(\mu_a^+\right)^2 + \left(\mu_a^-\right)^2 \right) - \frac{Q^2}{4} k \left(k - 2N_c'\right) + \frac{1}{2}\eta^2 - iQk \sum_{a=1}^{N_f} \left(\mu_a^+ + \mu_a^-\right) \\ & + \eta \sum_{a=1}^{N_f} \left(\mu_a^+ - \mu_a^-\right) + \frac{1}{2} \left(iQN_c' - \sum_{a=1}^{N_f} \left(\mu_a^+ + \mu_a^-\right) \right)^2 \Bigg)\end{aligned} \tag{3.31}$$

where $\mu_a^\pm$ are the real masses for the $U(N_f) \times U(N_f)/U(1)$ flavor symmetry.

Notice the exponential prefactors that can be interpreted as contact terms for the global symmetries [61, 62]. These are crucial since inside a quiver such symmetries are actually gauged, and the contact terms induce CS couplings for the gauge nodes that are adjacent to the quiver we dualized. This is expected from the brane perspective, since moving a $(1,1)$ through a (1,0) not only affects the CS level of the gauge node associated to the D3's suspended between them by changing its sign, but also those of the nodes associated to the adjacent intervals. Moreover, the gauge singlets produced with the duality might give mass to/produce adjoint chiral fields for the adjacent nodes, again compatibly with the fact that their CS levels change after the dualization.

3.3 Applying the algorithm

In this section we apply the algorithm to construct a web of dualities for the $U(2)$ SQCD with 4 flavors. In particular we compute the result of the sequence of an $\mathcal{S}$,$\mathcal{T}$ and $\mathcal{T}^T$ dualizations. Since the product of these operators is $\mathcal{S}(\mathcal{T}^T)^{-1}\mathcal{S}\mathcal{T}^{-1} = 1$ we expect that the final result of this dualizations to be the original theory, which indeed is the case as we will show next.

3.3.1 An $SL(2, \mathbb{Z})$ duality web

The web of dualities is depicted in Figure 3.22.

The first line of Figure 3.22 is the well-known $\mathcal{S}$-duality of the SQCD. Its derivation using the mirror dualization algorithm is schematically depicted in Figure 3.23 and we briefly describe it below. In the first row of Figure 3.23, theory A is chopped into QFT blocks. Applying the basic moves in Figure 3.10 and 3.11 we get the dualized theory in the second row of figure 3.23. Notice that, for the leftmost block, we apply the duality move in Figure 3.10 with a Cartan's reparametrization such that $\mathcal{S} \to \mathcal{S}^{-1}$ and viceversa (if one does not implement such a reparametrization, the first and second blocks would glue with $\mathcal{S}\mathcal{S}$ rather than as $\mathcal{S}\mathcal{S}^{-1}$). Now, as shown in the second and third row of figure 3.23, we apply the Hanany–Witten move of Figure 3.19. After these two iterations, we land on the theory B shown in the last row of figure 3.23.

To move from the top-right corner to the bottom-right of Figure 3.22 we applied a $(\mathcal{T}^T)^{-1}$ transformation, again by chopping the theory into QFT blocks and applying the correct duality moves. The frame that we obtain is the first out of the three on the bottom-right corner of Figure 3.22. This frame is engineered by a brane setup composed of NS and $(1,1)$-branes. We can perform HW moves swapping pairs of five-branes. As described in the end of Subsection 3.2.4, this corresponds to GK dualities in the quiver theory that move from one frame to another.

We then perform again $\mathcal{S}$-duality and a $\mathcal{T}^{-1}$ duality by again iterating the same strategy. Notice in particular the the brane setup before the $\mathcal{T}^{-1}$ duality is composed

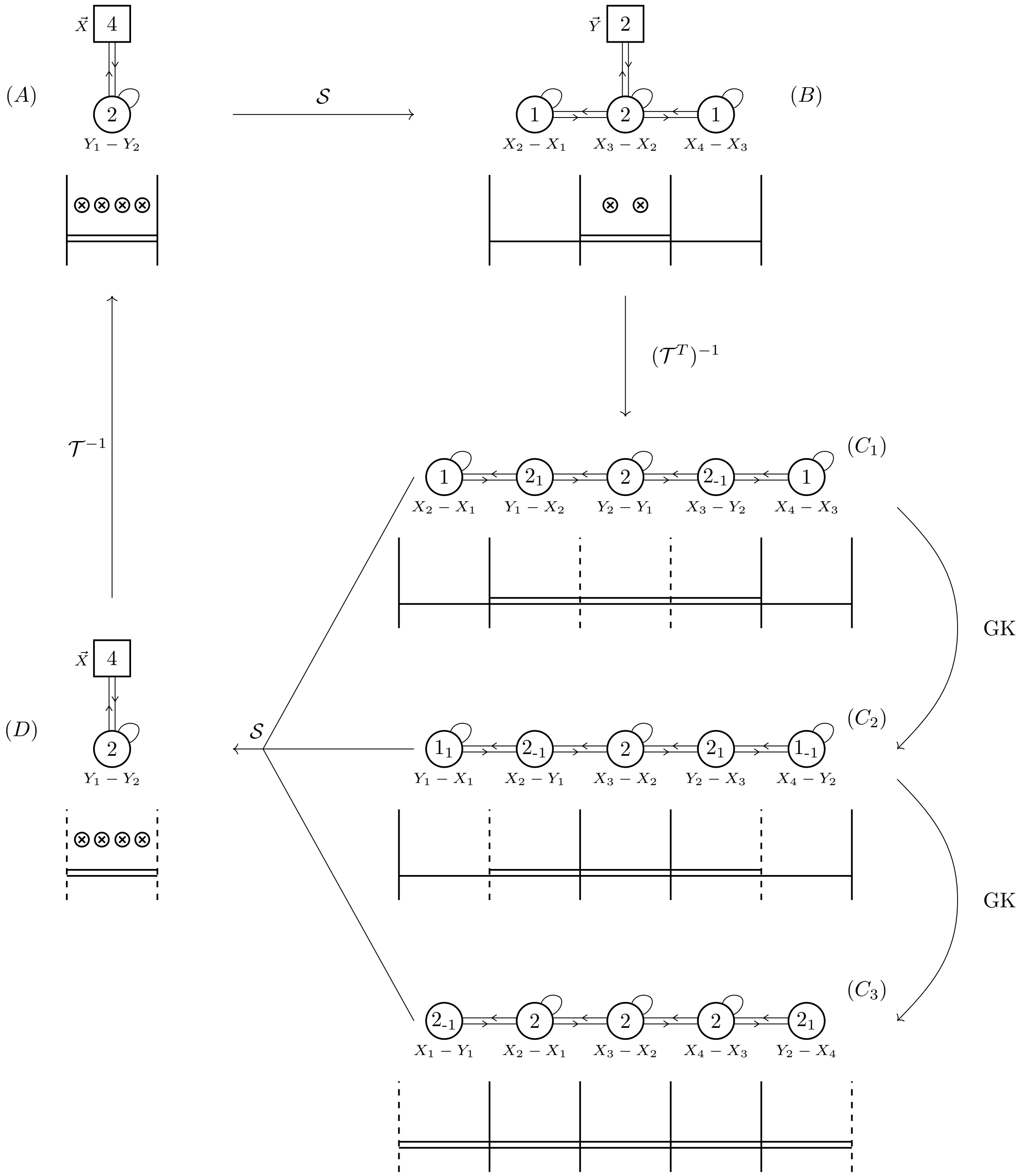


Figure 3.22: $SL(2,\mathbb{Z})$ duality web for the $U(2)$ SQCD with 4 flavors and its associated brane setup. We represent NS branes as vertical lines, D5 as crosses, $(1,1)$ as dashed lines and D3 as horizontal lines. On the bottom-right corner we move from one frame to another swapping NS and $(1,1)$-branes which in field theory consist in a Giveon-Kutasov duality. We apply the duality respectively on the second and fourth node, to move from the first to second frame, and on the first and fifth node, to move from the second to the third frame.

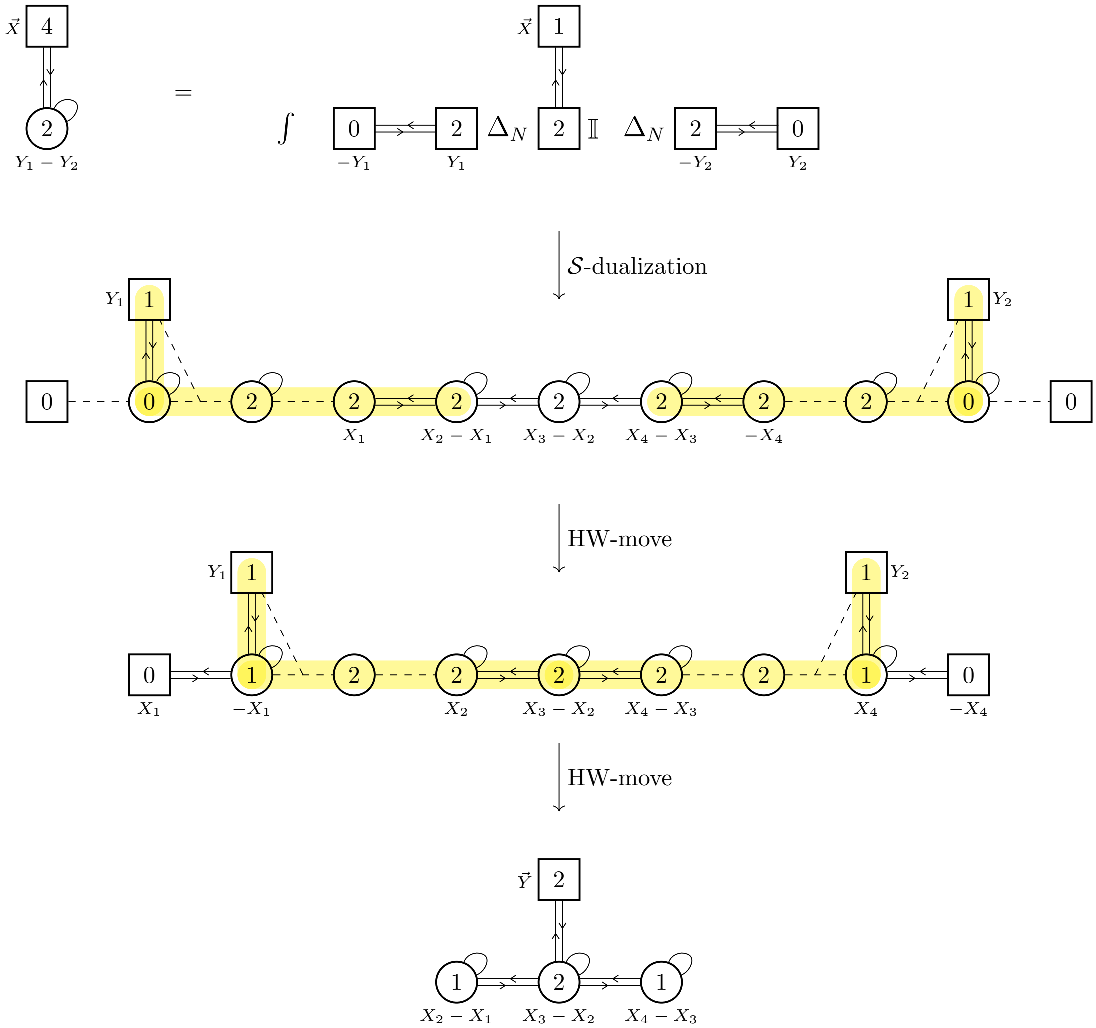


Figure 3.23: $\mathcal{S}$-dualization of the $U(2)$ SQCD with four flavors. To avoid cluttering we avoided giving some details such as BF couplings, extra singlets and the charges of each chiral. In the first line the integral and the Δ_N symbols indicate the frozen gauge interactions that are ristored after applying the basic duality moves. Notice that from the third line on we avoided giving the trivial rank 0 $\mathcal{S}$-wall on the l.h.s. and on the r.h.s. of the quiver.

only by $(1,1)$ and D5 branes. This setup is known to engineer the same SQCD as when using NS and D5 branes. The $\mathcal{T}^{-1}$ dualization thus leave this theory invariant.

Notice that the frames $C_{1,2,3}$ correspond to theories with CS couplings. In particular that in the first two frames when we integrate out the massive adjoint at the nodes where the CS coupling is turned on, a quartic superpotential for the bifundamentals is generated. These quartic superpotentials preserve the $U(1)_A$ symmetry which becomes part of the larger R-symmetry group since all these frames have enhanced $\mathcal{N} = 4$ supersymmetry as expected by the duality.

Indeed, more computations and details can be found in the original work [40].

3.3.2 $3d$ operators map

It is interesting to see how the operator map works in the $3d$ case to better understand the web of dualities. The global symmetry group of the three theories in the IR is

$$SU(4)_X \times SU(2)_Y \times U(1)_A \,, \tag{3.32}$$

but the UV manifest symmetries in the theories of the web are the following:

$$\begin{aligned} A = D: \quad & \underbrace{\frac{U(4)_X}{U(1)}}_{SU(4)_X} \times \frac{\prod_{j=1}^2 U(1)_{Y_j}}{U(1)} \times U(1)_A \,, \\ B: \quad & \underbrace{\frac{U(2)_Y}{U(1)}}_{SU(2)_Y} \times \frac{\prod_{i=1}^4 U(1)_{X_i}}{U(1)} \times U(1)_A \,, \\ C_i: \quad & \frac{\prod_{i=1}^4 U(1)_{X_i}}{U(1)} \times \frac{\prod_{j=1}^2 U(1)_{Y_j}}{U(1)} \times U(1)_A \,, \end{aligned} \tag{3.33}$$

where we are using the parametrization of the various symmetries that we obtained from the algorithm and that is specified in Figure 3.22, but we should remember that is an off-shell parametrization in $3d$ and that we should decouple a diagonal $U(1)$ both from the X and from the Y symmetries. In the following we keep the off-shell parametrization since it makes it easier to identify how the operators rearrange into representations of the enhanced symmetry and to map them across the various frames.

We now analyze each theory discussing its symmetries and the gauge invariant operators with the lowest dimension. Notice that operators with charges $(iQ - 2m_A)$ or $2m_A$ have R-charge 1 when we set the canonical parameterization $m_A = \frac{iQ}{4}$, which is the lowest R-charge possible in an interacting $3d\,\mathcal{N} = 4$ theory.

Let us start from Theory A, which is equivalent to Theory D. The manifest global symmetry group is given by

$$\frac{U(4)_X}{U(1)} \times \frac{\prod_{j=1}^2 U(1)_{Y_j}}{U(1)} \times U(1)_A \ , \tag{3.34}$$

where $\frac{U(4)_X}{U(1)} = SU(4)_X$ is the flavor symmetry, while $\frac{\prod_{j=1}^2 U(1)_{Y_j}}{U(1)}$ is the topological symmetry associated to the $U(2)$ gauge node. We work in a parameterization such that in theory A the hypermultiplets have R-charge 0 and $U(1)_A$ charge 1, consequently adjoint chirals have R-charge 2 and $U(1)_A$ charge -2. Although not being reported in Figure 3.22, the algorithm provides the parameterization of R-charges and $U(1)_A$

	$SU(4)_X \times \prod_{j=1}^{2} U(1)_{Y_j}$	$U(1)_A$	$U(1)_R$
$\mathrm{Tr}\, P\tilde{P}$	$(\mathbf{15}, 0, 0)$	2	0
$\mathrm{Tr}\, A$	$(\mathbf{1}, 0, 0)$	-2	2
$M^{(+)}$	$(\mathbf{1}, 1, -1)$	-2	2
$M^{(-)}$	$(\mathbf{1}, -1, +1)$	-2	2

Table 3.1: Gauge invariant operators in Theory A = Theory D. We denote by $M^{(\pm)} = M^{(\{\pm 1, 0\})}$ the fundamental monopoles of the $U(2)$ gauge group.

charges for theory D, that are the same as in theory A. The lowest dimension gauge invariant operators are listed in Table 3.1.

Notice that the $U(2)$ gauge node is balanced so according to [28] the topological symmetry is enhanced to $\frac{\prod_{j=1}^{2} U(1)_{Y_j}}{U(1)} \to SU(2)_Y$. In fact some of the operators in Table 3.1 combine to form representations of the IR global symmetry $SU(4)_X \times SU(2)_Y$ and provide the moment maps for it:

$$\mathrm{Tr}\, P\tilde{P} \quad \longrightarrow \quad (\mathbf{15}, 0) \quad = \quad (\mathbf{15}, \mathbf{1})\,, \tag{3.35}$$

$$\mathrm{Tr}\, A,\ M^{(\pm)} \quad \longrightarrow \quad (\mathbf{1}, 0, 0) \oplus (\mathbf{1}, 1, -1) \oplus (\mathbf{1}, -1, 1) \quad = \quad (\mathbf{1}, \mathbf{3})\,. \tag{3.36}$$

Notice that only the off-diagonal combination $U(1)_{Y_2} - U(1)_{Y_1}$ acts non-trivially and is the one that is enhanced to $SU(2)_Y$, while the diagonal combination $U(1)_{Y_1} + U(1)_{Y_2}$ acts trivially and should be decoupled as we previously mentioned.

In Theory B the manifest symmetry is

$$\frac{U(2)_Y}{U(1)} \times \frac{\prod_{i=1}^{4} U(1)_{X_i}}{U(1)} \times U(1)_A\,, \tag{3.37}$$

where $\frac{U(2)_Y}{U(1)} = SU(2)_Y$ is the flavor symmetry, while $\frac{\prod_{i=1}^{4} U(1)_{X_i}}{U(1)}$ after decoupling a diagonal $U(1)$ is a combination of the topological symmetries of the three gauge nodes which we parametrize as sepcified in Figure 3.22. Starting from the parameterization of theory A, the algorithm provides the R-charges and $U(1)_A$ charges in theory B, where all hypermultiplets have R-charge 1 and $U(1)_A$ charge -1, consequently adjoint chirals have R-charge 0 and $U(1)_A$ charge 2. We consider the operators in Table 3.2, which have the lowest scaling dimensions and provide the moment maps for the enhanced symmetry.

All the gauge nodes are balanced, so the three topological symmetries get enhanced to $\frac{\prod_{i=1}^{4} U(1)_{X_i}}{U(1)} \to SU(4)_X$. The operators combine to form representation of the IR global symmetry $SU(4)_X \times SU(2)_Y$ as

$$\begin{array}{r} M^{(\pm,0,0)},\ M^{(0,\pm,0)},\ M^{(0,0,\pm)} \\ M^{(\pm,\pm,0)},\ M^{(0,\pm,\pm)},\ M^{(\pm,\pm,\pm)} \\ \mathrm{Tr}\, A_1,\ \mathrm{Tr}\, A_2,\ \mathrm{Tr}\, A_3 \\ \end{array} \quad \longrightarrow \quad \begin{array}{l} (\mathbf{1}, \pm 1, \mp 1, 0, 0) \oplus (\mathbf{1}, 0, \pm 1, \mp 1, 0) \oplus (\mathbf{1}, 0, 0, \pm 1, \mp 1) \\ \oplus (\mathbf{1}, \pm 1, 0, \mp 1, 0) \oplus (\mathbf{1}, 0, \pm 1, 0, \mp 1) \oplus (\mathbf{1}, \pm 1, 0, 0, , \mp 1) \\ \oplus (\mathbf{1}, 0, 0, 0) \oplus (\mathbf{1}, 0, 0, 0) \oplus (\mathbf{1}, 0, 0, 0) \oplus \\ = \quad (\mathbf{15}, \mathbf{1})\,, \end{array} \tag{3.38}$$

$$\mathrm{Tr}\, P\tilde{P} \quad \longrightarrow \quad (\mathbf{3}, 0, 0, 0) \quad = \quad (\mathbf{1}, \mathbf{3})\,. \tag{3.39}$$

In the three C_i theories the manifest symmetry is

$$\frac{\prod_{i=1}^{4} U(1)_{X_i}}{U(1)} \times \frac{\prod_{j=1}^{2} U(1)_{Y_j}}{U(1)} \times U(1)_A\,, \tag{3.40}$$

	$SU(2)_Y \times \prod_{i=1}^{4} U(1)_{X_i}$	$U(1)_A$	$U(1)_R$
$\mathrm{Tr}\, P\widetilde{P}$	$(\mathbf{3},0,0,0,0)$	-2	2
$\mathrm{Tr}\, A_1$	$(\mathbf{1},0,0,0,0)$	2	0
$\mathrm{Tr}\, A_2$	$(\mathbf{1},0,0,0,0)$	2	0
$\mathrm{Tr}\, A_3$	$(\mathbf{1},0,0,0,0)$	2	0
$M^{(\pm,0,0)}$	$(\mathbf{1},\pm1,\mp1,0,0)$	2	0
$M^{(0,\pm,0)}$	$(\mathbf{1},0,\pm1,\mp1,0)$	2	0
$M^{(0,0,\pm)}$	$(\mathbf{1},0,0,\pm1,\mp1)$	2	0
$M^{(\pm,\pm,0)}$	$(\mathbf{1},\pm1,0,\mp1,0)$	2	0
$M^{(0,\pm,\pm)}$	$(\mathbf{1},0,\pm1,0,\mp1)$	2	0
$M^{(\pm,\pm,\pm)}$	$(\mathbf{1},\pm1,0,0,\mp1)$	2	0

Table 3.2: Gauge invariant operators in Theory B. We used the convention $M^{(\pm,\pm,\pm)} = M^{(\pm1,\{\pm1,0\},\pm1)}$ for the fundamental monopoles of the $U(1)\times U(2)\times U(1)$ gauge groups.

where all the symmetries, except for the axial symmetry, are topological. The central node is balanced, so we expect the enhancement $\frac{\prod_{j=1}^{2} U(1)_{Y_j}}{U(1)} \to SU(2)_Y$. On the other hand, if we look at the other gauge nodes, since they have non-zero CS level, their monopoles are not gauge invariant and have to be dressed with bifundamental fields. In the three theories C_i the hypermultiplets have the following charges. In theory C_1 the first and fourth hypermultiplets have R-charge 1 and $U(1)_A$ charge -1, while the second and third have R-charge 0 and $U(1)_A$ charge 1. In theory C_2 the first and fourth hypermultiplets have R-charge 0 and $U(1)_A$ charge 1, while the second and fourth have R-charge 1 and $U(1)_A$ charge -1. In theory C_3 all hypermultiplets have R-charge 1 and $U(1)_A$ charge -1. The charges of the adjoint chirals follow from the superpotential which couples cubically the adjoint to hypermultiplets. Notice that the nodes with non-zero CS-level have no adjoint chiral, in this case the superpotential couples quartically the hypermultiplets. The lowest dimension gauge invariant operators of Theory C_1 are listed in Table 3.3.
These operators rearrange into the representation of the IR global symmetry $SU(4)_X \times SU(2)_Y$ as follows:

$$\begin{array}{rcl}
\mathrm{Tr}\, A_1,\ \mathrm{Tr}\, A_5 & & \\
\mathrm{Tr}\,(Q_{2,3}\widetilde{Q}_{2,3}) & & (0,0,0,0,0,0)\oplus(0,0,0,0,0,0) \\
M^{(\pm,0,0,0,0)},\ M^{(0,0,0,0,\pm)} & & \oplus(0,0,0,0,0,0) \\
M^{(+,+,+,+,0)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4},\ M^{(-,-,-,-,0)}Q_{2,3}Q_{3,4} & \longrightarrow & \oplus(\pm1,\mp1,0,0,0,0)\oplus(0,0,0,0,\pm1,\mp1) \\
M^{(+,+,+,+,+)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4},\ M^{(-,-,-,-,-)}Q_{2,3}Q_{3,4} & & \oplus(\pm1,0,\mp1,0,0,0)\oplus(\pm1,0,0,\mp1,0,0) \\
M^{(0,+,+,+,0)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4},\ M^{(0-,-,-,0)}Q_{2,3}Q_{3,4} & & \oplus(0,\pm1,\mp1,0,0,0)\oplus(0,\pm1,0,\mp1,0,0) \\
M^{(0,+,+,+,+)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4},\ M^{(0-,-,-,-)}Q_{2,3}Q_{3,4} & & = \quad (\mathbf{15},\mathbf{1})\,,
\end{array} \tag{3.41}$$

$$\begin{array}{rcl}
\mathrm{Tr}\, A_3,\ M^{(0,0,\pm,0,0)} & \longrightarrow & (0,0,0,0,0,0)\oplus(0,0,0,0,\pm1,\mp1) \\
& & = \quad (\mathbf{1},\mathbf{3})\,.
\end{array} \tag{3.42}$$

After we have grouped operators in representations of the IR symmetry group it is

	$\prod_{i=1}^4 U(1)_{X_i} \times \prod_{j=1}^2 U(1)_{Y_j}$	$U(1)_A$	$U(1)_R$
$\mathrm{Tr}\, A_1$	$(0,0,0,0,0,0)$	2	0
$\mathrm{Tr}\, A_3$	$(0,0,0,0,0,0)$	-2	2
$\mathrm{Tr}\, A_5$	$(0,0,0,0,0,0)$	2	0
$\mathrm{Tr}\,(Q_{2,3}\widetilde{Q}_{2,3})$	$(0,0,0,0,0,0)$	2	0
$M^{(\pm,0,0,0,0)}$	$(\pm 1,\mp 1,0,0,0,0)$	2	0
$M^{(0,0,\pm,0,0)}$	$(0,0,0,0,\pm 1,\mp 1)$	-2	2
$M^{(0,0,0,0,\pm)}$	$(0,0,\pm 1,\mp 1,0,0)$	2	0
$M^{(+,+,+,+,0)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4}$	$(1,0,-1,0,0,0)$	2	0
$M^{(-,-,-,-,0)}Q_{2,3}Q_{3,4}$	$(-1,0,1,0,0,0)$	2	0
$M^{(+,+,+,+,+)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4}$	$(1,0,0,-1,0,0)$	2	0
$M^{(-,-,-,-,-)}Q_{2,3}Q_{3,4}$	$(-1,0,0,1,0,0)$	2	0
$M^{(0,+,+,+,0)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4}$	$(0,1,-1,0,0,0)$	2	0
$M^{(0-,-,-,0)}Q_{2,3}Q_{3,4}$	$(0,-1,1,0,0,0)$	2	0
$M^{(0,+,+,+,+)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4}$	$(0,1,0,-1,0,0)$	2	0
$M^{(0,-,-,-,-)}Q_{2,3}Q_{3,4}$	$(0,-1,0,1,0,0)$	2	0

Table 3.3: Gauge invariant operators in Theory C_1. We used the convention $M^{(\pm,\pm,\pm,\pm,\pm)} = M^{(\pm 1,\{\pm 1,0\},\{\pm 1,0\},\{\pm 1,0\},\pm 1)}$ for the fundamental monopoles of the $U(1) \times U(2)^3 \times U(1)$ gauge groups.

easy to see that they are mapped as follows:

$$\left\{\mathrm{Tr}\, P\tilde{P}\right\}_{A,D} \leftrightarrow \left\{\begin{array}{c} M^{(\pm,0,0)},\, M^{(0,\pm,0)},\, M^{(0,0,\pm)} \\ M^{(\pm,\pm,0)},\, M^{(0,\pm,\pm)},\, M^{(\pm,\pm,\pm)} \\ \mathrm{Tr}\, A_1,\, \mathrm{Tr}\, A_2,\, \mathrm{Tr}\, A_3 \end{array}\right\}_B \leftrightarrow \left\{\begin{array}{c} \mathrm{Tr}\, A_1,\, \mathrm{Tr}\, A_3 \\ \mathrm{Tr}\,(Q_{2,3}\widetilde{Q}_{2,3}) \\ M^{(\pm,0,0,0,0)},\, M^{(0,0,0,0,\pm)} \\ M^{(+,+,+,+,0)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4},\, M^{(-,-,-,-,0)}Q_{2,3}Q_{3,4} \\ M^{(+,+,+,+,+)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4},\, M^{(-,-,-,-,-)}Q_{2,3}Q_{3,4} \\ M^{(0,+,+,+,0)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4},\, M^{(0-,-,-,0)}Q_{2,3}Q_{3,4} \\ M^{(0,+,+,+,+)}\widetilde{Q}_{2,3}\widetilde{Q}_{3,4},\, M^{(0-,-,-,-)}Q_{2,3}Q_{3,4} \end{array}\right\}_{C_1} . \tag{3.43}$$

$$\left\{\mathrm{Tr}\, A,\ M^{(\pm)}\right\}_{A,D} \leftrightarrow \left\{\mathrm{Tr}\, P\tilde{P}\right\}_B \leftrightarrow \left\{\mathrm{Tr}\, A_3,\ M^{(0,0,\pm,0,0)}\right\}_{C_1} . \tag{3.44}$$

This operator map provides further evidence that the map of symmetries that results from the algorithm is the correct one.

In a similar way one can construct operators in the theories C_2 and C_3 and verify both that they rearrange into representations of the enhanced IR symmetry and that they are mapped correctly across the duality.

3.4 Conclusions

The dualization algorithm reviewed in this chapter is a twofold achievement.
It finally establishes the idea that generalized mirror dualities can be proved purely within a field theoretical framework. Moreover, it proves that mirror dualities follow from the much more fundamental Aharony duality. Even if the dualization algorithm is presented in this chapter as a sequence of manipulations on the $\mathbf{S}_b^3$ partition function, it can be easily extended to complete field theoretical statements.

An observation that is crucial for the following chapters is that having access to a field theoretical framework offers many benefits that are not accessible from a stringy point of view. Notably, it gives access to a complete mapping of the real parameters in the two theories and to the knowledge of the correct BF couplings involved in these dualities[4].

It is worth pointing out a second result presented in [40] that is not reviewed in the following thesis. It was found in [63] that there exists a class of 4d $\mathcal{N}=1$ SCFTs that exhibit generalized mirror-like dualities baptized $E_\rho^\sigma[USp(2N)]$. These theories typically have a USp gauge group with matter in the (bi)fundamental or in the antisymmetric representation. Their interest lies in the fact that they are tightly linked to the compactification of the E-string theory on Riemann surfaces [64, 65]. Moreover, these theories flow to $T_\rho^\sigma[SU(N)]$ theories upon dimensional reduction followed by suitable real mass deformations. In [40] it is established a $PSL(2,\mathbb{Z})$ group of dualities for $E_\rho^\sigma[USp(2N)]$ theories that upon dimensional reduction reduces to the $SL(2,\mathbb{Z})$ duality group of $T_\rho^\sigma[SU(N)]$ theories. It is proved also that these dualities originate from Intriligator-Pouliot duality, implying that they are also consequence of a more fundamental duality. The $PSL(2,\mathbb{Z})$ orbit of dualities have no string-theoretical interpretation yet, we leave this for future investigations.

Starting from the results presented in this chapter there are a number of possible genrealizations. The immediate generalization is to theories with non-unitary gauge groups, that is to $USp(2N)$ and $SO(N)$ gauge groups. This possibility is obstructed by the complexity of treating the partition function of $T_\sigma^\rho[G]$ where G is $USp(2N)$ or $SO(N)$, mostly because these theories do not exibit in general the full Cartan subgroup of the IR enhanced global symmetry, meaking the search for identities involving these partition functions much harder.
Another possible generalization consist in going beyond quiver with linear shape. The case of circular quivers is a possible generalization that is planned to be tackled in the near future. For these quivers the analysis is facilitated by the existance of Type IIB brane setups engineering them, allowing for a simpler generalization of the results presented in this chapter. Quivers with more complicated shape, such as star-shaped quiver, are typically expected to not enjoy a Lagrangian mirror, following the analysis of [66]. However, these theories could be analyzed using a local dualization techinque analogous to that presented in this chapter.

[4]It is important to point out that the analysis of the $\mathbf{S}_b^3$ partition function does not give access to information regarding discrete symmetries as background fields for them can not be turned on. However, we expect this strategy to extend straightforwardly to, for example, the superconformal index which is the partition function on $S^1 \times S^2$ or similar surfaces that are sensible to discrete symmetries.

Chapter 4

$\mathcal{N} = 2$: Brane setups with four supercharges and improved bifundamentals

The idea presented in this chapter can be simply stated. Once established the dualization algorithm for 3d $\mathcal{N} = 4$ theories, it is almost straightforward to generalize it to 3d $\mathcal{N} = 2$ theories. This is done simply by introducing new QFT blocks and new $SL(2, \mathbb{Z})$ basic moves relating them. Although the statement is simple, its realization is very non-trivial, as we discover that in order to dualize $\mathcal{N} = 2$ QFT blocks we need to introduce strongly-coupled SCFTs as elementary building blocks composing the theories.

The results presented in this chapter are taken from [41].

4.1 Introduction

In this chapter we leverage on the field-theoretical approach presented in the previous chapter to construct mirror dualities for 3d $\mathcal{N} = 2$ theories.

The strategy that we employ is as simple to formulate as it is tricky to achieve. It is in fact only necessary to extend the mirror dualization algorithm by introducing more QFT blocks, those that compose $\mathcal{N} = 2$ theories, along with basic duality moves relating them that can be used to dualize the theory. As an example, let us consider the $\mathcal{N} = 2$ adjoint SQCD, which has the same matter content as the $\mathcal{N} = 4$ one but lacks the cubic superpotential interaction between the fundamental hypermultiplets and the chiral adjoint. Due to the absence of this superpotential, F $\mathcal{N} = 2$ flavors contribute with a $U(F) \times U(F)$ global symmetry, which would be broken to the diagonal $U(F)$ by the $\mathcal{N} = 4$ superpotential. A single $\mathcal{N} = 2$ flavor block is therefore an hypermultiplet rotated by two $U(1)$ global symmetries. We then look for a basic duality move involving the action of the $\mathcal{S}$-wall on an $\mathcal{N} = 2$ flavor, which is a theory obtained by the diagonal gauging of two $FT[U(N)]$ theories with the addition of a flavor. It turns out that this theory is dual not to a free bifundamental hypermultiplet as in the $\mathcal{N} = 4$ case (see Figure 3.10) but instead to a strongly coupled $\mathcal{N} = 2$ SCFT called $FM[U(N)]$ theory (introduced in [67] and reviewed in Appendix C.2)[1]. This theory behaves as an *improved bifundamental* hypermultiplet as it supports a $S[U(N) \times U(N)] \times U(1)^2$

[1] This is a deformation of a duality called *braid* first presented in [64]. This duality was also proved to be a consequence of just Aharony duality in [42].

global symmetry, one extra $U(1)$ w.r.t. a bifundamental hypermultiplet, and contains in its spectrum a pair of operators in the bifundamental of $U(N) \times U(N)$.

Using this new basic duality we are then able to dualize the $\mathcal{N} = 2$ $U(N)$ adjoint SQCD. The resulting mirror dual is a linear quiver theory that is quasi-Lagrangian, in the sense that standard bifundamental hypermultiplets are replaced by copies of the *improved bifundamental* that are glued together via the gauging of emergent $U(N)$ global symmetries. For this duality we discuss the map between gauge invariant operators and global symmetries. In particular, we observe that this duality deserves the name *mirror* as it maps monopole operators into mesonic operators. Moreover, in the mirror we are able to observe at least the full Cartan subgroup of the global symmetry, that we recall is double that of the $\mathcal{N} = 4$ SQCD, thanks to the extra $U(1)$ symmetries carried by *improved bifundamentals*.

An interesting outcome of this approach is the natural interpretation of the dualization algorithm from the point of view of Type IIB brane setups preserving four supercharges. In fact, theories such as the $\mathcal{N} = 2$ adjoint SQCD are known to admit an engineering with linear Type IIB brane setups composed of NS, D5 and D3 branes in which the orientation of the five-branes is taken to be different from that of the eight supercharges case. These "rotated" five-branes are usually distinguished by a prime symbol (NS' and D5') and their presence breaks the supersymmetry preserved by the setup down to four supercharges. One could still perform $\mathcal{S}$-duality on such setups swapping NS-D5 branes as well as NS'-D5', and this gives a natural notion of a mirror dual theory. Even if the brane setup gives access to many properties of the engineered SCFT, it is a long standing open problem to determine the exact (quasi-)Lagrangian describing these SCFTs, outside of simple examples[2]. As we will review in detail towards the end of this chapter, the field-theoretical approach, in the form of the dualization algorithm, offers a natural solution to this problem. The conclusion is that an NS brane in a brane setup preserving four supercharges gives rise to an *improved bifundamental* instead of a standard bifundamental hypermultiplet. We test the consistency of this proposal by showing that two quivers arising from a pair of $\mathcal{S}$-dual brane setups, when constructed using this new dictionary, are IR dual to each other. Moreover we also show that if one restores $\mathcal{N} = 4$ supersymmetry our dualities correctly flow to known $\mathcal{N} = 4$ mirror pairs.

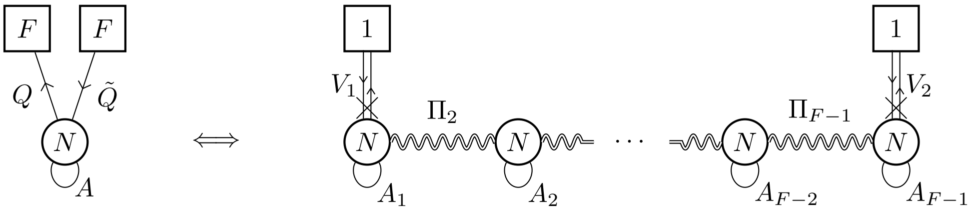
4.2 3d $\mathcal{N} = 2$ adjoint $U(N)$ SQCD and its mirror

In this section we present the mirror duality for the $\mathcal{N} = 2$ adjoint SQCD, which is depicted in Figure 4.1. The electric theory is an $\mathcal{N} = 2$ $U(N)$ gauge theory with a chiral field A in the traceless adjoint representation and F fundamental chirals Q and antifundamental chirals $\tilde{Q}$, with zero superpotential. The global symmetry group of the theory is:

$$SU(F)_U \times SU(F)_W \times U(1)_m \times U(1)_\tau \times U(1)_Y \,, \tag{4.1}$$

where we denote by U_j with $\sum_{j=1}^{N_f} U_j = 0$, W_j with $\sum_{j=1}^{N_f} W_j = 0$, m the real masses for the fundamental chirals, τ is the real mass for the adjoint chiral and Y is the FI parameter. The charges and representations for all the fields is given in table 4.1.

[2] It is worth pointing out that the brane approach offers a simple realization of Seiberg-like dualities in theories with four supercharges [4]. It is very different in the case of $\mathcal{S}$-duality, which is the problem studied in this chapter.



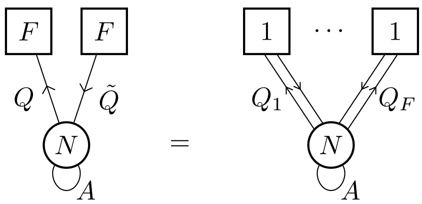
$\mathcal{W} = 0$ $\qquad\qquad$ $\mathcal{W} = \sum_{j=0}^{N-1}(Flip[V_1 A_1^j \tilde{V}_1] + Flip[V_2 A_{F-1}^j \tilde{V}_2]) + \mathcal{W}_{\text{gluing}}$

Figure 4.1: Mirror duality for the $\mathcal{N} = 2$ adjoint SQCD. The double wiggly-lines represent an $FM[U(N)]$ theory. In the picture the name reported beside the lines denotes the name of the chiral field. Whenever we have an hypermultiplet it is meant the presence of an anti-chiral field which is distinguished by a tilde. The names beside the $FM[U(N)]$ theories denote the operator in the bifundamental of the $U(N) \times U(N)$ symmetry.

	$U(1)_{R_0}$	$U(1)_\tau$	$U(1)_m$	$SU(F)_U \times SU(F)_W$
Q	1	0	-1	$\bar{\mathbf{F}} \times \mathbf{1}$
$\tilde{Q}$	1	0	-1	$\mathbf{1} \times \mathbf{F}$
A	0	1	0	$\mathbf{1} \times \mathbf{1}$

Table 4.1: List of the charges and representation of the fields in the electric theory.

It is useful also to parameterize the SQCD theory so that its manifest symmetry matches that of the mirror theory by combining pairs of fundamental/anti-fundamental chirals into flavors with axial-like mass B_j and vector-like mass X_j. The reparameterized theory is depicted in figure 4.2, along with a table with all the representation and charges of the fields after the reparameterization.



	$U(1)_{R_0}$	$U(1)_\tau$	$U(1)_{B_j}$	$U(1)_{X_j}$
Q_k	1	0	$-\delta_{jk}$	$-\delta_{jk}$
$\tilde{Q}_k$	1	0	$-\delta_{jk}$	δ_{jk}
A	0	1	0	0

Figure 4.2: Reparameterization of the electric theory. On the r.h.s. we also listed the abelian charges of all the fields of the reparameterized theory. The convention is to take the fields Q_j in the fundamental and $\tilde{Q}_j$ antifundamental representation of the gauge group.

The set of real masses for the vector-like symmetries can be taken such that: $\sum_{j=1}^F X_j = 0$, since the gauge group is $U(N)$. The $U(1)_m$ symmetry of the theory before the reparameterization is related to the axial masses as:

$$m = \frac{1}{F}\sum_{j=1}^F B_j \,. \tag{4.2}$$

We can also define a new set of axial masses as: $\tilde{B}_j = B_j - m$, so that $\sum_{j=1}^F \tilde{B}_j = 0$.

The real masses of the two parameterization are related as:

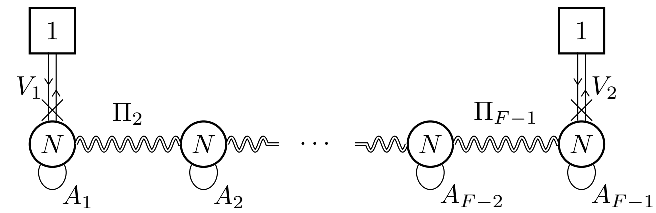
$$
\begin{aligned}
U_j &= X_j - \tilde{B}_j \\
W_j &= X_j + \tilde{B}_j \,,
\end{aligned}
\tag{4.3}
$$

while the symmetries recombine as:

$$
\begin{aligned}
\prod_{j=1}^{F} U(1)_{B_j} \times S\left[\prod_{j=1}^{F} U(1)_{X_j}\right] =& S\left[\prod_{j=1}^{F} U(1)_{\tilde{B}_j}\right] \times S\left[\prod_{j=1}^{F} U(1)_{X_j}\right] \times U(1)_m \\
&\to SU(N)_U \times SU(N)_W \times U(1)_m \,.
\end{aligned}
\tag{4.4}
$$

Dual quiver Let's now discuss the mirror theory. The main ingredient is the *improved bifundamental* which is identified with the $FM[U(N)]$ theory, a 3d $\mathcal{N}=2$ SCFT introduced in [67] which we describe in Appendix C.2. We denote this theory compactly by two wiggle lines connecting the two $U(N)$ nodes to visualize the two non-abelian $U(N)$ IR symmetries. In addition to them, the improved bifundamental has a $U(1)_\tau \times U(1)_\Delta$ abelian symmetry. The IR spectrum of the improved bifundamental includes two traceless adjoint operators and two bifundamental $(N, \bar{N})$, $(\bar{N}, N)$ operators $\Pi, \tilde{\Pi}$ of the two $U(N)$ symmetries and a matrix of singlets $\mathsf{B}_{n,m}$, with charges given in Table C.2. In particular the bifundamental operators carry charge $+1$ under the *axial* $U(1)_\Delta$ symmetry.

Our adjoint SQCD mirror dual is given by a linear quiver with $F-2$ improved bifundamental links and at each end of the quiver we have the the flavors $V_1, \tilde{V}_1$ and $V_2, \tilde{V}_2$:



$$
\mathcal{W} = \sum_{j=0}^{N-1} (Flip[V_1 A_1^j \tilde{V}_1] + Flip[V_2 A_{F-1}^j \tilde{V}_2]) + \mathcal{W}_{\text{gluing}}
\tag{4.5}
$$

The list of charges and representations for all the fields and bifundamental operators is given in Table 4.2.

	$U(1)_{R_0}$	$U(1)_\tau$	$U(1)_{B_j}$	$U(1)_Y$
$V_1, \tilde{V}_1$	0	$\frac{1-N}{2}$	δ_{j1}	∓ 1
$V_2, \tilde{V}_2$	0	$\frac{1-N}{2}$	δ_{jF}	0
$\Pi_k, \tilde{\Pi}_k$	0	0	δ_{jk}	0
A_k	0	1	0	0

Table 4.2: List of charges and representations. The Π_j operators are in the fundamental representation of the $(j-1)$-th and antifundamental of the j-th gauge groups. The fields V_1 and V_2 are in fundamental representation of the first and last gauge group, respectively.

The manifest UV global symmetry is:

$$
\prod_{j=1}^{F} U(1)_{B_j} \times \prod_{j=1}^{F-1} U(1)_{X_j - X_{j-1}} \times U(1)_\tau \times U(1)_Y \,,
\tag{4.6}
$$

Where $X_{j+1} - X_j$ is the FI parameter related to the $U(1)$ topological symmetry of the j-th gauge node. The parameters B_j for $j = 2, \cdots F-1$ are the real axial mass associated to the $U(1)_{B_j}$ symmetries of each improved bifundamental, while B_1 and B_F are the axial symmetries for the left and right vertical flavors, respectively. Notice that we can re-absorbe a $U(1)$ vector-like symmetry by a gauge transformation since all nodes are $U(N)$. We have the following symmetry enhancement in the IR

$$\prod_{j=1}^{F} U(1)_{B_j} \times \prod_{j=1}^{F-1} U(1)_{X_{j+1}-X_j} \to SU(N)_U \times SU(N)_V \times U(1)_m \,, \tag{4.7}$$

so that in the IR, the mirror dual theory has exactly the same global symmetry group of the electric theory.

The theory also contains singlets. To avoid introducing too many names for the singlet fields, we adopt the following notation. Given an operator X and a gauge singlet elementary field O_X we denote a superpotential term of the form $\mathcal{W} = O_X X$ as $Flip[X]$. In addition we refer to the flipper singlet O_X as $\mathcal{F}[X]$. In our mirror theory we have singlets $\mathcal{F}[V_1 A^j \tilde{V}_1]$ and $\mathcal{F}[V_2 A^j \tilde{V}_2]$, which flip the dressed mesons constructed with the left and right flavors, they are represented as crosses in the Figure 4.1.

In the mirror theory we have a string of consecutive improved bifundamentals which are glued by gauging a diagonal combination of the two $U(N)$ symmetries with the addition of an adjoint field A. More precisely we couple the adjoint operator A_L of the improved bifundamental on the left and the adjoint operator A_R of the improved bifundamental on the right to the extra adjoint field A as: $\mathcal{W} = A(\mathsf{A}_L - \mathsf{A}_R)$. We use also the short-hand notation $\mathcal{W}_{\text{gluing}}$ to collect all the superpotential terms coming from this procedure[3]. Notice that when we glue a string of improved bifundamentals, all the $U(1)_\tau$ symmetries are identified while the $U(1)_{B_j}$ symmetries are all preserved. These symmetries then recombine with the topological symmetries and enhance to match the global symmetry group of the electric theory as in equation (4.7).
At the level of $\mathbf{S}_b^3$ partition functions, the duality in Figure 4.1 implies the identity:[4]

$$Z_{\text{SQCD}}\left(\tau, \vec{B}, \vec{X}, Y\right) = Z_{\widetilde{\text{SQCD}}}\left(\tau, \vec{B}, \vec{X}, Y\right) \,. \tag{4.8}$$

On the l.h.s. we have the partition function of the $U(N)$ adjoint SQCD parameterized as in Figure 4.2, which is given as:

$$Z_{\text{SQCD}}\left(\tau, \vec{B}, \vec{X}, Y\right) = \int d\vec{Z}_N \Delta_N(\vec{Z};\tau) e^{2\pi i Y \sum_{j=1}^N Z_j} \prod_{j=1}^{N} \prod_{a=1}^{F} s_b\left(B_a \pm (Z_j - X_a)\right) \,. \tag{4.9}$$

Where $\Delta_N(\vec{Z},\tau)$ is the contribution of a $\mathcal{N}=2$ vector and a chiral adjoint of R-charge 0 and $U(1)_\tau$ charge 1:[5]

$$\begin{aligned} \Delta_N(\vec{Z};\tau) &= \Delta_N(\vec{Z}) s_b(i\tfrac{Q}{2} - \tau)^{N-1} \prod_{i<j=1}^{N} s_b\left(i\frac{Q}{2} \pm (Z_i - Z_j) - \tau\right) \,, \\ \Delta_N(\vec{Z}) &= \frac{1}{\prod_{i<j}^{N} s_b\left(i\frac{Q}{2} \pm (Z_i - Z_j)\right)} \,. \end{aligned} \tag{4.10}$$

[3]Notice that here we glue improved bifundamentals by turning on only $\mathsf{W}_{\text{gluing}}$. If one adds also a monopole superpotential of the type $\mathcal{W} = \mathfrak{M}^+ + \mathfrak{M}^-$, then two improved bifundamental theories fuse to an $\mathbb{I}$-wall as explained in appendix C.2.

[4]We follow the conventions summarized in Appendix A.

[5]Notice that in this chapter we adopt a different notation w.r.t. Chapter 3. The adjoint chiral is taken to be traceless and also the argument τ enters in a slightly different way.

The partition function of the mirror dual theory, given on the r.h.s. of Figure 4.1, is instead given as:

$$\begin{aligned} Z_{\widetilde{\text{SQCD}}}\left(\tau,\vec{B},\vec{X},Y\right) &= e^{2\pi i N Y X_1}\int\prod_{a=1}^{F-1} d\vec{Z}^{(a)}_N \Delta_N(\vec{Z}^{(a)};\tau)e^{2\pi i(X_{a+1}-X_a)\sum_{j=1}^N Z_j} \\ &\prod_{j=1}^{N}\Big[s_b\left(\frac{iQ}{2}-\frac{1-N}{2}\tau-B_1\pm(Z^{(1)}_j-Y)\right)s_b(-\frac{iQ}{2}+(j-N)\tau+2B_1) \\ &s_b\left(\frac{iQ}{2}-\frac{1-N}{2}\tau-B_F\pm Z^{(F-1)}_j\right)s_b(-\frac{iQ}{2}+(j-N)\tau+2B_F)\Big] \\ &\prod_{a=1}^{F-2} Z^{(N)}_{FM}\left(\vec{Z}^{(a)},\vec{Z}^{(a+1)},\tau,B_{a+1}\right)\,. \end{aligned} \tag{4.11}$$

The $\mathbf{S}^3_b$ partition function of the $FM[U(N)]$ theory is defined in Appendix C.2.

4.2.1 Comments on the $F = 1$ and $F = 2$ cases

The cases $F = 1, 2$ were already discussed in literature, in this section we wish to comment on how our result reconciles with these known results.

Let us start with the $F = 2$ case. In this case our mirror pair in Figure 4.1 has no improved bifundamental links and it reduces to a self-duality modulo flips:

$$\mathcal{W}=0 \qquad\qquad \mathcal{W}=\sum_{j=0}^{N-1}\left(Flip[P_1C^j\tilde{P}_1]+ +Flip[P_2C^j\tilde{P}_2]\right) \tag{4.12}$$

	$U(1)_{R_0}$	$U(1)_\tau$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_X$	$U(1)_Y$
$Q_1,\tilde{Q}_1$	1	0	-1	0	$\pm1/2$	0
$Q_2,\tilde{Q}_2$	1	0	0	-1	$\mp1/2$	0
$P_1,\tilde{P}_1$	0	$\frac{1-N}{2}$	1	0	0	$\mp1/2$
$P_2,\tilde{P}_2$	0	$\frac{1-N}{2}$	0	1	0	$\pm1/2$
A,C	0	1	0	0	0	0

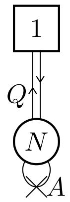
Table 4.3: Charges for the fields in the mirror $F = 2$ self-duality. In the electric theory the FI parameter for the topological symmetry is Y, while in the dual it is X.

The duality depicted in (4.12) was interpreted as a mirror duality in [21], which obtained it reducing the CSST self-duality modulo flips of $4d$ $\mathcal{N}=1$ $USp(2N)$ with antisymmetric and 8 fundamentals [68][6].

[6]A similar self-duality with $6N$ instead of $2N$ flipping fields on the r.h.s. can beobtained via se-

The $F = 1$ case can not be directly read from the mirror pair proposed in Figure 4.1, which is defined only for $F \geqslant 2$. However our dualization algorithm, which as we will see in section 4.3 allows us to prove the $F \geqslant 2$ duality, can be run also in the $F = 1$ case and the result produced is consistent with the earlier duality shown in figure (4.13) which was discussed in [21] and derived via sequential deconfinement in [14, 69].

$\mathcal{W} = \sum_{j=2}^{N} Flip[A^j]$



$\Leftrightarrow$

$3N$ chirals: R_j, S_j, T_j with:
$\mathcal{W} = \sum_{j,k,l=1}^{N} \delta_{j+k+l,N+2} R_j S_l T_k$

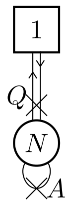
(4.13)

This duality relates the SQCD with one flavor where we flipped the tower of powers of the adjoint (which would all be below the unitarity bound), to a Wess-Zumino model. Notice that the superpotential on the Wess-Zumino side contains a number of terms growing quadratically with N. This superpotential was proposed in [21] and tested using sequential deconfinement in [69].

Starting from the the SQCD on the l.h.s. of the figure in (4.13), the algorithm yields on the dual side a collection of $3N$ chiral fields with a charge assignment which is indeed compatible with the superpotential given in (4.13).

It is useful to consider also a flipped version of this duality where on the electric side we flip the tower of dressed mesons:

$\mathcal{W} = \sum_{j=0}^{N-1} Flip[QA^j\tilde{Q}] +$
$+ \sum_{j=2}^{N} Flip[A^j]$



$\Leftrightarrow$ (Free Hyper)^N

(4.14)

In this case on the dual side we have N free hypers mapping to the tower of dressed monopoles $\mathfrak{M}^{\pm}_{A^j}$ in the electric SQCD side.

4.2.2 Operator Map

We now show how the gauge invariant operators of the SQCD theory are mapped into the mirror dual.

- In the electric theory we have the meson operator $Q\tilde{Q}$ in the $\bar{\mathbf{F}} \times \mathbf{F}$ representation of $SU(F)_U \times SU(F)_W$, with R-charge 2, m-charge -2 and zero τ-charge. This operator is mapped into the following collection of operators of the mirror theory:
  - Singlets $B^{(k)}_{1,1}$ of the (k)-th improved bifundamental theory, with $k = 2, \cdots F-1$, and singlets $\mathcal{F}[V_1 A_1^{N-1} \tilde{V}_1]$ and $\mathcal{F}[V_2 A_{F-1}^{N-1} \tilde{V}_2]$. These are F operators, each with R-charge 2 and m-charge -2.

quential deconfinement, [16]. As shown in [21], (4.12) is the 3d reduction of the CSST self-duality of $USp(2N)$ with antisymmetric and 8 fundamentals, while the sequential deconfinement method [16, 69] proves the IP-like self-duality of $USp(2N)$ with antisymmetric and 8 fundamentals and its 3d reductions.

- $F(F-1)$ monopole operators with topological charge given by strings of contiguous $+1$ (or -1) under the topological symmetries $U(1)_{X_{j+1}-X_j}$. This requires a careful analysis due to the presence of the improves bifundamentals. However it is possible to prove that these monopoles carry R-charge 2 and m-charge -2 (see Appendix E of [41]).

We can arrange all these operator into a matrix transforming in the $\mathbf{F}\times\bar{\mathbf{F}}$ representation of $SU(F)_U\times SU(F)_W$ which is naturally mapped to the electric meson. For example, for a theory with $F=4$, meaning that the mirror theory has 3 gauge nodes, this matrix is given as:

$$\begin{pmatrix} \mathcal{F}[V_1A_1^{N-1}\tilde{V}_1] & \mathfrak{M}^{(+,0,0)} & \mathfrak{M}^{(+,+,0)} & \mathfrak{M}^{(+,+,+)} \\ \mathfrak{M}^{(-,0,0)} & \mathsf{B}^{(2)}_{1,1} & \mathfrak{M}^{(0,+,0)} & \mathfrak{M}^{0,+,+)} \\ \mathfrak{M}^{(-,-,0)} & \mathfrak{M}^{(0,-,0)} & \mathsf{B}^{(3)}_{1,1} & \mathfrak{M}^{(0,0,+)} \\ \mathfrak{M}^{(-,-,-)} & \mathfrak{M}^{(0,-,-)} & \mathfrak{M}^{(0,0,-)} & \mathcal{F}[V_2A_3^{N-1}\tilde{V}_2] \end{pmatrix} \tag{4.15}$$

Where we denote with $\mathfrak{M}^{(i,j,k)}$ a monopole with topological charges i,j,k under the three topological symmetries.

We can also consider dressed mesons with powers of the adjoint in the electric theory. If we parameterize the SQCD as in Figure 4.2 the map works very intuitively as:

SQCD	Mirror
$Q_1A^l\tilde{Q}_1$	$\mathcal{F}[V_1A_1^{N-1-l}\tilde{V}_1]$
$Q_FA^l\tilde{Q}_F$	$\mathcal{F}[V_2A_{F-1}^{N-1-l}\tilde{V}_2]$
$Q_jA^l\tilde{Q}_j$ for $j=2,\dots,F-1$	$\mathsf{B}^{(j)}_{1,1+l}$
$Q_jA^l\tilde{Q}_k$ for $j\neq k$	$\mathfrak{M}^{(0,\dots,0,+,\dots,+,0,\dots,0)}_{A^l}$ non-null entries: j to $k-1$

(4.16)

- In the electric theory we then have the traces of powers of the adjoint field A^j, for $j=2,\dots,N$, that are only charged under the $U(1)_\tau$ symmetry with a charge of j. These operators are mapped into similar operators that can be built in the mirror theory. In the mirror we have an adjoint field for each gauge node, all with a charge 1 under the $U(1)_\tau$ symmetry. However, quantum effects relate the traces of powers of all these operators such that they are all identified, leaving only one independent set of operators with charges j under the $U(1)_\tau$ symmetry for $j=2,\dots,N$.

- Lastly, we also have monopoles in the SQCD theory. The lowest charged monopoles, with ±1 charge under $U(1)_Y$, have m-charge F, τ-charge $1-N$ and are singlets under all the other symmetries. These are mapped into long mesons $\tilde{V}_1\Pi_2\dots\Pi_{F-1}V_2$ and $V_1\tilde{\Pi}_2\dots\tilde{\Pi}_{F-1}\tilde{V}_2$ in the mirror theory. These have τ-charge $N-1$ and charge $+1$ under all the $U(1)_{B_j}$ symmetries, which implies that under the $U(1)_m$ symmetry it has charge F. Also, they have charges ±1 under $U(1)_Y$, which we recall is mapped into the topological symmetry of the SQCD theory.
Dressed monopoles of the SQCD theory are mapped similarly into dressed long mesons with the same level of dressing.

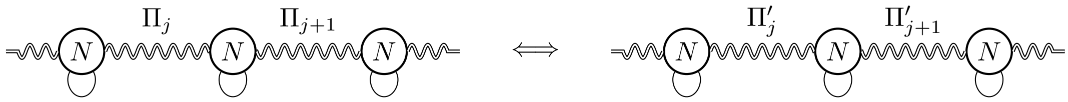
To conclude, let us mention that not all the holomorphic gauge invariant operators in the quiver side are mapped to the electric theory. Notable absent from the list of mapped operators are the gauge singlets $\mathsf{B}^{(j)}_{n,m}$ for $n \neq 1$. We claim that the holomorphic operators $\mathsf{B}^{(j)}_{n\neq 1,m}$ are trivial in the chiral ring of the magnetic theory, since there is no operator in the electric theory chiral ring with the correct global symmetries.

In particular, the triviality of the $\mathsf{B}^{(j)}_{2,1}$'s has interesting consequences. The $\mathsf{B}^{(j)}_{2,1}$'s, if turned on in the superpotential, would *iron* an improved bifundamental into a standard one (see (4.21)), providing an RG flow to the putative IR SCFT associated to the naive reading of the magnetic brane setup. Chiral ring stability tells us that adding to the superpotential chiral-ring-trivial operators has trivial consequences to the IR SCFT. Hence chiral ring stability tells us that our mirror and the naive mirror flow in the IR to the same SCFT. See Section 4.4.3 for more comments about the relation between our and the old proposals of 3d mirror duality with 4 supercharges.

4.2.3 Deformations and consistency checks

In this section we study the effect of some interesting deformations of our dual pair, providing non-trivial consistency checks of our duality.
Before discussing deformations we notice that in the magnetic theory, thanks to two *swapping* dualities, we are allowed to shuffle and reorder all the improved bifundamentals and the two vertical flavors.

The first duality allows us to *swap* two consecutive improved bifundamentals, that is under the duality the two $U(1)$ symmetries rotating the bifundamentals are exchanged. This duality is given in Appendix C.2, see C.32. Using this duality we get:[7]



Π_j	B_j
Π_{j+1}	B_{j+1}

Π'_j	B_{j+1}
Π'_{j+1}	B_j

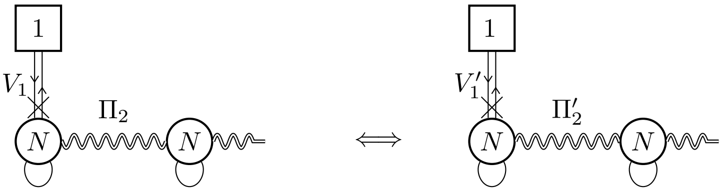
(4.17)

Notice that under this duality, the matrix $\mathsf{B}^{(j)}_{n,m}$ is mapped to $\mathsf{B}'^{(j+1)}_{n,m}$ while $\mathsf{B}^{(j+1)}_{n,m}$ is mapped to $\mathsf{B}'^{(j)}_{n,m}$. For more details about the duality involved see [41, 42].

A specialization of the previous duality allows us to exchange an improved bifundamental with a flavor. For example, we can exchange the left vertical flavor with the

[7] We list operators and their R-charge in a table beside the quiver representation of a theory. The R-charge is expressed as $R_0 + \sum_E q_E E$, where R_0 is the trial R-charge. Also, the sum runs over all the $U(1)_E$ global symmetries and we denote by E the mixing coefficient for $U(1)_E$ and by q_E the charge of the operator under the group. So with a slight abuse of notation we denote by E both the real mass and the mixing coefficient for $U(1)_E$.

first improved bifundamental:



$$\begin{array}{c|c} V_1 & \frac{1-N}{2}\tau + B_1 \\ \Pi_2 & B_2 \end{array} \qquad\qquad \begin{array}{c|c} V_1' & \frac{1-N}{2}\tau + B_2 \\ \Pi_2' & B_1 \end{array} \tag{4.18}$$

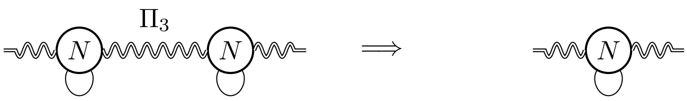
For more details about this duality see [41, 42]. Notice that under this duality the tower of flipping singlets $\mathcal{F}[V_1 A_1^k \tilde{V}_1]$ is mapped into part of the matrix of singlets of the improved bifundamental theory $\mathsf{B}^{(2)}_{1,k}$, and vice-versa. Instead, the rest of the singlet matrix is not mapped under this duality, this is consistent with our claim that the singlets that are not mapped are not in the chiral ring.

One can combine the two moves (4.17) and (4.18) to rearrange all the bifundamentals in any desired way. This property is important to discuss the deformations, as we show in detail below.

Shortening

The first deformation that we consider is given by complex mass terms in the electric theory. This means that we turn on a superpotential term as: $\delta\mathcal{W} = Q_j\tilde{Q}_j$ for any $j = 1, \dots, F$. For $j = 1, F$ this deformation is mapped respectively into $\mathcal{F}[V_1 A_1^{N-1}\tilde{V}_1]$ and $\mathcal{F}[V_2 A_{F-1}^{N-1}\tilde{V}_2]$ while for $j = 2, \cdots F-1$ it is mapped into $\mathsf{B}^{(j)}_{1,1}$, as it can be read from the table in (4.16).

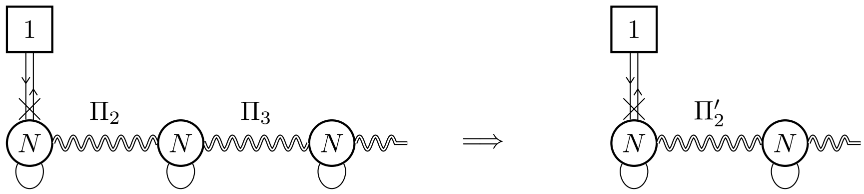
By means of the freedom of rearranging the improved bifundamentals, using the duality (4.17) we see that all the deformations for $j = 2, \dots, F-1$ are on the same footing. Also using the duality (4.18) the case $j = 1$ is equivalent to $j = 2$ and, analogously, $j = F$ is equal to $j = F-1$. Therefore we conclude that this deformation can be implemented on any improved bifundamental without any loss of generality. Let us then consider $j = 3$ for simplicity, the superpotential term $\delta\mathcal{W} = Q_3\tilde{Q}_3$ is mapped to $\delta\mathcal{W} = \mathsf{B}^{(3)}_{1,1}$. This deformation has the effect of transforming the improved bifundamental in an identity-wall, as explained in Appendix C.2 eq. (C.20) which identifies the two $U(N)$ symmetry groups which is connecting:



$$\mathcal{W} = \mathcal{W}_{\text{gluing}} + \mathsf{B}^{(3)}_{1,1} \qquad\qquad \mathcal{W} = \mathcal{W}_{\text{gluing}} \tag{4.19}$$

Intuitively, one can think that the linear superpotential term $\delta\mathcal{W} = \mathsf{B}^{(3)}_{1,1}$ has the effect of giving a VEV to the Π_3 and $\tilde{\Pi}_3$ operators. The effect of this VEV is to Higgs the $U(N) \times U(N)$ gauge syemmtry under which is charged down to the diagonal $U(N)$. This is supported from the fact that imposing the $\mathsf{B}^{(3)}_{1,1}$ operator to have R-charge 2 implies that the Π_3 operator has R-charge 0, as one would expect from an operator acquiring a VEV. The net effect of this deformation is then to shorten the sequence of improved bifundamentals by one unit. This is consistent with our duality, indeed the electric SQCD after the mass deformation has $F-1$ flavors and its mirror has one less improved bifundamental links.

In the electric theory we can also turn on a non-diagonal mass term of the type: $\delta\mathcal{W} = Q_j\tilde{Q}_k$, with $j \neq k$ which has also the effect of just removing one flavor. From the operator map (4.16), we see that this deformation is mapped into monopole superpotentials in the magnetic theory. Let us consider the case $\delta\mathcal{W} = Q_2\tilde{Q}_3$ without any loss of generality. This superpotential term is mapped to $\delta\mathcal{W} = \mathfrak{M}^{(0,+,0,\dots,0)}$ in the magnetic theory. We can now use a duality relating two improved bifundamentals glued with a monopole superpotential $\mathcal{W} = \mathcal{W}_{gluing} + \mathfrak{M}^{+}$ (and equivalently for $\mathcal{W} = \mathfrak{M}^{-}$) with a single improved bifundamental (eq. (C.6) in [41]).

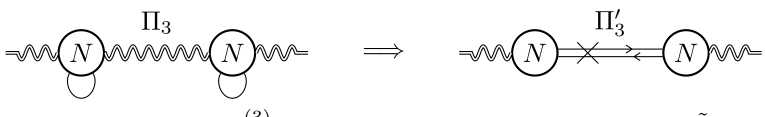


$$\mathcal{W} = \mathcal{W}_{\text{gluing}} + \mathfrak{M}^{(0,+,0\dots,0)} \qquad\qquad \mathcal{W} = \mathcal{W}_{\text{gluing}} \tag{4.20}$$

We can then conclude that this deformation has the effect of shortening the sequence of improved bifundamentals by one unit. So our duality passes also this consistency check.
Notice that instead the deformation $\delta\mathcal{W} = Q_2\tilde{Q}_3 + \tilde{Q}_2Q_3$ which corresponds to integrating out two flavors in the electric theory, maps to $\delta\mathcal{W} = \mathfrak{M}^{(0,+,0,\dots,0)} + \mathfrak{M}^{(0,-,0,\dots,0)}$ in the magnetic theory. We can now use the fact that two improved bifundamentals glued with a monopole superpotential $\mathcal{W} = \mathcal{W}_{\text{gluing}} + \mathfrak{M}^{+} + \mathfrak{M}^{-}$ fuse to an identity-wall (see eq. (C.19)), to conclude that this deformation has the effect of shortening the sequence of improved bifundamentals by two units, as expected.

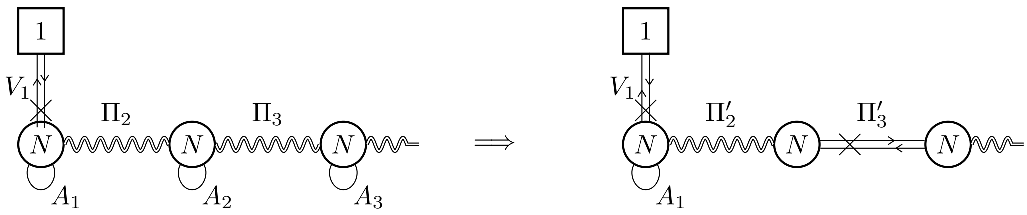
Ironing

The second set of deformations that we consider are cubic terms for the flavors in the SQCD: $\delta\mathcal{W} = Q_jA\tilde{Q}_j$. Following the operator map in(4.16) we see that those are mapped in either $\mathcal{F}[V_1A_1^{N-2}\tilde{V}_1]$, $\mathcal{F}[V_2A_{F-1}^{N-2}\tilde{V}_2]$ for $j=1,F$ and into $\mathsf{B}_{1,2}^{(j)}$ for $j = 2,\dots F-1$. As before, using the swapping dualities (4.17) and (4.18) to rearrange improved bifundamentals and flavors, we can focus on the case $j=3$ for simplicity without any loss of generality where the deformation $\delta\mathcal{W} = Q_3A\tilde{Q}_3$ is mapped to $\delta\mathcal{W} = \mathsf{B}_{1,2}^{(3)}$. This deformation has the effect of ironing an improved bifundamental into an ordinary bifundamental of charge $\tau/2$ coupled to two extra adjoint fields, as reviewed in Appendix C.2 eq. (C.21). These extra adjoint fields give mass to the adjoint fields in $\mathcal{W}_{\text{gluing}}$ to its left and right and the bifundamental is then coupled to adjoint operators inside the improved bifundamentals. We summarize graphically this picture below:



$$\mathcal{W} = \mathcal{W}_{\text{gluing}} + B_{1,2}^{(3)} \qquad\qquad \begin{aligned}\mathcal{W} = \Pi'_3(\mathsf{A}_L + \mathsf{A}_R)\tilde{\Pi}'_3 + \\ + Flip[\Pi'_3\tilde{\Pi}'_3]\end{aligned} \tag{4.21}$$

One can consider also a non diagonal superpotential term: $\delta\mathcal{W} = Q_j A \tilde{Q}_k$ which is mapped to a superpotential term for the magnetic theory given by dressed monopole operators. Again by consistency this deformation should result into the ironing of an improved bifundamental link. Indeed using the swapping dualities (4.17) and (4.18), without any loss of generality, we can consider the case $\delta\mathcal{W} = Q_2 A \tilde{Q}_3$. This superpotential term is mapped to the superpotential $\delta\mathcal{W} = \mathfrak{M}^{(0,+,0,\dots,0)}_{A_2}$ involving a dressed monopole. In this case we can use a duality relating two improved bifundamentals glued with $\mathcal{W} = \mathcal{W}_{gluing} + \mathfrak{M}^+_A$ (and analogously for $\mathcal{W} = \mathfrak{M}^-_A$) and an improved bifundamental glued to an ordinary bifundamental coupled to the adjoint operators of the improved bifundamental theories (see [41] eq. (D.9)). We summarise graphically this picture below.



$$\mathcal{W} = \mathcal{W}_{\text{gluing}} + \mathfrak{M}^{(0,+,0\dots,0)}_{A_2} \qquad \begin{aligned} \mathcal{W} = {} & \Pi'_3(\mathsf{A}_L + \mathsf{A}_R)\tilde{\Pi}'_3 \\ & + Flip[\Pi'_3\tilde{\Pi}'_3] \end{aligned} \tag{4.22}$$

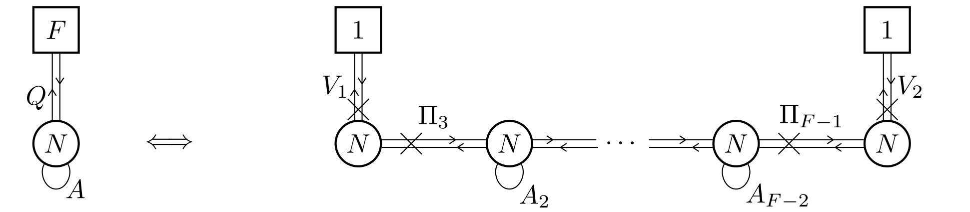
We can then conclude that any generic cubic superpotential term $\delta\mathcal{W} = Q_j A \tilde{Q}_k$ in the electric theory leads to the ironing of a single improved bifundamental link.

Flow to the $\mathcal{N} = 4$ mirror symmetry

If we turn on the superpotential $\delta\mathcal{W} = \sum_{j=1}^F Q_j A \tilde{Q}_j$, that is we couple all flavors to the adjoint chiral, we reach the $\mathcal{N} = 4$ $U(N)$ SQCD. It is then an interesting consistency check to show how our mirror dual reduces to the known $\mathcal{N} = 4$ of the SQCD for $F \geqslant 2N$ [8]. The effect of the deformation on the mirror theory is to iron all the improved bifundamentals. Keeping track of extra adjoints appearing in the ironing duality (C.21), we observe that each node has an adjoint of charge $2 - \tau$ which couples to the bifundamentals to its right and its left and also all the bifundamentals are flipped. We denote these couplings as $\mathcal{W}^{\text{partial}}_{\mathcal{N}=4}$. On the first and last node the adjoint has become massive and now the vertical flavors are coupled to an adjoint operator built from the square of the bifundamentals. On the mirror side we also turn on linear superpotentials for the flipping fields $\mathcal{F}[V_1(\Pi_2\tilde{\Pi}_2)^{N-2}\tilde{V}_1]$ and $\mathcal{F}[V_2(\Pi_{F-1}\tilde{\Pi}_{F-1})^{N-2}\tilde{V}_2]$.

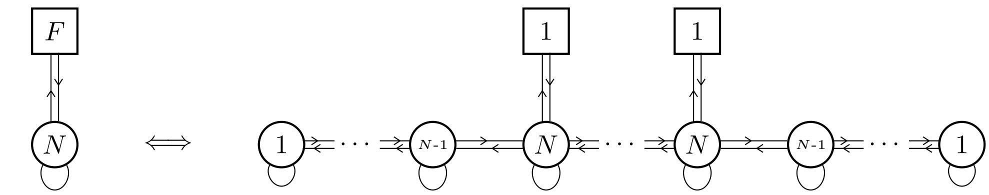
[8] It is possible to show that for $F < 2N$ our dual reproduces also the results for the bad SQCD found in [48]. However, this analysis is beyond the scope of this work.

The resulting duality is depicted below.



$$\mathcal{W} = Q A \tilde{Q} \qquad \mathcal{W} = \sum_{j=0}^{N-1} (Flip[V_1(\Pi_2\tilde{\Pi}_2)^j \tilde{V}_1] + Flip[V_2(\Pi_{F-1}\tilde{\Pi}_{F-1})^j \tilde{V}_2])$$
$$+\mathcal{F}[V_1(\Pi_2\tilde{\Pi}_2)^{N-2}\tilde{V}_1] + \mathcal{F}[V_2(\Pi_{F-1}\tilde{\Pi}_{F-1})^{N-2}\tilde{V}_2] + \mathcal{W}^{\text{partial}}_{\mathcal{N}=4} \tag{4.23}$$

The EOMs for the two singlets $\mathcal{F}[V_1(\Pi_2\tilde{\Pi}_2)^{N-2}\tilde{V}_1]$ and $\mathcal{F}[V_2(\Pi_{F-1}\tilde{\Pi}_{F-1})^{N-2}\tilde{V}_2]$ yield VEVs for the dressed mesons. By carefully studying the effect of sequential Higgsing triggered by these VEVs (see for example [40]), one can show that on each side of the quiver a tail of gauge nodes with increasing ranks from 1 to N is reconstructed. We also have a plateau of $F - 2N - 1$ gauge nodes of rank N with two flavors on the two ends. The result is depicted below and indeed coincides with the known mirror dual of the $\mathcal{N} = 4$ $U(N)$ SQCD [5]:



$$\mathcal{W} = \mathcal{W}_{\mathcal{N}=4} \qquad \mathcal{W} = \mathcal{W}_{\mathcal{N}=4} \tag{4.24}$$

Notice that in the picture above we have rearranged the singlets flipping the bifundamentals so that all the adjoint chirals are tracefull.

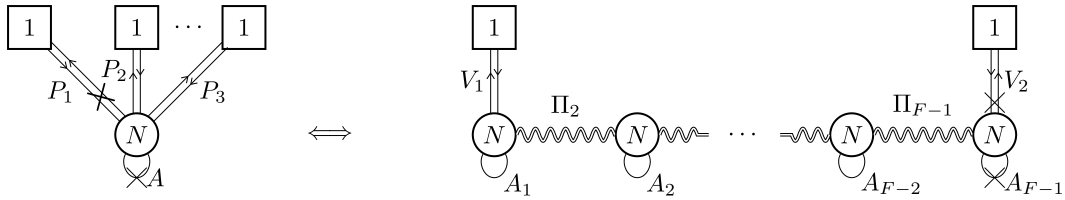
Higgsing $\leftrightarrow$ massive flavor and sequential confinement

We now consider another deformation which consists in giving a mass to any of the two vertical flavors in the mirror theory (4.5).

Actually if we want to turn on a mass term we first need to *move* the flippers[9], meaning that we consider a modified version of the mirror pair where on the electric side we have the superpotential $\mathcal{W} = Flip[Q_1 A_1^j \tilde{Q}_1]$ and on the mirror side the first vertical flavor has no flippers. It is also convenient to introduce $N - 1$ singlets that on the l.h.s. flip the traces of powers of the adjoint chiral. On the r.h.s. we recall that the traces of powers of all the adjoint chirals are identified via quantum relations, therefore we can pick any adjoint, say the last one A_{F-1}, and flip its traces with the effect of

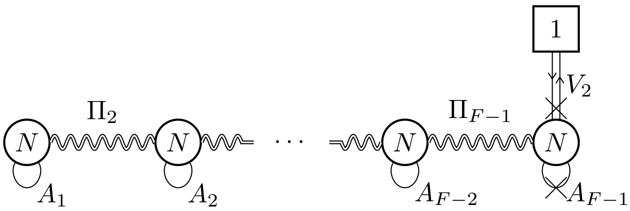
[9] The operation of moving the flippers, simply means that on the magnetic side where we have flipping terms of the type $\mathcal{W} = Flip[X] = O_X X$ we add a mass deformation for the flipper $\delta\mathcal{W} = O_X \tilde{O}_X$ which removes the flipping terms. The effect of this mass deformation on the electric side is then worked out using the operator map.

flipping all the traces of powers of the adjoint chirals.



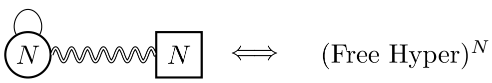
$$\mathcal{W} = \sum_{j=0}^{N-1} Flip[P_1 A^j \tilde{P}_1] + + \sum_{j=2}^{N} Flip[\mathrm{Tr} A^j] \qquad \mathcal{W} = \mathcal{W}_{\text{gluing}} + \sum_{j=0}^{N-1} (Flip[V_2 A_{F-1}^j \tilde{V}_2]) + + \sum_{j=2}^{N} Flip[\mathrm{Tr} A_{F-1}^j] \tag{4.25}$$

We can now can turn on the mass term $\delta\mathcal{W} = V_1\tilde{V}_1$ on the r.h.s. of (4.25). After this deformation we are left with the following theory:



$$\mathcal{W} = \mathcal{W}_{\text{gluing}} + \sum_{j=0}^{N-1} Flip[V_2 A_{F-1}^j \tilde{V}_2] + \sum_{j=2}^{N} Flip[\mathrm{Tr} A_{F-1}^j] \tag{4.26}$$

Now we use the fact that an improved bifundamental gauged on one side *confines* to N free hypers (see [41] eq. (C.9)):



$$\mathcal{W} = \mathcal{W}_{\text{gluing}} \tag{4.27}$$

Using this fact we can sequentially confine all the improved bifundamentals in (4.26) into a total of $(F-2)\times N$ hypers. We are then left just with a $U(N)$ adjoint SQCD with one flipped flavor ($\mathcal{W} = \sum_{j=0}^{N-1} Flip[V_2 A_{F-1}^j \tilde{V}_2]$) and $(F-2)\times N$ free hypers.

Using the duality (4.14) we claim that that the $U(N)$ adjoint SQCD with one flipped flavor is dual to N free hypers. So in conclusion on the mirror side of the duality in (4.25), after the mass deformation for the first flavor we have just $(F-1)\times N$ free hypers.

Now let's go back to the electric theory in (4.25). Using the operator map we see that in the electric theory the mass term $\mathcal{W} = V_1\tilde{V}_1$ maps to $\mathcal{F}[Q_1 A^{N-1}\tilde{Q}_1]$ inducing a VEV for $Q_1 A^{N-1}\tilde{Q}_1$. This is a VEV for a meson dressed $N-1$ times which Higgses completely the theory leaving $(F-1)\times N$ free hypers. So also this consistency check is passed.

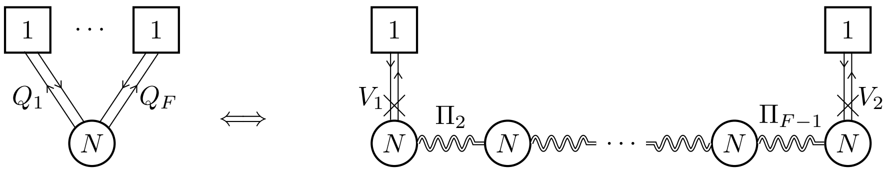
4.2.4 Flowing to $U(N)$ SQCD without adjoint

The last deformation we consider is turning on a mass term for the adjoint A in the electric $U(N)$ SQCD. As discussed in Section 4.2.2, $\mathrm{Tr}\, A^j$ maps to $\mathrm{Tr}\, A_I^j$ in the mirror quiver. In the mirror quiver, for each j, there are $F-1$ holomorphic operators $\mathrm{Tr}\, A_I^j$, one for each node. However only one combination is non-zero in the chiral ring.[10] Hence, we turn on masses for all the adjoints

$$\delta\mathcal{W} = \mathrm{Tr}\big(A^2\big) \quad \Longleftrightarrow \quad \delta\mathcal{W} = \sum_{I=1}^{F-1} \mathrm{Tr}\big(A_I^2\big)\,. \tag{4.28}$$

This deformation breaks the $U(1)_\tau$ symmetry and triggers an RG flow to a new dual pair.

On the left hand side the effect is to simply remove the adjoint. On the right hand side we remove the $F-1$ adjoints A_I, and the superpotential now includes the adjoint operators of the improved bifundamentals $A^{(I)}_{R/L}$.



$$\mathcal{W}=0 \qquad\qquad \mathcal{W} = \sum_{j=0}^{N-1}\big(Flip[V_1(\mathsf{A}_L^{(2)})^j\tilde{V}_1] + Flip[V_2(\mathsf{A}_R^{(F-1)})^j\tilde{V}_2]\big) + $$
$$+\sum_{I=2}^{F-2}\mathsf{A}_R^{(I)}\mathsf{A}_L^{(I+1)} \tag{4.29}$$

The list of charges is given in Table 4.4.

	$U(1)_{R_0}$	$U(1)_{B_j}$	$U(1)_{X_j}$	$U(1)_Y$
$Q_k, \tilde{Q}_k$	1	$-\delta_{j,k}$	$\mp\delta_{j,k}$	0
$V_1, \tilde{V}_1$	$\frac{1-N}{2}$	$\delta_{1,j}$	0	∓ 1
$V_2, \tilde{V}_2$	$\frac{1-N}{2}$	$\delta_{F,j}$	0	0
$\Pi_k, \tilde{\Pi}_k$	0	$\delta_{j,k}$	0	0

Table 4.4: Charges of the fields in the mirror duality for the $U(N)$ SQCD in (4.29). In the first block are listed the fields in the SQCD, while in the second block are listed those of the mirror description.

The global symmetry on the l.h.s. in (4.29) is given by:

$$SU(F)_U \times SU(F)_W \times U(1)_m \times U(1)_Y\,, \tag{4.30}$$

where the two $SU(F)$ and the $U(1)_m$ global symmetries are obtained from the $U(1)_{B_j} \times U(1)_{X_j}$ as usual with the redefinitions in eqs. (4.2), (4.3).
In the mirror theory, on the r.h.s. in (4.29), the UV global symmetry is:

$$\prod_{j=1}^{F} U(1)_{B_j} \times \prod_{j=1}^{F-1} U(1)_{X_{j+1}-X_j} \times U(1)_Y\,, \tag{4.31}$$

where $U(1)_{X_{j+1}-X_j}$ are the topological symmetries of the $F-1$ gauge nodes. In the IR the global symmetry enhances to the group in (4.30).

[10] This follows from the superpotential $\mathcal{W}_{\text{gluing}}$ and of the chiral ring relation $\mathrm{Tr}\big(\mathsf{A}_L^k\big) = \mathrm{Tr}\big(\mathsf{A}_R^k\big), k = 2, \dots, N$ in the $FM[U(N)]$ theory, see [42].

The chiral ring generators in the SQCD side are the F^2 mesons $Q\tilde{Q}$, with R-charge $2-2m$ and the 2 monopoles $\mathfrak{M}^{\pm}$, with R-charge $Fm-N+1$. The mapping of the mesons is very similar to (4.15), for instance if $F=4$:

$$\mathrm{Tr}\left(Q\tilde{Q}\right) \Longleftrightarrow \begin{pmatrix} \mathcal{F}[V_1(\mathsf{A}_L^{(2)})^{N-1}\tilde{V}_1] & \mathfrak{M}^{(+,0,0)} & \mathfrak{M}^{(+,+,0)} & \mathfrak{M}^{(+,+,+)} \\ \mathfrak{M}^{(-,0,0)} & \mathsf{B}_{1,1}^{(2)} & \mathfrak{M}^{(0,+,0)} & \mathfrak{M}^{0,+,+)} \\ \mathfrak{M}^{(-,-,0)} & \mathfrak{M}^{(0,-,0)} & \mathsf{B}_{1,1}^{(3)} & \mathfrak{M}^{(0,0,+)} \\ \mathfrak{M}^{(-,-,-)} & \mathfrak{M}^{(0,-,-)} & \mathfrak{M}^{(0,0,-)} & \mathcal{F}[V_2(\mathsf{A}_R^{(3)})^{N-1}\tilde{V}_2] \end{pmatrix}, \quad (4.32)$$

The SQCD monopoles $\mathfrak{M}^{\pm}$ map to long mesons $V_1\Pi_2\dots\Pi_{F-1}\tilde{V}_2$ and $V_2\tilde{\Pi}_{F-1}\dots\tilde{\Pi}_2\tilde{V}_1$. It is easy to check that the charge assignements given in Table 4.4 are consistent with the mapping.

We now comment about the fate of the operators on the quiver side that were mapping to dressed mesons in the duality with adjoint. These operators are the $\mathcal{F}[V_1(\mathsf{A}_L^{(2)})^j\tilde{V}_1]$, $\mathsf{B}_{1,n}^{(a)}$ $\mathcal{F}[V_2(\mathsf{A}_R^{(F-1)})^j\tilde{V}_2]$ and the dressed monopoles and dressed long mesons (monopoles and long mesons, in the quiver side, cannot be dressed by the explicit adjoint fields, since they are massive, but we can consider dressing with the adjoints inside the improved bifundamental theories that is the A's). Such operators do not exist in the SQCD side of (4.29), while candidate dressed operators appear in the quiver side, so the duality implies that the dressed operators in the quiver are holomorphic operators set to zero in the chiral ring. We would like to understand this feature directly in the quiver side without invoking the duality.

We can explain why the quiver operators along the diagonal in (4.32) are set to zero in the quiver chiral ring using the logic of [70], where it is shown that when a flipper field is flipping an operator below the unitarity bound (hence the flipper has $R>\frac{3}{2}$), it is zero on the chiral ring as a consequence of quantum effects, e.g. giving a VEV to such a flipper leads to a theory with no supersymmetric vacuum.

On the electric side, we know that the superconformal R-charge is such that[11]

$$\frac{1}{2} < R[Q\tilde{Q}] < 1\,. \quad (4.33)$$

The left inequality is the unitarity bound, the right inequality follows from the fact in absence of superpotential the interactions decrease the R-charge with respect to the free theory, where $R[Q]=\frac{1}{2}$. The inequality (4.33) implies that on the quiver side, $\frac{1}{2} < R[\mathcal{F}[\mathsf{A}^{N-1}\tilde{V}] < 1$, while all the operators $\mathcal{F}[V\mathsf{A}^h\tilde{V}], h=0,1,\dots,N-2$, have R-charge greater than $\frac{3}{2}$ (recall $R[\mathsf{A}]=1$). Following the logic of [70], we learn such flippers are holomorphic operators which are zero in the chiral ring of the quiver theory.

The same argument works for the $\mathsf{B}_{1,k}$ operators with $k=2,\dots,N-1$, which were mapping to dressed mesons in the duality with the adjoint. Such operators, when viewed in the Lagragian UV completion of the improved bifundamental are flippers (see Figure C.1), see Table C.2, hence they are flippers with $R>\frac{3}{2}$, so they must be zero in the chiral ring.

4.3 Derivation via the $\mathcal{N}=2$ algorithm

In this section we generalize the dualization algorithm introduced in [56, 40] (see Chapter 2) for $\mathcal{N}=4$ linear quivers to the =2 case and show how to construct the mirror

[11] We are considering the region $F \geqslant N$.

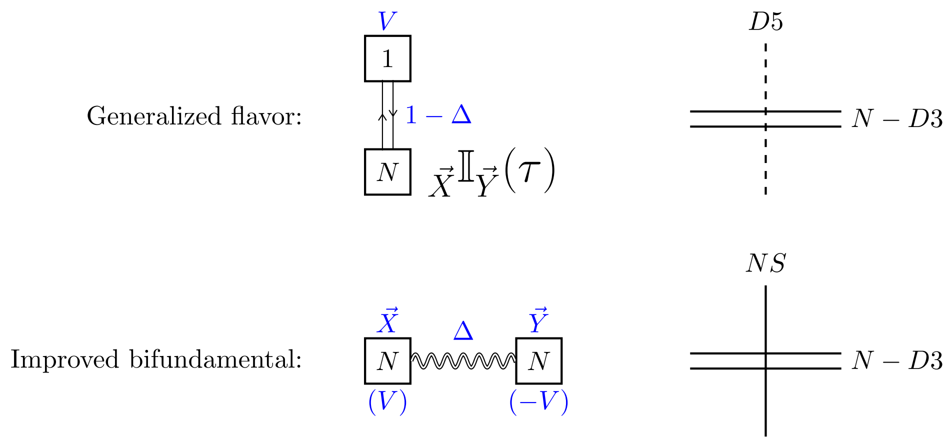


Figure 4.3: Definition of the generalized blocks. In the picture we write in blue the parameterization of the two theories. To the generalized flavor block we assign trial R-charge 1 and charge -1 under the axial symmetry $U(1)_\Delta$, while V denotes the real mass parameter for its vector-like symmetry. $\vec{X}, \vec{Y}$ denote the Cartans of two $U(N)$ flavor groups. The improved bifundamental block, with trial R-charge 0 and Δ-charge 1, is defined with background FI couplings for the two $U(N)$ groups. The FI parameters are denoted by the $(\pm V)$. Notice that D5 branes are depicted as dashed lines instead of crosses as in Chapter 3, this is useful to indicate when the brane is rotated.

dual of a generic $\mathcal{N}=2$ linear quiver with $U(N)$ gauge nodes, improved bifundamentals and generalized flavors.

The idea of the algorithm builds on the observation [28, 71, 54] that on linear or circular brane setups, $\mathcal{S}$-duality can act locally on each five-brane creating an $\mathcal{S}$-duality wall on its right and an $\mathcal{S}^{-1}$-duality wall on its left: D5 $= \mathcal{S}$ NS $\mathcal{S}^{-1}$ and $\overline{\text{NS}} = \mathcal{S}$ D5 $\mathcal{S}^{-1}$. The dualization algorithm implements in field theory this local action of $\mathcal{S}$-duality.

We first define the basic $\mathcal{N}=2$ QFT blocks and the basic $\mathcal{N}=2$ duality moves. We then explain the steps of the algorithm and apply them to the example of the $\mathcal{N}=2$ adjoint SQCD.

4.3.1 Generalized QFT blocks and basic moves

The generalized QFT blocks

The generalized matter blocks are depicted in Figure 4.3. To a D5 brane with N D3 branes stretching on the left and right we associate a generalized flavor block, which consist in a flavor with $U(1)_\Delta \times U(1)_V$ symmetry together with the identity operator $_{\vec{X}}\mathbb{I}_{\vec{Y}}(\tau)$ which identifies the Cartans $\vec{X}$ and $\vec{Y}$ of two $U(N)$ symmetries.
To a NS brane with N D3 branes stretching on the left and right we associate an improved bifundamental block given by an $FM[U(N)]$ theory with background FI couplings for the two $U(N)$ global symmetries.
The $\mathbf{S}_b^3$ partition functions of the QFT blocks are given by:

$$
\begin{aligned}
Z_{\text{D5}}^{(N)}(\vec{X},\vec{Y},\tau,\Delta,V) &= \prod_{j=1}^{N} s_b(\Delta \pm (X_j - V))\, {}_{\vec{X}}\mathbb{I}_{\vec{Y}}(\tau)\,, \\
Z_{\text{NS}}^{(N)}(\vec{X},\vec{Y},\tau,\Delta,V) &= e^{2\pi i V \sum_{j=1}^{N}(X_j - Y_j)} Z_{FM}^{(N)}(\vec{X},\vec{Y},\tau,\Delta)\,,
\end{aligned}
\tag{4.34}
$$

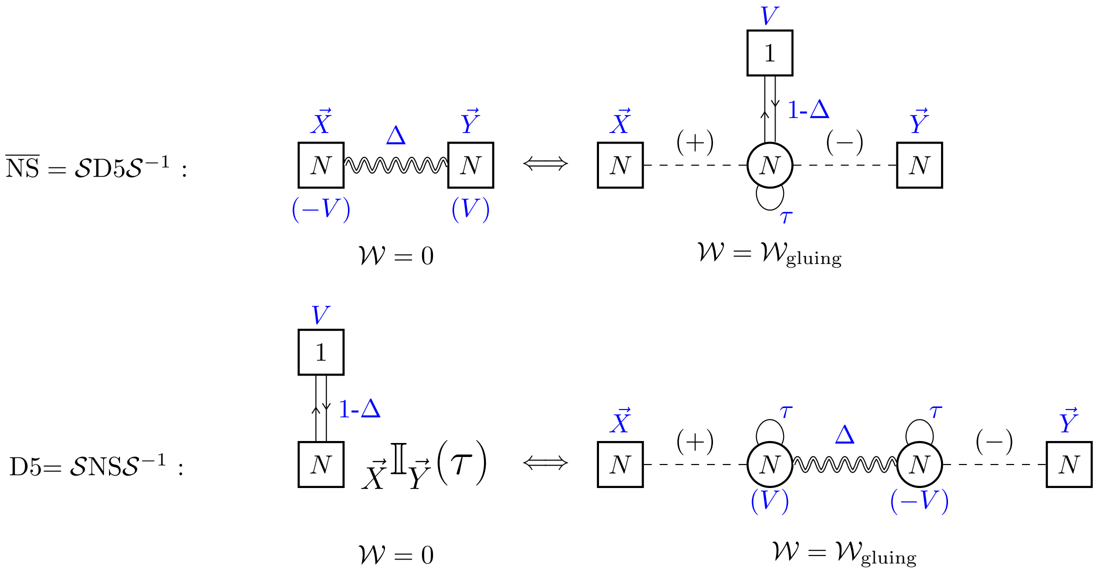


Figure 4.4: Basic $\mathcal{S}$-duality moves for the $\mathcal{N}=2$ QFT blocks. In the first line a flavor block acted by an $\mathcal{S}$-wall on on the left and by an $\mathcal{S}^{-1}$-wall on the right is dualized to an improved bifundamental. On the r.h.s. $\mathcal{W}_{\text{gluing}}$ couples the adjoint chiral to the adjoint operators of the two $\mathcal{S}$-walll theories. Similarly in the second line we have the $\mathcal{S}$-dualization of the improved bifundamental into a flavor block. On the r.h.s. $\mathcal{W}_{\text{gluing}}$ couples the adjoint chirals to the adjoint operators of the improved bifundamental and of the $\mathcal{S}$-wall theories.

where $Z_{FM}^{(N)}$ is defined in Appendix C.2 in eq. (C.14). The identity operator instead is defined as follows:

$$
{}_{\vec{X}}\mathbb{I}_{\vec{Y}}(\tau) = \frac{1}{\Delta_N(\vec{X};\tau)} \sum_{\sigma \in S_N} \prod_{j=1}^{N} \delta(X_j - Y_{\sigma(j)})\,. \tag{4.35}
$$

Where $\Delta_N(\vec{X},\tau)$ is defined as in (4.10).

Basic duality moves

The last ingredient necessary for the definition of the algorithm is given by the *basic duality moves*. They realize at the field theory level the local action of $\mathcal{S}$-duality on each five-brane. The two basic moves are given in Figure 4.4.

In the duality move in the first line of Figure 4.4, we see how by acting with an $\mathcal{S}$-wall on the left and $\mathcal{S}^{-1}$-wall on the right of a generalized fundamental block we obtain an improved bifundamental block. The superpotential $\mathcal{W}_{\text{gluing}}$ couples the adjoint chiral to the two adjoint moment map present in the two $\mathcal{S}$-wall theories, $\mathsf{A}_L, \mathsf{A}_R$, as $\mathcal{W}_{\text{gluing}} = a(\mathsf{A}_L + \mathsf{A}_R)$. As we gauge together two $\mathcal{S}$-walls with the addition of a flavor, supersymmetry is broken from $\mathcal{N}=4$, as in an isolated $\mathcal{S}$-wall, to $\mathcal{N}=2$. In particular the $\mathcal{N}=4$ R-symmetry is broken to $U(1)_R$ and its flavor subgroup (that in Chapter 3 was named $U(1)_A$) is identified with the $U(1)_\tau$ symmetry rotating the adjoint chiral due to the superpotential. The flavor does not enter the superpotential and indeed is rotated by a $U(1)_V \times U(1)_\Delta$ symmetry.

One can recover the $\mathcal{N}=4$ basic moves of [56, 40], by adding on the r.h.s. a cubic superpotential coupling the flavor f to the moment maps as $\delta\mathcal{W} = f(\mathsf{A}_L - \mathsf{A}_R)\tilde{f}$, therefore making the theory $\mathcal{N}=4$. This deformation is mapped on the l.h.s. to the $\mathsf{B}_{2,1}$ singlet of the improved bifundamental theory which has the effect of ironing it to a $U(N)\times U(N)$ bifundamental hypermultiplet, as shown (C.24). In this way we recover exactly the duality in Figure 3.10.
In the duality move in the second line of Figure 4.4, instead the $\mathcal{S}$-dualization of an improved bifundamental block gives the generalized fundamental block. The superpotential $\mathcal{W}_{\text{gluing}}$ couples the two adjoint chirals to the adjoint operators inside the $\mathcal{S}$-walls and of improved bifundamentals.

The first $\mathcal{N}=2$ duality move can be derived by taking suitable real mass deformations of the 3d braid duality of [64] (as explained in [41] Appendix C). The second $\mathcal{N}=2$ duality move can actually be obtained by acting on the left and right hand side of the the first duality move with $\mathcal{S}$ and $\mathcal{S}^{-1}$ and using the fusion to identity property $\mathcal{S}\mathcal{S}^{-1}=1$, hence the braid duality is the fundamental duality move.[12] Moreover it has been shown in [42] that the braid duality can be demonstrated by induction by assuming only the elementary Seiberg-like dualities. Hence all the $\mathcal{N}=2$ mirror dualities following from the algorithm are demonstrated to be consequence Seiberg-like dualities only.

As partition function identities the basic moves are:[13]

$$\begin{aligned}
Z^{(N)}_{\text{NS}}(\vec{X},\vec{Y},\tau,\Delta,-V) &= \int \prod_{a=1}^{2}\left(d\vec{Z}^{(a)}_N \Delta_N(\vec{Z}^{(a)};\tau)\right) Z^{(N)}_{\mathcal{S}}(\vec{X},\vec{Z}^{(1)};\tau) \\
&\qquad Z^{(N)}_{\text{D5}}(\vec{Z}^{(1)},\vec{Z}^{(2)},\tau,\Delta,V) Z^{(N)}_{\mathcal{S}^{-1}}(\vec{Z}^{(2)},\vec{Y};\tau)\,, \\
Z^{(N)}_{\text{D5}}(\vec{X},\vec{Y},\tau,\Delta,V) &= \prod_{a=1}^{2}\left(d\vec{Z}^{(a)}_N \Delta_N(\vec{Z}^{(a)};\tau)\right) Z^{(N)}_{\mathcal{S}}(\vec{X},\vec{Z}^{(1)};\tau) \\
&\qquad Z^{(N)}_{\text{NS}}(\vec{Z}^{(1)},\vec{Z}^{(2)},\tau,\Delta,V) Z^{(N)}_{\mathcal{S}^{-1}}(\vec{Z}^{(2)},\vec{Y};\tau)\,.
\end{aligned} \tag{4.36}$$

It is useful to regard the matter blocks and the $\mathcal{S}$ generator as matrices with two indexes $\vec{X}$ and $\vec{Y}$ for their two $U(N)$ symmetries. Multiplying these matrices corresponds to gauging $U(N)$ symmetries using the integration measure $\Delta_N(\vec{Z},\tau)$, defined in equation (4.10), containing both the contribution of a $\mathcal{N}=2$ vector multiplet and an extra adjoint chiral with $+1$ charge under a $U(1)_\tau$ symmetry. Notice that the $U(1)_\tau$ symmetries in the matter blocks and in the $\mathcal{S}$-duality walls are all identified, this is because when we gauge $U(N)$ nodes we always turn on $\mathcal{W}_{\text{gluing}}$.

Focusing on the first duality only, notice that on the r.h.s. the $U(1)_V$ symmetry can be reabsorbed by a $U(1)$ gauge transformation and therefore it acts trivially on

[12] Notice that also the fusion to identity property in Figure 3.3 follows from the first move which can be regarded as an S-confining duality, similar to $4d$ $\mathcal{N}=1$ $SU(N)$ SQCD with $N+1$ flavors. Turning on a mass for a flavor one flow to $SU(N)$ SQCD with N flavors whose low energy dynamics is well known to be governed by a quantum deformed moduli space, over which a part of the global symmetry is spontaneously broken. In the same way we can obtain (3.3) by giving a mass to the flavor in the first duality move to go from a confining duality to a quantum deformed moduli space where the $U(N)\times U(N)$ global symmetry is broken to the diagonal.

[13] In this chapter we adopt a different notation w.r.t. Chapter 3. In particular, in the UV completion of the $\mathcal{S}$-wall we take the bifundamental to have R-charge 0 and $U(1)_\tau$ charge $+1$, while adjoint chirals have R-charge 2 and $U(1)_\tau$ charge -2. Therefore the actual partition function for the $\mathcal{S}$-wall $Z^{(N)}_{\mathcal{S}}(\vec{X},\vec{Y};\tau)$ is obtained from (C.3) setting $m_A = \frac{iQ}{2}-\frac{\tau}{2}$. Moreover in this chapter the adjoint chirals are taken to be traceless instead than tracefull as in Chapter 3.

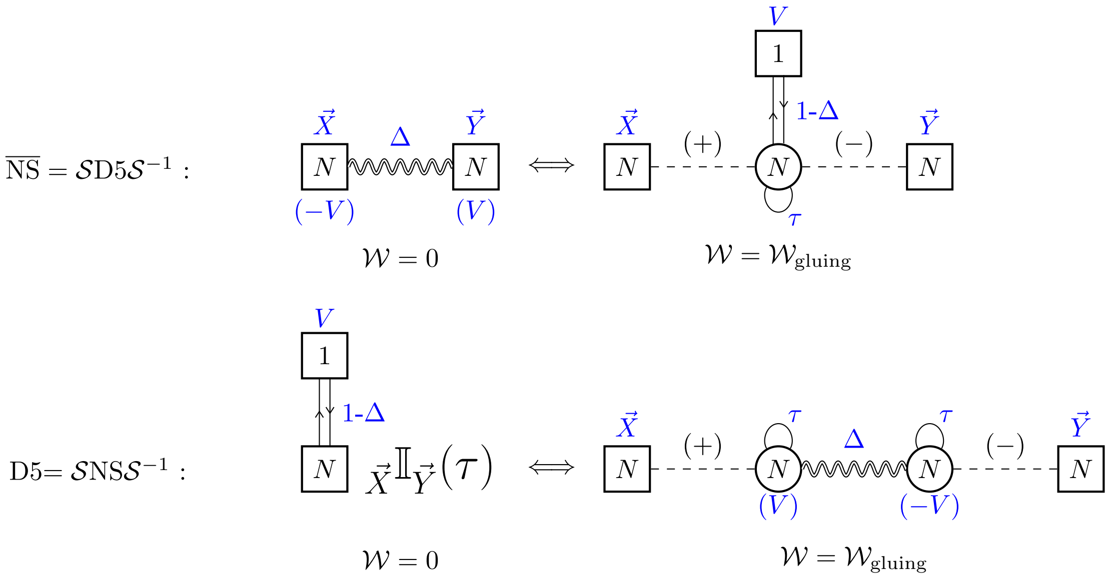


Figure 4.5: Asymmetric basic duality move relating a trivial $U(N)\times U(0)$ bifundamental to trivial flavor block.

the theory. In fact, on the l.h.s. the V parameter appears just as a background FI term and therefore it is not associated to any symmetry acting on the theory. This feature recurs many times, we find useful to give the dualities writing explicitly also the redundant parameters because they become physical when the duality is used as a local dualization inside a bigger theory.

So far we have only considered improved bifundamentals with $U(N) \times U(N)$ non-abelian symmetry. To describe more general theories, corresponding to brane setups with non-constant number of $D3$ branes, we would need an improved bifundamentals with $U(N) \times U(M)$ non-abelian symmetry and its $\mathcal{S}$-dual which we do not have at the moment. We plan to focus on this generalization in future works.

In this work we only need the $M = 0$ case. The $U(N) \times U(0)$ bifundamental is a trivial theory consisting only of a background FI term for a $U(N)$ global symmetry, its $\mathcal{S}$-dualization to a trivial flavor block acted by a trivial $\mathcal{S}$-wall on its left and an asymmetric $\mathcal{S}^{-1}$-wall on its right is shown in Figure 4.5. The definition of the asymmetric $\mathcal{S}$-wall is given in Appendix C.1. This duality move corresponds to the dualization of a D5 brane into an NS brane, with 0 $D3$ branes on the left and N on the right. The partition function identity associated to this duality is given by:

$$e^{-2\pi i V \sum_{j=1}^{N} X_j} = Z^{(N)}_{\mathcal{S}^{-1}}\big(\vec{X}, \{\frac{N-1}{2}\tau + V, \dots, \frac{1-N}{2}\tau + V\}; \tau\big) \prod_{j=2}^{N} s_b\big(\frac{iQ}{2} - j\tau\big) . \quad (4.37)$$

Where the partition function of the trivial $\mathcal{S}$-wall and of the trivial matter is equal to one.

Useful combined moves

It is convenient to also define some combined duality moves that are not fundamental and are obtained by composing several basic moves. This corresponds to the idea of

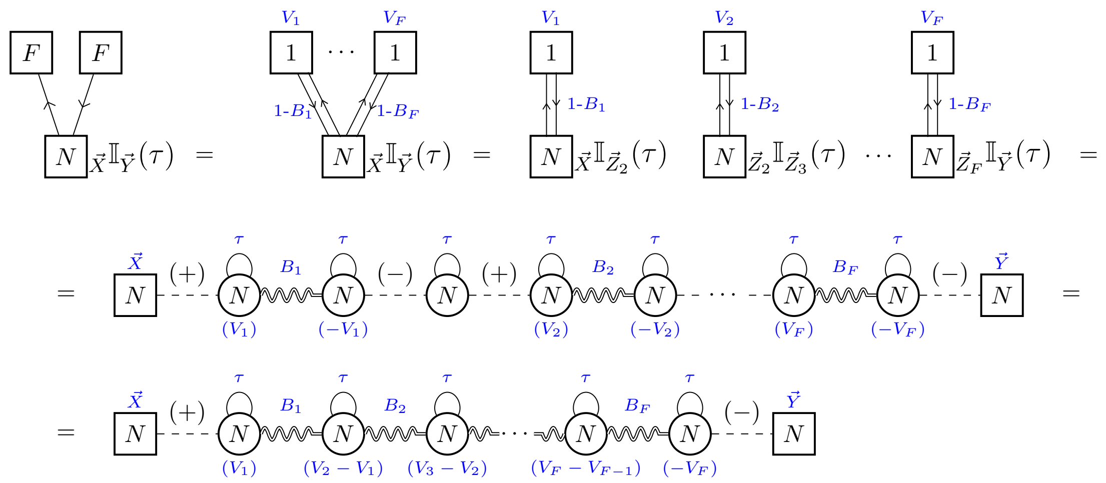


Figure 4.6: $\mathcal{S}$-dualization of a block of F $\mathcal{N}=2$ flavors. In the first step we reparameterize the $U(F)\times U(F)$ flavors as a set of F fundamental anti-fundamental pairs of flavors. In the second step we cut the block of F flavours into F generalized flavor blocks. We then dualize each block to an improved bifundamental block and glue together the results to reach the theory in the second line. Implementing the fusion to identity $\mathcal{S}\mathcal{S}^{-1}=1$ we reach the final frame which is given by a string of F improved bifundamentals with an $\mathcal{S}$-wall on the left and an $\mathcal{S}^{-1}$-wall on the right.

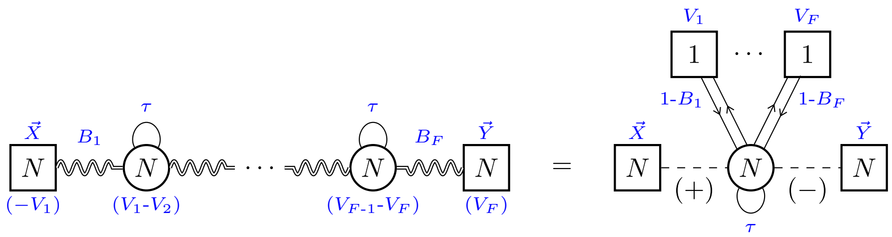


Figure 4.7: $\mathcal{S}$-dualization of a block of F improved bifundamentals.

acting on a set of many 5-branes at the same time, instead of acting on a single one.

For example, it can be useful to consider the $\mathcal{S}$-dualization of a block of F $\mathcal{N}=2$ flavors as schematically shown in Figure 4.6. Similarly it can be useful to dualize a string of consecutive improved bifundamentals, as shown in Figure 4.7. This second move can be obtained starting from the duality in Figure 4.6 by acting on the left and right with a $\mathcal{S}$ and $\mathcal{S}^{-1}$ operators and using the fact that $\mathcal{S}\mathcal{S}^{-1}=1$. As partition function identities, the two combined duality moves corresponds to:

$$\begin{aligned} Z^{(N)}_{F-\text{D5}}(\vec{X},\vec{Y},\tau,\vec{B},\vec{V}) = \int \prod_{a=1}^{F+1} \left(d\vec{Z}^{(a)}_N \Delta_N(\vec{Z}^{(a)};\tau)\right) Z^{(N)}_{\mathcal{S}}(\vec{X},\vec{Z}^{(1)};\tau) \\ \prod_{a=1}^{F} Z^{(N)}_{\text{NS}}(\vec{Z}^{(a)},\vec{Z}^{(a+1)},\tau,B_a,V_a) Z^{(N)}_{\mathcal{S}^{-1}}(\vec{Z}^{(F+1)},\vec{Y};\tau)\,, \end{aligned} \tag{4.38}$$

$$\int \prod_{a=1}^{F-1} \left(d\vec{Z}_N^{(a)} \Delta_N(\vec{Z}^{(a)};\tau)\right) Z_{\text{NS}}^{(N)}(\vec{X},\vec{Z}^{(1)},\tau,B_1,-V_1)$$

$$\prod_{a=2}^{F-1} Z_{\text{NS}}^{(N)}(\vec{Z}^{(a-1)},\vec{Z}^{(a)},\tau,B_a,-V_a) Z_{\text{NS}}(\vec{Z}^{(F-1)},\vec{Y},\tau,B_F,-V_F) =$$

$$= \int \prod_{a=1}^{2} \left(d\vec{Z}_N^{(a)} \Delta_N(\vec{Z}^{(a)};\tau)\right) Z_{\mathcal{S}}^{(N)}(\vec{X},\vec{Z}^{(1)};\tau) Z_{F-\text{D5}}^{(N)}(\vec{Z}^{(1)},\vec{Z}^{(2)},\tau,\vec{B},\vec{V}) Z_{\mathcal{S}^{-1}}^{(N)}(\vec{Z}^{(2)},\vec{Y};\tau)\,, \tag{4.39}$$

with

$$Z_{F-\text{D5}}^{(N)}(\vec{X},\vec{Y},\tau,\vec{B},\vec{V}) = \prod_{j=1}^{N}\prod_{a=1}^{F} s_b(B_a \pm (X_j - V_a))\, {}_{\vec{X}}\mathbb{I}_{\vec{Y}}(\tau)\,. \tag{4.40}$$

$\mathcal{N} = 2$ mirror dualization algorithm

Now that all the ingredients are at disposals, namely the $\mathcal{N} = 2$ QFT blocks and the $\mathcal{N} = 2$ basic duality moves, we are able to run the mirror dualization algorithm on $\mathcal{N} = 2$ theories. For example, we can dualize the adjoint $U(N)$ SQCD to obtain the mirror presented in Section 4.2. The explicit computation can be found in Section 3.3 of [41].

4.4 Brane setups with four supercharges and improved bifundamentals

The $\mathcal{N} = 2$ algorithm discussed in the previous sections, allows us to advance our understanding of Hanany-Witten brane setup with 4 supercharges. In this section we make a proposal for the 3d $\mathcal{N} = 2$ gauge theory living on brane setups composed of $D3$, NS and D5' branes. As we will see, our proposal differs from the *naive* quiver gauge theory in that the bifundamentals are improved instead of standard.

Let us start by defining the Hanany-Witten brane setup we are interested in. There are D3 branes stretching along the 6 direction between NS branes or NS' branes, that are different by the way the fill the space. Flavors are added inserting D5 or so called D5' branes, that also differ from the way the fill the space. The list of branes and the way they fill the space is listed in the table below.

	0	1	2	3	4	5	6	7	8	9
$D3$	x	x	x				x			
NS	x	x	x	x	x	x				
D5	x	x	x					x	x	x
NS'	x	x	x	x					x	x
D5'	x	x	x		x	x		x		

(4.41)

If all the branes are present the system preseves 3d $\mathcal{N} = 2$ supersymmetry. The $U(1)^2$ symmetry rotating the 45 and 89 directions becomes the

$$U(1)_R \times U(1)_\tau \tag{4.42}$$

symmetry in the IR QFT. $U(1)_R$ is the $\mathcal{N} = 2$ R-symmetry while $U(1)_\tau$ is an additional global symmetry always present in the QFT's associated to brane setups with the branes in (4.41)[14]. If only D3, NS and D5 (or D3, NS' and D5') are present, the system preserves 8 supercharges and the low energy theory living on it is well known to be a $\mathcal{N} = 4$ quiver with standard bifundamental matter. For instance the 3d $\mathcal{N} = 4$ $U(N)$ theory with no flavors is associated to a Type IIB brane setup with N D3 branes stretching between 2 NS branes. We can add F flavors, adding D5 or equivalently D5' branes. The D5 branes preserve the 8 supersymmetries, while the D5' break half of the superymmetry, from $\mathcal{N} = 4$ to $\mathcal{N} = 2$.

In $\mathcal{N} = 2$ language, N D3 branes stretching between 2 NS branes with F D5 branes in the middle provide adjoint $U(N)$ with F flavors and a cubic superpotential coupling the flavor to the adjoint (equivalently, we could use N D3 branes stretching between 2 NS' branes with F D5' branes in the middle). On the other hand, N D3 branes stretching between 2 NS branes with F D5' branes in the middle,[15] as in the left of the setup in Figure (4.43), give rise to $U(N)$ with an adjoint and F flavors and a *vanishing* superpotential, $\mathcal{W} = 0$ [4, 72, 73]. We can exclude a superpotential counting the motion of the D3 brane segments as follows. In the F flavors case there are $N(F-1)$ D5'-D5' segments (providing $N(F-1)$ quaternionic directions) and $2N$ D5'-NS segments (providing $2N$ complex directions), so there must be a branch in the moduli space of vacua of the theory of complex dimension $2NF$. Such a branch exists if $\mathcal{W} = 0$, parameterized by NF Q's, NF $\tilde{Q}$'s, N^2 A's minus N^2 gauge symmetries, but a non zero superpotential, e.g. of the form $(Q\tilde{Q})^2$, would lift part of these $2NF$ complex directions.[16]

The $U(N)$ adjoint SQCD with F flavors, $\mathcal{W} = 0$, is precisely the theory we studied in the previous sections, for which we found the mirror dual with $F-1$ gauge groups linked by improved bifundamentals. Now, as shown in picture (4.43) we apply Type IIB $\mathcal{S}$-duality to its associated brane setup. Modulo rotating the branes, [17] the $\mathcal{S}$-dual

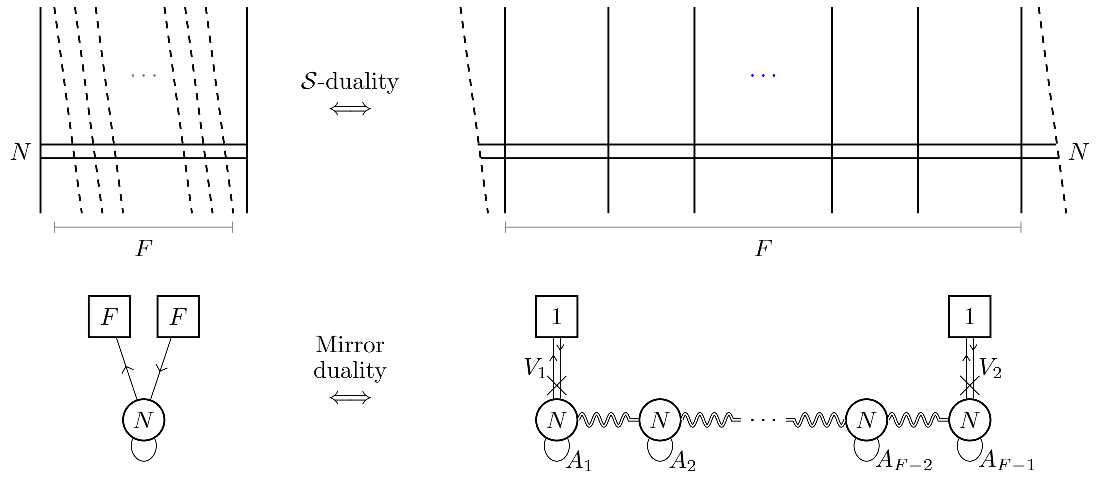
[14]One can break this $U(1)_\tau$ symmetry rotating some 5-branes to generic angles along the 45 and 89 directions, without breaking the $\mathcal{N} = 2$ supersymmetry. In this paper we do not study such configurations, but they should be obtained turning on superpotential deformations from the setups we study.

[15]Equivalently, we could use N D3 branes stretching between 2 NS' branes with F D5 branes in the middle.

[16]One can also argue for the absence of a cubic superpotential of the type $AQ\tilde{Q}$ noticing that the $D3$ branes, when moving along the 45 directions (which corresponds to a vev for the adjoint A), remain in contact with the D5' branes, hence the D3-D5 strings (which correspond to the flavor fields) remain at zero length, so the flavors remain massless.

[17]For convenience, in the pictures we always present the action of $\mathcal{S}$-duality combined with the rotation acting by NS'→NS and D5→D5'. Clearly the QFT description is invariant under this rotation.

setup is N D3 branes stretching between 2 D5' branes with F NS branes in the middle.



$\mathcal{W} = 0 \qquad \mathcal{W} = \sum_{j=0}^{N-1}(Flip[V_1 A_1^j \tilde{V}_1] + Flip[V_2 A_{F-1}^j \tilde{V}_2]) + \mathcal{W}_{\text{gluing}}$

(4.43)

Looking at the web of dualities depicted in (4.43), it is natural to propose that the IR QFT associated to the brane setup on the right hand side is our mirror SQCD quiver obtained via the dualization algorithm, with improved, instead of standard bifundamentals (in Section 4.4.3 we will comment on the relation between our proposal and previous ones). Building on this observation and on the $\mathcal{N} = 2$ algorithm perspective, for an $\mathcal{N} = 2$ brane setup made of a constant number of $D3$ branes stretching between an arbitrary sequence of NS and D5' branes, we formulate the following

The IR QFT associated to N D3 branes stretching along an arbitrary ordered sequence of $(g+1)$ NS branes and F D5' branes consists of a linear quiver with g $U(N)$ adjoint nodes, $g-1$ improved bifundamentals and a total of F flavors distributed among the g nodes, according to the position of the D5' branes. The superpotential is $\mathcal{W}_{gluing}$, which couples the adjoint of each $U(N)$ node to the adjoint operators of the nearby improved bifundamentals. The flavors do not enter the superpotential, the only exception is if at the beginning (or at the end) of the sequence of 5-branes there is a single D5' brane, then the dressed mesons made with the associated flavor are flipped.

We claim that this proposal is consistent with $\mathcal{S}$-duality, that is two improved quivers corresponding to $\mathcal{S}$-dual brane setups are mirror dual, and one can construct the dual using the $\mathcal{N} = 2$ algorithm. We provide a few examples in Section 4.4.1. The improved quiver theories associated to these brane setups have interesting patterns of symmetry enhancement and, as we discuss in 4.4.2, we have a notion of *balanced nodes* leading to symmetry enhancement, in analogy with the $\mathcal{N} = 4$ case [28].

Let us discuss some special sequences of five-branes.
If at the beginning of the sequence of five-branes there are $h > 1$ NS branes, as we show in Section 4.4.2, the associated theory is *ugly* and we can sequentially confine a string of $h-1$ improved bifundamentals generating $(h-1)N$ free hypers. So the interacting part of the theory is associated to the set up where the first $h-1$ NS branes have been removed. In particular, our proposal for the theory associated to a sequence of h NS branes, a $U(N)^{h-1}$ improved quiver with no flavors flows in the IR to hN free hypers, exactly as the *bad* $\mathcal{N} = 4$ $U(N)^{h-1}$ quiver theory with standard bifundamentals.

Analogously, by $\mathcal{S}$-duality, if at the beginning of the sequence of 5-branes there are $h > 1$ D5' branes, the QFT is given by $(h-1)N$ free hypers plus the QFT associated the brane setup where the first $h-1$ D5' have been removed.
The last example is the short sequence D5'-NS-D5', this sequence is not associated to an improved bifundamental (which in our prescription always connects gauge nodes) but to an improved bifundamental where both the $U(N)$ symmetries are broken to $U(1)$s. This deformation reduces the improved bifundamental to the Wess-Zumino model on the r.h.s. of (4.13) which is indeed mirror dual to the SQCD with one flavor associated to the $\mathcal{S}$-dual brane setup NS-D5'-NS.

Let us also mention that we actually understand some instances of more general situations.
We can describe brane systems where an arbitrary number of D5's sit on top of an NS, that is the NS and the D5' form a (p, q)-web of rectangular shape. These are discussed in Section 5 of [41] by following the logic of [74] from the Abelian to the non-Abelian case, we propose the QFT corresponding to the $\mathcal{S}$-dual (p, q)-web, that is many NS's sitting on top of a D5'. This situation will not be describe in this thesis, we refer to [41] for any detail.
We can turn real mass deformations in our quivers to generate Chern-Simons interactions and/or theories with chiral matter (different number of fundamentals vs anti-fundamentals). The corresponding brane setup might include (p, q) 5-branes and non-rectangular (p, q)-webs.

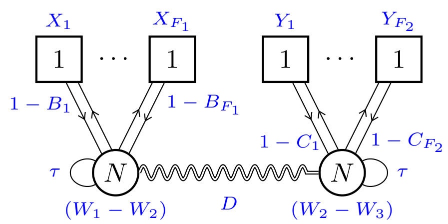
Let us conclude saying that the most general 3d $\mathcal{N} = 2$ setup would involve all four types of 5-branes (NS, NS', D5, D5') and a non-constant number of $D3$ branes along the brane setup. To describe such setups we need a new object, an improved bifundamental, with non-abelian global symmetry $S[U(N_1) \times U(N_2)]$. We plan to investigate it in the future.

4.4.1 More examples of $\mathcal{N} = 2$ mirror quivers

In this subsection we consider brane setups with N D3 branes stretched along the sequence $\mathrm{NS} - (\mathrm{D5}')^{F_1} - \mathrm{NS}^K - (\mathrm{D5}')^{F_2} - \mathrm{NS}$, which is mirror to $\mathrm{D5}' - \mathrm{NS}^{F_1} - (\mathrm{D5}')^K - \mathrm{NS}^{F_2} - \mathrm{D5}$. We write down the associated QFT's, discuss the chiral rings and global symmetries, and prove the IR duality between them.

Electric theory with 2 nodes

Let us start from the simplest example, $K = 1$, corresponding to the brane setup in the top left corner of Figure 4.8. According to our proposal the associated theory is the two nodes improved quiver on the bottom left corner. As in the case of the SQCD, it is convenient to reparameterize the electric theory as:



(4.44)

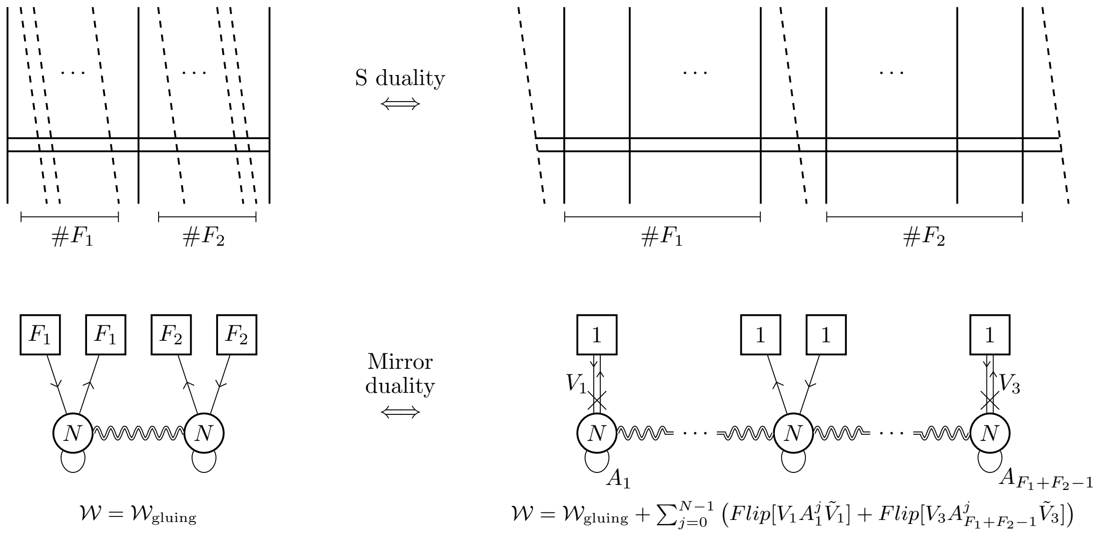


Figure 4.8: In the top left corner we have the electric brane set up with three NS branes, F_1 D5' branes in the first interval and F_2 in the second and a N $D3$ branes stretching from the first NS to the third. The associated quiver theory in the bottom left corner has two gauge nodes linked by an improved bifundamental, F_1 flavor on the first node and F_2 on the second. On the top right corner we have the S-dual brane setup and on the bottom right corner its associated quiver description. The leftmost and rightmost flavors are associated to the D5' located outside the NS branes hence the corresponding dressed mesons are flipped. We denoted by V_a, with $a = 1, 2, 3$ the flavors from left to right and by A_n the adjoint of the n-th gauge node. The two quiver theories are mirorr dual.

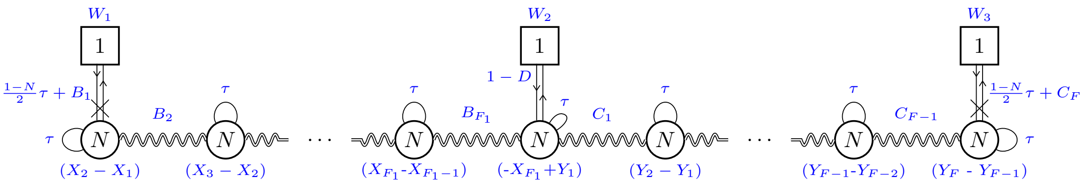
The global symmetry group of this theory is given by:[18]

$$S[U(F_1)^2 \times U(F_2)^2] \times U(1)_D \times U(1)_{W_1-W_2} \times U(1)_{W_2-W_3} \times U(1)_\tau \,, \tag{4.45}$$

Where the parameterization of (4.44) recombines as:

$$\begin{aligned} \prod_{j=1}^{F_1} \left(U(1)_{B_j} \times U(1)_{X_j}\right) &= U(F_1)^2 \,, \\ \prod_{j=1}^{F_2} \left(U(1)_{C_j} \times U(1)_{Y_j}\right) &= U(F_2)^2 \,. \end{aligned} \tag{4.46}$$

At this point we run the dualization algorithm and find the mirror dual quiver theory:



(4.47)

This computation is given explicitly in [41] (Appendix F). Where $F = F_1 + F_2$. As expected, the mirror dual quiver (4.47) coincides with the quiver in bottom right corner of Figure 4.8 (with a different parameterization of the central flavor) which we wrote down starting from the $\mathcal{S}$-dual brane configuration and applying our proposal. The manifest global symmetry group is given by:

$$\begin{aligned} &S\left[\prod_{j=1}^{3} U(1)_{W_j}\right] \times U(1)_D \times \prod_{j=1}^{F_1} U(1)_{B_j} \times \prod_{j=1}^{F_2} U(1)_{C_j} \times \\ &\times \prod_{j=1}^{F_1-1} U(1)_{X_{j+1}-X_j} \prod_{j=1}^{F_2-1} U(1)_{Y_{j+1}-Y_j} \times U(1)_{Y_1 - X_{F_1}} \times U(1)_\tau \,. \end{aligned} \tag{4.48}$$

The pattern of symmetry enhancement is similar to the SQCD case, in particular we observe that the topological and axial symmetries enhance as:

$$\begin{aligned} \prod_{j=1}^{F_1} U(1)_{B_j} \times \prod_{j=1}^{F_1-1} U(1)_{X_{j+1}-X_j} &\to S[U(F_1)^2] \,, \\ \prod_{j=1}^{F_2} U(1)_{C_j} \times \prod_{j=1}^{F_2-1} U(1)_{Y_{j+1}-Y_j} &\to S[U(F_2)^2] \,. \end{aligned} \tag{4.49}$$

The complete IR global symmetry of the mirror theory is then:

$$S[U(F_1)^2] \times S[U(F_2)^2] \times U(1)_{Y_1 - X_{F_1}} \times U(1)_D \times S\left[\prod_{j=1}^{3} U(1)_{W_j}\right] \times U(1)_\tau \,, \tag{4.50}$$

which upon a redefinition of some $U(1)$ factors, matches precisely with the IR global symmetry of the original theory in eq. (4.45).

Notice that the pattern of symmetry enhancement in the mirror theory is quite non-trivial but thanks to the parameterization obtained from the dualization algorithm, it is easier to collect together operator with the same R-charge and therefore construct representations of the emergent symmetries.

[18] We can factorise a $U(1)$ vector-like factor from $U(F_1)^2 \times U(F_2)^2$ by a gauge transformation. This consists in imposing the constraint: $\sum_{j=1}^{F_1} X_j + \sum_{j=1}^{F_2} Y_j = 0$.

Operator Map

The operator map works as follows:

- In the electric theory we can build mesonic operators in the $\bar{F}_1 \times F_1$ bifundamental. In the magnetic theory these are mapped into a collection of monopoles and singlets. In particular, using the results of Appendix C.2 we can check that monopoles $\mathfrak{M}^{\pm(0,\dots,0,1,\dots,1,0,\dots,0|0|0,\dots,0)}$, with topological charge given by strings of contiguous 1 (or -1) under the topological symmetries $U_{X_{j+1}-X_j}$ of the $F_1 - 1$ nodes on the l.h.s. of the central node, they all have the same R-charge. We can then collect these $F_1(F_1-1)$ monopoles with the $\mathsf{B}_{1,1}^{(j)}$ singlet in the $F_1 - 1$ improved bifundamentals on the left of the central flavor plus the flipper of the left flavor $\mathcal{F}[V_1 A_1^{N-1} \tilde{V}_1]$ in a matrix transforming in the $\bar{F}_1 \times F_1$ bifundamental of the emergent $U(F_1)^2$ symmetry.

- Similarly have an electric mesonic operators in the $\bar{F}_2 \times F_2$ bifundamental to $F_2(F_2-1)$. This is mapped to a collection of monopoles $\mathfrak{M}^{\pm(0,\dots,0|0|0,\dots,0,1,\dots,1,0,\dots,0)}$, with topological charge given by strings of contiguous 1 (or -1) under the topological symmetries $U_{Y_{j+1}-Y_j}$ of the $F_2 - 1$ nodes on the r.h.s. of the central node, and the $\mathsf{B}_{1,1}^{(j)}$ singlets in the $F_2 - 1$ improved bifundamentals on the right of the central flavor plus the flipper of the right flavor $\mathcal{F}[V_3 A_{F_1+F_2-1}^{N-1} \tilde{V}_3]$. Collecting all these operators we can assemble a matrix transforming in the bifundamental $\bar{F}_2 \times F_2$ of the emergent $U(F_2)^2$ symmetry.

- Electric long mesons in the $F_1 \times \bar{F}_2$ and $\bar{F}_1 \times F_2$, involving the improved bifundamental, are mapped into magnetic monopole operators $\mathfrak{M}^{\pm(0,\dots,0,1,\dots,1|1|1,\dots,1,0,\dots,0)}$ with topological charge given by a string of ± 1 extending from the central node, to the left and to the right. Using the results in Appendix C.2 we can check that all these $2F_1 \times F_2$ operators have the same R-charge and can be assembled into two matrices. Collecting all the positively charged monopoles we assemble a matrix transforming in the $\bar{F}_1 \times F_2$ which therefore maps to the corresponding electric mesons. Similarly, the negatively charged monopoles form a matrix mapping to the $F_1 \times \bar{F}_2$ mesons.

- We also have electric monopoles charged under the topological symmetry of the left gauge node $\mathfrak{M}^{\pm(1,0)}$. The positively charged one is mapped into the long meson in the magnetic theory built by joining the chirals $\tilde{V}_1$ and V_2 with the string of improved bifundamental operators connecting them. The negatively charged monopole is instead mapped in the conjugate long meson built by similarly joining V_1 and $\tilde{V}_2$.

- Similarly, we have electric monopoles charged under the topological symmetry of the right gauge node $\mathfrak{M}^{\pm(0,1)}$ These are mapped respectively into the long meson in the magnetic theory built by joining the chirals $\tilde{V}_2$ and V_3 with the string of improved bifundamentals connecting them and its conjugate.

- Electric monopoles charged under both the topological symmetries $\mathfrak{M}^{\pm(1,1)}$ are respectively mapped into long mesons built by joining $\tilde{V}_1$ and V_3 with the string of improved bifundamental connecting them and its conjugate.

- The singlets $\mathsf{B}_{n,m}$ (with R-charge $2n - 2D + (m-n)\tau$) contained in the improved bifundamental of the electric theory are mapped into magnetic dressed mesons

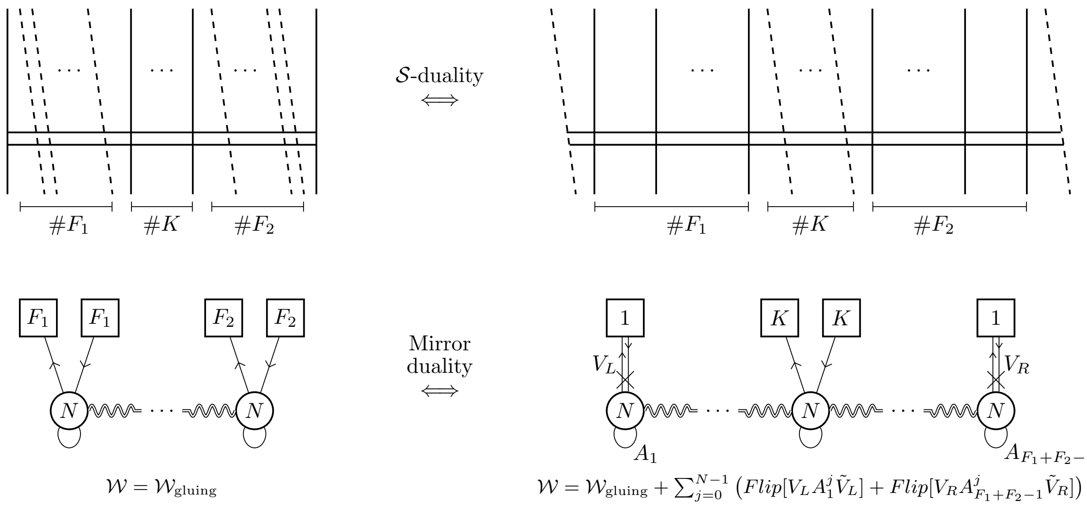


Figure 4.9: On the top left corned the electric brane set up with $K+2$ NS branes, F_1 D5' branes in the first interval and F_2 in the last and N $D3$ branes. The associated quiver theory, in the bottom left corner, has $K+1$ gauge nodes linked by K improved bifundamental, F_1 flavor on the first node and F_2 on the last. On the top right corner the S-dual brane setup and on the right bottom corner its associated quiver theory. The leftmost and rightmost flavors are associated to the D5' located outside the NS branes hence the corresponding dressed mesons are flipped. The two quiver theories are mirror dual.

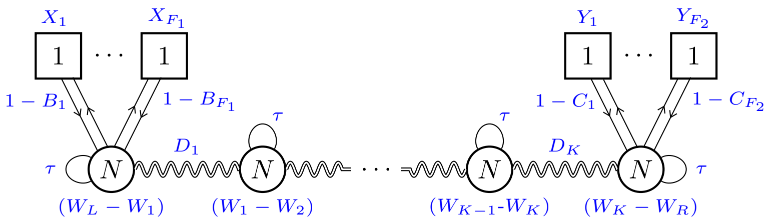
obtained from the central flavor: $V_2 \mathbf{A}^{n-1} A^{m-1} \tilde{V}_2$. Where A is the adjoint at the central node while $\mathbf{A}$ is the moment map of the improved bifundamental to the right or to the left of the central node, that are identified due to the F-term relations coming from the field A. Notice that in the electric quiver all the $\mathsf{B}_{n,m}$'s are non trivial in the chiral ring, while in the mirror quiver the $\mathsf{B}_{n\neq 1,m}$'s of each improved bifundamental are trivial in the chiral ring. This is consistent with the fact that operators of the type $V_b \mathbf{A}_b^{n-1} A_b^{m-1} \tilde{V}_b$ for $n \neq 1$ (with the flavor V_b living at the boundary of an improved quiver) are zero in the chiral ring, because the moment map operator $\mathbf{A}_b$ attached to a boundary node is set to zero by the F-terms of A_b, the adjoint of the boundary node.

All the presented operators can be also dressed with powers of the adjoint, unlike in the $\mathcal{N}=4$ theories where the cubic superpotential sets them to zero. The generalization of the map to dressed operators is straightforward. For mesonic operators in the electric theory, their dressed version is mapped into a collection of dressed monopoles plus a set of singlets. For any electric mesons dressed with $j < N$ powers of an adjoint, we consider singlets in the magnetic theory that are given by: the $\mathsf{B}^{(j)}_{1,j+1}$ singlets in the improved bifundamental theories and the flip of the flavors dressed j times $Flip[V_a A^j \tilde{V}_a]$. Analogously, dressed monopoles in the electric theory are mapped into dressed mesons in the magnetic theory.

Electric theory with $K+1$ nodes

We now consider the electric brane setup on the top left corner of figure 4.9 and its associated quiver theory with $K+1$ gauge nodes linked by K improved bifundamental.

It is convenient to consider the following parametrization:



$$(4.51)$$

The manifest global symmetry group of the theory is given by:

$$S[U(F_1)^2 \times U(F_2)^2] \times \prod_{j=1}^{K} U(1)_{D_j} \times \prod_{j=1}^{K-1} U(1)_{W_j - W_{j+1}} \times U(1)_{W_L - W_1} \times U(1)_{W_K - W_R} \times U(1)_\tau \,, \tag{4.52}$$

where $U(F_1)^2$ and $U(F_2)^2$ are realised as:

$$\begin{aligned} \prod_{j=1}^{F_1} \left(U(1)_{X_j} \times U(1)_{B_j} \right) &= U(F_1)^2 \,, \\ \prod_{j=1}^{F_2} \left(U(1)_{Y_j} \times U(1)_{C_j} \right) &= U(F_2)^2 \,. \end{aligned} \tag{4.53}$$

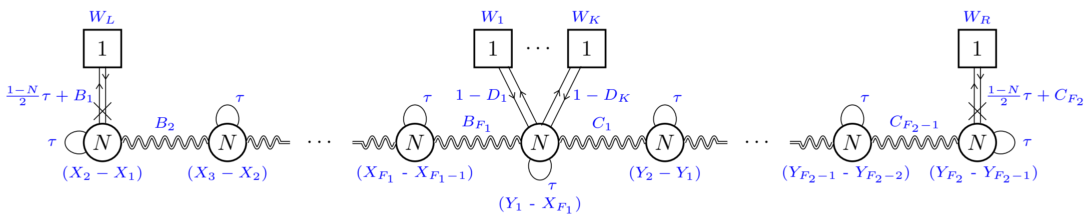
In this case also in the electric theory we have a non-trivial symmetry enhancement in the IR where the $K-1$ topological symmetries $U(1)_{W_j - W_{j+1}}$ with $j = 1, \dots K-1$, together with the K symmetries $U(1)_{D_j}$, associated to the improved bifundamentals, enhance as:

$$\prod_{a=1}^{K} U(1)_{D_a} \times \prod_{j=1}^{K-1} U(1)_{W_j - W_{j+1}} \to S[U(K)^2] \,. \tag{4.54}$$

Hence the full IR global symmetry of the theory is:

$$S[U(F_1)^2 \times U(F_2)^2] \times S[U(K)^2] \times U(1)_{W_L - W_1} \times U(1)_{W_K - W_R} \times U(1)_\tau \,, \tag{4.55}$$

Now we run the dualization algorithm and find the mirror dual quiver theory with the following parameterization:



$$(4.56)$$

This computation is given in [41] (Appendix F). The set of K flavor on the central node can be reparameterized so that they are rotated by a $U(K)^2$ symmetry obtained as:

$$\prod_{j=1}^{K} \left(U(1)_{D_j} \times U(1)_{W_j} \right) = U(K)^2 \,, \tag{4.57}$$

so that the manifest global symmetry of the theory is given by:

$$
\begin{aligned}
&S[U(K)^2 \times U(1)_{W_L} \times U(1)_{W_R}] \times \prod_{j=1}^{F_1} U(1)_{B_j} \times \prod_{j=1}^{F_2} U(1)_{C_j} \times \\
&\times \prod_{j=1}^{F_1-1} U(1)_{X_{j+1}-X_j} \times \prod_{j=1}^{F_2-1} U(1)_{Y_{j+1}-Y_j} \times U(1)_\tau \,.
\end{aligned}
\tag{4.58}
$$

It is trivial to check that, after the reparameterization, the theory in (4.56) coincides with the quiver in the bottom right corner in the Figure 4.9 which we wrote applying our proposal to the $\mathcal{S}$-dual brane setup.
Similarly to the previous example, the $U(1)_{B_j}$ and $U(1)_{C_j}$ symmetries acting on each improved bifundamental and the topological symmetries recombine to produce the enhanced IR symmetry:

$$
\begin{aligned}
&\prod_{j=1}^{F_1} U(1)_{B_j} \times \prod_{j=1}^{F_1-1} U(1)_{X_{j+1}-X_j} \to S[U(F_1)^2]\,, \\
&\prod_{j=1}^{F_2} U(1)_{C_j} \times \prod_{j=1}^{F_2-1} U(1)_{Y_{j+1}-Y_j} \to S[U(F_2)^2]\,.
\end{aligned}
\tag{4.59}
$$

The complete IR global symmetry of the mirror theory is then:

$$
S[U(F_1)^2] \times S[U(F_2)^2] \times U(1)_{Y_1 - X_{F_1}} \times S[U(K)^2 \times U(1)_{W_L} \times U(1)_{W_R}] \times U(1)_\tau \,,
\tag{4.60}
$$

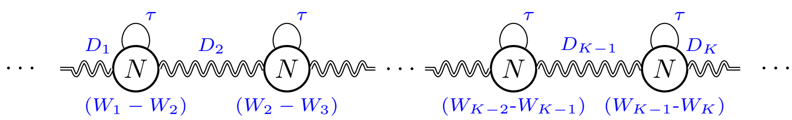
that upon a redefinition of some $U(1)$ factors, it matches precisely with the IR global symmetry of the original theory in (4.55).

4.4.2 3d $\mathcal{N} = 2$ improved quivers: the good, the bad and the ugly

In this section we extend the $\mathcal{N} = 4$ quivers notion of balanced ($N_F = 2N_C$), good ($N_F \geqslant 2N_C$), ugly ($N_F = 2N_C - 1$), and bad ($N_F < 2N_C - 1$) nodes of [28], to the $\mathcal{N} = 2$ improved quivers with constant ranks. We expect that a similar story holds for $\mathcal{N} = 2$ improved quivers with non-constant ranks.

Comments on symmetry enhancement: balancing condition

Looking back at all the theories presented up to this point, namely the 1, 2 and K node examples together with their mirror duals, a recurring pattern of global symmetry enhancement can be seen, we now want to collect all these hints to formulate a general rule to recognize the enhancement of symmetries.
Let us start from the K nodes example which is given in (4.51). Let us isolate from the theory the sequence of improved bifundamentals:

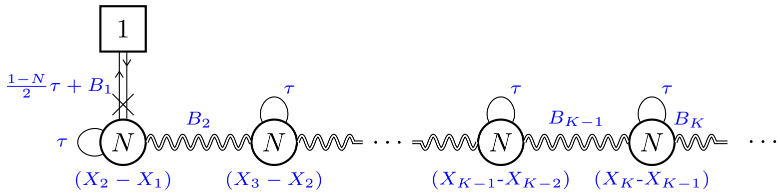


(4.61)

This structure of K improved bifundamentals gives a global symmetry enhancement obtained from the K $U(1)_{D_a}$ axial symmetries associated to the improved bifundamentals and the $K-1$ $U(1)_{W_j - W_{j+1}}$ topological symmetries. Together these symmetries enhance to a $U(K)^2/U(1)$ non-abelian global symmetry group. Let's take the Cartan's of $U(K)^2$ to be $\vec{M}$ and $\vec{N}$, we have the following relations:

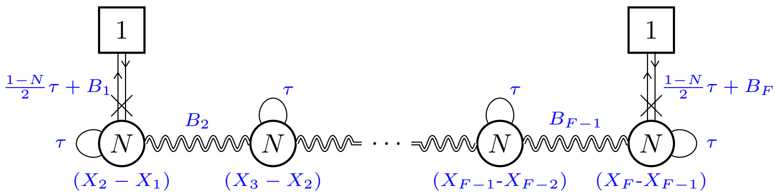
$$\begin{aligned} M_a &= W_a - D_a\,, \\ N_a &= W_a + D_a\,, \end{aligned} \tag{4.62}$$

from which we notice that the $U(1)_{D_a}$ symmetries parameterize the axial-like and the $U(1)_{W_a}$ the vector-like subgroup of $U(K)^2$.
We then look at the mirror theory given in (4.56), both on the left and on the right of the quiver we observe a string of improved bifundamental ending with a single flipped flavor. Let's focus on the left part only and isolate the following structure:



(4.63)

We observe that the $U(1)_{B_a}$ and $U(1)_{X_{j+1}-X_j}$ symmetries enhance to a $U(F_1)^2/U(1)$, where the parameterization is given analogously to that in (4.62). The same enhancement happens also in the mirror of the two node theory in (4.47).
Let us now look finally to the mirror of the SQCD in Figure 4.1, we have a string of improved bifundamentals ending on both sides with a flipped flavor:



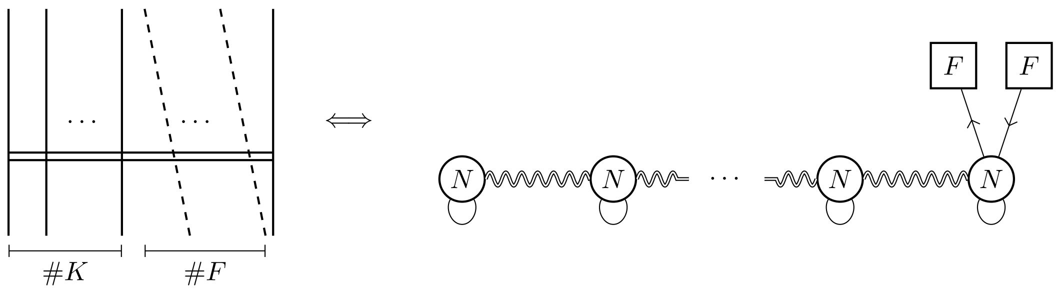
(4.64)

As discussed in Section 4.2 we have symmetry enhancement involving all the $U(1)_{B_i}$ and $U(1)_{X_i}$ symmetries.
Collecting all these observations we can give a definition for a balanced node:

A node is balanced if it joins two improved bifundamentals or if it joins an improved bifundamental to a flipped flavor.

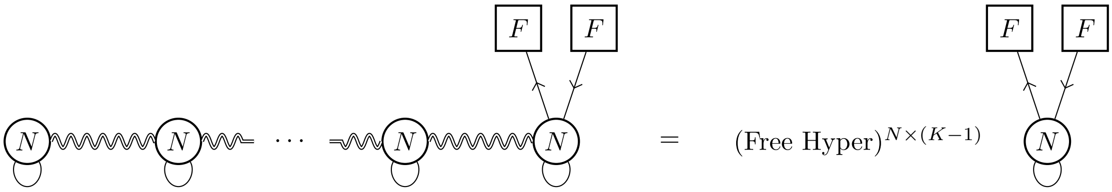
Ugly and Bad quivers

We now study the following brane setup and its QFT description:



(4.65)

We focus on the QFT. We have a sequence of $K-1$ improved bifundamentals ending with F flavors on the last node. We use the fact that an improved bifundamental with one $U(N)$ symmetry gauged *confines* into N free hypers, as reported in (4.27), to sequentially confine all the improved bifundamentals in (4.65). So the QFT associated to this brane setup is given by $(K-1)\times N$ free hypers and a $U(N)$ adjoint SQCD with F flavors:

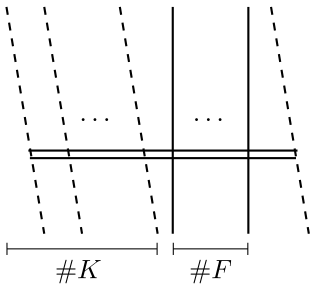


(4.66)

We thus call a node attached only to an improved bifundamental an *ugly* node, in analogy with $\mathcal{N}=4$ $U(N)$ with $2N-1$ flavors whose monopole has $\Delta=\frac{1}{2}$ and is a free field.

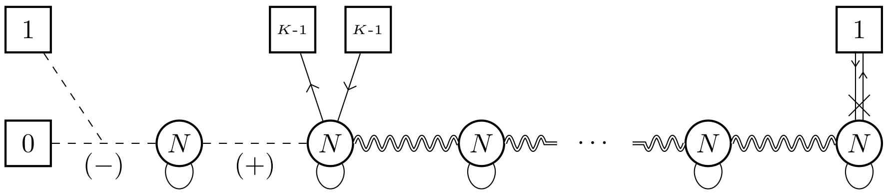
A $U(N)$ node attached to two improved bifundamentals with $F\geqslant 0$ flavors, or a $U(N)$ node attached to one improved bifundamental with $F\geqslant 1$ flavors is *good* and does not provide free decoupled fields.

Let's now consider the $\mathcal{S}$-dual configuration of (4.65), which is given by:



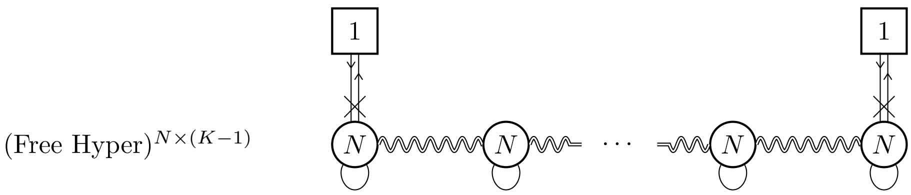
(4.67)

We can find the QFT description of this setup by applying the dualization algorithm to the electric theory. After we have dualized each block composing the theory, we find the following intermediate step:



(4.68)

We now implement the asymmetric $\mathbb{I}$-wall on the left given by the two $\mathcal{S}$-walls glued together, which, repeating the steps in Section 4.3.1, leads to the following theory:

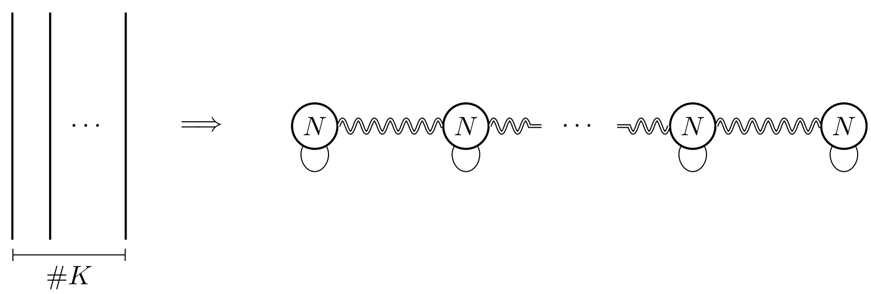


(4.69)

More precisely, implementing the asymmetric $\mathbb{I}$-wall has the effect of Higgsing completely the second $U(N)$ gauge node in (4.68) down to a flavor $U(1)$. Let us now

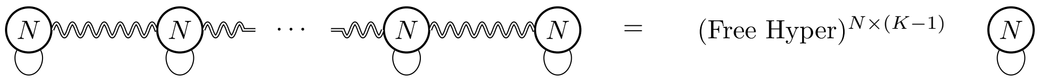
analyze separately the effect of this Higgsing on the first improved bifundamental and on the $K-1$ flavors to show how the result (4.69) is obtained. The first improved bifundamental becomes asymmetric (see Appendix C.2), using the duality (C.27) it is dual to a flipped flavor for the third $U(N)$ gauge node. By taking this effect into account, we see that we have a string of $F-2$ improved bifundamentals with a flipped flavor on the two sides, which is the theory depicted on the right side of (4.69). After the Higgsing, the $K-1$ flavors for the second $U(N)$ gauge node in (4.68) become just a set of $2N(K-1)$ chirals that do not interact, therefore these are $N(K-1)$ free hypermultiplets, as written on the lh.s. of (4.69).
We can compare the result for the mirror theory in (4.69) with that of the electric theory in (4.66) to see that in both frames we have $K-1$ set of N free hypers times an interacting theory. We can notice that the two interacting theories are mirror dual to each other by means of the duality proposed in 4.1. This analysis suggest the following recipe: given a $\mathcal{N}=2$ brane setup (with constant number of N D3 branes) starting with a sequence of K NS or D5' branes, the first $K-1$ branes decouple from the theory giving a set of N free hypers.

Let us now consider the following brane setup and its QFT description:



(4.70)

We notice that this setup preserves eight supercharges and not just four. Therefore, we expect that our prescription gives the same result as the $\mathcal{N}=4$ case. As in the previous case we can sequentailly confine all the improved bifundamentals producing $(K-1)\times N$ free hypers:



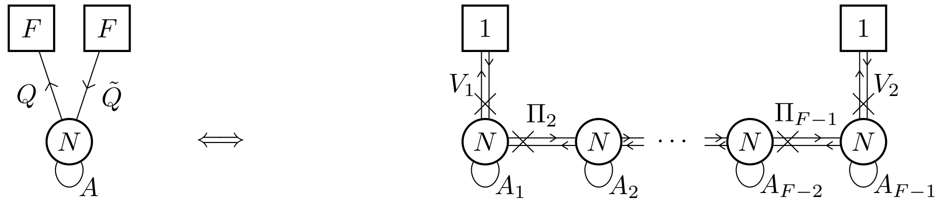
(4.71)

In addition to the free hypers we are left with a $\mathcal{N}=4$ $U(N)$ pure SYM theory which, as shown in [48, 75], is a bad theory described by N free hypers. All in all, the theory (4.70) is just given as $K\times N$ free hypers and is a *bad* theory.

4.4.3 Comments on previous proposals of $\mathcal{N}=2$ mirror dualities

Non-abelian 3d mirror symmetry with 4 supercharges has been discussed in [33, 72, 76, 21, 77].

In this subsection we focus on the case of $U(N)$ SQCD, and compare our proposal for the mirror dual with the *naive* proposal by [21] depicted in Figure 4.10. The naive mirror dual is a quiver theory with $F-1$ $U(N)$ gauge nodes linked by standard flipped bifundamental fields that come coupled to the adjoint fields via cubic superpotential terms. Moreover we have two towers of singlets flipping the meson built from the two vertical flavors dressed with the adjoint fields, exactly as in the theories described in this paper. The proposal in Figure 4.10 was based on a *naive* reading on the magnetic brane setup of (4.43), analogous to [33, 72, 76, 21, 77]. A very similar proposal (with standard bifundamentals) for the mirror of 3d $\mathcal{N} = 2$ $U(N)$ SQCD without adjoint appeared before in [76]. Both for $SU(N)$ and $U(N)$, as was already noticed, the naive proposals suffer from a mismatch in the number of UV global symmetries, namely the mirror quiver has much fewer global symmetries than the SQCD, which, having zero superpotential enjoies a chiral $U(F)^2/U(1)$ symmetry. For instance, the naive mirror dual of Figure 4.10 has UV global symmetry $U(1)^{F-1} \times U(1)_\tau \times U(1)$, only half of the Cartans of the electric theory.



$\mathcal{W} = 0$

$$\mathcal{W} = \sum_{j=2}^{F-2} A_j(\Pi_j\tilde{\Pi}_j + \Pi_{j+1}\tilde{\Pi}_{j+1}) + \sum_{j=2}^{F-1} Flip[\Pi_j\tilde{\Pi}_j] + \sum_{j=0}^{N-1}(Flip[V_1 A_1^j \tilde{V}_1] + Flip[V_2 A_{F-1}^j \tilde{V}_2])$$

Figure 4.10: Naive proposal for the mirror pair of the $\mathcal{N} = 2$ $U(N)$ adjoint SQCD. The superpotential $\mathcal{W}\mathcal{N} = 4$ in the mirror theory contains all the superpotential terms coupling each adjoint field to the bifundamentals besides it.

One argument in support of the proposals in [76, 21] was provided in [76, 77] showing that the naive mirror pairs can be obtained by starting from a well established $\mathcal{N} = 4$ mirror pair by turning on a superpotential deformations to land on the $\mathcal{N} = 2$ dualities of [76, 21]. Because of this strategy, the resulting mirror theory inherits from the $\mathcal{N} = 4$ superpotential the cubic couplings between bifundamentals and the adjoint fields. However this strategy also produces additional superpotential terms. In the case of $U(N)$ with adjoint there is an additional $\mathcal{W} = V_1 A_1^N \tilde{V}_1 + V_2 A_{F-1}^N \tilde{V}_2$. These terms are zero in the chiral ring of the theory on the right of figure 4.10, hence they violate the chiral stability condition [70]. Simply removing these two terms, if $F > 2$, breaks the degeneracy between the monopole operators that are supposed to map to the electric mesons, hence rendering the mapping of the operators problematic.

As already mentioned, our mirror dual can not be deformed to the naive one of Figure 4.10. In order to do such a deformation, we would need to iron all our improved bifundamentals to standard ones. To do so we have two options.

The first one corresponds to adding linearly to the superpotential the singlet $\mathsf{B}_{1,2}^{(k)}$ as we did in section 4.2.3 when discussing the $\mathcal{N} = 4$ limit. Keeping track of the adjoints appearing in the ironing duality (C.24) we would then find that all nodes, apart from the first and the last one, have an adjoint of charge $2 - \tau$ which couples to the bifundamentals to its right and to its left. So we reach a theory different from the mirror dual in Figure 4.10.

To reach a mirror theory where also the first and last gauge node have an adjoint we could consider the second option to iron the improved bifundamentals by adding linearly to the superpotential the singlet $\mathsf{B}_{2,1}^{(k)}$, which have the effect of ironing the improved bifundamentals to bifundamentals without any extra adjoint field as shown in (C.24). Therefore, if we use this deformation on all the improved bifundamentals we reach exactly the theory in the r.h.s. of Figure 4.10. However, as we noticed when discussing the operator map in our SQCD mirror pair in Section 4.2.2, the $\mathsf{B}_{2,1}^{(k)}$ singlets (unlike $\mathsf{B}^{(k)}_{1,2}$ which map to dressed mesons) are trivial in the chiral ring, they can not map to any operator in the $U(N)$ SQCD chiral ring.

The general lesson is that our mirrors with improved bifundamentals and the naive mirrors with standard bifundamentals (or the mirrors obtained deforming $\mathcal{N} = 4$ dualities) differ by turning on or off in the superpotential holomorphic operators which are zero in the chiral ring, hence they provide different UV completions of the same IR SCFT.

While both the naive and our mirrors are *correct*, the mirrors with improved bifundamentals discussed in this paper are more *useful*, since they encode a full rank UV global symmetry and allow us to study the IR SCFT's in a transparent way. In particular the present technology allows us to compute the superconformal index, the $\mathbf{S}^3_b$ partition functions, the chiral ring and the moduli space of vacua of the IR SCFT's using our UV quivers with improved bifundamentals. Moreover, the $\mathcal{N} = 2$ algorithm proves that the UV improved quivers associated to $\mathcal{S}$-dual brane setup flow to the same IR SCFT.

4.5 Conclusions

In this chapter we have presented a novel $\mathcal{N} = 2$ mirror duality for non-Abelian gauge theories. Notice that although Abelian examples are widely understood in the literature [34, 37], the non-Abelian case, prior to this work, was understood only in sporadic instances, for example with low fixed amount of flavors.

One of the results provided by this work is to offer a natural interpretation of the duality presented as the SCFTs engineered by Type IIB brane setups preserving four supercharges. From this point of view, the name mirror duality follows from the fact that it is the field theory realization of $\mathcal{S}$-duality acting on the brane setup.

More generalizations of this result are contained in [40], as for example theories with $SU(N)$ gauge group, which lead to a magnetic quiver for the 4d $\mathcal{N} = 1$ $SU(N)$ SQCD, and to brane setups involving (p, q)-branes.

Still, this result has limitations at the moment, we now comment on some of them and how to possibly generalize further the results presented. One of the evident limitations is that the quiver theories have gauge groups with constant ranks N. However, since the improved bifundamental theory has a natural "asymmetric" generalization, similar to that of the $FT[U(N)]$ theory (see Appendix C.1 and C.2), we expect to be able to further extend this result simply by taking into consideration these asymmetric improved bifundamentals. This would be an interesting generalization since such theories should describe brane setups with non-constant number of D3 branes.
Moreover, the brane setup considered only involves one type of five-brane, in the sense that it contains only NS and D5' (or similarly NS' and D5) but not NS and NS' at the same time. This situation is believed to lead to theories without adjoint chiral fields and it also leads to a natural problem, which is how to implement the Hanany-Witten move between NS and D5-brane inside brane setups preserving four supercharges, thus involving improved bifundamentals. This problem is yet open to be explored and it is left for future investigations.
Another possible generalization is to consider $\mathcal{N} = 2$ theories with more general matter, in fact, with four supercharges it is possible to write a Lagrangian that has also CS interaction and/or chiral matter content. This is expected to be possible straightforwardly since basic moves for different types of matter contents, such as chiral flavors, are already known and were presented in [42], along with new *improved bifundamen-*

tals. Moreover, in $\mathcal{N}=2$ theories it is possible to produce non-zero CS levels starting from theories with zero CS level by performing suitable real mass deformations. We expect to be able to study them using the techniques of [46, 78] and, in fact, the results presented in Chapter 4 offer a generalization in this direction.
A last possible generalization is to consider more improved bifundamental, for example those in [42]. This could give access to more $\mathcal{N}=2$ mirror dualities similar in spirit to those presented in this chapter. With the set of improved bifundamentals at hand it is possible to dualize $U(N)$ CS-QCD theories with chiral matter content and quiver theories with $USp(2N)$ and $U(N)$ gauge groups. Th3 investigation for more improved bifundamentals, for example with orthogonal global symmetries that could allow us to include orthogonal gauge groups in the class of $\mathcal{N}=2$ theories for which we can compute a mirror dual.

Chapter 5

$\mathcal{N} = 2$: SUSY breaking deformations from $\mathcal{N} = 4$

This chapter is based on [43, 44]. We continue seeking 3d $\mathcal{N} = 2$ mirror dualities. Differently from Chapter 4, where the idea was taking strong inspiration from Type IIB brane setups and $\mathcal{S}$-duality, this time we follow a different path. The idea this time is that we can derive $\mathcal{N} = 2$ mirror dualities starting from $\mathcal{N} = 4$ ones and performing a SUSY breaking deformation.

5.1 Introduction

In this chapter we pursue a second possible way to construct $\mathcal{N} = 2$ mirror dualities. This idea consist of starting from $\mathcal{N} = 4$ mirror dualities to obtain $\mathcal{N} = 2$ mirror dualities by performing a suitable SUSY breaking deformation. This deformation is constructed starting from any $\mathcal{N} = 4$ theory $\mathcal{T}_{\mathcal{N}=4}$ of our choice and rewriting it in the $\mathcal{N} = 2^*$ language [37]. We therefore have in general a real mass associated to each global symmetry. In particular we rewrite the $SU(2)_H \times SU(2)_C$ $\mathcal{N} = 4$ R-symmetry as a $U(1)_R$ $\mathcal{N} = 2$ R-symmetry times its commutant which is a $U(1)_\tau$ global symmetry[1]. We then aim to perform in $\mathcal{T}_{\mathcal{N}=4}$ a real mass deformation associated to $U(1)_\tau$ by giving a non-zero VEV to the background vector multiplet associated to this symmetry. This deformation is also followed by a suitable CB deformation in order to flow to a non-trivial interacting $\mathcal{N} = 2$ SCFT $\mathcal{T}_{\mathcal{N}=2}$ as a result of the deformation.

We will explain in depth this deformation in the bulk of the chapter, the problem which we aim to describe now is that it is quite a non-trivial problem to understand which is the deformation in $\mathcal{T}^\vee_{\mathcal{N}=4}$ that drives us to the dual vacuum, that is to an interacting SCFT $\mathcal{T}^\vee_{\mathcal{N}=2}$ that is dual to $\mathcal{T}_{\mathcal{N}=2}$. In fact, it is not known in general how to map $\mathcal{N} = 2$ vacua at various points of the CB across mirror duality. To solve this problem we employ mainly a strategy based on the analysis $\mathcal{S}^3_b$ partition function [8, 12]. The idea is that the partition function is in general a function of the real mass parameters, in particular the real mass associated to $U(1)_\tau$ that by abuse of notation we also call τ; the real mass deformation can then be performed by considering the limit $\tau \to +\infty$ and studying the asymptotic behavior of the partition function. If this deformation is followed by a suitable CB deformation, we find that the original partition

[1] Notice that in this chapter we stick with the notation of Chapter 4 and not Chapter 3, where for example the commutant symmetry was named $U(1)_A$.

function $Z_{\mathcal{T}_{\mathcal{N}=4}}(\tau)$ behaves as:

$$Z_{\mathcal{T}_{\mathcal{N}=4}}(\tau) = e^{iH(\tau)} Z_{\mathcal{T}_{\mathcal{N}=2}} \tag{5.1}$$

where we have a highly oscillating prefactor depending on the divergent parameter τ and the partition function of the theory $\mathcal{T}_{\mathcal{N}=2}$ living in the vacuum, that does not depend on τ.
We can do the same on the partition function of the mirror theory $Z_{\mathcal{T}^{\vee}_{\mathcal{N}=4}}(\tau)$, which for a generic CB deformation produces an highly oscillating phase $H'(\tau)$. The claim is that the dual vacuum is the one that reproduces the same phase, namely $H(\tau)$.

With this strategy we are then able to effectively identify the dual vacuum and thus produce a duality between $\mathcal{N} = 2$ theories that inherits properties of the original mirror duality such as the mapping between mesonic and monopole operators and the exchange of flavor and topological symmetries. Indeed, the result obtained can be tested in many ways such as matching the spectrum, global symmetries, and superconformal indices and also we can check the existence of the considered deformations in the $\mathcal{N} = 2$ 1-loop vacuum equations [46].

A surprising result that we find is that the $\mathcal{N} = 2$ mirror duality relates a non-Abelian gauge theory with a purely Abelian description. This Abelian dual comes in the shape of a planar quiver with self and mixed-CS interactions, cubic superpotential terms associated to the faces of the quiver and also a monopole superpotential that ensures the correct enhanced topological symmetry.

Even though we are able to derive various $\mathcal{N} = 2$ mirror dualities by starting from various $\mathcal{N} = 4$ mirror dualities and considering a variety of real mass deformations, we also aim to understand the underlying phenomena behind these new dualities. As a first attempt in this direction, we try to generalize the $\mathcal{N} = 4$ dualization algorithm of Chapter 3 to the class of $\mathcal{N} = 2$ theories considered in this chapter. This is a task that we achieve by simply taking the real mass deformation of the basic QFT blocks and duality moves described in Chapter 3. With this construction we understand the existence of a duality wall, which is the real mass deformed theory obtained starting from the $\mathcal{S}$-wall, that we baptize $G[U(N)]$. This theory satisfies $SL(2,\mathbb{Z})$ relations and can be thought as the generator of planar Abelian $\mathcal{N} = 2$ mirror dualities. With this new tool at hand we construct many more examples of planar Abelian mirror dualities and also study the phenomenon of symmetry enhancement in these theories.

5.2 A Planar Abelian Dual for CS-SQCD$_3$ with Fundamental Matter

In this section, we explicitly derive the $\mathcal{N} = 2$ mirror of $U(N)_{(-\frac{F}{2}+N,-\frac{F}{2})}$ SQCD with F fundamental chiral multiplets by starting from the $\mathcal{N} = 4$ mirror duality for the $U(N)$ SQCD with F flavors.

Our main tool is the $\mathbf{S}^3_b$ partition function [18, 19, 20] which allows us to follow the effect of the real mass deformations efficiently. We verify these results by checking that the resulting vacuum satisfies the F- and D-term equations [46].

5.2.1 The $\mathcal{N} = 4$ Mirror Pair

We start from the mirror duality for the $U(N)$ $\mathcal{N} = 4$ SQCD depicted in Figure 5.1.

To study possible deformation breaking supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ we work in the so-called $\mathcal{N} = 2^*$ set-up [37] taking $U(1)_R = U(1)_{C+H}$ as the R-symmetry

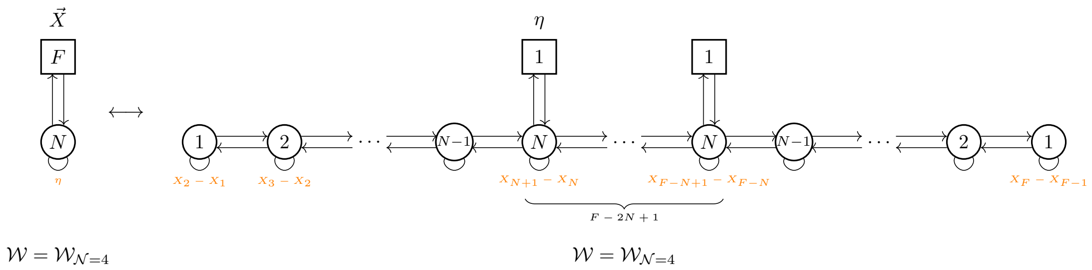


Figure 5.1: Mirror duality for $\mathcal{N} = 4$ SQCD with gauge group $U(N)$ and F flavors. Notice that the theories are depicted in the $\mathcal{N} = 2$ notation. Round nodes denote gauge groups and square nodes denote flavor groups, while arrows denote chiral multiplets charged under the nodes that they connect. Black labels denote fugacities associated to flavor global symmetries while orange labels denote the FI of the corresponding gauge group.

group, where $U(1)_C \in SU(2)_C$ and $U(1)_H \in SU(2)_H$ of the non-abelian $\mathcal{N} = 4$ R-symmetry $SU(2)_H \times SU(2)_C$. The commutant of $U(1)_R$, which is $U(1)_\tau = U(1)_{C-H}$, is a global symmetry from the point of view of the $\mathcal{N} = 2$ theory.

However, we have the freedom to redefine $U(1)_R$ up to abelian flavor symmetries, meaning that we can shift the R-charge by $U(1)_\tau$ transformations. Using this freedom we pick a convention in which we assign trial R-charge 1 and $U(1)_\tau$ charge $-1/2$ to the fundamental and anti-fundamental chiral fields while adjoint chiral fields have R-charge 0 and $U(1)_\tau$ charge 1. The charges of the fields under the symmetries of the theories are reported in the tables below where we adopt an $\mathcal{N} = 2$ notation denoting by Φ, Q, $\tilde{Q}$ the adjoint and the fundamental/antifundamental chirals of the SQCD and on the mirror side denoting respectively by Φ_J, $b_{J,J+1}$, $\widetilde{b}_{J+1,J}$, the adjoints and the bifundamental chirals and the left and right flavors $q, \tilde{q}$ and $p, \tilde{p}$.

	$SU(F)$	$U(1)_\tau$	$U(1)_R$	$U(1)_\eta$
Φ	1	1	0	0
Q	$\overline{\square}$	$-\frac{1}{2}$	1	0
$\tilde{Q}$	$\square$	$-\frac{1}{2}$	1	0

(5.2)

	$SU(F)$	$U(1)_\tau$	$U(1)_R$	$U(1)_\eta$
Φ_J	1	-1	2	0
$b_{J,J+1}, \widetilde{b}_{J,J+1}$	1	$\frac{1}{2}$	0	0
$q, \widetilde{q}$	1	$\frac{1}{2}$	0	∓ 1
$p, \widetilde{p}$	1	$\frac{1}{2}$	0	0

Notice that the R-charges and the $U(1)_\tau$ charges in the mirror theory are obtained from the definitions given before, taking into account that mirror symmetry swaps $SU(2)_C \leftrightarrow SU(2)_H$.

Global symmetries and Operator Map

The global symmetry of the $U(N)$ SQCD with F flavors, depicted on the l.h.s. of (5.1), is $SU(F)\times U(1)_\eta\times U(1)_\tau$[2], where $U(1)_\eta$ is the topological symmetry. On the other hand, the manifest global symmetry of the mirror theory, depicted on the r.h.s. of (5.1), is $\prod_{j=1}^{F-1} U(1)_{X_{j+1}-X_j}\times U(1)_\eta\times U(1)_\tau$, where the first term is the collection of topological symmetries, while $U(1)_\eta$ is a flavor symmetry. In the IR this global symmetry enhances, in particular all the topological symmetries combine to give an enhanced $SU(F)$ global symmetry group. We therefore see that the IR global symmetries of the two theories are in agreement and topological symmetries are mapped into flavor symmetries and vice-versa, as expected in a mirror duality.

It is also instructive to see how the operators of the electric theory are mapped to those of the mirror theory.

- The fundamental monopoles $\mathfrak{M}^{\pm}$ of the SQCD are mapped into the long mesons on the mirror quiver side.
- The electric mesons, in the traceless adjoint of $SU(F)$, are mapped to a collection of monopole operators in the mirror theory. The latter is a traceless adjoint matrix of $SU(F)$ constructed with the $F-1$ traces of the adjoints and with monopoles ($\mathfrak{M}^{\cdots,0,\pm1,0,\cdots}$) with flux ±1 under one of the $F-1$ gauge nodes, $(F-2)$ monopoles ($\mathfrak{M}^{\cdots,\pm1,\pm1,0,\cdots}$) carrying GNO flux ±1 under two adjacent gauge nodes, $(F-3)$ monopoles ($\mathfrak{M}^{\cdots,\pm1,\pm1,\pm1,\cdots}$) carrying GNO flux $+1$ under three adjacent gauge nodes, and so on with the final monopole operators carrying GNO flux ±1 under all $F-1$ gauge nodes.

5.2.2 Real Mass Deformation to the $\mathcal{N}=2$ Chiral-Planar Mirror Pair

We now move on the SUSY-breaking deformation. We perform a real mass deformation for the $U(1)_\tau$ symmetry, meaning that we turn on a non-zero VEV for the real scalar component of the background vector multiplet associated to the $U(1)_\tau$ symmetry. The VEV induces a mass term for every chiral superfield proportional to its charge under the symmetry. Recalling that $U(1)_\tau$ is a flavor subgroup of the $\mathcal{N}=4$ R-symmetry, the deformation performed has the effect of breaking the R-symmetry from $SU(2)_C\times SU(2)_H\to U(1)_R$, indeed this means that supersymmetry is broken from $\mathcal{N}=4$ to $\mathcal{N}=2$.
As we shall discuss in depth, in the most generic vacua this deformation triggers a flow to a TFT. However, we will try to suitably follow this deformation by moving on special points of the Coulomb branch, so to reach an interacting vacua. Both the SQCD and its mirror can have many possible interacting vacua that can be reached and the task of understanding how these are mapped correctly among them is the problem we attempt to solve in the following section. We will start by giving a qualitative description of the deformation of both the SQCD and its mirror in the following subsection, discussing the features of the resulting $\mathcal{N}=2$ mirror duality. In the following subsections we will show how this duality can be derived from two perspectives: the $\mathbf{S}^3_b$ partition function and the 1-loop corrected EOM.

[2]The actual faithful global symmetry of the theories is $PSU(F)\times U(1)_\eta\times U(1)_\tau$. We will not discuss the features related to the discrete factors of the global symmetry group. Moreover, the $U(1)_\eta$ symmetry enhances to $SU(2)$ in the IR if $F=2N$. Indeed in the mirror theory for $F=2N$ there is a manifest $SU(2)$ flavor symmetry instead of $U(1)$. This will not have an effect in the computations that we will perform and we refer to the topological symmetry of the SQCD as being generically $U(1)_\eta$.

Let us first discuss the SQCD side, on the l.h.s. of Figure 5.1. As already commented, a real mass for the $U(1)_\tau$ symmetry naively renders every chiral field massive, and no massless fields are left in the deep IR. We then choose to move on the Coulomb branch in such a way to reach an interacting vacuum where all the fundamental chiral fields Q remain massless while the adjoint Φ and the F anti-fundamental chiral fields $\widetilde{Q}$ acquire a mass, negative and positive respectively, and are integrated out[3]. The procedure of integrating out chiral fields produces CS interactions due to fermionic modes becoming massive. The resulting theory is the $\mathcal{N}=2$ $U(N)_{-\frac{F}{2}+N,-\frac{F}{2}}$ SQCD depicted on the l.h.s. of Figure 5.2. Here, $U(N)_{k,k+lN}$[4] denotes a $U(N)$ gauge group with CS terms[5]:

$$-i\frac{k}{4\pi}\int \mathrm{tr}\,(A\wedge dA) - i\frac{l}{4\pi}\int \mathrm{tr}\,(A)\wedge \mathrm{tr}\,(dA) + \text{SUSY completion} \tag{5.3}$$

where A is the $U(N)$ gauge field. Each massive antifundamental multiplet, with negative mass, contributes as $(-\frac{1}{2},-\frac{1}{2})$ to the CS level, and the massive adjoint chiral, with positive mass, contributes as $(N,0)$. Notice in particular that we have $l=-1$.

We now discuss the result of the deformation in the mirror theory, depicted on the r.h.s. of Figure 5.1. As already pointed out, many possible interacting vacua can be reached with the $U(1)_\tau$ real mass, each lying at different points of the Coulomb branch. We claim that the vacuum dual to the chiral SQCD theory described above is a special Coulomb branch point where each gauge node $U(k)$ of the original $\mathcal{N}=4$ mirror is Higgsed to its maximal torus $U(1)^k$. Also, some components of the chiral fields in the original theory remain massless. The complete description of the theory is depicted as a quiver on the r.h.s. of Figure 5.2.
A few comments regarding the planar dual on the r.h.s. of Figure 5.2 are in order:

- There is a cubic superpotential term for every triangular loop. There is one term with -1 $(+1)$ coefficient for each clock-wise (anti-clock-wise) closed triangle. The cubic terms in the superpotential are a remnant of the cubic $\mathcal{N}=4$ superpotential that are preserved by the deformation. We denote these superpotential terms in short as $\mathcal{W}_{\text{planar}}$. The diagonal/vertical chiral multiplets have R-charge $0/+2$, compatible with the superpotential.

- $\mathcal{W}_{\text{monopole}}$ are the monopole terms in the superpotential generated by the Polyakov mechanism [79] due to the Higgsing of a $U(k)$ gauge symmetry to $U(1)^k$. The exact collection of terms can be obtained with the following schematic rule. For each vertical arrow, there is a linear superpotential for the monopole with GNO flux -1 and $+1$ under the nodes connected by the arrow, from top to bottom. As a consequence of these monopole superpotential terms, the topological symmetry is broken to $U(1)^{F-1}$ which is expected to enhance in the infrared to $SU(F)$, thus matching the global symmetry on the electric side. The FI terms of the $U(1)$ gauge nodes are compatible with the monopole superpotential terms. With the choice of trial R-symmetry described above, the α^{th} gauge node in the I^{th}

[3] One can also consider different interacting vacua where a certain number of fundamental and antifundamental fields remain massless. While this problem is tractable, we will tackle the case of SQCD theories with more general matter content in a different, algorithmic way in Section 5.5.

[4] We follow the standard notation for Chern-Simons levels: $U(N)_{(k_1,k_2)} = \frac{SU(N)_{k_1}\times U(1)_{Nk_2}}{\mathbb{Z}_N}$. To avoid anomalies $k_1\in\mathbb{Z}$ and the $U(1)$ level must be of the form $k_2=k_1+lN$, with $l\in\mathbb{Z}$.

[5] Notice that we will adopt a different notation for the CS level as that of Chapter 3, namely we will swap the sign of the convention.

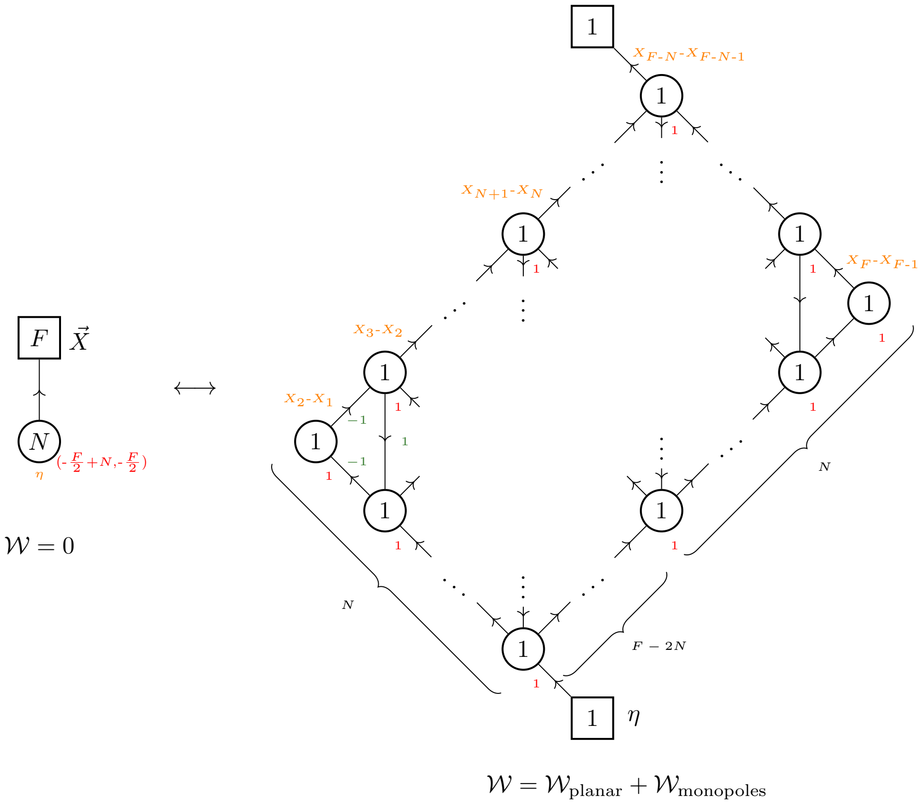


Figure 5.2: On the l.h.s. it is reported the $\mathcal{N}=2$ $U(N)_{-\frac{F}{2}+N,-\frac{F}{2}}$ SQCD with F fundamental fields obtained from the real mass deformation of the $\mathcal{N}=4$ SQCD. On the r.h.s. it is reported the planar mirror-like dual of the SQCD. Note that every gauge node has a CS level (red labels) 1 and there are mixed CS terms (green labels) for every pair of nodes connected by a line, with level $-1/+1$ if the line is diagonal/vertical. FI terms are indicated in orange. On the SQCD side we assign trial R-charge 1 to the fundamental chirals and on the planar side the diagonal chiral fields have trial R-charge 0 and the vertical ones 2. The diagonal chirals pointing from SE to NW connect N nodes, while the diagonal chirals pointing from SW to NE connect $F-N$ nodes.

N	F	r	Index
2	4	1/5	$1+\eta\, x^{4/5}+\eta^2\, x^{8/5}-16x^2+\eta^3\, x^{12/5}+\eta^4\, x^{16/5}+(\eta^5+88)x^4+O(x^{21/5})$
2	5	1/5	$1+\eta x+\left(\eta^2-25\right)x^2+\eta^3x^3+\left(\eta^4+250\right)x^4+\left(\eta^5-100\eta\right)x^5+O\left(x^6\right)$
2	6	1/5	$1+\eta x^{6/5}-36x^2+\eta^2x^{12/5}+\eta^3x^{18/5}+558x^4+\eta^4x^{24/5}+O\left(x^{26/5}\right)$
3	6	1/7	$1+\eta x^{6/7}+\eta^2x^{12/7}-36x^2+\eta^3x^{18/7}+\eta^4x^{24/7}+558x^4$ $+\eta^5x^{30/7}-225\eta x^{34/7}+O\left(x^{36/7}\right)$

Table 5.1: Index of $U(N)_{(-\frac{F}{2}+N,-\frac{F}{2})}$ SQCD with F fundamental chiral multiplets. The chiral multiplets are assigned trial R-charge $1-r$. We checked that the indices match with the corresponding mirror abelian quiver in Figure 5.2.

column of the planar quiver ($\alpha=1,\dots,|G^{(I)}|$, and $|G^{(I)}|$ is the rank of the corresponding gauge node in the $\mathcal{N}=4$ quiver, or equivalently, the number of $U(1)$ gauge nodes in the I^{th} column of the $\mathcal{N}=2$ quiver) has FI terms equal to $X_{I+1}-X_I+\frac{iQ}{4}(\delta_{\alpha,1}-\delta_{\alpha,|G^{(I)}|})$. In Figure 5.2 we only reported the X_I-deopendent part of the FIs.

- CS interactions are present. Self and mixed-interactions can be encoded in a matrix k_{ab} such that:

$$-i\frac{k_{ab}}{4\pi}\int A_a\wedge dA_b+\text{SUSY completion} \tag{5.4}$$

where A_a is the vector field of the a-th $U(1)$ gauge field. Indeed, by construction k_{ab} is symmetric. For the theory in Figure 5.2 the rule is that each gauge node has (self) CS interaction with level 1, denoted by the red labels in Figure 5.2. Nodes connected by diagonal/vertical lines have a mixed-CS interaction with level $-1/+1$, that is $k_{ab}=\mp\frac{1}{2}$. Mixed CS levels are denoted by the green labels along the corresponging line in Figure 5.2.

Superconformal Index, global symmetries and operator map

We now comment on the features of the duality described before, highlighting the tests that this duality passes.

The first test that we performed consists in the matching of the numerical expansion of the Superconformal Index. This guarantees that a number of properties are correctly reproduced by the dual theory such as: the presence of conserved currents associated to the correct global symmetry, the properties of the operators in the chiral ring and more. We performed this test for $N=2$, $F=4,5,6$ and $N=3\,,F=6$, the resulting indices are reported in Table 5.1. Notice that in order to perform the matching it is important to have the mirror theory parameterized as in Figure 5.2. In particular the mixing between the R-symmetry and the topological symmetries is crucial for the enhancement of the global symmetry to be seen from the Superconformal Index.

We now proceed by describing the matching of the global symmetries and of the chiral ring generators.

The global symmetry of the SQCD theory is, up to discrete factors, $SU(F)\times U(1)_\eta$. On the mirror side we can match the rank of the global symmetry as follows. In the absence of the monopole superpotential there is a $U(1)$ topological symmetry for each $U(1)$ gauge node. The monopole superpotential breaks the topological symmetries associated to the nodes in a column of the quiver to a diagonal $U(1)$. Therefore there is a $U(1)^{F-1}$ unbroken topological symmetry associated to the $F-1$ columns of the planar quiver. Furthermore there is a single $U(1)_\eta$ flavor symmetry which is preserved by the planar superpotential. In total the rank of the global symmetry if F, matching the SQCD side. The mirror-like duality implies that the $U(1)^F$ symmetry of the quiver theory is enhanced to $SU(F)\times U(1)$.

The chiral ring of the planar theory is generated by the meson built out of F diagonal bifundamentals starting from the $U(1)_\eta$ flavor node and ending at the other flavor node, this operator has trial R-charge 0 and and $U(1)_\eta$ charge 1. There are many possible paths which are all identified by the F-terms due to the planar superpotential $\mathcal{W}_{\text{planar}}$.

We expect this meson to be mapped to some monopole operator generating the SQCD chiral ring, as observed from the Index. The bare monopole operators $\mathfrak{M}^{\pm 0\ldots 0}$ are gauge variant but can be dressed to obtain gauge-invariant operators (as can be verified by following the discussion on monopole charges in Appendix B). We argue that the relevant operators are dressed with the gaugino as follows. In a monopole background, i.e. when we turn on GNO fluxes, the gauge group is effectively broken and the breaking pattern depends on the magnetic flux. In the case of fundamental monopoles the breaking pattern is $U(N)\to U(1)\times U(N-1)$. Consequently, all gauge variant operators decompose into representations of the residual group. The branching rules for irreducible representations of $U(N)\supset U(N-1)\times U(1)$ are given below:

$$\begin{aligned}\square_N &= (\square_{N-1},1)\oplus(\mathbf{1},1-N)\\ \square_N\otimes\overline{\square}_N &= (\square_{N-1}\otimes\overline{\square}_{N-1},0)\oplus(\square_{N-1},N)\oplus(\overline{\square}_{N-1},-N)\oplus(\mathbf{1},0)\end{aligned}\tag{5.5}$$

where $(\overline{\square}_K)$ $\square_K$ denotes the (anti-) fundamental representation of $U(K)$ and $(\cdot,\cdot)$ denotes the representation under the residual group $U(N-1)\times U(1)$. We see, in particular, that the gaugino of $U(N)$ decomposes into the gaugino of $U(N-1)$, a singlet, a fundamental operator and an anti-fundamental operator. In general, for a $U(N)$ gauge theory with F fundamentals and generic CS levels $(k,k+lN)$ the bare monopoles $\mathcal{M}^{\pm,0,\ldots,0}$ transform in a representaton of $U(N-1)$ with $(N-1)$-ality of $\mp l(N-1)$ and with charge $(-\frac{F}{2}\mp k\mp l)$ of $U(1)$ [78]. Dressing with $N-1$ powers of the fundamental component of the gaugino compensates these charges to:

$$Q_{U(1)}(\mathcal{M}^{\pm,0,\ldots,0})=(N-1)-\left(\frac{F}{2}\pm k\pm l\right)\,.\tag{5.6}$$

In the case studied in this section we have $k=-\frac{F}{2}+N$ and $l=-1$ and the monopole $\mathcal{M}^{+,0,\ldots,0}$ can be dressed $N-1$ times with the gaugino components in the fundamental representation of $U(N-1)$ to obtain a gauge invariant operator[6]. We expect that this operator is the generator of the chiral ring of $U(N)$ SQCD with levels $\left(-\frac{F}{2}+N,-\frac{F}{2}\right)$ and F fundamentals, and is mapped on the planar side to the meson discussed above. To support this claim, we compute the R-charge of this dressed monopole. The fundamental monopole $\mathfrak{M}^{+,0,\ldots,0}$ has R-charge:

$$R(\mathfrak{M}^{+,0,\ldots,0})=(1-R_Q)\frac{F}{2}-(N-1)=-(N-1)\,,\tag{5.7}$$

where $R_Q=1$ is the trial R-charge of the fundamental chirals. The factor $-(N-1)$ is the contribution to the R-charge coming from the zero modes of the gaugini. Dressing

[6]If instead $k=\frac{F}{2}-N$ and $l=1$ the $\mathcal{M}^{-,0,\ldots,0}$ monopole dressed $N-1$ times with the gaugino is gauge invariant.

$N-1$ times with the gaugino contributes an additional factor of $N-1$ to the R-charge:

$$R(\mathfrak{M}^{+,0,\dots,0}\lambda^{N-1}) = 0\,, \tag{5.8}$$

thus matching the gauge invariant meson of the planar theory.

It is interesting to notice that by carefully analyzing the SQCD theory, it is possible to find that there is also a monopole with high topological charge which is naturally gauge invariant, without requiring any dressing. This monopole is $\mathcal{M}^{+,+,\dots,+}$, that has topological charge N and trial R-charge 0. On the other hand in the theories discussed here, namely for $k = -F/2 + N$ and $l = -1$, we find that the dressed fundamental monopole $\mathfrak{M}^{+,0,\dots,0}\lambda^{N-1}$ is in the chiral ring as well. While this is non-trivial to establish from the SQCD point of view, the presence of this operator is manifest in the mirror dual where it is mapped to the single mesonic operator with $U(1)_\eta$ charge 1. Then the monopole $\mathcal{M}^{+,+,\dots,+}$ is the N-th power of the fundamental monopole in the chiral ring.

The view from $\mathbf{S}^3_b$ partition function

We now show how to derive the chiral-planar mirror duality for the SQCD, depicted in Figure 5.2, via an analysis of the $\mathbf{S}^3_b$ partition function [18, 19, 20] following the strategies of [8, 12].

We start from the $\mathbf{S}^3_b$ partition functions of the two theories in Figure 5.1. The partition function of the SQCD is:

$$Z(\vec{X},\eta,\tau) = \int \prod_{\alpha=1}^{N} du_\alpha \,\Delta_N(\vec{u},\tau)\, e^{2\pi i\eta\sum_{\alpha=1}^N u_\alpha} \prod_{j=1}^{F} s_b\Big(\frac{\tau}{2} \pm (u_\alpha - X_j)\Big)\,; \tag{5.9}$$

while that of its mirror is:

$$\begin{aligned}\hat{Z}(\eta,\vec{X},\tau) = e^{2\pi i\eta\sum_{j=1}^N X_j} &\int \prod_{I=1}^{F-1}\Big(\prod_{\alpha=1}^{|G^{(I)}|} dz_\alpha^{(I)} e^{2\pi i z_\alpha^{(I)}(X_{I+1}-X_I)} \Delta_{|G^{(I)}|}(\vec{z}^{(I)}, iQ-\tau)\Big)\\ &\prod_{I=1}^{F-2}\Big[\prod_{\alpha=1}^{|G^{(I)}|}\prod_{\beta=1}^{|G^{(I+1)}|} s_b\Big(\frac{iQ}{2} - \frac{\tau}{2} \pm (z_\alpha^{(I)} - z_\beta^{(I+1)})\Big)\Big]\\ &\prod_{\alpha=1}^{|G^{(N)}|} s_b\Big(\frac{iQ}{2} - \frac{\tau}{2} \pm (z_\alpha^{(N)} - \eta)\Big) \prod_{\alpha=1}^{|G^{(F-N)}|} s_b\Big(\frac{iQ}{2} - \frac{\tau}{2} \pm z_\alpha^{(F-N)}\Big)\,;\end{aligned} \tag{5.10}$$

where in the partition function of the mirror theory $\vec{z}^{(I)}$ denotes the fugacity of the I^{th} gauge node whose rank is $|G^{(I)}|$. We also use the short notation for a $\mathcal{N}=4$ vector multiplet:

$$\Delta_N(\vec{z},\tau) = \frac{1}{N!}\,\frac{\prod_{\alpha,\beta=1}^N s_b(\frac{iQ}{2} - \tau + z_\alpha - z_\beta)}{\prod_{\beta<\alpha}^N s_b(\frac{iQ}{2} \pm (z_\alpha - z_\beta))} \tag{5.11}$$

To write the $\mathbf{S}^3_b$ we followed the conventions of [80, 20], wherein a chiral field of R-charge r and charge $+1$ under an abelian flavor symmetry $U(1)_x$ contributes to the partition function as $s_b(\frac{iQ}{2}(1-r) - x)$, and $Q = b + b^{-1}$ is the squashing parameter.

Indeed, since the SQCD and its mirror are IR-dual the two partition functions must be equal:

$$Z(\vec{X},\eta,\tau) = \hat{Z}(\eta,\vec{X},\tau)\,. \tag{5.12}$$

The two partition functions match as functions of the real mass parameters $\vec{X}, \tau, \eta$. Here $\vec{X}$ is the set of real mass associated to the $SU(F)$ flavor symmetry of the SQCD and they thus satisfy $\sum_{j=1}^{F} X_j = 0$. From the mirror point of view the X_j masses parameterize the $F-1$ topological symmetries in a convenient way so that they reflect how the Cartan subgroup of $SU(F)$ maps across the duality. Also, η is the real mass associated to the $U(1)$ topological/flavor symmetry of the SQCD/mirror[7]. Finally, τ is the real mass of the $U(1)$ flavor subgroup of the $\mathcal{N}=4$ R-symmetry.

At the level of the partition function, the real mass deformation can be implemented as follows [8, 12]: we perform the $\tau \to +\infty$ limit combined with a suitable shift in gauge fugacities. Let us start the analysis considering the SQCD. The gauge shift performed is:

$$\vec{u} \to \vec{u} + \frac{\tau}{2} \tag{5.13}$$

accompanied by a shift of the FI parameter:

$$\eta \to \eta - \frac{F}{2}\tau\,. \tag{5.14}$$

Performing the limit $\tau \to +\infty$ takes us to the non-trivial Coulomb branch vacuum where the fundamental chiral fields remain massless. The limit is implemented using the asymptotic behavior of the double sine function [81]:

$$\lim_{\xi \to \pm\infty} s_b(x+\xi) \sim e^{\pm \frac{\pi i}{2}(x+\xi)^2}\,. \tag{5.15}$$

In general, if x is a global parameter then the limit produces highly oscillating phases as well as finite phases corresponding to background terms for global symmetries such as BF couplings. If instead x is a gauge parameter, the limit produces CS interactions and divergent FI parameters together with a highly oscillating phase. The redefinition of η is chosen to cancel exactly every divergent FI parameter produced by the limit. In the end, we obtain a highly oscillating phase $e^{i\Phi(\tau,\eta)}$ independent of the gauge fugacities which therefore factorizes outside the integral:

$$Z(\vec{X},\eta,\tau) \to e^{\frac{i\pi}{2}\Phi(\tau,\eta)} Z(\vec{X},\eta)\,, \tag{5.16}$$

with

$$\Phi(\tau,\eta) = -N^2\tau^2 + 2N\tau\left(\eta + \frac{iQ}{2}N\right)\,, \tag{5.17}$$

and[8]:

$$\begin{aligned} Z(\vec{X},\eta) =& e^{-\frac{N\pi i}{2}\sum_{j=1}^{F} X_j^2} \int \prod_{\alpha=1}^{N} du_\alpha \, \frac{1}{\prod_{\alpha<\beta} s_b\left(\frac{iQ}{2} \pm (u_\alpha - u_\beta)\right)} \\ & e^{2\pi i \eta \sum_\alpha u_\alpha + \pi i (\sum_{\alpha=1}^{N} u_\alpha)^2 + \pi i (\frac{F}{2} - N)\sum_\alpha u_\alpha^2} \prod_{j=1}^{F} s_b\left(X_j - u_\alpha\right)\,. \end{aligned} \tag{5.19}$$

[7] Indeed, for $F = 2N$ the parameter η becomes the real mass of the $SU(2)$ enhanced topological symmetry. This subtlety does not spoil any of the analysis done in this section

[8] Our sign convention is as follows - a $U(N)$ gauge node with FI parameter η and CS level (k_1, k_2) contributes to the $\mathbf{S}_b^3$ partition function as, with $\vec{u}$ denoting the gauge fugacity:

$$e^{2\pi i \eta \sum_j u_j}\, e^{-\pi i k_1 \sum_j u_j^2}\, e^{-\frac{\pi i (k_2 - k_1)}{N}(\sum_j u_j)^2}. \tag{5.18}$$

A mixed CS (BF) level k_{ij} contributes to the $\mathbf{S}_b^3$ partition function as: $e^{-\pi i k_{ij} u_i u_j}$. If $k_1 = k_2 \equiv \kappa$, we specify the CS level as (κ).

The phase will be crucial in determining the mirror dual theory. Indeed, as explained in [8, 12] the vacua related by duality under the deformation must have the same oscillating phase. In general, different shifts in the gauge fugacities (corresponding to moving to different points on the Coulomb branch) give different oscillating phases so there can be multiple vacua on the electric and magnetic sides and the match of the phases is a necessary criterion to map vacua related by duality.

We now move to the analysis of the partition function of the mirror theory.

We study the limit partition function of the mirror theory $\check{Z}(\eta, \vec{X}, \tau)$ for large τ and with $\eta \to -\frac{F}{2}\tau + \eta$. There is a large number of possible $z_\alpha^{(I)}$ shifts that can be performed, each corresponding to different points on the Coulomb branch and each producing (possibly) a different highly oscillating phase times the partition function of an interacting theory describing the vacuum. We look for a vacuum that reproduces the same oscillating prefactor as the SQCD. Schematically this means that:

$$Z(\vec{X}, \eta, \tau) = \check{Z}(\eta, \vec{X}, \tau) \quad \xrightarrow{\tau \to +\infty} \quad e^{\frac{i\pi}{2}\Phi(\tau)} Z(\vec{X}, \eta) = e^{\frac{i\pi}{2}\Phi(\tau)} \check{Z}(\eta, \vec{X}) \tag{5.20}$$

implying the equality of the partition function of the two $\mathcal{N} = 2$ theories living in the vacua. We found that the correct shift for the $z_\alpha^{(I)}$ is such that:

$$\begin{aligned}
& z_\alpha^{(I)} - z_{\alpha+1}^{(I)} \to z_\alpha^{(I)} - z_{\alpha+1}^{(I)} - \tau \\
& z_\alpha^{(I)} - z_\alpha^{(I+1)} \to z_\alpha^{(I)} - z_\alpha^{(I+1)} + \frac{\tau}{2}, \qquad I < F - N \\
& z_\alpha^{(I)} - z_\alpha^{(I+1)} \to z_\alpha^{(I)} - z_\alpha^{(I+1)} - \frac{\tau}{2}, \qquad I \geqslant F - N \\
& z_1^{(N)} \to z_1^{(N)} - \frac{F-1}{2}\tau \,.
\end{aligned} \tag{5.21}$$

As a result, each gauge node is fully Higgsed to a Cartan subgroup and the chiral adjoint and bifundamentals decompose accordingly. Most of these fields are massive and are integrated out, resulting in the $\mathcal{N} = 2$ quiver gauge theory shown on the r.h.s. of Figure 5.2.

Notice that when we performed the τ limit, the shift of the gauge fugacities in (5.21) is not invariant under a Weyl transformation S_k associated to any gauge group $U(k)$. Therefore if we take a different shift by reshuffling the k gauge parameters associated to a gauge symmetry we obtain the same result after the limit. Each of these $k!$ possibilities represent a different point of the Coulomb branch where the correct magnetic theory lives. One should then sum over all these vacua, all reproducing the same highly oscillating prefactor, effectively producing a $k!$ factor that cancel that at the denominator in the definition of the $\Delta_{(N)}$ measure in (5.11).

By taking the real mass limit on the partition function on the mirror side we obtain the partition function of the quiver:

$$
\begin{aligned}
\hat{Z}(\eta,\vec{X}) =& e^{2\pi i\eta(-(2N-1)\frac{iQ}{4}+\sum_{i=1}^{N}X_i)}e^{-\frac{i\pi}{2}\eta^2}\int\prod_{I=1}^{F-1}\prod_{\alpha=1}^{|G^{(I)}|}dz_\alpha^{(I)}e^{2\pi i z_\alpha^{(I)}(X_{I+1}-X_I)}e^{-i\pi(z_\alpha^{(I)})^2}\\
&\prod_{I=1}^{F-1}\prod_{\alpha}s_b\left(-\frac{iQ}{2}+z_{\alpha+1}^{(I)}-z_\alpha^{(I)}\right)e^{-i\pi z_\alpha^{(I)}z_{\alpha+1}^{(I)}}\\
&\prod_{I=1}^{N-1}\prod_{\alpha}s_b\left(\frac{iQ}{2}+z_\alpha^{(I)}-z_\alpha^{(I+1)}\right)s_b\left(\frac{iQ}{2}-z_\alpha^{(I)}+z_{\alpha+1}^{(I+1)}\right)e^{i\pi z_\alpha^{(I)}z_{\alpha+1}^{(I+1)}}\\
&\prod_{I=N}^{F-1}\prod_{\alpha}s_b\left(\frac{iQ}{2}-z_\alpha^{(I)}+z_\alpha^{(I+1)}\right)s_b\left(\frac{iQ}{2}+z_\alpha^{(I)}-z_{\alpha+1}^{(I+1)}\right)e^{i\pi z_\alpha^{(I)}z_{\alpha+1}^{(I+1)}}\\
&s_b\left(\frac{iQ}{2}-\eta+z_1^{(N)}\right)s_b\left(\frac{iQ}{2}-z_N^{(F-N)}\right)e^{i\pi z_1^{(N)}\eta}
\end{aligned}
\tag{5.22}
$$

Where the products run over all the α such that the fugacities z in the corresponding double sine functions are within the ranges described above. Notice that in the first line are encoded the background terms, that are not included in the Figure 5.2.

The view from the semiclassical EOM

We now show that the flow from $\mathcal{N}=4$ to $\mathcal{N}=2$ mirror dualities described in Subsection 5.2.2 is consistent with the equation of motion of the $\mathcal{N}=4$ theories. In particular, we show that the vacua corresponding to the shifts in gauge fugacities satisfies the F-terms and D-terms equations. Furthermore, we show that fluctuations around these vacua reproduce the EOM of the resulting $\mathcal{N}=2$ theories. This proves to be an important check of our proposal.

We start our analysis by first considering the $\mathcal{N}=4$ $U(N)$ SQCD with F flavors. Recall that, as an $\mathcal{N}=2$ supersymmetric theory, it has F (anti-)fundamental fields Q ($\tilde{Q}$) and an adjoint chiral multiplet Φ. Additionally, we also turn on a real FI parameter η. Upon diagonalizing the real scalar σ in the vector multiplet, the equations for the F- and D-terms are as follows (note that each field represents a matrix)[9] [46]:

$$
\begin{aligned}
&\sum_{i=1}^{F}Q^i_\alpha\tilde{Q}^\beta_i=0\,,\qquad \sum_{\alpha=1}^{N}Q^i_\alpha\Phi^\alpha_\beta=0\,,\qquad \sum_{\alpha=1}^{N}\Phi^\beta_\alpha\tilde{Q}^\alpha_i=0\,,\\
&(\sigma_\alpha-X_i-\tfrac{\tau}{2})Q^i_\alpha=0\,,\qquad (X_i-\sigma_\alpha-\tfrac{\tau}{2})\tilde{Q}^\alpha_i=0\,,\qquad [\sigma,\Phi]^\beta_\alpha+\tau\Phi^\beta_\alpha=0\,,\\
&[\Phi,\Phi^\dagger]^\beta_\alpha+\sum_{i=1}^{F}(Q^{i,\dagger}_\alpha Q^i_\beta-\tilde{Q}^\alpha_i\tilde{Q}^{\beta,\dagger}_i)=\frac{\delta^\beta_\alpha}{2\pi}\Big[-\eta+\frac{1}{2}\sum_{i=1}^{F}\big(|\sigma_\alpha-X_i-\frac{\tau}{2}|-|X_i-\sigma_\alpha-\frac{\tau}{2}|\big)+\\
&+\frac{1}{2}\sum_{\gamma=1}^{N}(|\sigma_\alpha-\sigma_\gamma+\tau|-|\sigma_\gamma-\sigma_\alpha+\tau|)\Big]\,.
\end{aligned}
\tag{5.23}
$$

Where the values of $Q^i_\alpha,\tilde{Q}^\alpha_i,\Phi^\alpha_\beta$ are the Vacuum Expectation Values (VEVs) of the (scalar component of) chiral fields of the theory. σ_α is the VEV for the real scalar in the $\mathcal{N}=2$ vector multiplet, noting that in principle it is a matrix in the adjoint

[9] N.B.: Latin letters $i,j=1,\dots,F$ run over flavor indices, and Greek letters $\alpha,\beta=1,\dots,N$ run over gauge indices. The sum over indices is never implied.

representation of the $U(N)$ gauge symmetry but we can always perform a gauge transformation such that it is diagonal and we employ the single index notation for the diagonal components $\sigma_\alpha^\beta = \delta_\alpha^\beta \sigma_\alpha$. Lastly, the real parameters X_i, τ, η are the real masses associated to the $SU(F)/U(1)_\tau/U(1)_\eta$ global symmetries, i.e. VEVs of the real scalars in background $\mathcal{N} = 2$ vector multiplets of the global symmmetries. Since X_i parameterizes a $SU(F)$ symmetry they satisfy $\sum_{i=1}^F X_i = 0$.
The set of equations in (5.23) represent the following: In the first line the F-term constraint imposed by the $\mathcal{N} = 4$ superpotential $\mathcal{W} = \sum_{i=1}^F \sum_{\alpha,\beta=1}^N Q_\alpha^i \Phi_\beta^\alpha \tilde{Q}_i^\beta$; in the second line the mass term for the chiral fields; in the third and last line the 1-loop corrected D-term equation.

We now consider a mass deformation for the axial symmetry $U(1)_\tau$, which corresponds to taking a positive large value of τ in (5.23). While it would be interesting to study the most general solution to the vacuum equations under this deformation, we do not attempt to solve this problem here. Instead, we look for a vacuum where the fundamental fields remain massless. In order to do so we consider a large τ real mass, and set the real masses X_i to zero.

The effective mass of the field Q_α^i is given by $(\sigma_\alpha - \frac{\tau}{2})$, therefore we set $\sigma_\alpha = \frac{\tau}{2}$. The first line of equations in (5.23) then requires that $\tilde{Q}_i^\alpha = \Phi_\beta^\alpha = 0$. Also, although the condition $\sigma_\alpha = \frac{\tau}{2}$ is enough to solve the first equation in the first line, we also require Q_α^i to be zero, in order for these fields to not acquire any VEV and to therefore be light degrees of freedom of the theory in the vacuum. The D-term equation therefore reads:

$$0 = \frac{1}{2\pi}\left[-\eta - \frac{F}{2}\tau \right] \longrightarrow \eta = -\frac{F}{2}\tau \tag{5.24}$$

Therefore, we are performing a real mass deformation for both the $U(1)_\tau$ and $U(1)_\eta$ symmetries, keeping fixed the combination $\frac{F}{2}U(1)_\tau + U(1)_\eta$. This corresponds to the deformation considered in the previous subsection.

Now to conclude, we would like to study the theory that lives in the vacuum solution that we found. To do so, we study small fluctuations around the solution:

$$\begin{aligned} Q_\alpha^i &= \check{Q}_\alpha^i \\ X_i &= \check{X}_i \\ \sigma_\alpha &= \tfrac{\tau}{2} + \check{\sigma}_\alpha \\ \eta &= -\frac{F}{2}\tau + \check{\eta} \end{aligned} \tag{5.25}$$

Notice that the equations in the second line of (5.23) impose that the $\tilde{Q}$ and Φ fields do not fluctuate, as expected since they do not describe light degrees of freedom in the chosen vacuum.
Expanding the equations in the light modes we get:

$$\begin{aligned} &(\check{\sigma}_\alpha - \check{X}_i)\check{Q}_\alpha^i = 0 \\ &\sum_{i=1}^F Q_\alpha^{i,\dagger} Q_i^\beta = \frac{\delta_{\alpha\beta}}{2\pi}\Big[-\underbrace{\check{\eta}}_{=\eta_{eff}} + \underbrace{\Big(-\frac{F}{2} + N\Big)\check{\sigma}_\alpha - \sum_{\gamma=1}^N \check{\sigma}_\gamma}_{\text{CS interactions}} + \frac{1}{2}\sum_{i=1}^F |\check{\sigma}_\alpha - \check{X}_i| \Big] \end{aligned} \tag{5.26}$$

Which can be interpreted as the EOM of a $U(N)$ gauge theory with F massless fundamental chirals and with CS level $(-\frac{F}{2}+N, -\frac{F}{2})$ by comparing with the generic formula:

$$\sum_{i=1}^{F} Q_\alpha^{i,\dagger} Q_i^\beta = \frac{\delta_{\alpha\beta}}{2\pi}\Big[- \eta_{\text{eff}} + k_{\text{eff}}\sigma_\alpha + l_{\text{eff}} \sum_{\gamma=1}^{N} \sigma_\gamma + \frac{1}{2}\sum_{i=1}^{F} |\sigma_\alpha - \check{X}_i| \Big] \tag{5.27}$$

where the CS level is $(k, k+lN)$.

We now turn to the mirror of the $\mathcal{N}=4$ $U(N)$ SQCD theory in Figure 5.1. Studying the vacuum of this theory using the equations of motion is a highly non-trivial task and is not attempted here. The difficulty of this task lies clearly in the high number of equations that must be analysed. Another issue is that for the same problem, i.e. same values of the real mass parameter in input, there can be multiple solutions and therefore multiple possible vacua.

Instead, we employ mirror duality to know how the real mass parameters are mapped across the duality so that we can fix correctly the problem we attempt to study. The system of equations for the mirror theory are[10]:

$$\begin{aligned}
&(\sigma_i^{(a)} - \sigma_j^{(a+1)} + \tfrac{\tau}{2}) b_{i,j}^{(a,a+1)} = 0\,; \qquad (-\sigma_i^{(a)} + \sigma_j^{(a+1)} + \tfrac{\tau}{2}) \tilde{b}_{i,j}^{(a,a+1)} = 0 \\
&(\sigma_i^{(N)} - \eta + \tfrac{\tau}{2}) q_i = 0\,; \qquad (-\sigma_i^{(N)} + \eta + \tfrac{\tau}{2}) \tilde{q}_i = 0 \\
&(\sigma_i^{(F-N)} + \tfrac{\tau}{2}) p_i = 0\,; \qquad (-\sigma_i^{(F-N)} + \tfrac{\tau}{2}) \tilde{p}_i = 0 \\
&[\sigma^{(a)}, \Phi^{(a)}]_i^j - \tau \Phi^{(a)}{}_i^j = 0
\end{aligned} \tag{5.28}$$

All the σ's are taken to be diagonal after an appropriate gauge fixing. Also, we have the set of F-terms equations, that we do not attempt to report here since they will not be important for the following discussion. And lastly we have the D-terms equations that are:

$$\begin{aligned}
&[\Phi^{(a)}, \Phi^{(a),\dagger}]_{i,j} + \ldots (\text{sum over } b^{(a,a+1)}\text{'s and eventual } q,p) = \frac{\delta_{i,j}}{2\pi} F_i^{(a)}\,, \\
&F_i^{(a)} = X_{a+1} - X_a + \sum_{k=1}^{a+1} \big(|\sigma_i^{(a)} - \sigma_k^{(a+1)} + \frac{\tau}{2}| - | - \sigma_i^{(a)} + \sigma_k^{(a+1)} + \frac{\tau}{2}| \big) + \\
&+ \sum_{k=1}^{a-1} \big(- |\sigma_k^{(a-1)} - \sigma_i^{(a)} + \frac{\tau}{2}| + | - \sigma_k^{(a-1)} + \sigma_i^{(a)} + \frac{\tau}{2}| \big) + \\
&+ \sum_{k=1}^{a} \big(|\sigma_i^{(a)} - \sigma_k^{(a)} - \tau| - | - \sigma_i^{(a)} + \sigma_k^{(a)} - \tau| \big)\,.
\end{aligned} \tag{5.29}$$

The solution that we are looking for have non-zero positive value of τ, the value of η is fixed to $-\frac{F}{2}\tau$ and $X_a = 0$. Notice that the value of the external parameters is fixed by the analysis of the SQCD. We observe that there is a solution for the unknown variables

[10]The index between the round brackets labels the number of the gauge node. The latin indices are used for gauge symmetries.

$\sigma_i^{(a)}$ obtained by taking:

$$\begin{aligned}
&\sigma_\alpha^{(a)} - \sigma_{\alpha+1}^{(a)} = -\tau \\
&\sigma_\alpha^{(a)} - \sigma_\alpha^{(a+1)} = +\frac{\tau}{2}, \qquad a < F-N \\
&\sigma_\alpha^{(a)} - \sigma_\alpha^{(a+1)} = -\frac{\tau}{2}, \qquad a \geqslant F-N \\
&\sigma_1^{(N)} = -\frac{F-1}{2}\tau
\end{aligned} \tag{5.30}$$

This corresponds to the vacuum considered in the previous section (5.21), where the VEVs for σ's indicate the pattern of Higgsing. The light DOFs can be found by computing the masses of the various fields from the expression (5.28), which gives the same answer as the analysis of the $\mathbf{S}_b^3$ partition function. One can also try to perform a perturbative expansion of the equation of motion, as we did for the SQCD, to find that this analysis reproduces correctly all the properties of the chiral-planar mirror duality of Figure 5.2, such as the self and mixed-CS interactions. The planar superpotential are the F-term relations surviving in the vacuum. On the other hand, the monopole superpotential can not be inferred easily from the analysis of the EOM.

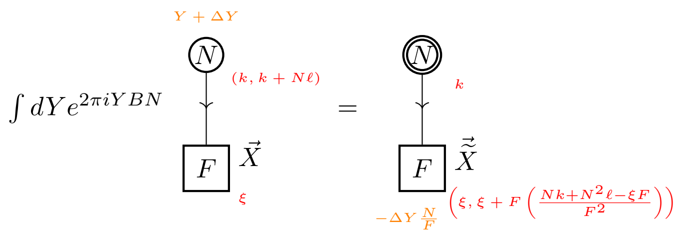
To summarize, we have shown that the vacua corresponding to the real mass flows considered in the previous Section satisfy the EOM of the $\mathcal{N}=4$ theory. This, combined with the matching of the $\mathbf{S}_b^3$ partition function and the Superconformal Index discussed above provide a strong check of the chiral-planar mirror duality for $\mathcal{N}=2$ SQCD.

As a technical aside we would like to comment on the similarities and differences between the analysis of the $\mathbf{S}_b^3$ partition function and the EOM. It can be checked that, under a real mass deformation τ, the requirement that the $\mathbf{S}_b^3$ partition function factorizes into a divergent phase and a finite integral is equivalent to solving the D-term equations up to orders $\mathcal{O}(\tau^0)$. This is due to the fact that large shifts in the gauge fugacities appearing in the $\mathbf{S}_b^3$ partition function correspond to moving on the Coulomb branch far away from the origin. On the other hand, the $\mathbf{S}_b^3$ partition function is not sensitive to motion on the Higgs branch, therefore solving the other EOM provide an additional check of the deformation considered above. When there are multiple vacua the analysis of the EOM does not provide a general prescription to understand the mapping of vacua across the duality. The divergent phase coming from the $\mathbf{S}_b^3$ partition function is sensitive to the matter that becomes massive at a given vacuum, and therefore can be leveraged to constrain the mapping of vacua across the duality.

As an example, both the $\mathcal{N}=4$ SQCD and its mirror have a *chiral-like* and a *planar-like* vacuum under the axial mass deformation. In [37], where the $\mathcal{N}=2^*$ framework was introduced, the axial mass deformation appeared to lead to *chiral-chiral* dualities. In the previous Section we showed that the *chiral-like* vacuum of one theory can be mapped to the *planar-like* vacuum of the mirror theory because the divergent prefactors in the $\mathbf{S}_b^3$ partition function cancel between the two. Such a cancellation does not happen for pairs of *chiral-like* vacua or *planar-like* vacua, showing that such a mapping of vacua is not consistent. Therefore, our results represent an example where refined precision tools such as the computation of the $\mathbf{S}_b^3$ partition function can be leveraged to understand qualitative properties of dualities between QFTs.

5.2.3 A Planar Dual for $SU(N)_k$ and $U(N)_{(k,k+\ell N)}$ SQCD

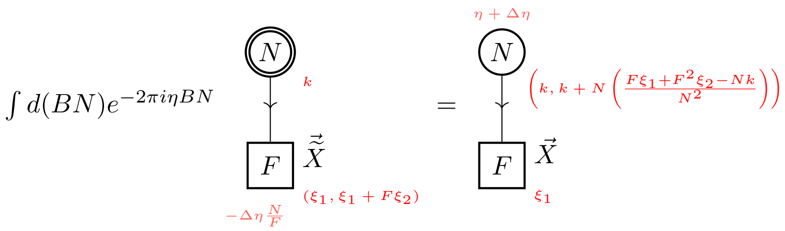
In this paper, we primarily consider dualities involving $U(N)_{(k,k-N)}$ gauge theories (that is, theories where the abelian CS level ℓ is -1). One can obtain an $SU(N)_k$ CS theory by introducing a background BF term between the topological symmetry $U(1)_Y$ and a new $U(1)_B$ and then gauging the topological symmetry. This operation is referred to as *ungauging* of the diagonal $U(1) \subset U(N)$. In the notation used throughout this paper, this ungauging operation consists in



$$, \qquad \widetilde{X}_a = X_a + B \tag{5.31}$$

wherein the double circle denotes a $SU(N)$ gauge symmetry. In the $U(N)$ SQCD the flavor symmetry is $SU(F)$ and thus $\sum_{j=1}^{F} X_j = 0$ and ξ is a possible background CS for the flavor symmetry. In the $SU(N)$ theory $\sum_{j=1}^{F} \tilde{X}_j = FB$, where $U(1)_B$ is a baryonic symmetry.

Conversely, starting from SQCD with gauge group $SU(N)$ one can gauge the baryonic symmetry $U(1)_B$ to obtain a unitary gauge group:



$$, \qquad \widetilde{X}_a = X_a + B \tag{5.32}$$

Where ξ_1 and ξ_2 are possible background CS levels for the $U(F)$ global symmetry of the SQCD on the r.h.s. and $-\Delta\eta\frac{N}{F}$ is a background FI term. Consistently, one can check that by performing both transformations (5.31) and (5.32), a theory is mapped back to itself.

We may apply the ungauging operation (5.31) to the chiral-planar mirror duality of SQCD in Figure 5.2. On the electric side we obtain the $SU(N)_{-F/2+N}$ SQCD. On the mirror side, the topological symmetry corresponds to a flavor symmetry acting on the bottom flavor node. Therefore, the operation described above corresponds to gauging the flavor node itself. In order to correctly perform this gauging it is important to keep track of possible background terms in the duality that are not provided in Figure 5.2, but can be read from the partition function given in (5.22). The resulting chiral-planar duality is given in Figure 5.3 and was first presented in [43][11].

By gauging the baryonic symmetry in Figure 5.3, using the strategy in (5.32), we revert to the duality for $U(N)_{(k,k+N)}$ SQCD. At this stage, we may also introduce a CS term at level $\Delta\ell$ for the baryonic symmetry prior to gauging, resulting in

[11] Notice that the new gauge node is a $U(1)_{1/2}$ with one chiral and thus it can be confined. This results in a chiral that is not charged under any gauge symmetry and it is coupled to the theory as a flipper of a gauge invariant monopole. This does not trigger a sequential confinement except for the case of $N = 2$, see Subsubsection 2.3.1 of [44].

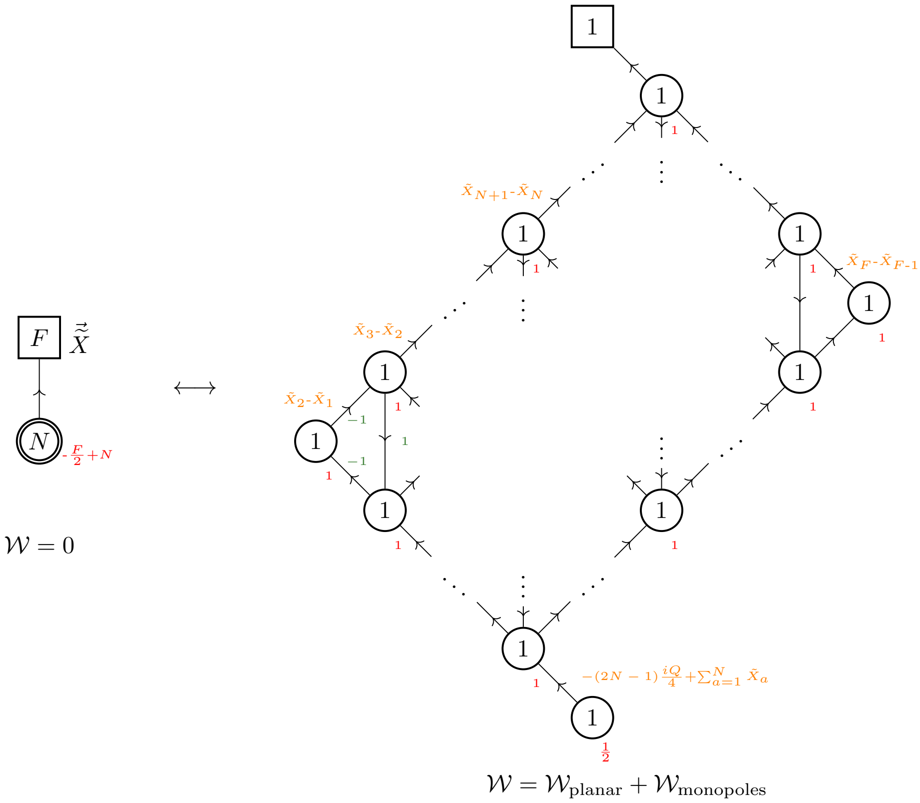
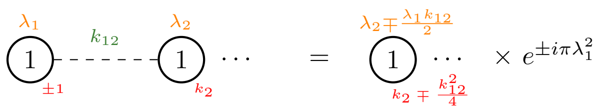


Figure 5.3: The $\mathcal{N}=2$ planar mirror dual of $SU(N)_{-\frac{F}{2}+N}$ SQCD with F fundamentl chiral multiplets.

a $U(N)_{(k,k+N(1+\Delta\ell))}$ gauge theory. The gauging/ungauging operations together with background CS terms correspond to Witten's $SL(2,\mathbb{Z})$ action [82] applied to the topological symmetry of $U(N)$. On the mirror side, this procedure introduces a new $U(1)_{\Delta_\ell}$ gauge node coupled to the rest of the quiver by a BF term, the result is the duality in Figure 5.4.

For $\Delta\ell=0,\pm1$ the quiver can be further simplified by explicitly integrating over the additional gauge node. For $\Delta\ell=0$ the integration results in a delta function which further fixes the gauge field of the node on its right, which becomes a flavor node and we recover the original mirror for $U(N)_{-F/2+N,-F/2}$. For $\Delta\ell=\pm1$ the additional gauge node is a $U(1)_{\pm1}$ sector, which is almost trivial and its path integral can be performed exactly as [31, 82]:



(5.33)

In particular, for $\Delta\ell=1$ we obtain a planar abelian dual for $U(N)_{(k,k)}$. Integrating out the additional node as above shifts the CS level of the node on its right by -1. Then the bottom-most node is a $U(1)_{-1/2}$ gauge node with one fundamental, which can be locally dualized to a chiral field, resulting in the duality shown in Figure 5.5.

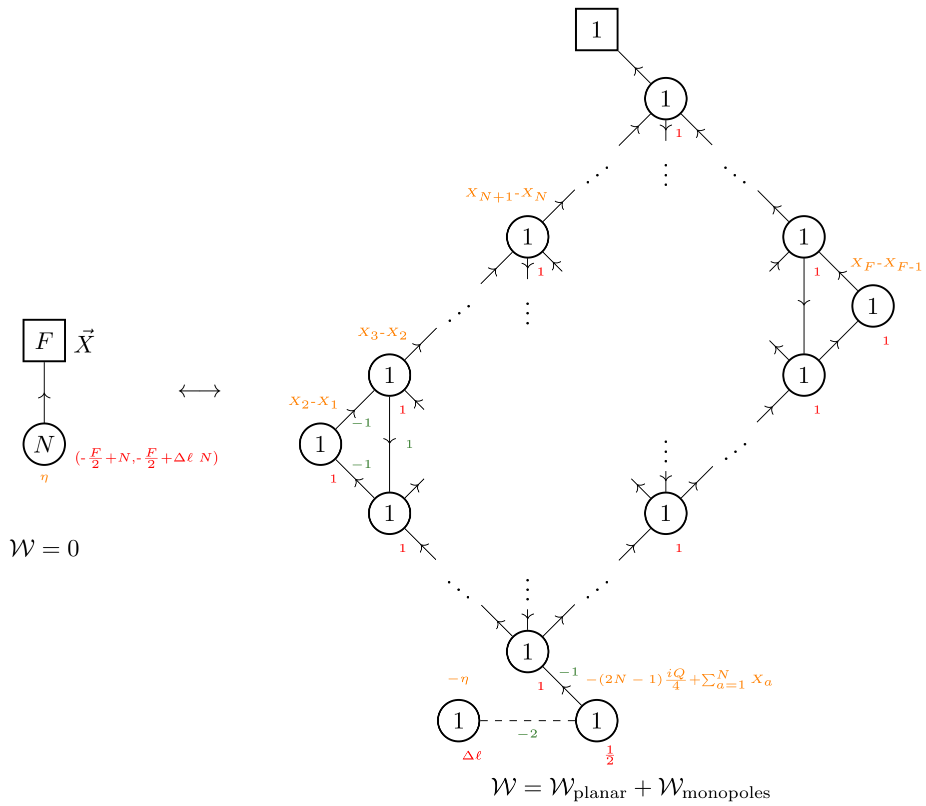


Figure 5.4: The $\mathcal{N}=2$ planar mirror dual of $U(N)_{(-\frac{F}{2}+N,-\frac{F}{2}+N\Delta\ell)}$ SQCD with F fundamental chiral multiplets.

5.3 A Chiral-Planar duality web

In this section we discuss an interesting web of chiral-planar dualities which originates from the $\mathcal{N}=4$ $FT[U(N)]$ theory[12].

In Subsection 5.3.1 starting from the mirror self-duality of the $FT[U(N)]$ theory [57] and performing a real mass deformation, we obtain an $\mathcal{N}=2$ duality relating a linear, chiral quiver gauge theory to a planar abelian theory.

The $FT[U(N)]$ theory is also know to have another self-dual frame (modulo flips), the so called flip-flip dual frame [57]. In Subsection 5.3.2 we start from this flip-flip duality and performing real mass deformations, we obtain two $\mathcal{N}=2$ dualities: a chiral-chiral duality, relating two chiral quivers and a planar-planar duality, relating two planar quivers.

Notice that starting from the flip-flip duality we don't expect to find a chiral-planar duality, since this is a peculiarity of mirror dualities exchanging flavor and topological symmetries. As shown in [57] the flip-flip duality is not a mirror duality but is more akin to an Aharony duality, and indeed it is possible to prove it by iterative local applications of Aharony duality.

One can look at Appendix C.1 for a review of the $FT[U(N)]$ theory, in what follows we will just report some properties for future references.

The $FT[U(N)]$ SCFT admits a UV completion given by a quiver theory that we

[12]This theory only differs from the $T[SU(N)]$ theory introduced in [28] by the presence of a singlet field in the adjoint of the $SU(N)$ symmetry flipping the Higgs branch moment map.

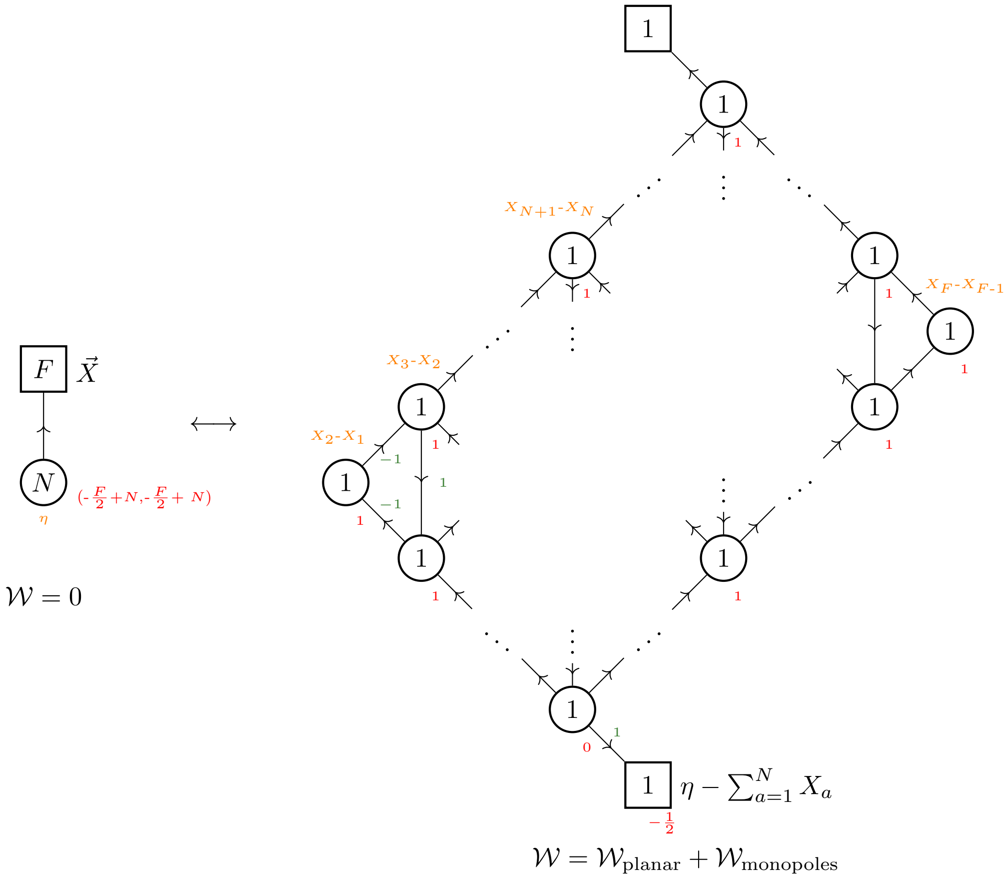
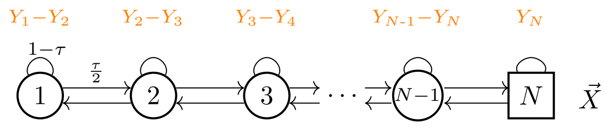


Figure 5.5: The $\mathcal{N}=2$ planar mirror dual of $U(N)_{(-F/2+N,-F/2+N)}$ SQCD with F fundamental chiral multiplets. In the quiver we omitted the mixing of the topological symmetries with the R-symmetry, in particular the bottom-most gauge node has FI parameter $X_{N+1}-X_N-\frac{iQ}{8}$. The bottom chiral fundamental has trial R-charge $-N+1$.

parameterize as follows:



$$\mathcal{W}=\mathcal{W}_{\mathcal{N}=4} \tag{5.34}$$

The UV global symmetry is $SU(N)_{\vec{X}}\times\prod_{j=1}^{N-1}U(1)_{Y_j-Y_{j+1}}\times U(1)_\tau$, which enhances to $SU(N)_{\vec{X}}\times SU(N)_{\vec{Y}}\times U(1)_\tau$ in the IR.

The $FT[U(N)]$ admits a second UV completion, as shown in Figure 5.6. In each completion only one of the two global $SU(N)$ symmetries is realized manifestly as the flavor symmetry rotating the last bifundamental. The second $SU(N)$ global symmetry instead enhances from the $N-1$ topological symmetries due to all the nodes being balanced. The two UV completions are related by mirror duality, in this case a self-duality.

The $\mathbf{S}_b^3$ partition function of the first UV completion theory is defined recursively

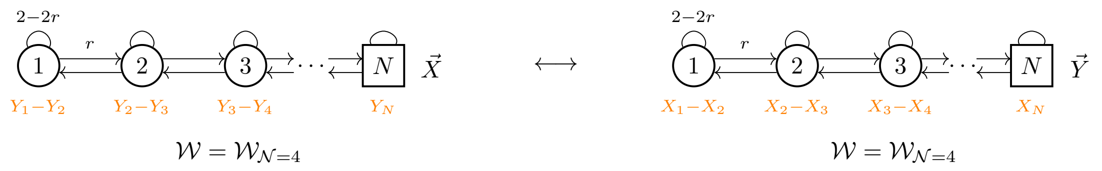


Figure 5.6: The $FT[U(N)]$ quiver gauge theory is self-mirror under the exchange of the manifest and topological symmetries. We also indicate the assignment of trial R-charges for the bifundamental and adjoint chiral fields. Notice that the bifundamental fields have trial R-charge r and adjoint chiral fields have trial R-charge $2-2r$ on both sides of the duality. As a consequence, the two moment maps $A_{\vec{X}}, A_{\vec{Y}}$ have both trial R-charge $2-r$.

as follows:

$$\begin{aligned} Z_{FT[U(N)],I}(\vec{X},\vec{Y},\tau) := e^{2\pi i Y_N \sum_j^N X_j} \prod_{j,k=1}^{N} s_b\left(\tau - \frac{iQ}{2}(1-2r) + X_j - X_k\right) \\ \int \frac{\prod_{\alpha=1}^{N-1} du_\alpha \, e^{-2\pi i Y_N u_\alpha}}{(N-1)! \prod_{\alpha<\beta} s_b\left(\frac{iQ}{2} \pm (u_\alpha - u_\beta)\right)} \\ s_b\left(\frac{iQ}{2}(1-r) - \frac{\tau}{2} \pm (u_\alpha - X_j)\right) \\ Z_{FT[U(N-1)],I}(\vec{u}, \{Y_i\}_{i=1}^{N-1}, \tau)\,. \end{aligned} \tag{5.35}$$

with

$$Z_{FT[U(1)],I}(X,Y,\tau) := e^{2\pi i XY} \; s_b\left(\tau - \frac{iQ}{2}(1-2r)\right). \tag{5.36}$$

The partition function of the second UV completion can be obtained by simply swapping $\vec{X} \leftrightarrow \vec{Y}$.

$$Z_{FT[U(N)],II}(\vec{X},\vec{Y},\tau) = Z_{FT[U(N)],I}(\vec{Y},\vec{X},\tau)\,, \tag{5.37}$$

Notice in the first UV completion the first slot $FT[U(N)](\cdot\ ,\cdot;\tau)$ is for the manifest $U(N)$, the second slot for emergent $U(N)$. On the other hand in the second UV completion the second slot is for the manifest and the first one for the emergent $U(N)$.

Since both the UV completions describe the same SCFT, their partition functions are equal and we can define:

$$Z_{FT[U(N)],I}(\vec{X},\vec{Y},\tau) = Z_{FT[U(N)],II}(\vec{Y},\vec{X},\tau) := Z_{FT[U(N)]}(\vec{X},\vec{Y},\tau)\,, \tag{5.38}$$

Furthermore, the partition function satisfies:

$$Z_{FT[U(N)]}(\vec{X},\vec{Y},\tau) = Z_{FT[U(N)]}(-\vec{X},-\vec{Y},\tau) \tag{5.39}$$

as can be seen through a redefinition of the gauge fields.

For the rest of the paper, we consider the $FT[U(N)]$ theory with a $U(N)_{\vec{X}} \times U(N)_{\vec{Y}}$ global symmetry instead (in practice, this corresponds to relaxing the constraints $\sum_i X_i = \sum_i Y_i = 0$ on the fugacities in the $\mathbf{S}_b^3$ partition function).

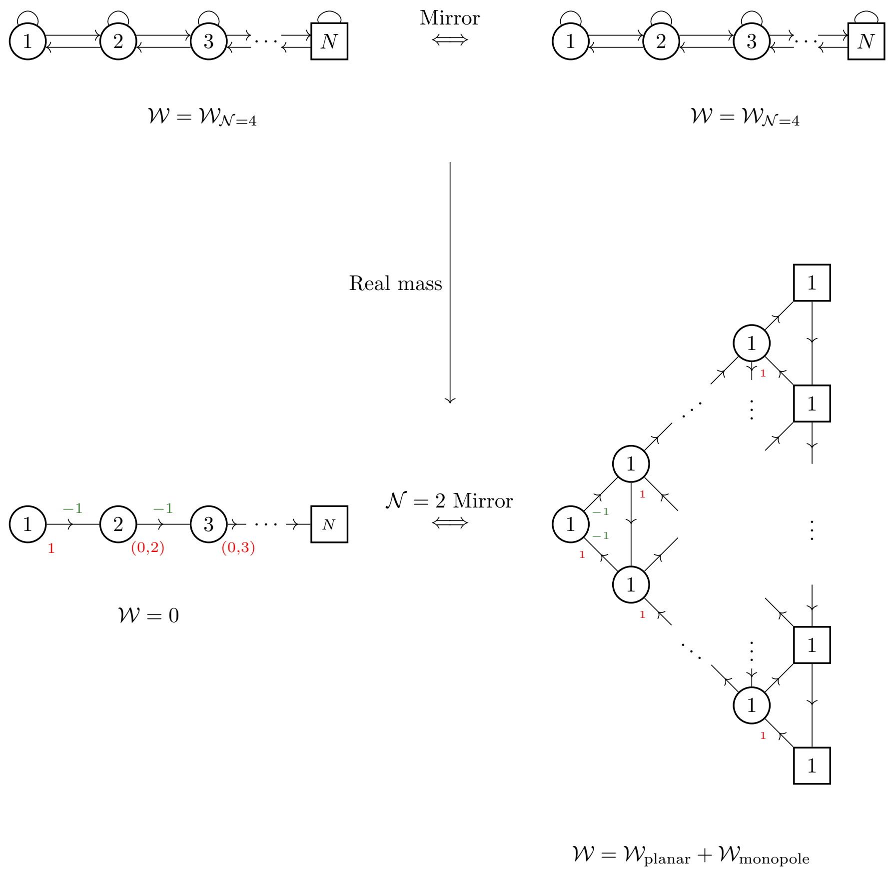


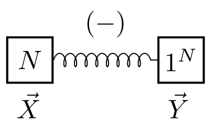
Figure 5.7: On the top row the mirror self-duality relates the two UV completions of the $FT[U(N)]$ theory. The real mass deformation yield a new $\mathcal{N} = 2$ mirror duality relating the two UV completions of the $G[U(N)]$ theory depicted in the second row.

5.3.1 A Chiral-Planar mirror dual

Starting from the self-mirror pair relating the two UV completion of the $FT[U(N)]$ SCFT in Figure 5.6, we will now consider a real mass deformation to produce a new duality relating a chiral quiver to a planar quiver which is depicted in Figure 5.7.

As in the previous section, if on the electric side we implement the real mass deformation landing on a chiral non-abelian quiver theory, on the mirror dual side, the matching vacuum we flow to, requires to move to a point on the Coulomb where all the gauge symmetries are broken to their maximal torus subgroup. The chiral and the planar quivers on the bottom of Figure 5.7 are IR dual and can be regarded as two UV completions of a new SCFT which we name $G[U(N)]$ with $U(1)^{N-1} \times SU(N)$ global symmetry, which we denote in compact form as:

$$G[\vec{X},\vec{Y}] =$$



(5.40)

We stress that in our notation the first argument of the $G[\cdot\,;\cdot]$ theories corresponds to fugacities for an $SU(N)$ global symmetry, while the second argument corresponds to a $U(1)^{N-1}$ global symmetry. Also, the label "$(-)$" inherited from the $FT[U(N)]$ theory is put for later convenience.

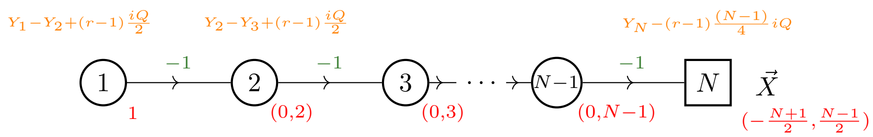
We now describe in detail this deformation beginning from the first UV completion of the $FT[U(N)]$, the electric theory, where the $U(N)_X$ is manifest. We consider the real mass deformation defined at the level of the fugacities by

$$\begin{cases} Y_J \to Y_J + \frac{\tau}{2}(2J - N - 1), \qquad J = 1, \dots, N \\ \vec{X} \to \vec{X} + \frac{\tau}{2} \end{cases}, \qquad \tau \to +\infty \tag{5.41}$$

In addition to this we also perform the following shifts in gauge fugacities:

$$\vec{u}^{(J)} \to \vec{u}^{(J)} - \frac{\tau}{2}(N - 1 - J). \tag{5.42}$$

Here $\vec{u}^{(J)}$ are the fugacities of the J-th gauge group, starting from the left. Integrating out the massive fields we obtain the following $\mathcal{N} = 2$ quiver gauge theory:



$$\mathcal{W} = 0 \tag{5.43}$$

All the chirals in the theory have trial R-charge r which should be possible, in principle, to fix by performing F-extremization to determine the superconformal R-charge. However, we will not perform F-extremization in this paper and keep r to be generic. Notice also that the FI parameters depend on the value of r. This is the first UV completion of the $G[U(N)]$ theory with UV symmetry $U(1)_Y^{N-1} \times SU(N)_{\vec{X}}$.

The $\mathbf{S}_b^3$ partition function in the $\tau \to +\infty$ limit is given by:

$$\lim_{\tau \to +\infty} Z_{FT[U(N)],I}(\vec{X}, \vec{Y}, \tau) = \lim_{\tau \to +\infty} e^{\frac{\pi i}{2} H^{\text{chiral}}(\vec{X}, \vec{Y}, \tau)} Z_{G[U(N)],I}(\vec{X}, \vec{Y}) \tag{5.44}$$

where $Z_{G[U(N)],I}(\vec{X}, \vec{Y})$ is the partition function of the first UV completion of the $G[U(N)]$ theory in (5.43) and:

$$H^{\text{chiral}}(\vec{X}, \vec{Y}, \tau) = H^{\text{chiral}}_{\text{div}}(\vec{X}, \vec{Y}, \tau) + H^{\text{chiral}}_{\text{res}} \tag{5.45}$$

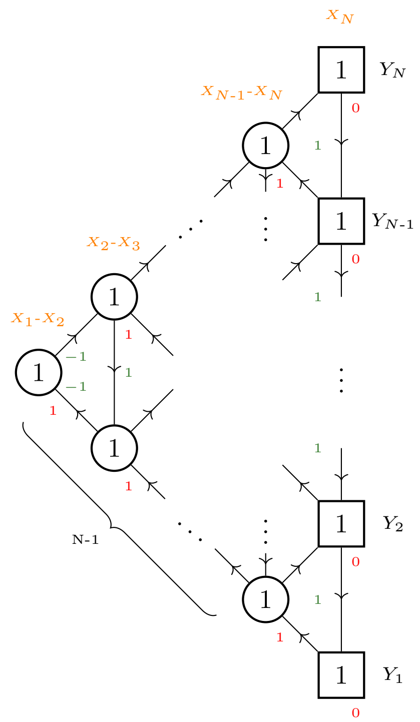
$$\begin{aligned} H^{\text{chiral}}_{\text{div}}(\vec{X}, \vec{Y}, \tau) =& \frac{\tau^2}{4}\left(\frac{4}{3}N^3 + 2N^2 + \frac{2}{3}N\right) \\ &+ \frac{\tau}{2}\left(8\sum_{J=1}^{N} J Y_J - 4\sum_{J=1}^{N} Y_J - iQ(N + N^2) + iQr\left(\frac{2}{3}N^3 + 2N^2 + \frac{4}{3}N\right)\right), \end{aligned} \tag{5.46}$$

and $H^{\text{chiral}}_{\text{res}}$ is a finite term independent on the global parameters $\vec{X}$ and $\vec{Y}$, that only depends on the squashing parameter Q.

We now consider the second UV completion of the $FT[U(N)]$ theory, the mirror theory. As in the case of the planar mirror of the chiral SQCD discussed in the previous section, here in addition to the real mass (5.41) to flow to the vacumm dual to the chiral quiver we also need to perform the following shifts in gauge fugacity:

$$\omega_\alpha^{(J)} \to \omega_\alpha^{(J)} + \tau\left(\alpha - \frac{J+1}{2}\right) \tag{5.47}$$

and take the $\tau \to +\infty$ limit. Here $\omega_\alpha^{(J)}$ is the α-th fugacity of the J-th gauge group, starting from the left. The resulting $\mathcal{N} = 2$x planar quiver theory in Figure 5.8 is IR dual to the chiral theory in Figure (5.43), and provides the second UV completion of the $G[U(N)]$ SCFT.



$$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$$

Figure 5.8: The $\mathcal{N} = 2$ planar mirror dual of the $G[U(N)]$ theory. CS levels are indicated in red and the level of mixed CS interactions is indicated in green. On top of each column we indicate in orange the FIs of all the gauge nodes of the column, up to shifts that are discussed in the text.

A few comments regarding the superpotential of the quiver in Figure 5.8 are in order:

- There is a cubic superpotential term for every closed triangle. This superpotential originates from the cubic $\mathcal{N} = 4$ interactions that are not broken by the real mass deformation. We denote these superpotential terms as $\mathcal{W}_{\text{planar}}$. The assignment of trial R-charges is r for each diagonal chiral field and $2-r$ for each vertical one. Indeed this assigment of R-charges is compatible with the planar superpotential. Notice also that the FI parameters depend on the value of r, encoding a mixing between the topological and R-symemtries as described in Subsection 5.2.2.
- Due to adjoint Higgsing of the gauge nodes, there are monopole terms in the superpotential [79]. There is a linear monopole superpotential for each monopole with charges $+1/-1$ under the upper/lower gauge nodes connected by a vertical line. We denote these linear monopole superpotential terms as $\mathcal{W}_{\text{monopole}}$. These are essential for the topological symmetry to correctly enhance to $SU(N)$, as we will comment later.

- In the planar theory there are self and mixed CS interactions. They are such that each gauge node has CS level $+1$ and each pair of gauge nodes connected by a diagonal/vertical line have a mixed CS interaction with level $-1/+1$. All these interactions arise from the integrated fermionic modes that became massive upon real mass deformation.

- The are shifts in the FIs associated to the mixing of the topological symmetry of the various nodes and the R-symmetry. The FIs of the gauge nodes in the J-th colum from the right are:

$$\text{FIs: } X_J - X_{J+1} - \frac{iQr}{4},\ X_J - X_{J+1},\ \ldots,\ X_J - X_{J+1} + \frac{iQr}{4} \tag{5.48}$$

from top to bottom, while the background FIs for the j-th flavor node associated to Y_j are:

$$\text{FIs: } X_N + \frac{iQr}{2}(2j - N - 1) + \delta_{j,1}\frac{iQ}{2} - \delta_{j,N}\frac{iQ}{2} \tag{5.49}$$

where $j = 1$ corresponds to the top flavor node in Figure 5.8 and $j = N$ corresponds to the bottom node. In Figure 5.8 we suppressed these shifts and only report the contribution to the FIs that depends on X_i.

The $\mathbf{S}_b^3$ partition function in the $\tau \to +\infty$ limit is given by:

$$\lim_{\tau\to\infty} Z_{FT[U(N)],II}(\vec{X}, \vec{Y}, \tau) = \lim_{\tau\to\infty} e^{\frac{\pi i}{2} H^{\text{planar}}(\vec{X},\vec{Y},\tau)} Z_{G[U(N)],II}(\vec{X}, \vec{Y}) \tag{5.50}$$

where $Z_{G[U(N)],II}(\vec{X}, \vec{Y})$ is the partition function of the second UV completion of the $G[U(N)]$ theory in (5.8) and:

$$H^{\text{planar}}(\vec{X}, \vec{Y}, \tau) = H^{\text{planar}}_{\text{div}}(\vec{X}, \vec{Y}, \tau) + H^{\text{planar}}_{\text{res}} \tag{5.51}$$

Here $H^{\text{planar}}_{\text{res}}$ is a constant phase independent of the global parameters $\vec{X}$ and $\vec{Y}$, that only depends on the squashing parameter Q. $H^{\text{planar}}_{\text{div}}$ is a divergent phase that is equal to $H^{\text{chiral}}_{\text{div}}$ and therefore it cancels against the chiral dual theory, providing a strong check that the theories in 5.8 and (5.43) are dual. We found also that $H^{\text{planar}}_{\text{res}} = H^{\text{chir}}_{\text{res}}$, therefore taking the limit $\tau + \infty$ we find:

$$Z_{G[U(N)],I}(\vec{X}, \vec{Y}) = Z_{G[U(N)],II}(\vec{X}, \vec{Y}) \tag{5.52}$$

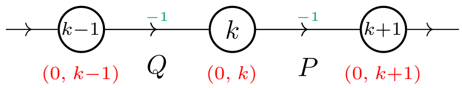
Global Symmetries and Operator Map

We now comment on the duality map between global symmetries and the chiral rings of the two theories.

Let us start from the global symmetries. As already stated at the beginning of the section, the $G[U(N)]$ SQCFT has a $U(1)^{N-1} \times SU(N)$ global symmetry. In the chiral UV completion (5.43) , the two global symmetries are realized manifestly in the UV. $SU(N)$ is the flavor symmetry while $U(1)^{N-1}$ are the topological symmetries of the $N-1$ nodes. In the planar UV completion, in Figure 5.8, instead the $U(1)^{N-1}$ is realized as the flavor symmetry that is unbroken by the planar superpotential and up to gauge transformations. The $SU(N)$ global symmetry is instead obtained from the enhancement of the $N-1$ topological symmetries. We recall that the presence of $\mathcal{W}_{\text{monopole}}$ has the effect of breaking the $U(1)^k$ topological symmetries in a colum with

k gauge nodes down to a diagonal $U(1)$, such that effectively each column of gauge nodes contributes with a single $U(1)$ topological symmetry.

Let us now move to the operator map. The chiral ring of the planar $G[U(N)]$ theory in Figure 5.8 is generated by the $N-1$ vertical bifundamental fields connecting adjacent vertical flavor nodes. These can be taken as the only indpendent chiral ring generators up to F-term relations imposed by $\mathcal{W}_{\text{planar}}$.
We expect these operators to be mapped to monopole operators in the chiral $G[U(N)]$ theory (5.43). This is the case, but the fundamental monopoles are gauge variant and must be appropriately dressed by appropriate powers of fundamental chiral multiplets. Focusing on a section of the chiral $G[U(N)]$ quiver:



(5.53)

we notice that the fundamental monopole of the $U(k)$ node can be dressed with Q^{k-1} to ensure gauge invariance of the resulting operators. The $N-1$ dressed fundamental monopoles of the chiral $G[U(N)]$ theory generate the chiral ring of the theory. We can also calculate that they have the correct charges to be mapped to the $N-1$ vertical bifundamental chirals connecting adjacent vertical flavor nodes in the mirror, which can be verified also via the superconformal index Assume that each bifundamental in the electric theory $(Q,\ P\dots)$ has trial R-charge r. Consequently, the vertical bifundamental chirals connecting adjacent vertical flavor nodes in the mirror planar theory, that the monopoles map to, have R-charge $2-2r$. The dressed monopole of the $k-$th gauge node in the chiral theory $\text{Tr}(Q^{k-1}\mathfrak{M}^{0,\dots,0,-,0,\dots,0})$ has R-charge (see Appendix B):

$$\begin{aligned} R(Q^{k-1}\mathfrak{M}^{0,\dots,0,-,0,\dots,0}) =& -(r-1)+\frac{1}{2}(k-1)(1-r)+\frac{1}{2}(k+1)(1-r)-(k-1)+(k-1)r= \\ =&(1-r)+k(1-r)-(k-1)+(k-1)r= \\ =&2-2r \end{aligned} \tag{5.54}$$

where the first factor comes from the mixing between the topological symmetry and the R-symmetry[13], while the second and third terms in (5.54) are the bifundamental fermions contributions, the $-(k-1)$ is the contribution from the gaugini, and $(k-1)r$ is the contribution from the dressing with Q^{k-1}. So the $N-1$ fundamental dressed monopoles map to the $N-1$ vertical bifundamental chirals connecting adjacent vertical flavor nodes in the mirror. In the rest of the paper we denote the gauge-invariant dressed monopoles as $\mathcal{M}_d$, and we omit the dressing fields.

5.3.2 Flip-Flip chiral-chiral and planar-planar dualities

The $FT[U(N)]$ theory enjoys another UV completion known as the Flip-Flip dual [57]. The Flip-Flip dual quiver $ffFT[U(N)]$ is obtained starting from the $FT[U(N)]$ theory with two extra sets of singlets which flip the operators A_X, A_Y:

$$\mathcal{W}_{ffFT} = \mathcal{W}_{FT} + O_X A_{\vec{X}} + O_Y A_{\vec{Y}} \tag{5.55}$$

Since in the $FT[U(N)]$ theory the meson matrix is already flipped by the singlet a_N, this further flip give mass to a_N. As a result the spectrum of the Flip-Flip theory

[13]The mixing can be extracted as the coefficient of $\frac{iQ}{2}$ in the FI of the k-th node: $Y_k - Y_{k+1} + \frac{iQ}{2}(r-1)$. The contribution to the charge is then obtained by multiplying it by the topological charge.

consist in a the adjoint $SU(N)_X$ matrix is realised as the meson matrix while adjoint matrix $SU(N)_Y$ coincides with the monopole flip matrix O_Y.

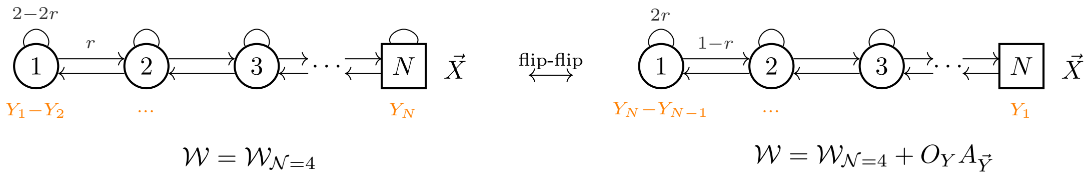


Figure 5.9: The flip-flip duality for the FT[U(N)] theory. On the electric side the bifundamentals have $U(1)_\tau$ charge $\frac{1}{2}$ and trial R-charge r, while on the magnetic side the bifundamentals have $U(1)_\tau$ charge $-\frac{1}{2}$ and trial R-charge $1-r$, as enphasized by the labels in black.

The Flip-Flip duality is represented by the following equality of $\mathbf{S}_b^3$ partition functions:

$$Z_{ffFT[U(N)]}(\vec{X},\vec{Y},\tau) = \prod_{j,k=1}^{N} s_b\left(\frac{iQ}{2}(2r-1)+\tau+X_j-X_k\right) s_b\left(\frac{iQ}{2}(2r-1)+\tau+Y_j-Y_k\right)$$
$$Z_{FT[U(N)]}(\vec{X},\vec{Y},-\tau)\Big|_{r\to 1-r} \tag{5.56}$$

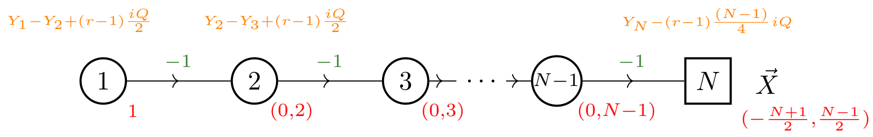
As shown in [63] it is possible to prove the flip-flip duality by iterative applications of the Aharony duality. The idea of the proof is the following. Starting from the left-most $U(1)$ node in the $FT[U(N)]$ theory (whose adjoint chiral is just a singlet) we apply Aharony duality [7] which leaves the rank invariant but gives mass to the adjoint chiral field of the adjacent $U(2)$, hence we can apply again the Aharony duality on it. This will remain a $U(2)$ node but the dualization will alaso give mass to the the adjoint chiral field of the adjacent $U(3)$ node. We continue applying iteratively the Aharony duality until we reach the last $U(N-1)$ node. Notice that since every $U(k)$ node sees $2k$ flavors, ranks do not change when we apply the Aharony duality. We then perform another sequence of dualizations starting from the leftmost $U(1)$ node and stopping at the second last node $U(N-2)$. In the third sequence of dualization we start again from the $U(1)$ node and proceed along the tail stopping at the $U(N-3)$ node. We iterate this procedure for a total of $N-1$ times, applying Aharony duality $N(N-1)/2$ times. The singlet fields flipping the mesons and the monopoles appearing in the Aharony duality reconstruct the singlet matrix O_Y and give mass to the singlet matrix O_X.

We will now show that we can derive similar self-dualities (modulo flips) for the two $G[U(N)]$ UV completions. In analogy with the $\mathcal{N}=4$ case we denote these dualities as flip-flip dualities. These dualities map chiral theories to chiral theories and planar theories to planar theories, in contrast to the mirror-like dualities described in Section 5.3. This is expected because the $\mathcal{N}=4$ flip-flip does not exchange flavor and topological symmetries, therefore in the $\mathcal{N}=2$ case the $SU(N)$ and $U(1)^{N-1}$ factors of the global symmetries of $G[U(N)]$ will not be exchanged.

Chiral-Chiral flip-flip duality

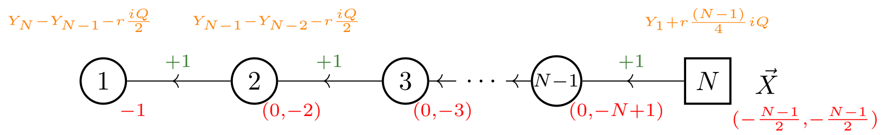
Starting from the flip-flip duality in Figure 5.9, we take the real mass deformation 5.41 that takes the $FT[U(N)]$ on the l.h.s. to the first UV completion of $G[U(N)]$, the chiral UV completion.

Contrary to the case discussed in the previous section, where we started from the two UV completions related by mirror duality, now the same deformation can be taken on the dual side in 5.9. Indeed in this case we find that in the dual vacuum is again a chiral quiver, so we obtain the following new chiral-chiral duality relating the first chiral UV completion of the $G[U(N)]$ theory (5.43):



$$\mathcal{W} = 0 \tag{5.57}$$

to its flip-flip chiral dual:



$$\mathcal{W} = t_1\mathfrak{M}_d^{-,0,\dots,0} + t_2\mathfrak{M}_d^{0,-,0,\dots,0} + \dots \tag{5.58}$$

Notice that in the flip-flip dual theory there is a superpotential containing flippers t_i for all the dressed monopoles $\mathcal{M}_d^{0,\dots,-,\dots,0}$ with negative GNO flux under a single gauge node. In addition all the arrows and CS coupling signs are flipped and the FIs are reshuffled. The bifundamental fields in the flip-flip theory have trial R-charge $1-r$. On the the flip-flip side there is also a background CS term at level $-\frac{1}{2}$ for each $U(1)$ factor in the $U(1)_Y^{N-1}$ symmetry group.

Under the duality map the dressed monopoles with negative charge under the topological symmetries of the electric theory map to the set of singlets t_i in the flip-flip dual.

Also in this case, similarly as for the $FT[U(N)]$ theory, the duality can be demonstrated by iterative applications of local dualities as shown in Appendix B of [44]. Now at each step we need a to locally apply the ciral Giveon-Kutasov duality for $U(N)_{1,1}$ with $[N-1, N+1]$ flavors [10, 12, 83].

Planar-Planar flip-flip duality

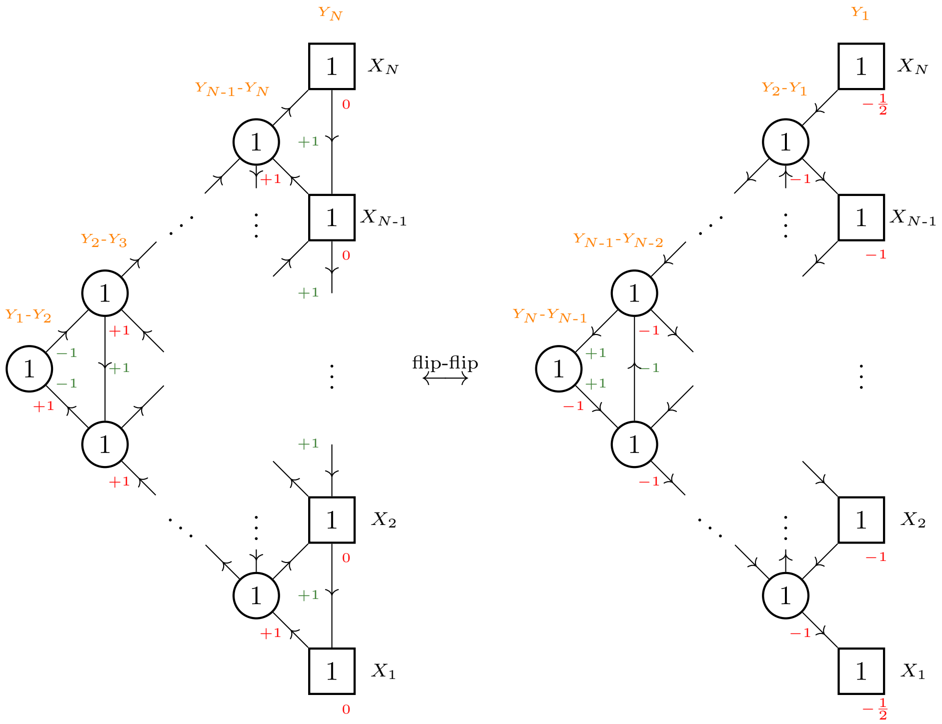
Let's start again from the duality in Figure 5.9. Now we take the real mass deformation that takes the $FT[U(N)]$ on the l.h.s. to the second UV completion of $G[U(N)]$, the planar UV completion. So we perform the real mass deformation:

$$\begin{cases} X_J \to X_J + \frac{\tau}{2}(2J-N-1), & J=1,\dots,N \\ \vec{Y} \to \vec{Y} + \frac{\tau}{2}N \end{cases}, \qquad \tau \to +\infty \tag{5.59}$$

and we also perform the following shifts in gauge parameters:

$$\omega_\alpha^{(J)} \to \omega_\alpha^{(J)} + \tau\left(\alpha - \frac{J+1}{2}\right) \tag{5.60}$$

Also in this case we can implement the same deformation on the dual side and we land on a pair of dual planar abelian quivers given in Figure 5.10.



$\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$ $\qquad$ $\mathcal{W} = \mathcal{W}_{\text{planar}} + \mathcal{W}_{\text{monopole}}$

Figure 5.10: Self-duality *modulo flips* displayed by the planar $G[U(N)]$ theory. The background CS terms for the flavor symmetry are different on the two sides of the duality, furthermore on the r.h.s. there is a background CS term at level $-N$ for the emergent $SU(N)_Y$ global symmetry. Flowing from the $\mathcal{N}=4$ *flip-flip* this is generated from integrating out the monopole flippers O_Y in Figure 5.9. On the r.h.s. there are also shifts in the FI terms, discussed in the main text, which are not reported in the figure.

Notice that in the flip-flip dual theory all the arrows are flipped and the FIs are reshuffled. Furthermore, CS and mixed CS levels have opposite sign with respect to the electric theory and the trial R-charge of the chirals is $1-r$. To avoid clutter here we set $r=0$, therefore the electric diagonal bifundamentals have trial R-charge 0 and the magnetic diagonal bifundamentals have trial R-charge 1. On the the r.h.s. there is also a background CS term at level $-N$ for the emergent $SU(N)_Y$ symmetry, as well as shifts for the FIs. On the flip-flip side the FIs for the gauge node of the J-th column from the right are:

$$\text{FI: } Y_J - Y_{J-1} + \frac{iQ}{4},\ Y_J - Y_{J-1},\ \ldots,\ Y_J - Y_{J-1},\ Y_J - Y_{J-1} - \frac{iQ}{4} \tag{5.61}$$

from top to bottom. In Figure 5.10 we only reported the Y-dependent part of the FIs.

The operator map is very simple: the vertical string of $N-1$ singlets in the last column of the electric theory maps to the $N-1$ mesons of the flip-flip theory theory constructed along the shortest path connecting two adjacent $U(1)$ flavor nodes.

5.4 Towards an Algorithmic Approach: $\mathcal{S}-$walls, QFT Building Blocks, and Basic Duality Moves

Starting from the $\mathcal{N}=4$ $U(N)$ SQCD with $F \geqslant 2N$ hypers we can also consider a more general real mass deformation, w.r.t. the one discussed in the previous section, where we partially break the $SU(F)$ global symmetry to obtain an $\mathcal{N}=2$ $U(N)$ SQCD with n_f fundamental and n_a anti-fundamental chiral multiplets on the electric side:

$$\begin{array}{c}\mathcal{N}=4 \quad U(N)\\ F \text{ hypermultiplets}\end{array} \quad \longrightarrow \quad \begin{array}{c}\mathcal{N}=2 \quad U(N)_{-\frac{F}{2}+N,-\frac{F}{2}}\\ {[n_f,n_a]} \text{ chiral fields}, \qquad F=n_f+n_a\end{array} \tag{5.62}$$

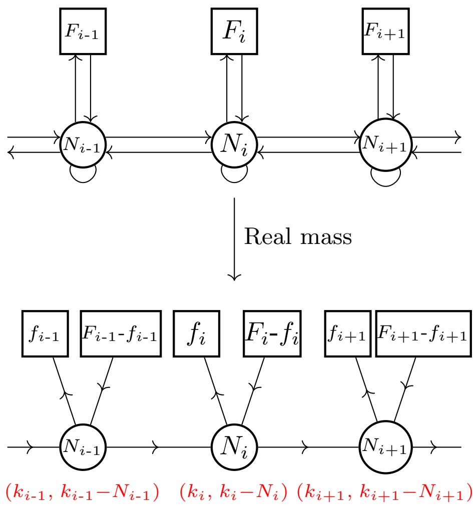
Here we consider flows that preserve the maximum amount of chiral matter multiplets $F=n_f+n_a$, while giving mass to the adjoint chiral field. Therefore, the number of (anti-) fundamentals n_f and n_a satisfy $F=n_f+n_a$ but are otherwise unconstrained and depend on the specific mass deformation that we consider. On the other hand, the choice of CS levels **is not arbitrary** and is fixed by the parent $\mathcal{N}=4$ theory: each massive (anti)fundamental multiple contributes as $(-\frac{1}{2},-\frac{1}{2})$ to the CS level and the massive adjoint chiral contributes as $(N,0)$. In the CS-level notation: $(k,k+lN)$, this corresponds to $k=-\frac{F}{2}+N$ and $l=-1$.

One can generalize to arbitrary values of l by applying Witten's $SL(2,\mathbb{Z})$ action on both sides of the duality as discussed in Subsection 5.2.3.

We can also turn more general real masses and integrate out more chirals to access a more general range of CS levels. Notice that this can also be achieved by first performing the real mass deformation that preserves the maximum number of chirals, discussed here, and then performing additional real masses for the resulting $\mathcal{N}=2$ theories. We will address this generalization in future work [84].

One can then consider the same limit on the mirror side and try to identify the dual vacuum. In general, this is a non-trivial exercise as there are multiple potential vacua corresponding to possible ways of moving on the Coulomb branch of the $\mathcal{N}=4$ mirror (which, in turn, match how we shift gauge fugacities in the $\mathbf{S}^3_b$ partition function).

More generally, one can start on the electric side with a linear unitary $\mathcal{N}=4$ quiver with gauge group $\prod_{N_i} U(N_i)$. We consider a real mass deformation, paired with a VEV for the scalar in the $\mathcal{N}=2$ vector multiplets, such that the adjoint chiral fields and only one of the $\mathcal{N}=2$ chiral multiplets in each matter hypermultiplet acquires a real mass. Such a deformation preserves the maximum total amount of chiral fields. Integrating out the massive field we obtain a chiral linear quiver:



(5.63)

where $f_i \leqslant F_i$ is the number of fundamental flavors for the i-th node, and $F_i - f_i$ is the number of antifundamental flavors. The CS levels of the i-th gauge node k_i are fixed by the parent theory:

$$k_i = -\frac{N_{i-1} + N_{i+1} + F_i}{2} + N_i \,. \tag{5.64}$$

One can also consider more general real mass flows such that in the resulting $\mathcal{N} = 2$ quiver some of the bifundamental chiral fields point to the left. As before, one can implement the same limit in the mirror dual quiver theory to reach the dual vacuum. This remains a non-trivial exercise.

Clearly, it is desirable to have a more systematic way to determine the planar mirror dual of a generic flavored $\mathcal{N} = 2$ quiver without the need to resort to $\mathbf{S}_b^3$ partition functions. We do so by adapting the idea of the local dualization algorithm of [56, 40].

5.4.1 The chiral-planar $\mathcal{S}$-wall, fusion to Identity and gluing rules

In this section we propose, in analogy $\mathcal{N} = 4$ algorithm, to identify $G[U(N)]$ as the "chiral-planar $\mathcal{S}$-wall" and prove its fusion to Identity properties.

More precisely we define:[14]

$$\begin{aligned} \mathcal{S}\text{-wall}: \quad & \mathcal{S}[\vec{X}, \vec{Y}] = G[-\vec{X}, \vec{Y}] = \underset{\vec{X}}{\boxed{N}} \overset{(+)}{\rightsquigarrow} \underset{\vec{Y}}{\boxed{1^N}} \\ \mathcal{S}^{-1}\text{-wall}: \quad & \mathcal{S}^{-}[\vec{X}, \vec{Y}] = G[\vec{X}, \vec{Y}] = \underset{\vec{X}}{\boxed{N}} \overset{(-)}{\rightsquigarrow} \underset{\vec{Y}}{\boxed{1^N}} \end{aligned} \tag{5.65}$$

Both theories admit (at least) two UV completions whose difference lies in the direction of the arrows and the sign of FIs. For convenience the UV completions of both theories are reported in Figures 5.11 and 5.12.

Notice that in the chiral UV completion of $\mathcal{S}$ the arrow direction is inverted w.r.t. $\mathcal{S}^-$, where it emanates from the manifest $U(N)$, and the FI at each node are opposite: $Y_{i+1} - Y_i$. In the planar UV completion of $\mathcal{S}$ the FI's associated to each string are $X_{i+1} - X_i$ rather than $X_i - X_{i+1}$ as in $\mathcal{S}^-$.

In our notation the first argument of the $Z_{\mathcal{S}^\pm}[\cdot\,;\cdot]$ theories corresponds to fugacities for an $U(N)$ global symmetry, while the second argument corresponds to a $U(1)^{N-1}$ global symmetry.

We will now prove the fusion to identity properties of the $\mathcal{N} = 2$ $\mathcal{S}$-wall.

[14] Notice that in this chapter we will name in short the $G[U(N)]$ theory just as the $\mathcal{S}$-wall. This theory is clearly different to the $FT[SU(N)]$ theory which is the $\mathcal{N} = 4$ $\mathcal{S}$-wall of [28] in Chapter 2 and 3. To distinguish the two we will refer to the $FT[U(N)]$ theory as the $\mathcal{S}_{\mathcal{N}=4}$-wall in this chapter.

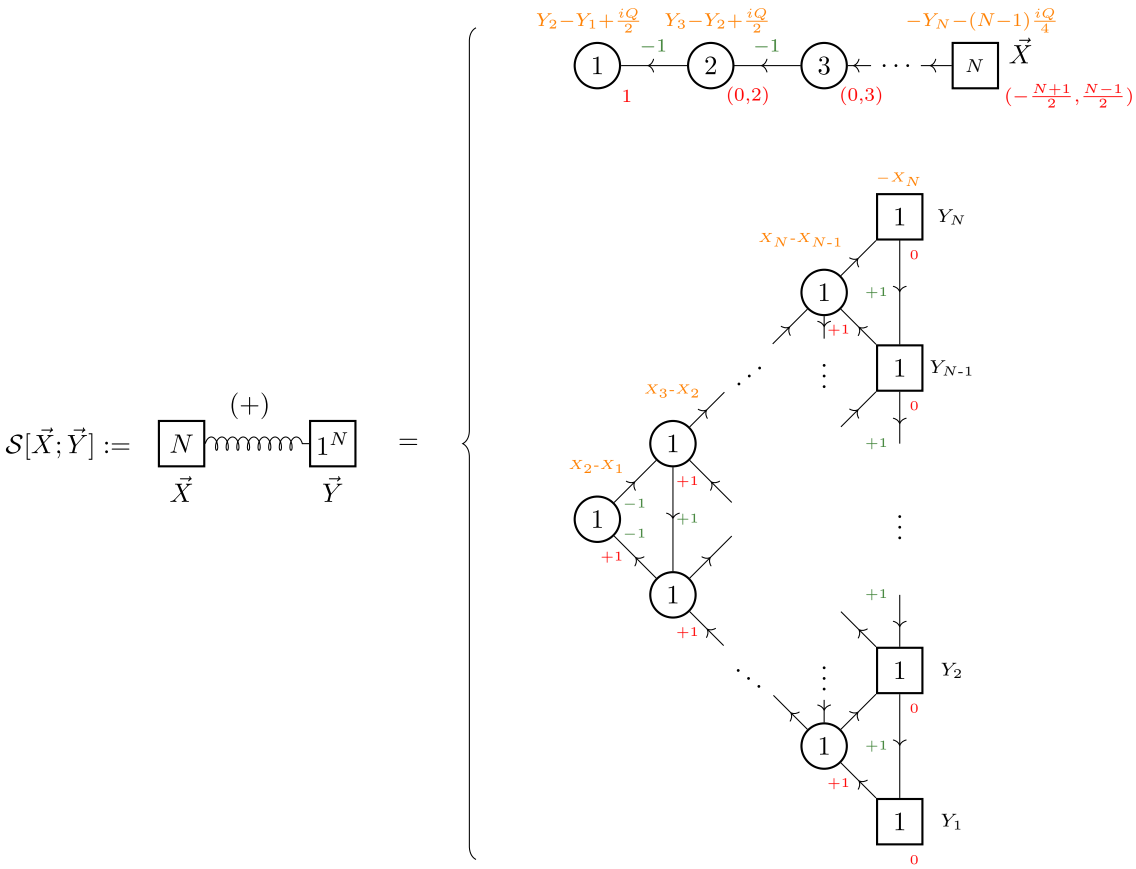


Figure 5.11: The planar and chiral UV completions of the $\mathcal{N}=2$ chiral-planar $\mathcal{S}$-wall coinciding with $G[-\vec{X};\vec{Y}]$ are shown here. Mixed CS interactions have been suppressed for brevity. Here, the trial R-charge r is set to 0. Therefore, in the chiral completion, on top, each bifundamental has trial R-charge 0. In the planar completion the vertical/diagonal chirals have trial R-charge 2/0.

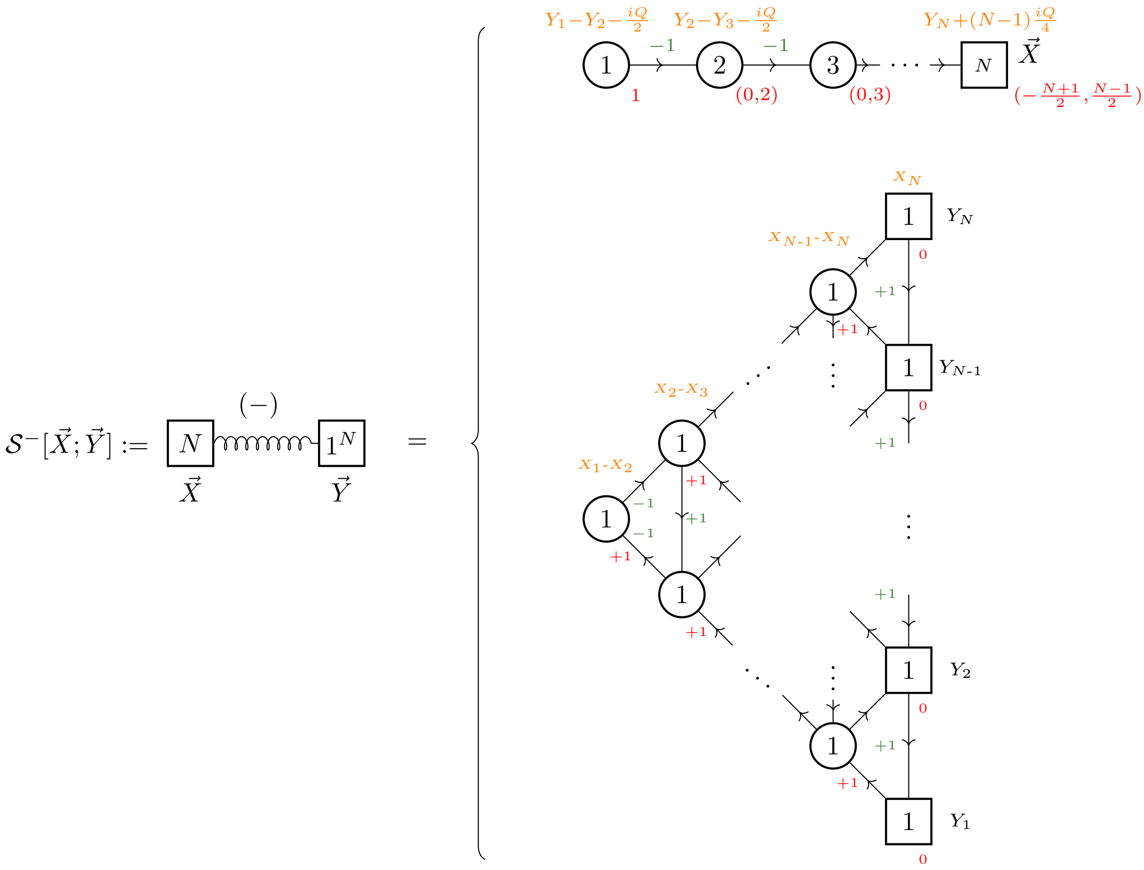


Figure 5.12: The planar and chiral UV completions of the $\mathcal{N} = 2$ $\mathcal{S}$-wall coinciding with $G[-\vec{X};\vec{Y}]$ are shown here. Mixed CS interactions have been suppressed for brevity. Here, the trial R-charge r is set to 0 and the trial R-charges of the chiral fields can be read as in the previous figure.

Planar Fusion to Identity

Our starting point is the $\mathcal{N}=4$ fusion to identity where we consider the UV completion where we gauge the left and right $\mathcal{S}_{\mathcal{N}=4}$-wall by gauging their manifest $U(N)$ symmetries.

We consider the large mass deformation:

$$\begin{cases} X_j \to X_j + N\frac{\tau}{2}\,, \\ Y_j \to Y_j + N\frac{\tau}{2}\,, \end{cases} \qquad j = 1, \dots, N \tag{5.66}$$

which preserves both the (emergent) $U(N)_{X,Y}$ global symmetries. We also perform the following shifts in gauge fugacities:

$$\begin{cases} u_\alpha^{(I)} \to u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha - I - 1)\,, & \text{for } I = 1, \dots, N \\ u_\alpha^{(I)} \to u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha + I - 2N - 1)\,, & \text{for } I = N+1, \dots, 2N-1 \end{cases} \tag{5.67}$$

with $\tau \to \infty$, where $u_\alpha^{(I)}$ are the gauge fugacities for the I-th gauge node in the quiver (starting from the left), while $\alpha = 1, \dots, |G^{(I)}|$ with $|G^{(I)}|$ being the rank of the I-th gauge node.
This triggers a flow to the $\mathcal{N}=2$ planar fusion to identity depicted in eq. 5.13. On the l.h.s. we glue two $\mathcal{S}$-walls in their planar abelian UV completion by gauging a diagonal $U(1)^N$ symmetry. On the r.h.s. we find an Identity-wall identifying the $U(N)_X$ and $U(N)_Y$ manifest symmetries.

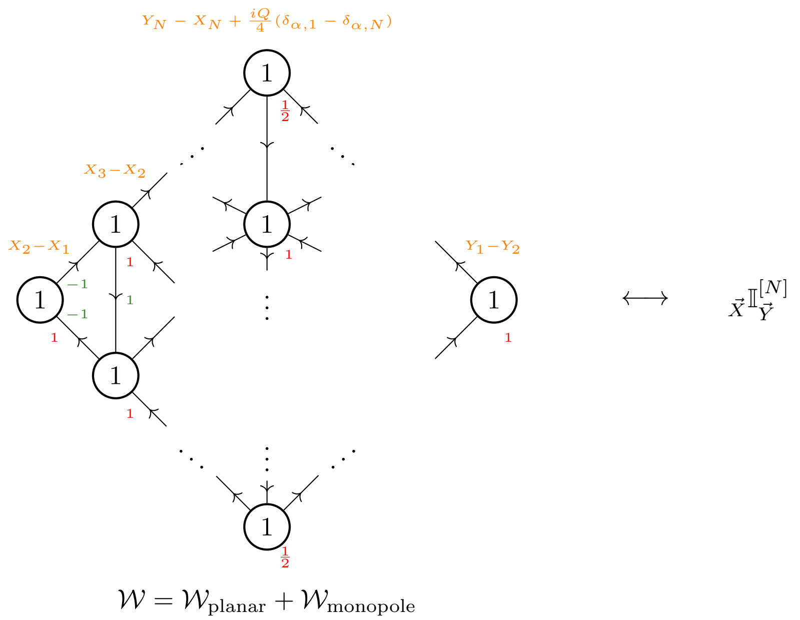


Figure 5.13: The planar fusion to identity observed upon gluing a $\mathcal{S}$ and $\mathcal{S}^-$ wall. Note that the $\delta_{\alpha,j}\frac{iQ}{4}$ indicates the j^{th} node of the middle column has a shifted FI term, counting from below.

We encode the fusion to identity in the following partition function identity:

$$\int d\vec{Z} \Delta^{[1^N]}(\vec{Z}) Z_{\mathcal{S}}(\vec{X}, \vec{Z}) Z_{\mathcal{S}^-}(\vec{Y}, \vec{Z}) = \frac{1}{N! \Delta^{[N]}(\vec{X})} \sum_{\sigma \in S_N} \delta\big(X_i - Y_{\sigma(i)}\big) \equiv {}_{\vec{X}}\mathbb{I}_{\vec{Y}}^{[N]}\,. \tag{5.68}$$

On the l.h.s. we integrate over the abelian set of fugacities $\vec{Z}$ entering the second slot of $Z_{\mathcal{S}^\pm}[\cdot\,;\cdot]$. The **Planar Measure** $\Delta^{[1^N]}(\vec{Z})$ in (5.68) is defined as a limit of the $\mathcal{N}=4$ gluing measure $\Delta_{(N)}(\vec{Z},\tau)$ (5.11):

$$\lim_{\tau\to+\infty}\Delta_{(N)}(\vec{Z},\tau)\Big|_{Z_i\to Z_i+\frac{\tau}{2}(2i-N-1)}=\Delta^{[1^N]}(\vec{Z})\times(\text{divergent phase})\,. \tag{5.69}$$

Notice the shift on the gauge variables is consistent with eq. (5.67) for $\vec{Z}=\vec{u}^{(I=N)}$. We obtain:

$$\Delta^{[1^N]}(\vec{Z})=e^{i\pi H^{[1^N]}(\vec{Z})}\prod_{i=1}^{N-1}s_b\Big(\frac{iQ}{2}-Z_i+Z_{i+1}\Big) \tag{5.70}$$

with

$$H^{[1^N]}(\vec{Z}):=\frac{iQ}{2}(Z_1-Z_N)-\frac{1}{2}(Z_1^2+Z_N^2)-\sum_{i=2}^{N-1}Z_i^2+\sum_{j=1}^{N-1}Z_jZ_{j+1}\,. \tag{5.71}$$

Note that the contribution from the vector multiplet of $U(N)$ has disappeared, as expected from Higgsing the gauge group down to its maximal torus $U(1)^N$. The limit for the $U(N)$ adjoint gives rise to a string of vertical chiral multiplets connecting adjacent $U(1)^N$ nodes and flowing upward. This string of vertical chiral fields couples to the vertical chiral fields (flowing downward) connecting the $U(1)^N$ nodes of the left and right $\mathcal{S}$-walls. As a result two strings of vertical bifundamental (flowing in opposite directions) give mass to each other and we are left with a single string of vertical bifundamentals in the middle flowing downward. This is the planar version of the $\mathcal{N}=4$ gluing $a(A_L+A_R)$.

The quadratic form $H^{[1^N]}(\vec{Z})$ encodes the CS levels and the BF couplings and FI shifts for the middle column of $U(1)^N$ gauge nodes.

On the r.h.s. of the identity (5.68), we identify the $\mathcal{N}=2$ **Chiral Identity-wall**:

$$_{\vec{X}}\mathbb{I}^{[N]}_{\vec{Y}}\equiv\frac{1}{N!\Delta^{[N]}(\vec{X})}\sum_{\sigma\in S_N}\delta(X_i-Y_{\sigma(i)}) \tag{5.72}$$

identifying the $U(N)_X$ and $U(N)_Y$ global symmetries. The **Chiral Measure** $\Delta^{[N]}(\vec{X})$ is defined by starting again from the definition of the $\mathcal{N}=4$ gluing measure $\Delta_{(N)}(\vec{X},\tau)$ in (5.11), then shifting the flavor fugacities $\vec{X}$ as in (5.66) and taking the limit $\tau\to+\infty$. We obtain (again after suppressing the diverging prefactor):

$$\Delta^{[N]}(\vec{X})=e^{i\pi H^{[N]}(\vec{X})}\prod_{\alpha=1}^{N}\prod_{\beta<\alpha}\frac{1}{s_b(\frac{iQ}{2}\pm(X_\alpha-X_\beta))}\,; \tag{5.73}$$

with

$$\begin{aligned}H^{[N]}(\vec{X})&:=-(N-1)(\sum_{i=1}^{N}X_i^2)+2\sum_{i<j}^{N}X_iX_j=\\&=-N\sum_{i=1}^{N}X_i^2+\Bigg(\sum_{i=1}^{N}X_i\Bigg)^2\,,\end{aligned} \tag{5.74}$$

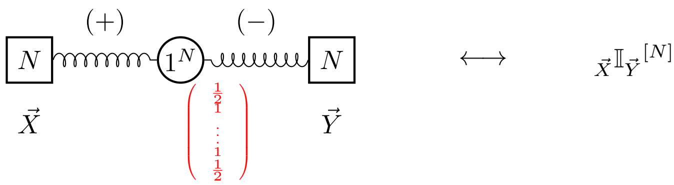
which corresponds to a CS coupling at level $(N,0)$.

Observe that in the integrand on the l.h.s. of (5.68) appears the Planar Measure $\Delta^{[1^N]}(\vec{Z})$, while at the denominator of the r.h.s. of the same identity appears the Chiral

Measure $\Delta^{[N]}(\vec{X})$. This is consistent with our claim that mirror symmetry sends the $\mathcal{N}=2$ *chiral* deformation of an $\mathcal{N}=4$ theory into the $\mathcal{N}=2$ *planar* deformation of its $\mathcal{N}=4$ mirror dual and vice versa, and so the same happens for the basic moves.

Although we suppressed the divergent contributions to avoid clutter, we have checked that the divergent pre-factors on l.h.s. and r.h.s. of the partition function identity and cancel-out.

We schematically depict this planar fusion to identity as:



(5.75)

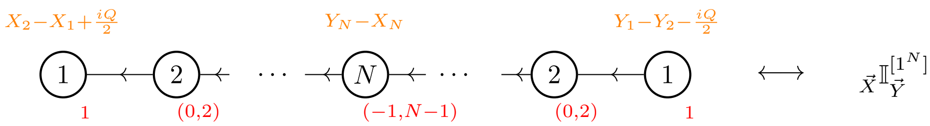
Where the vector in red indicates the CS levels for the middle column of $U(1)^N$ nodes in (5.13). Remember however that there also BF couplings and FI shitfs given by the planar measure (5.71).

The Chiral Fusion to Identity

Starting again from the $\mathcal{N}=4$ fusion to identity (3.3) where we glue the left and right $\mathcal{S}_{\mathcal{N}=4}$-wall by gauging their manifest $U(N)$ symmetries, we perform a large mass deformations that breaks the $U(N)_{X,Y}$ global symmetries to their Cartan. This is encoded in the following shift in fugacities:

$$\begin{cases} \vec{u}^{(I)} \to \vec{u}^{(I)} - \frac{\tau}{2} I\,, & \text{for } I = 1, \dots, 2N-1 \\ X_j \to X_j + \frac{\tau}{2}(2j-N)\,, & \text{for } j = 1, \dots, N \\ Y_j \to Y_j + \frac{\tau}{2}(2j-N)\,, & \text{for } j = 1, \dots, N \end{cases} \tag{5.76}$$

With this limit we flow to the $\mathcal{N}=2$ operator identity which admits the following Lagrangian UV completion:



(5.77)

The associated partition function identity reads:

$$\int d\vec{Z}\Delta^{[N]}(\vec{Z})Z_{\mathcal{S}}(\vec{Z},\vec{X})Z_{\mathcal{S}^{-}}(\vec{Z},\vec{Y}) = \frac{1}{\Delta^{[1^N]}(\vec{X})}\prod_{i=1}^{N}\delta(X_i - Y_i) \equiv {}_{\vec{X}}\mathbb{I}_{\vec{Y}}^{[1^N]}\,, \tag{5.78}$$

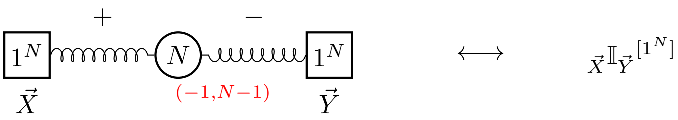
where we integrate over the first slot of $Z_{\mathcal{S}}$ corresponding to the non-abelian symmetry. On the l.h.s. we have the chiral integration measure $\Delta^{[N]}(\vec{Z})$ defined in 5.73 which carries the factor $H^{[N]}(\vec{Z})$ contributing a CS coupling at level $(N,0)$. The left and right $\mathcal{S}$-walls also carry a background (now becoming dynamical) CS level $(-(N+1)/2,(N-1)/2)$ so the overal CS level of the middle $U(N)$ node is $(-1,N-1)$ in the the Lagrangian UV completion (5.77). On the r.h.s. of the identity we identify the $\mathcal{N}=2$ **Planar Identity-wall**:

$${}_{\vec{X}}\mathbb{I}_{\vec{Y}}^{[1^N]} \equiv \frac{1}{\Delta^{[1^N]}(\vec{X})}\prod_{i=1}^{N}\delta(X_i - Y_i)\,, \tag{5.79}$$

identifying the two sets of $U(1)^N$ global symmetries of the left and right $\mathcal{S}$-walls.

Although we suppressed the divergent contributions to avoid clutter, we checked that the divergent pre-factors on l.h.s. and r.h.s. of the partition function identity cancel out.

We schematically depict this chiral fusion to identity as:



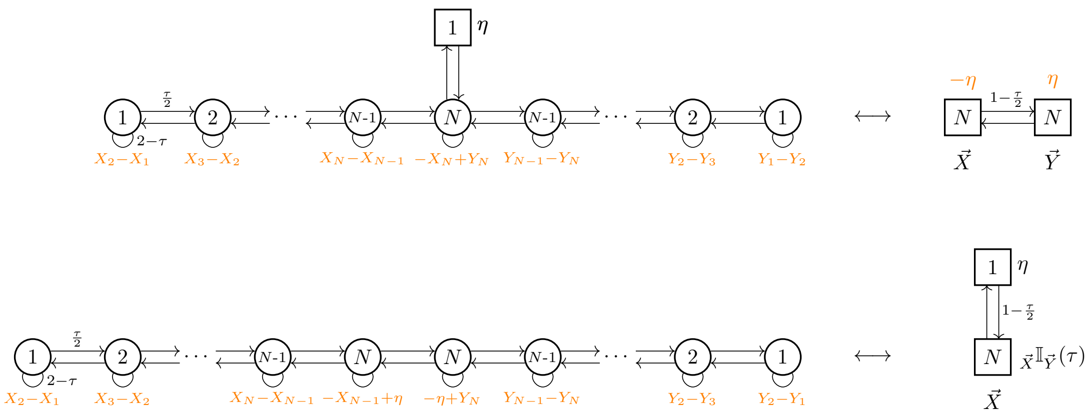
(5.80)

The two dualities schematically depicted in (5.75) and (5.80) can be interpreted as the field theory equivalent of the multiplication between two operators $\mathcal{S}$ and $\mathcal{S}^{-1}$ giving an identity as a result.

5.4.2 Basic Dualities moves

The two dualities schematically depicted in (5.75) and (5.80) can be interpreted as the field theory equivalent of the multiplication between two operators $\mathcal{S}$ and $\mathcal{S}^{-1}$ giving an identity as a result. This statement is analogous to that for the $\mathcal{S}_{\mathcal{N}=4}$ theory. To extend further the idea that the $\mathcal{S}$-wall theory is an oporator we need to define on which object it acts and how. The objects will be $\mathcal{N}=2$ QFT blocks, that are simple WZ models that can be glued together to form generic $\mathcal{N}=2$ theories. As we will show in a moment, it is possible to define an action of the $\mathcal{S}$-wall theory on QFT-blocks. We refer to these identities as *basic duality moves.*

To derive the $\mathcal{N}=2$ duality moves we start from the $\mathcal{N}=4$ basic moves which we repeat below for convenience in their Lagrangian form:



(5.81)

Chiral Bifundamental $\mathcal{S}$-dualization

Our starting point is the $\mathcal{N}=4$ basic move in the first line of (5.81). We perform the real mass deformations specified by the following shifts of fugacities, on both sides:

$$\begin{cases} \vec{X} \to \vec{X} + N\frac{\tau}{2} \\ \vec{Y} \to \vec{Y} + (N+1)\frac{\tau}{2} \\ u_\alpha^{(I)} \to u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha - I - 1), & \text{for } I = 1, \dots, N \\ u_\alpha^{(I)} \to u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha + I - 2N - 1), & \text{for } I = N+1, \dots, 2N-1 \\ \eta \to \eta - \frac{N}{2}\tau \end{cases} \tag{5.82}$$

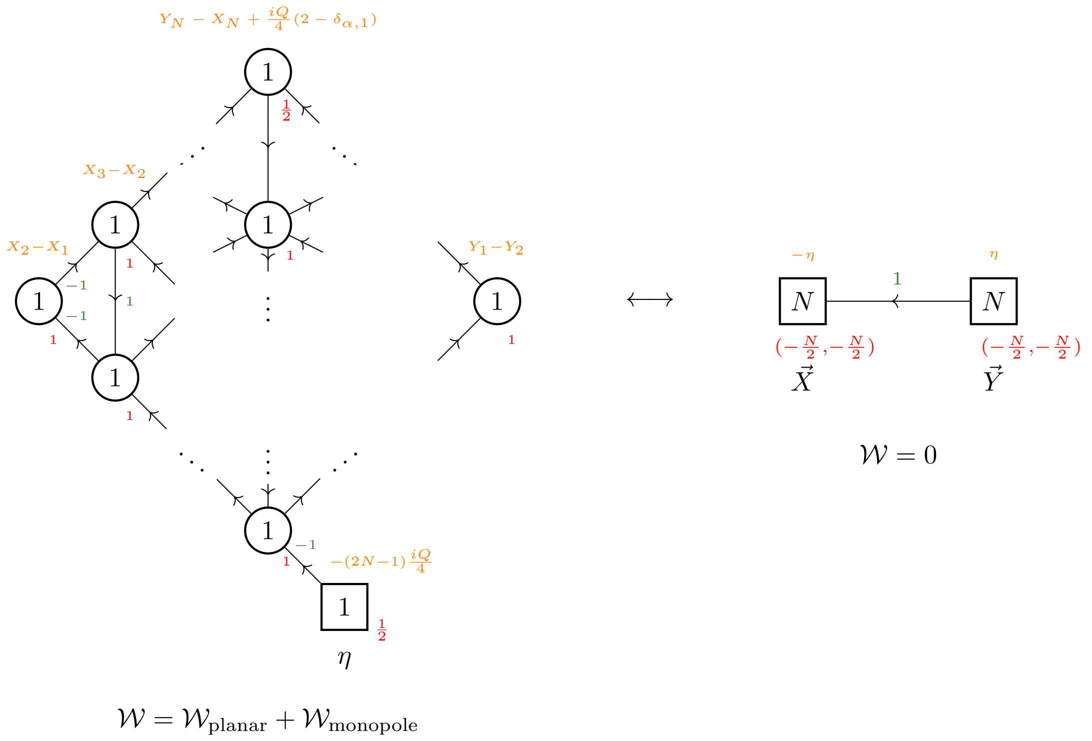
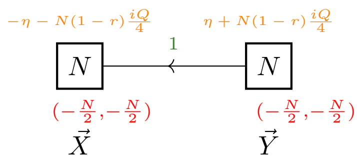


Figure 5.14: A chiral bifundamental on the r.h.s. is $\mathcal{S}$-dualized into planar flavor sandwiched between two $\mathcal{S}$-walls. As usual, on the planar side the FIs in orange are those of all the nodes in the corresponding column below. The term $\delta_{\alpha,1}$ indicates that there is a shift valid only for the 1-st node in the middle column, counting from below.

where $\vec{X}$ and $\vec{Y}$ are fugacities of the two $U(N)$ global symmetries (which are preserved by the deformations), $\vec{u}^{(I)}$ are gauge fugacities for the I-th gauge node in the quiver (where we start to count from the left), and η is the fugacity for the flavor. The flow generated by these deformations yields the duality in Figure 5.14. We interpret this duality as a *basic move* relating a *chiral* bifundamental to a *planar* flavor sandwiched between two $\mathcal{S}$-walls.

We define the *left-pointing* chiral bifundamental block with trial R-charge r as:

$$Z^{[N]}_{\text{bif, left}}(\vec{X},\vec{Y},\eta;r) := \tag{5.83}$$



whose $\mathbf{S}^3_b$ partition function is given by:

$$Z^{[N]}_{\text{bif, left}}(\vec{X},\vec{Y},\eta;r) := e^{2\pi i\sum_j X_j(-\eta-N(1-r)iQ)}\, e^{2\pi i\sum_j Y_j(\eta+N(1-r)iQ)} \\ e^{\frac{N\pi i}{2}\sum_j(X_j^2+Y_j^2)-\pi i\sum_{j,k}X_jY_k} \prod_{j,k=1}^{N} s_b\left(\frac{iQ}{2}(1-r)+X_k-Y_j\right). \tag{5.84}$$

Then on the r.h.s. of Figure 5.14 we identify the chiral bifundamental block[15]

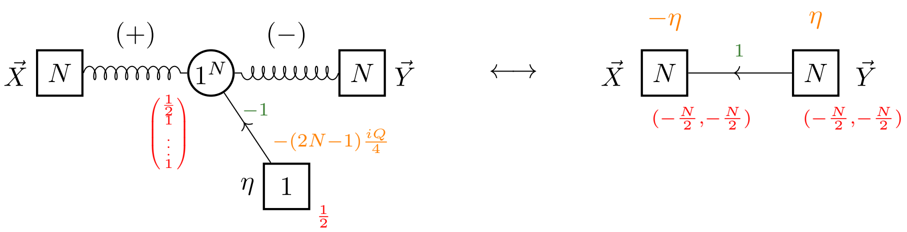
[15]Recall that in this Section we assign trial R-charge 0 to the diagonal bifundamentals of the $\mathcal{S}$-wall

$Z^{(N)}_{\text{bif, left}}(\vec{X},\vec{Y},\eta;1)$, and the duality in Figure 5.14 corresponds to the following identity between partition functions:

$$Z^{[N]}_{\text{bif, left}}(\vec{X},\vec{Y},\eta;1)=\int d\vec{Z}\Delta^{[1^N]}(\vec{Z})e^{-\frac{i\pi}{2}(\eta^2-2\eta Z_1+Z_1^2+iQ[(2N-1)\eta-Z_1-2\sum_{j=2}^N Z_j])}$$
$$Z_{\mathcal{S}}(\vec{X},\vec{Z})s_b\left(\frac{iQ}{2}+Z_1-\eta\right)Z_{\mathcal{S}^-}(\vec{Y},\vec{Z})\,. \tag{5.85}$$

On the r.h.s. the two $\mathcal{S}$-walls are glued as in the case of the planar fusion to identity in (5.75) except for the CS level of the bottom node being 1 instead of 1/2 and different FI terms.

We write the move depicted in Figure 5.14 in compact form as:



(5.86)

Where in the figure above we specify the shift of the CS levels of the N $U(1)$ nodes in the middle column by a red column vector.

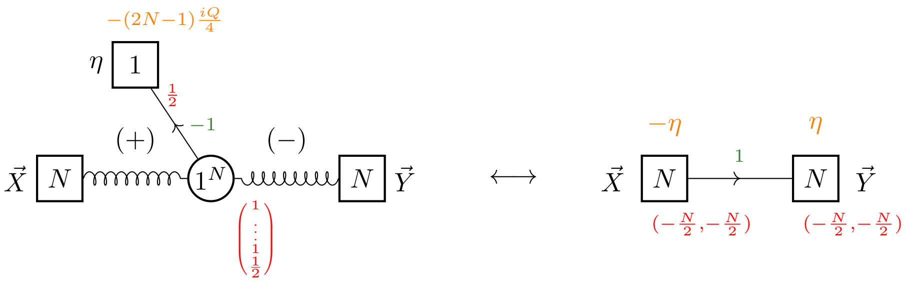
Similarly, we can define a right-pointing bifundamental block, whose $\mathbf{S}^3_b$ partition function is given by:

$$Z^{[N]}_{\text{bif, right}}(\vec{X},\vec{Y},\eta;r):=Z^{[N]}_{\text{bif, left}}(-\vec{X},-\vec{Y},-\eta;r)\,. \tag{5.87}$$

Also, an equivalent definition is:

$$Z^{[N]}_{\text{bif, right}}(\vec{X},\vec{Y},\eta;r):=Z^{[N]}_{\text{bif, left}}(\vec{Y},\vec{X},-\eta;r)\,. \tag{5.88}$$

whose basic move in short notation is:



(5.89)

Where in this notation it is meant that the fundamental chiral is connected to the topmost node of the cental column of gauge nodes, instead of the bottom one as in Figure 5.14. The corresponding partition function identity is:

$$Z^{[N]}_{\text{bif, right}}(\vec{X},\vec{Y},\eta;1)=\int d\vec{Z}\Delta^{[1^N]}\left(\vec{Z}\right)e^{-\frac{i\pi}{2}(\eta^2-2\eta Z_N+Z_N^2+iQ[-(2N-1)\eta+Z_1+2\sum_{j=2}^N Z_j])}$$
$$Z_{\mathcal{S}}(\vec{X},\vec{Z})s_b\left(\frac{iQ}{2}-Z_N+\eta\right)Z_{\mathcal{S}^-}(\vec{Y},\vec{Z}) \tag{5.90}$$

theory, which in turn fixed the trial R-charge of the bifundamental on the r.h.s. of Figure 5.14 to 1.

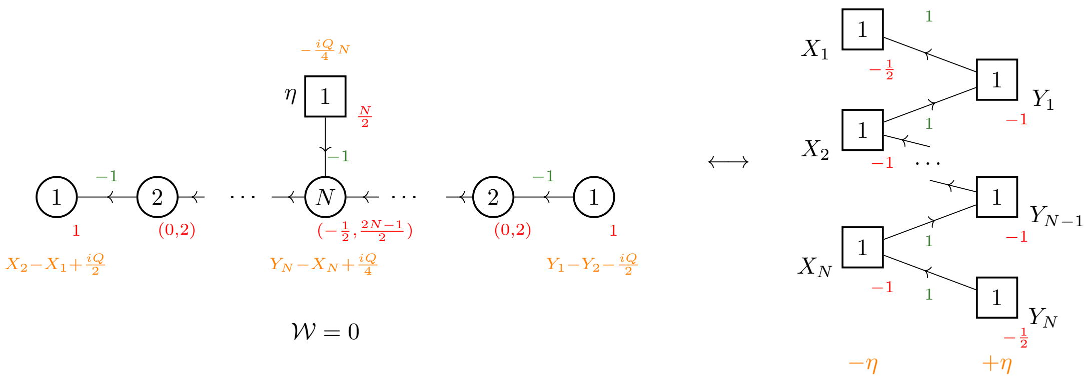


Figure 5.15: A planar bifundamental on the r.h.s. is $\mathcal{S}$-dualized into a flavor sandwiched between two $\mathcal{S}$-walls. As usual, on the planar side the background FIs in orange are those of all the nodes in the column below.

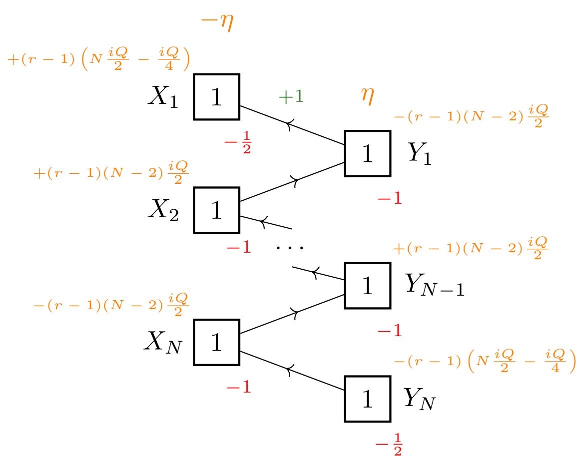
Planar Bifundamental $\mathcal{S}$-dualization

Starting again from the first $\mathcal{N} = 4$ basic move in (5.81), we perform the real mass deformation specified by the following shift of fugacities:

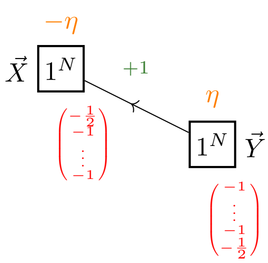
$$\begin{cases} X_j \to X_j + (2j - N)\frac{\tau}{2} \\ Y_j \to Y_j + \frac{\tau}{2}(2j - N + 1) \\ \vec{u}^{(I)} \to \vec{u}^{(I)} - \frac{\tau}{2}I, \quad \text{for } I = 1, \dots, 2N-1 \\ \eta \to \eta - \frac{N+1}{2}\tau\,. \end{cases} \tag{5.91}$$

The flow generated by these mass deformations break the emergent $U(N)_{X,Y}$ symmetries and yelds the duality depicted in Figure 5.15. We define the left-pointing planar bifundamental block with trial R-charge r as:

$$Z^{[1^N]}_{\text{bif, left}}(\vec{X}, \vec{Y}, \eta; r) :=$$



$$:=$$



(5.92)

In the second line we introduced a short notation wherein we suppressed the specific shift in the background FI of each single node, leaving only the common η dependence manifest. Also the background mixed-CS interaction in green is meant to be present for each bifundamental linking two $U(1)$ symmetries while the background CS levels are given as a colum in red. The contribution to the $\mathbf{S}_b^3$ partition function is given below:

$$
\begin{aligned}
Z^{[1^N]}_{\text{bif, left}}(\vec{X},\vec{Y},\eta;r) := {} & e^{\frac{\pi i}{2}(X_1^2+Y_N^2)}\, e^{\pi i(\sum_{j=2}^N X_j^2+\sum_{j=1}^{N-1} Y_j^2)}\, e^{2\pi i\eta\sum_j(Y_j-X_j)}\, e^{-\pi i\sum_j Y_j(X_j+X_{j+1})} \\
& e^{2\pi i\sum_J X_J(r-1)(N+2-2J)\frac{iQ}{2}}\, e^{-2\pi i\sum_J Y_J(r-1)(N-2J)\frac{iQ}{2}}\, e^{2\pi i(Y_N\frac{iQ}{4}-X_1\frac{iQ}{4})} \\
& \prod_{j=1}^{N} s_b\left(\frac{iQ}{2}-r\frac{iQ}{2}+X_j-Y_j\right)\prod_{j=1}^{N-1} s_b\left(\frac{iQ}{2}-r\frac{iQ}{2}+Y_j-X_{j+1}\right).
\end{aligned}
\tag{5.93}
$$

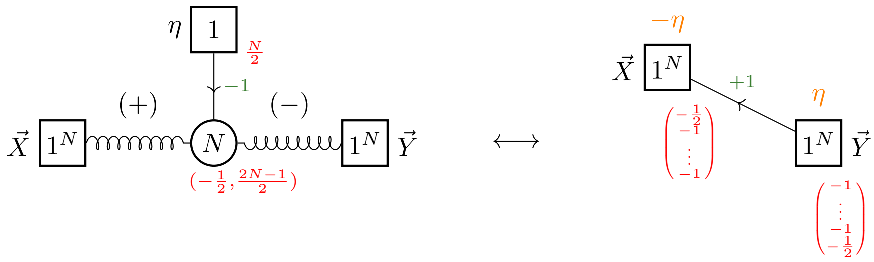
On the r.h.s. of Figure 5.15 we recognize the left-pointing chiral bifundamental block[16] $Z^{[1^N]}_{\text{bif, left}}(\vec{X},\vec{Y},\eta;1)$.

The $\mathbf{S}_b^3$ partition function identity corrresponding to the duality in Figure 5.15 is:

$$
\begin{aligned}
Z^{[1^N]}_{\text{bif, left}}(\vec{X},\vec{Y},\eta;1) = {} & \int d\vec{Z}\Delta^{[N]}(\vec{Z})e^{-\frac{i\pi}{2}[N\eta^2+iQ(N\eta-\sum_{j=1}^N Z_j)+\sum_{j=1}^N(Z_j^2-2\eta Z_j)]} \\
& Z_{\mathcal{S}}(\vec{Z},\vec{X})\prod_{j=1}^{N} s_b\left(\frac{iQ}{2}+Z_j-\eta\right) Z_{\mathcal{S}^-}(\vec{Z},\vec{Y})\,.
\end{aligned}
\tag{5.94}
$$

On the r.h.s. the two $\mathcal{S}$-walls are glued as in the case of the chiral fusion to identity in (5.77). There is only a slight modification for the CS level of the middle $U(N)$ node to which the flavor is attached, which has now CS level $(-1/2,(2N-1)/2)$ and a shifted FI.

We can write the move in Figure 5.15 in compact form as:

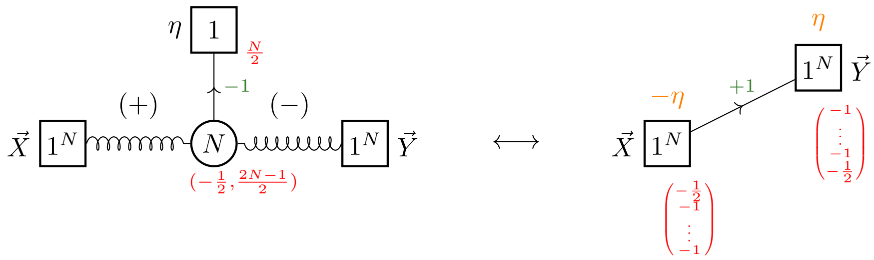


(5.95)

Similarly, we can define a right-pointing planar bifundamental block, whose $\mathcal{S}_b^3$ partition function is given by:

$$
Z^{[1^N]}_{\text{bif, right}}(\vec{X},\vec{Y},\eta;r) := Z^{[1^N]}_{\text{bif, left}}(\vec{Y},\vec{X},-\eta;r)
\tag{5.96}
$$

and whose basic move in short notation is:



(5.97)

[16]Recall that in this Section we assign trial R-charge 0 to the bifundamentals of the $\mathcal{S}$-wall theory, which in turn fixed the trial R-charge of the bifundamental on the r.h.s. of Figure 5.15 to 1.

Notice that in short form the right-pointing planar bifundamental has opposite slope. The corresponding partition function identity is:

$$Z^{[1^N]}_{\text{bif, right}}(\vec{X},\vec{Y},\eta;1) = \int d\vec{Z}\Delta^{[N]}(\vec{Z})e^{-\frac{i\pi}{2}[N\eta^2+iQ(-N\eta+\sum_{j=1}^N Z_j)+\sum_{j=1}^N(Z_j^2-2\eta Z_j)]}$$
$$Z_{\mathcal{S}}(\vec{Z},\vec{X})\prod_{j=1}^N s_b\left(\frac{iQ}{2}-Z_j+\eta\right)Z_{\mathcal{S}^-}(\vec{Z},\vec{Y})\,. \qquad (5.98)$$

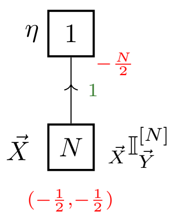
Chiral fundamental $\mathcal{S}$-dualization

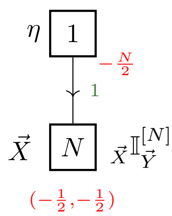
We now discuss the *inverse* basic moves. A possible way to obatain them is by "multiplying" an $\mathcal{S}$ and $\mathcal{S}^{-1}$-wall respectively on the right and left sides of the basic moves in Figures 5.14 and 5.15 and using the appropriate *fusion to identity* properties. However, we will follow a different route which is to start back from the inverse $\mathcal{N}=4$ basic move in the second line of (5.81) and perform a suitable mass deformation.

Let us start from the basic move for a chiral fundamental. Starting from (5.81) we take the real mass deformation

$$\begin{cases} X_j, Y_j \to X_j, Y_j \\ u_\alpha^{(I)} \to u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha - I), \quad \text{for } I=1,\dots,N \\ u_\alpha^{(I)} \to u_\alpha^{(I)} + \frac{\tau}{2}(2\alpha + I - 2N - 2), \quad \text{for } I = N+1,\dots,2N \\ \eta \to \eta + \frac{\tau}{2} \end{cases} \qquad (5.99)$$

where again $u_\alpha^{(m)}$ represents the set of parameters for the m-th gauge node in the quiver (starting from the left), X_j, Y_j are the parameters for the global $U(N)$ symmetries (unbroken by the deformation) and η is the FI parameter. The resulting basic move is shown in Figure 5.16.
On the l.h.s. we identify the **chiral Flavor Block** consisting of a single chiral in the anti-fundamental of $U(N)$, equipped with a chiral identity wall. Together with this block we can also define an analogous fundamental chiral flavor block. The two blocks can be defined graphically as:

$Z^{[N]}_{\text{flavor,fund}}(\vec{X};\eta) :=$ 

$Z^{[N]}_{\text{flavor,antif}}(\vec{X};\eta) :=$ 

(5.100)

The contribution to the $\mathbf{S}_b^3$ partition function is given below[17]:

$$Z^{[N]}_{\text{flavor,fund}}(\vec{X},\eta) = e^{\frac{i\pi}{2}\left[N\eta^2+\sum_{j=1}^N(X_j^2-2\eta X_j)\right]}\prod_{j=1}^N s_b\left(\eta - X_j\right){}_{\vec{X}}\mathbb{I}^{[N]}_{\vec{Y}} \qquad (5.101)$$

$$Z^{[N]}_{\text{flavor,antif}}(\vec{X},\eta) = e^{\frac{i\pi}{2}\left[N\eta^2+\sum_{j=1}^N(X_j^2-2\eta X_j)\right]}\prod_{j=1}^N s_b\left(-\eta + X_j\right){}_{\vec{X}}\mathbb{I}^{[N]}_{\vec{Y}} \qquad (5.102)$$

[17]The CS contact term indicated for the global $U(N)$ symmetry in the $\mathbf{S}_b^3$ partition function has been calculated by including the contribution from the identity wall as well.

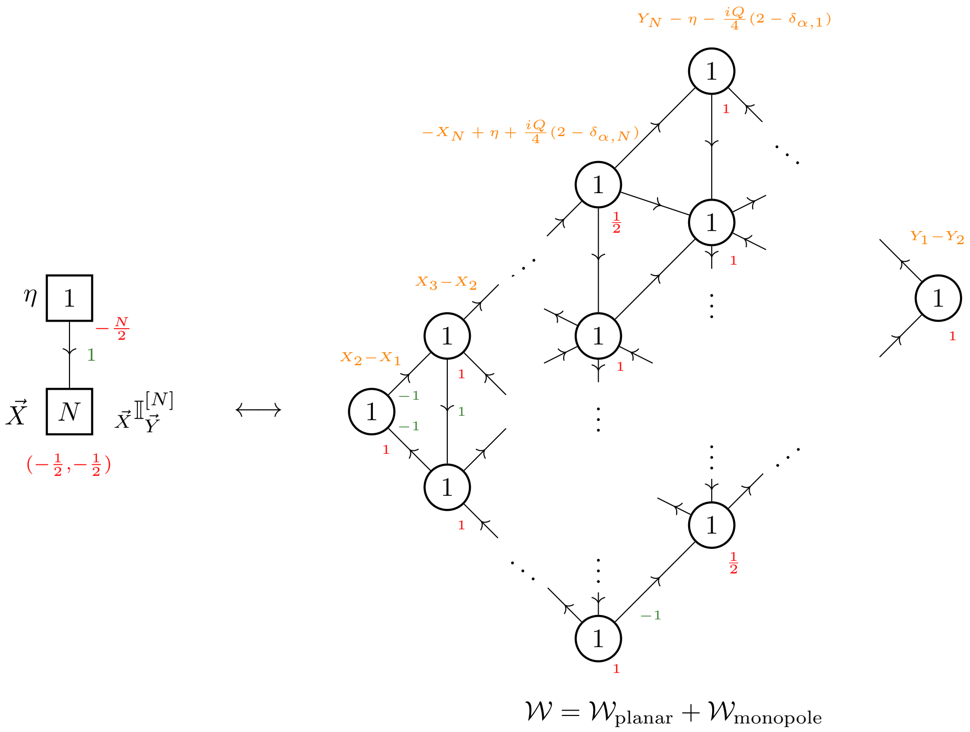


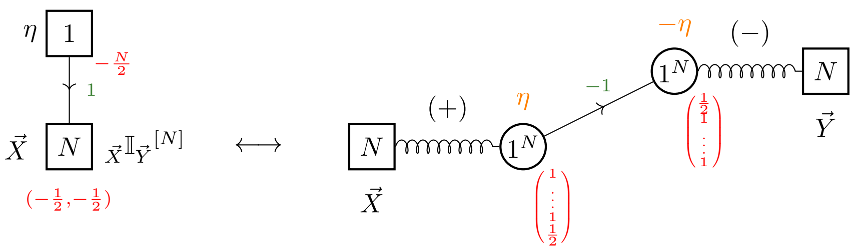
Figure 5.16: The planar mirror dual of the chiral anti-fundamental flavor block is shown here. Mixed CS interactions have been suppressed for brevity. Notice that w.r.t. the fusion to identity property in Figure 5.13, altough being very similar, in the basic move the planar quiver has an extra column of height N in the middle. As usual, on the planar side we give the FI parameter of all the nodes in a colum on top of it. The term $\delta_{\alpha,i}$ implies a shift in the FI of the i-th gauge node in the column, counting from below.

We emphasize the CS level for the $U(N)_{\vec{X}}$ symmetry in (5.100) refers to the contribution coming from the single chiral that has been integrated out. The overall background CS level for this symmetry has to take into account also the shift $(-N, 0)$ encoded in the identity wall, as previously mentioned so that the total CS level for the chiral block is $(-N - \frac{1}{2}, -\frac{1}{2})$.

In the duality in Figure 5.16 we recognize a chiral flavor block on the l.h.s. and the corresponding $\mathbf{S}^3_b$ partition function identity is:

$$Z^{[N]}_{\text{flavor,fund}}(\vec{X},\eta) = \int d\vec{Z}\, d\vec{W}\; Z_{\mathcal{S}}(\vec{X},\vec{Z})\; Z^{[1^N]}_{\text{bif, right}}(\vec{Z},\vec{W},-\eta;0)\; Z_{\mathcal{S}^-}(\vec{Y},\vec{W})\,. \tag{5.103}$$

Where on the r.h.s. we recognize a planar bifundamental sandwiched between two $\mathcal{S}$-walls. In short notation the duality in Figure 5.16 can be depicted as:



(5.104)

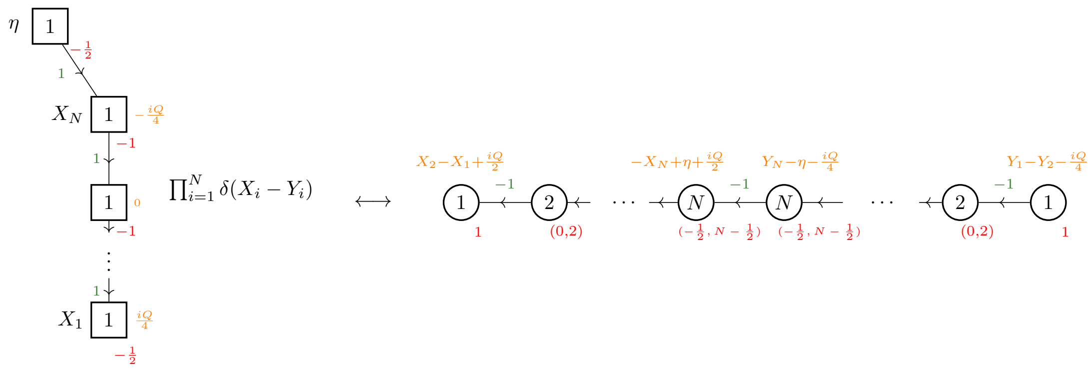
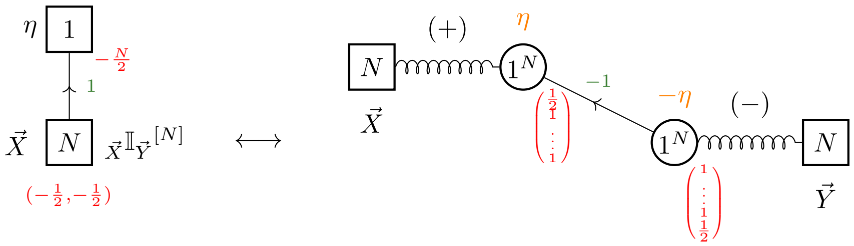


Figure 5.17: The planar mirror dual of the chiral anti-fundamental flavor block is shown here.

There is also a variant of this move, corresponding to the $\mathcal{S}$-dualization of a chiral anti-fundamental.



(5.105)

Which corresponds to the $\mathbf{S}_b^3$ partition function identity:

$$Z^{[N]}_{\text{flavor,antif}}(\vec{X},\eta) = \int d\vec{Z}\, d\vec{W}\; Z_{\mathcal{S}}(\vec{X},\vec{Z})\; Z^{[1^N]}_{\text{bif, left}}(\vec{Z},\vec{W},-\eta;0)\; Z_{\mathcal{S}^-}(\vec{Y},\vec{W})\,. \tag{5.106}$$

Planar fundamental $\mathcal{S}$-dualization

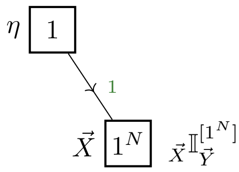
Starting again from the $\mathcal{N}=4$ basic move in the second line of (5.81), we perform the mass deformation defined by the following deformation breaking the $U(N)_{X,Y}$ symmetries:

$$\begin{cases} X_j \to X_j - (2N-2j+1)\frac{\tau}{2} \\ Y_j \to Y_j - (2N-2j+1)\frac{\tau}{2} \\ \vec{u}^{(I)} \to \vec{u}^{(I)} + (2N-I+1)\frac{\tau}{2}, \quad \text{for } I = 1,\dots,2N \\ \eta \to \eta \end{cases} \tag{5.107}$$

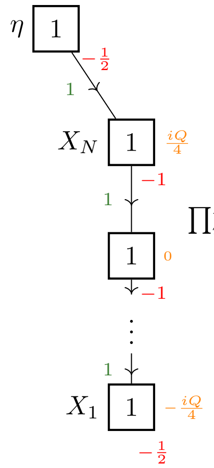
This limit yelds the duality in Figure 5.17. On the l.h.s. we identify the anti-fundamental

Planar flavor block for which we introduce the compact notation:

$$Z^{[1^N]}_{\text{flavor,antif}}(\vec{X},\eta) :=$$



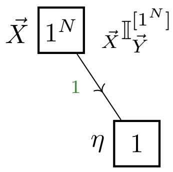
$$:=$$



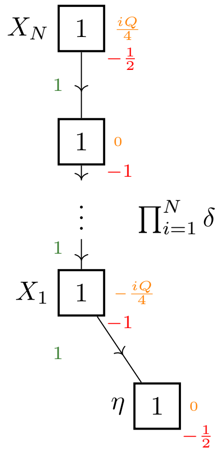
$$\prod_{i=1}^{N}\delta\left(X_i - Y_i\right) \tag{5.108}$$

Notice that all the arrows point downward. Similarly we define the planar fundamental block as:

$$Z^{[1^N]}_{\text{flavor,fund}}(\vec{X};\eta) :=$$



$$:=$$



$$\prod_{i=1}^{N}\delta\left(X_i - Y_i\right) \tag{5.109}$$

Notice that for the fundamental planar block the chiral is attached to the bottom of the string of chirals belonging to the Identity-wall. The contribution to the $\mathbf{S}^3_b$ partition function is given below:

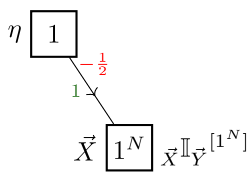
$$Z^{[1^N]}_{\text{flavor,fund}}(\vec{X};\eta) = e^{\frac{i\pi}{2}(\eta^2 - 2\eta X_1 + X_1^2)}\ s_b\left(\eta - X_1\right) {}_{\vec{X}}\,\mathbb{I}^{[1^N]}_{\vec{Y}} \tag{5.110}$$

$$Z^{[1^N]}_{\text{flavor,antif}}(\vec{X};\eta) = e^{\frac{i\pi}{2}(\eta^2 - 2\eta X_N + X_N^2)}\ s_b\left(X_N - \eta\right) {}_{\vec{X}}\,\mathbb{I}^{[1^N]}_{\vec{Y}} \tag{5.111}$$

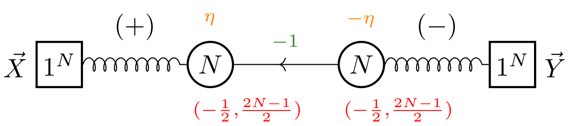
The duality in Figure 5.17 corresponds to the $\mathbf{S}^3_b$ partition function identity:

$$Z^{[1^N]}_{\text{flavor,antif}}(\vec{X};\eta) = \int d\vec{Z}\, d\vec{W}\ Z_{\mathcal{S}}[\vec{Z};\vec{X}]\ Z^{[N]}_{\text{bif, left}}(\vec{Z};\vec{W};-\eta;0)\ Z_{\mathcal{S}^-}[\vec{W};\vec{Y}] \tag{5.112}$$

We represent the previous move in compact form as:

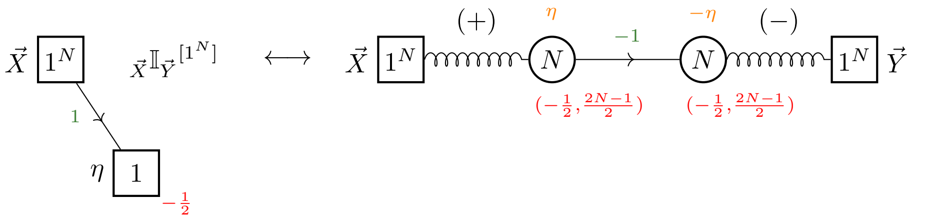


$$\longleftrightarrow$$



(5.113)

There is also a variant of this move that is the $\mathcal{S}$-dual of a planar fundamental block.



(5.114)

Which corresponds to the following identity between partition functions:

$$Z^{[1^N]}_{\text{flavor,fund}}(\vec{X};\eta) = \int d\vec{Z}\; d\vec{W}\; Z_{\mathcal{S}}[\vec{Z};\vec{X}]\; Z^{[N]}_{\text{bif, right}}(\vec{Z};\vec{W};-\eta;0)\; Z_{\mathcal{S}^-}[\vec{W};\vec{Y}]\,. \tag{5.115}$$

5.5 The Chiral-Planar dualization Algorithm at work

In this section we use the algorithm introduced in the previous section to produce new examples of the chiral-planar $\mathcal{N}=2$ mirror dualities.

Along the lines of [40], we define the steps of the algorithmic dualization of a non-abelian chiral $\mathcal{N}=2$ quiver to the be the following:

- **Step 1:** The quiver is chopped into its basic QFT blocks that can be either bifundamentals or (anti-)fundamental chiral fields. This step is performed by freezing the gauge interactions that will be reintroduced later.
- **Step 2:** Each QFT block is dualized via the basic duality moves described in Section 5.4.2. These identities, or *basic moves* are summarized for reference in Table 5.2.
- **Step 3:** The dualized blocks are glued back together by turning back on the gauge interactions frozen at the first step. The pairs $\mathcal{S}^{-1}\mathcal{S}$ fuse to Identity-walls as described in Section 5.4.1.

Each of these steps will be illustrated in more detail in the following sections with explicit examples.

We will use the algorithm to dualize chiral $\mathcal{N}=2$ quivers into abelian-planar quivers. Although the same algorithm can be applied in reverse — starting from a planar abelian quiver to obtain its chiral dual — we do not discuss this case in the present work.

The current version of the algorithm has some limitations that we list below.

- We can dualize chiral quivers with fixed CS levels. A $U(N)$ node with F chirals will have CS level $(k, k+lN)$ with $k = -\frac{F}{2} + N$ and $l = -1$. Moreover, each pair of gauge nodes connected by a bifundamental chiral will be also coupled via a mixed-CS interaction.
- We can dualize only chiral quivers with constant ranks $U(N)$. To algorithmically dualize quivers with non-constant ranks we would need to introduce asymmetric blocks and the chiral-planar version of the Hanany-Witten duality moves derived in [40].

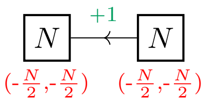
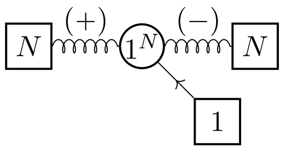
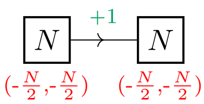
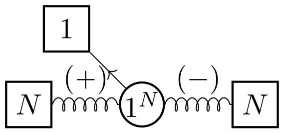
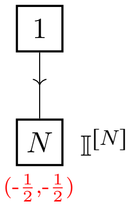
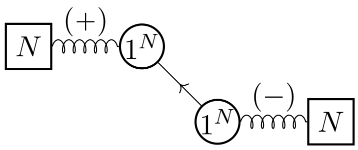
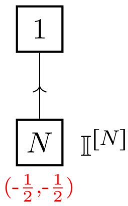
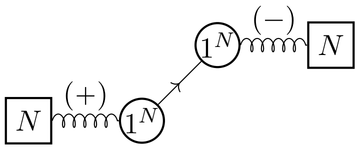
<table>
<tr><th>QFT block</th><th>$\mathcal{S}$-dualization</th><th>$\mathbf{S}_b^3$ p.f. identity</th></tr>
<tr><td></td><td></td><td>Fig. 5.14
Eq. (5.85)</td></tr>
<tr><td></td><td></td><td>Eq. (5.90)</td></tr>
<tr><td></td><td></td><td>Fig. 5.16
Eq.(5.103)</td></tr>
<tr><td></td><td></td><td>Eq. (5.106)</td></tr>
</table>

Table 5.2: Summary of the basic moves that are relevant to run the algorithm. In the first column are given the basic QFT blocks, in the second column their $\mathcal{S}$-duals. In the last column we give the reference to the $\mathbf{S}_b^3$ partition function identity and to the extended quiver form. We avoided to give all the CS-interactions and FI terms, we only provided those that are dynamical either in the starting chiral theory or in the reulting planar theory. Notice that in order to keep track of all the details more information are needed, see Section 5.4 for a complete discussion.

- We also limit ourselves to cases where each gauge node sees $[n_f, n_a]$ chirals, where $F = n_f + n_a$ with

$$[n_f, n_a] = \begin{cases} [N + F_1,\ N + F_2] \\ [2N + F_1,\ F_2] \\ [F_1,\ 2N + F_2] \end{cases} \qquad F_1, F_2 \geqslant 0 \tag{5.116}$$

The algorithm can be further extended to overcome the current limitations, but we leave this for future work. Some of these limitations can be overcomed by considering a richer class of $\mathcal{N} = 4$ QFT blocks to begin with and also by considering further real mass deformations of the $\mathcal{N} = 2$ mirror dualities to unlock more generic CS levels (see the discussion in Section 6 of [44]).

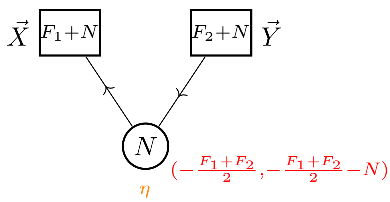
It is important to emphasize that, even within its current scope, the algorithm already produces a rich variety of examples that capture essential features of the new type of mirror duality proposed in this paper. These will be discussed in detail in the following sections.

5.5.1 Chern-Simons SQCD with fundamental and anti-fundamental Matter

As a first example of the application of the algorithm, we consider the case of SQCD with $[n_f, n_a]$ flavors, where $F = n_f + n_a$ satisfying the constraint in eq. (5.116). The second and third cases are analogous, therefore we explicitly present only the first and second case in eq. (5.116).

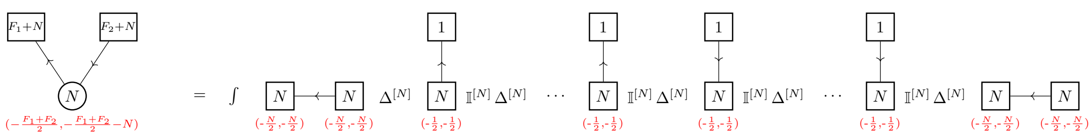
$U(N)$ CS-SQCD with $[F_1 + N, F_2 + N]$ Chiral Multiplets

We start from the first case in (5.116), therefore we consider $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2} - N)}$ SQCD with $[F_1 + N, F_2 + N]$ chiral multiplets, depicted in short as:



(5.117)

To apply our algorithm we begin by chopping the theory into chiral QFT blocks using the basic ingredients defined in Section 5.4. We do so by considering F_1 chiral fundamental blocks, F_2 chiral anti-fundamental blocks and two chiral bifundamental blocks, in complete analogy with the $\mathcal{N} = 4$ SQCD case (see Chapter 2) where the two bifundamentals account effetively for $2N$ flavors. The QFT block decomposition is schematically depicted below.



(5.118)

In this schematic picture the act of multiplication between two consecutive blocks is understood. The measure of integration between two blocks is given by a $U(N)$ vector field $\Delta^{[N]}$ which is precisely the measure defined in (5.73). To avoid cluttering we suppressed the labeling of global symmetry parameters.

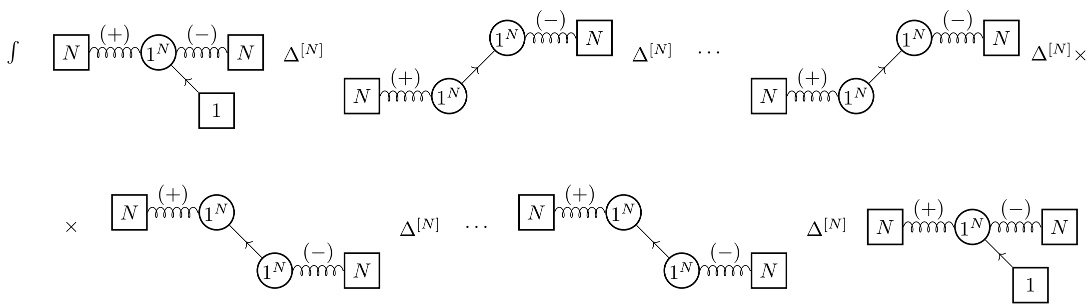
Let's first check that gluing back the QFT blocks we recover the $U(N)_{(-\frac{F_1+F_2}{2},-\frac{F_1+F_2}{2}-N)}$ SQCD. The gluing procedure consist in identifying and gauging two $U(N)$ global symmetries, one coming from each of the two blocks glued together. This means that there must be a total of (#QFT blocks − 1) integrations to account for the gluing. Taking into account the presence of $F_1 + F_2$ Identity-walls coming from flavor blocks, each freezing an integration, we are left with a single dynamical, i.e. non-frozen, $U(N)$ gauge symmetry.

We recall that each QFT block, Identity-wall and measure of integration comes together with CS interactions. In the schematic figure we give explictly only those coming from the QFT-blocks and leave the remaning implicit. The resulting CS level upon gauging and implementing the Identity-walls can be computed as follows. Each factor of $\Delta^{[N]}$ contributes as $(N, 0)$ to the CS level. Each flavor block consists of a (anti-)fundamental chiral field, with a CS level $(-\frac{1}{2}, -\frac{1}{2})$, and an identity, contributing as $(N, 0)$, for a total contribution of $(-N - \frac{1}{2}, -\frac{1}{2})$. Lastly, a bifundamental chiral contributes with a CS level of $(-\frac{N}{2}, -\frac{N}{2})$. Therefore the total CS level is:

$$\underbrace{2(-\frac{N}{2},-\frac{N}{2})}_{\text{bifundamental blocks}} + \underbrace{(F_1+F_2)(-N-\frac{1}{2},-\frac{1}{2})}_{\text{flavor blocks}} + \underbrace{(F_1+F_2+1)(N,0)}_{\text{chiral measure}} = (-\tfrac{F_1+F_2}{2}, -\tfrac{F_1+F_2}{2} - N) \tag{5.119}$$

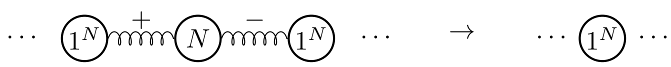
which matches the CS level of the SQCD theory we are considering.

Now we dualize each QFT block in the decomposition in (5.118) by exploiting the basic moves derived in Section 5.4 and summarized in Table 5.2, we obtain:



(5.120)

Now we can use the chiral fusion to identity 5.77 to remove pairs of $\mathcal{S}$-walls glued through their $U(N)$ symmetry. Schematically this operation can be represented as:



(5.121)

This allows us to remove all but the first and last $\mathcal{S}$-walls. The resulting generalized

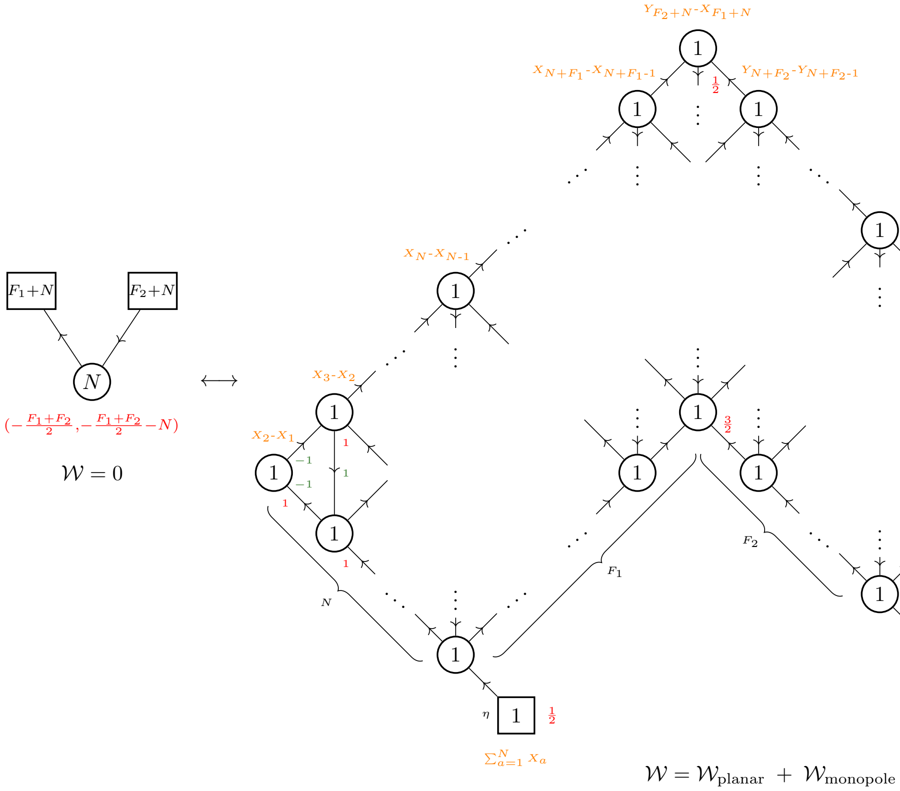
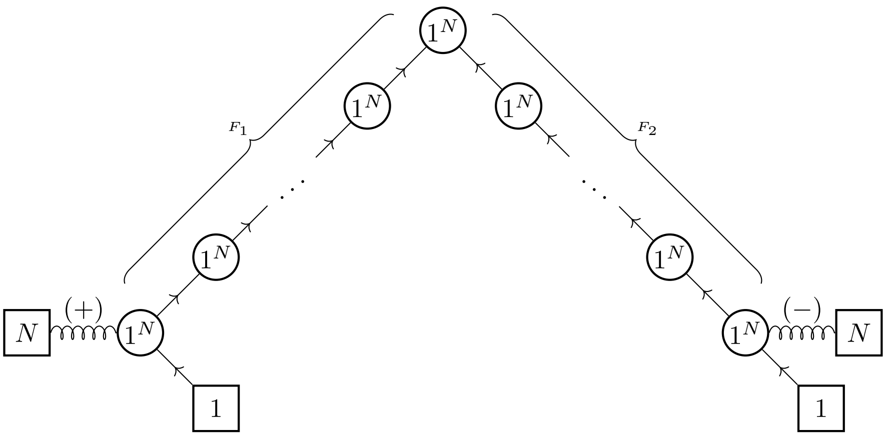


Figure 5.18: The $\mathcal{N}=2$ planar mirror dual of $U(N)_{(-\frac{F_1+F_2}{2}, -\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1+N, F_2+N]$ chiral multiplets. The superpotential terms and (mixed) Chern-Simons interactions are as described in Section 5.2.

quiver in short notation is:



(5.122)

The generalized quiver in (5.122) can finally be written in the explicit Lagrangian form by using the planar UV completion of the $\mathcal{S}$-walls and the definition of the planar bifundamental and fundamental blocks. The result is depicted in Figure 5.18.

Let us comment some of the details of the duality in Figure 5.18.

The global symmetry of the SQCD theory is, in general, $S[U(F_1+N)\times U(F_2+N)]\times U(1)$, where the first factor is the flavor symmetry while the second one is the topological symmetry[18]. In the mirror dual the $U(1)$ symmetry is the only flavor symmetry, taking into account gauge transformations and superpotential constraints.

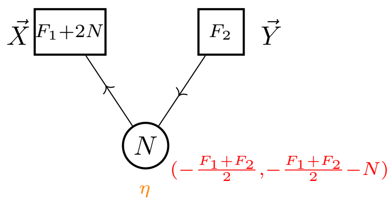
The $\mathcal{W}_{\text{monopole}}$ superpotential contains a term that is linear in the monopole with $-1/+1$ magnetic flux under two nodes connected by a vertical line, which reduce to a single $U(1)$ the topological symmetry of each column. So the UV topological symmetry is $U(1)^{F_1+F_2+N-1}$ which enhances in the IR to $S[U(F_1+N)\times U(F_2+N)]$, so that the global symmetry of the mirror quiver matches that of the SQCD.

In the SCQD the chiral ring is generated by mesonic operators constructed from the fundamentals Q_i and antifundamentals $\tilde{Q}_j$ that form a bifundamental representation of $SU(F_1+N)\times SU(F_2+N)$. These are mapped to monopole operators that are gauge invariant in the planar theory. For example the meson $Q_{F_1+N}\tilde{Q}_{F_2+N}$ is mapped to the monopole with flux $+1$ under the topmost gauge node of the abelian dual. The other mesons are mapped to monopoles with two strings of $+1$ fluxes starting at the topmost node and propagating along the two upper diagonals of the planar quiver. There are $(F_1+N)(F_2+N)$ such monopoles which reproduce the bifundamental representation of the enhanced $SU(F_1+N)\times SU(F_2+N)$ global symmetry.

Notice that in this case, the planar dual never exhibits a mesonic chiral ring generator. This predicts that, for all values of F_1 and F_2, there are no dressed gauge-invariant monopoles in the SQCD theory that would parametrize a non-compact moduli space.

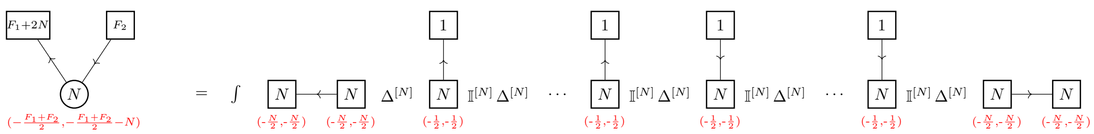
$U(N)$ CS-SQCD with $[F_1+2N, F_2]$ Chiral Multiplets

We now consider the second case in (5.116) (which is analogous to the third one), the $U(N)_{(-\frac{F_1+F_2}{2},-\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1+2N, F_2]$ chiral multiplets:



(5.123)

We decompose this theory into fundamental blocks according to the rules described in the previous section:

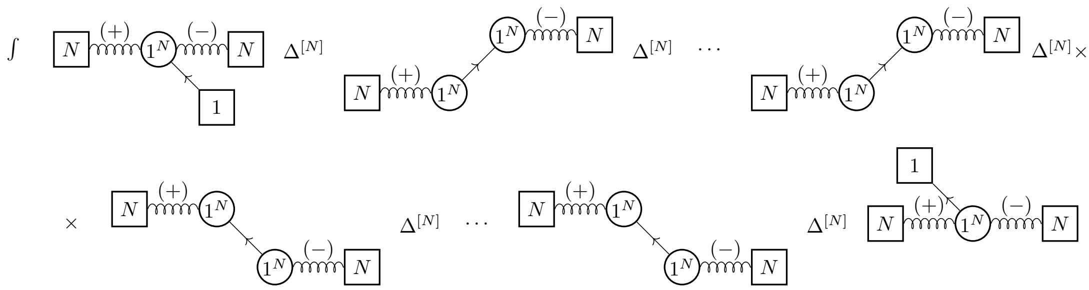


(5.124)

where the diagonal gauging of adjacent $U(N)$ fugacities is understood. The difference w.r.t. (5.118) is that the last bifundamental is right-pointing, so that it contributes as N extra fundamentals instead of anti-fundamentals. We dualize each block using the

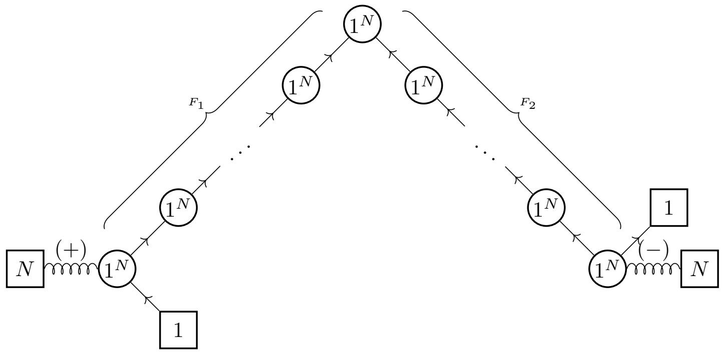
[18]For special cases, namely $F_1 = F_2 = 0$ the topological symmetry enhances to $SU(2)$. This can also be understood by considering the Aharony-like dual for the theory [85], which is an abelian gauge theory, and then taking the mirror dual to that theory [37]. We will comment on this feature later in Section 5.5.2 and study other chiral theories exhibiting enhancement of the topological symmetry.

basic moves in Table 5.2:



(5.125)

The dualized QFT blocks can be glued back reintroducing gauge interactions with $\Delta^{[N]}$ measure, we also implement Identity-walls and obtain:



(5.126)

This quiver can be written explicitly in Lagrangian form using the definitions of the planar blocks, the result in depicted in Figure 5.19.

The features of this duality are very similar w.r.t. the duality in Figure 5.18, so that the map between global symmetries and chiral ring generators works similarly as that explained at the end of Section 5.5.1[19].

However, in this case there can be an extra chiral ring generator which is a mesonic operator from the point of view of the planar dual. It is possible to deduce that such operator exists only if $F_2 < N$. This operator maps to a gauge invariant (dressed) monopole operator with positive magnetic charge of the SQCD. It is possible to verify, for example through a Superconformal Index expansion, that in the SQCD this operator exists for the same range of F_2.

[19] In this case there can not be any enhancement of the topological symmetry for special values of F_1 and F_2 differently from the case of the SQCD with $[F_1 + N, F_2 + N]$ flavors.

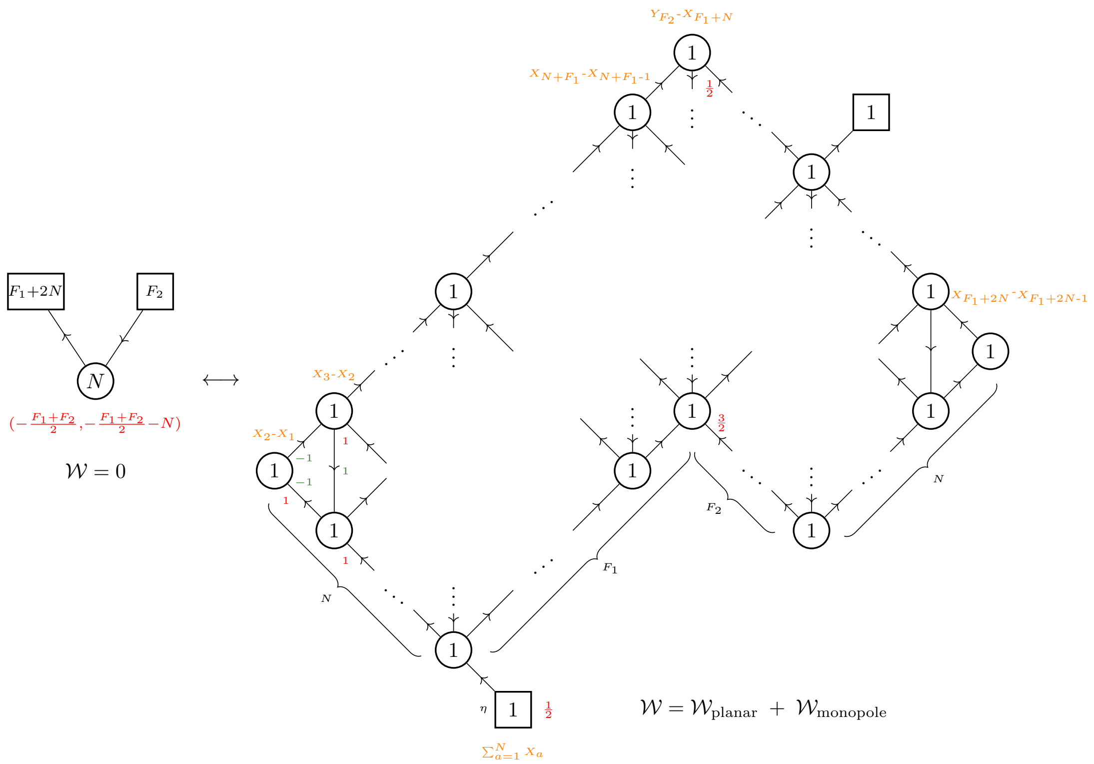
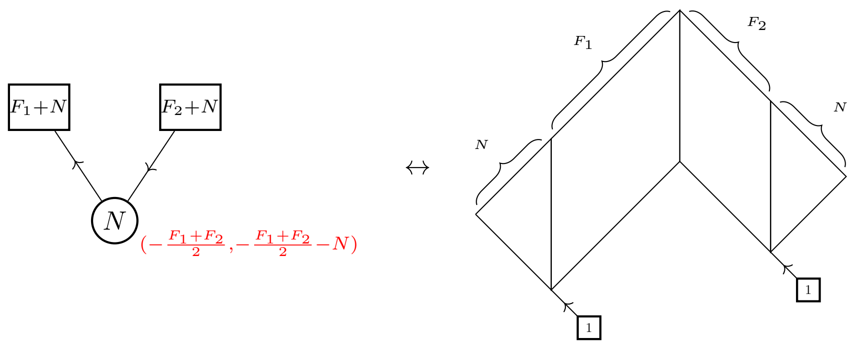
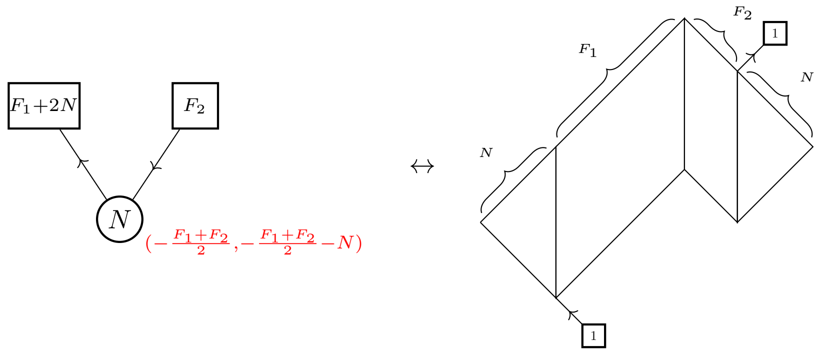


Figure 5.19: The $\mathcal{N}=2$ planar mirror dual of $U(N)_{(-\frac{F_1+F_2}{2},-\frac{F_1+F_2}{2}-N)}$ SQCD with $[F_1+2N, F_2]$ chiral multiplets. The superpotential terms and (mixed) Chern-Simons interactions are as described in Section 5.2.

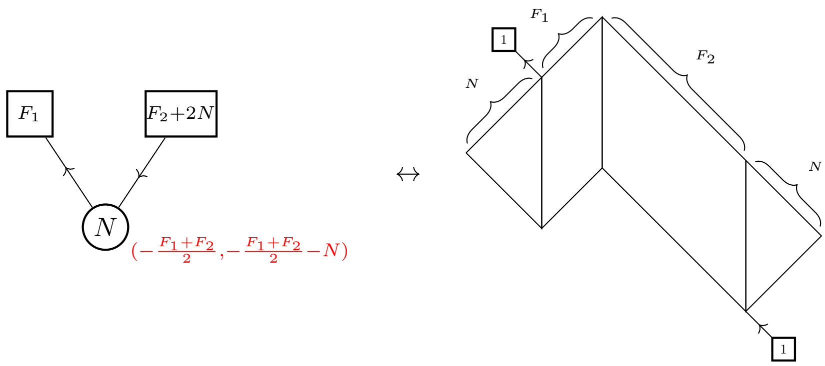
To summarize, the duals of the SQCD theories considered in this Section can be schematically depicted as:



(5.127)



(5.128)



(5.129)

Notice that for high enough number of fundamentals and antifundamentals an SQCD theory can admit multiple mirror duals among the ones presented above. As an example, $U(N)$ SQCD with $n_f \geqslant 2N$ fundamentals and $n_a \geqslant N$ antifundamentals admits two inequivalent abelian planar duals, schematically (5.127) and (5.128). It would be interesting to understand whether these two duals, in the range of n_f and n_a mentioned above, can be connected by performing local Aharony-like dualities, in the spirit of the analysis performed in Appendix B of [44] for the case of the flip-flip duality, but we leave this to future work.

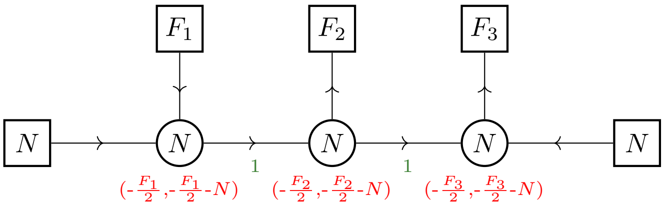
It is possible to derive planar abelian duals for SQCD theories with $SU(N)$ gauge group or with $U(N)$ gauge group and CS levels $(k, k+lN)$ for any l by applying Witten's $SL(2,\mathbb{Z})$ action [82] on both sides of the dualities described above. This was described in detail for the case of SQCD with only fundamental chirals in Section 5.2.3, the generalization to SQCD with both fundamentals and antifundamentals is analogous and we do not carry it out explicitly.

5.5.2 Topological symmetry enhancement in chiral quivers

A "local" balancing condition

We now move to an example of a quiver theory to show an interesting pattern of topological symmetry enhancement.

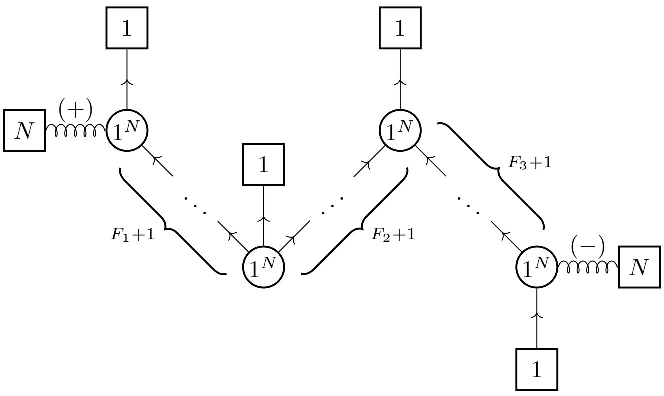
We consider the following quiver:



(5.130)

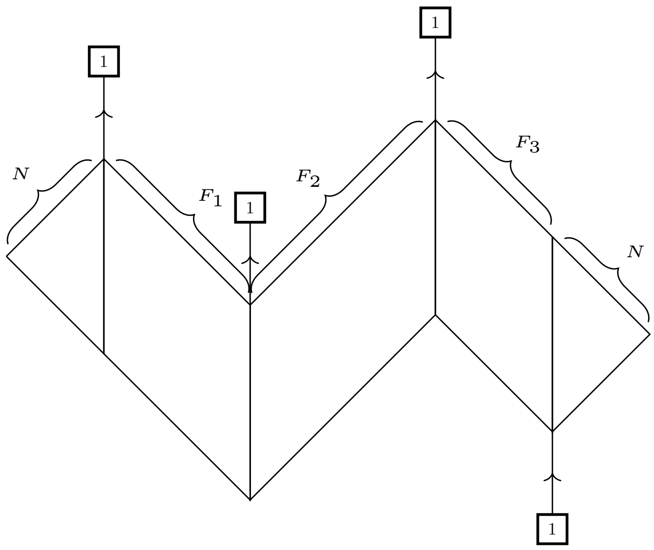
Notice that, as emphasized before, the CS levels for each gauge group are not arbitrary and are fixed by the amount of matter fields that each node sees following (5.64). The quiver presented here serves as a specific example, but it is straightforward to modify many of its properties. For instance, the directions of the bifundamental arrows can be flipped, and the flavor content can be generalized by considering different numbers of fundamentals and anti-fundamentals. However, for the sake of concreteness, we focus on this particular choice in the current discussion.

The mirror theory can be obtained by applying the algorithmic procedure described above, resulting in the following planar quiver. Here we only report the compact notation for the mirror quiver for ease of readability:



(5.131)

Therefore the planar quiver has the following structure:



(5.132)

If we slightly modify the starting quiver in (5.130) — for instance, by reversing the direction of a bifundamental arrow or swapping a fundamental flavor with an

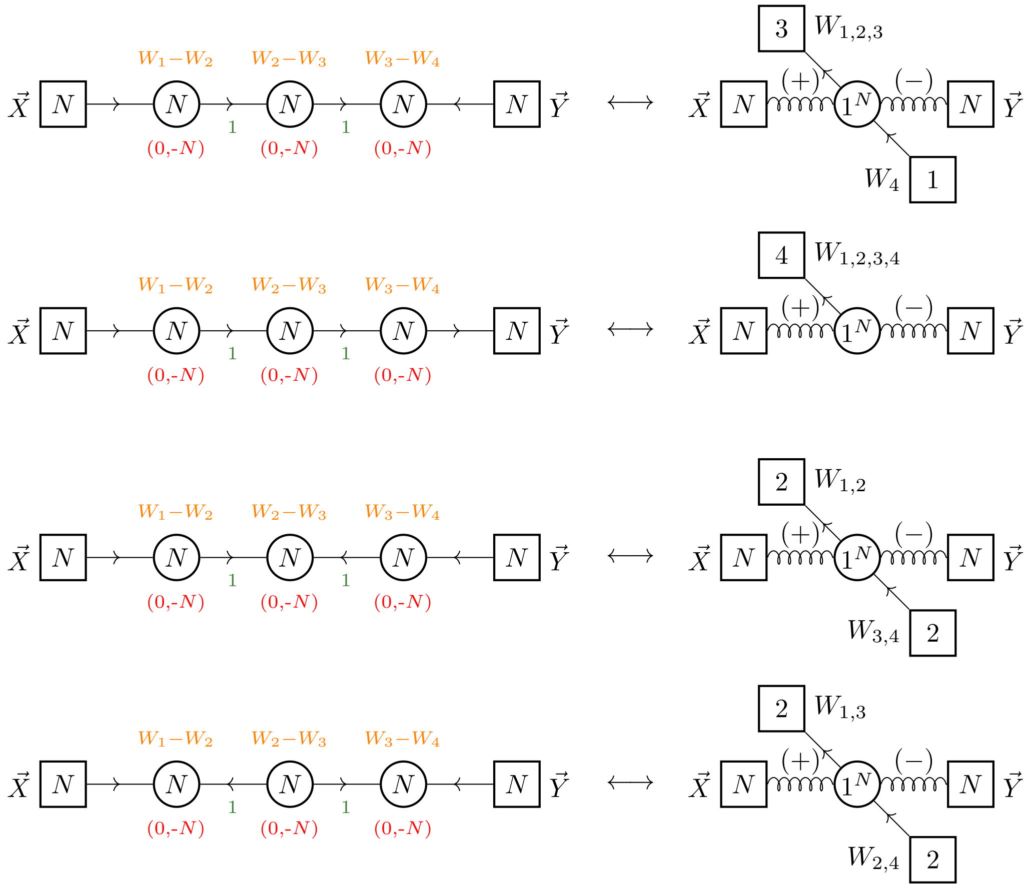


Figure 5.20: Chiral-Planar mirror dualities for a quiver with three gauge nodes. The planar dual of each chiral quiver makes the enhancement of the topological symmetry manifest. In general, the topological symmetry enhances to $S[U(N_L) \times U(N_R)]$, where N_L and N_R are the numbers of left- and right-pointing bifundamentals, respectively. This enhancement occurs independently of whether bifundamentals with the same orientation are consecutive. Furthermore, examining the last two dualities, we observe that all theories with the same values of N_L and N_R are dual to each other.

anti-fundamental — the resulting quiver in (5.131) changes accordingly. For example changing the arrow of a bifundamental changes a fundamental flavor attached to the top node of a column, with an anti-fundamental attached to the bottom node of the same column. Or also, changing a flavor from fundamental to anti-fundamental changes the slope of a planar bifundamental.

It is interesting to consider the special case where each gauge group has exactly $2N$ chiral (anti-)fundamentals, that is the case of $F_1 = F_2 = F_3 = 0$. In this case the mirror planar quiver does not have sequences of planar bifundamentals, and all the flavor nodes are attached to the central column. The result is depicted in the first line of Figure 5.20. This leads to the flavor symmetry of the planar mirror theory to increse, in this case from $U(1)^4/U(1)$ to $SU(3) \times U(1)$.
Changing the orientation of the arrow produces different dualities whith possibly different enhancements of the global symmetry. Some other possibilities are depicted in Figure 5.20. This, as far as we know, is a non-trivial prediction regarding the enhancement of topological symmetries in $\mathcal{N} = 2$ quiver theories.

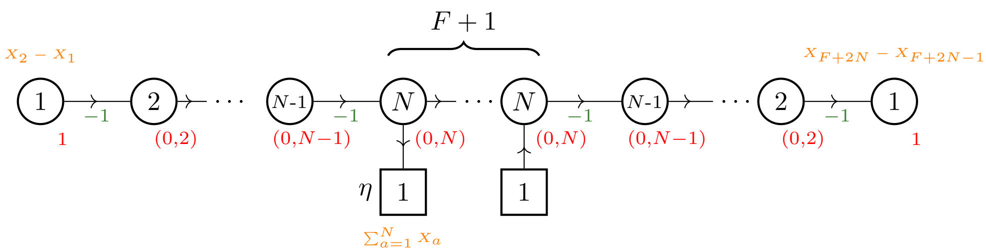
This suggests us that a rule to predict the enhancement of global symmetries.

Whenever a $U(N)$ gauge node sees exactly $[N,N]$ flavors the topological symmetry enhances to at least $SU(2)$. When in a quiver more than one gauge node satisfied this condition then the enhancement is bigger. For example in Figure 5.20, it is shown a collection of linear quivers with $n=3$ nodes of rank N, with N_R bifundamental chirals pointing to the right and N_L pointing to the left, with $N_R+N_L=n+1$. Then the chiral-planar mirror duality predicts that the $U(1)^n$ topological symmetry enhances to $S[U(N_r)\times U(N_\ell)]$. The limiting case with $n=1$ corresponds to SQCD with $U(N)_{(0,-N)}$ gauge group and $[N,N]$ flavors, where the topological symmetry enhances from $U(1)$ to $SU(2)$. This is a special case of the dualities already discussed in Subsubection 5.5.1. It is important to notice that the enhancement occurs for any quiver with the same values of N_R and N_L, regardless of the ordering of the bifundamentals since the planar mirror dual is invariant under this choice. This fact implies a duality among all the quivers with the same value of N_L and N_R, one such example is provided in Figure 5.20.

A "non-local" balancing condition

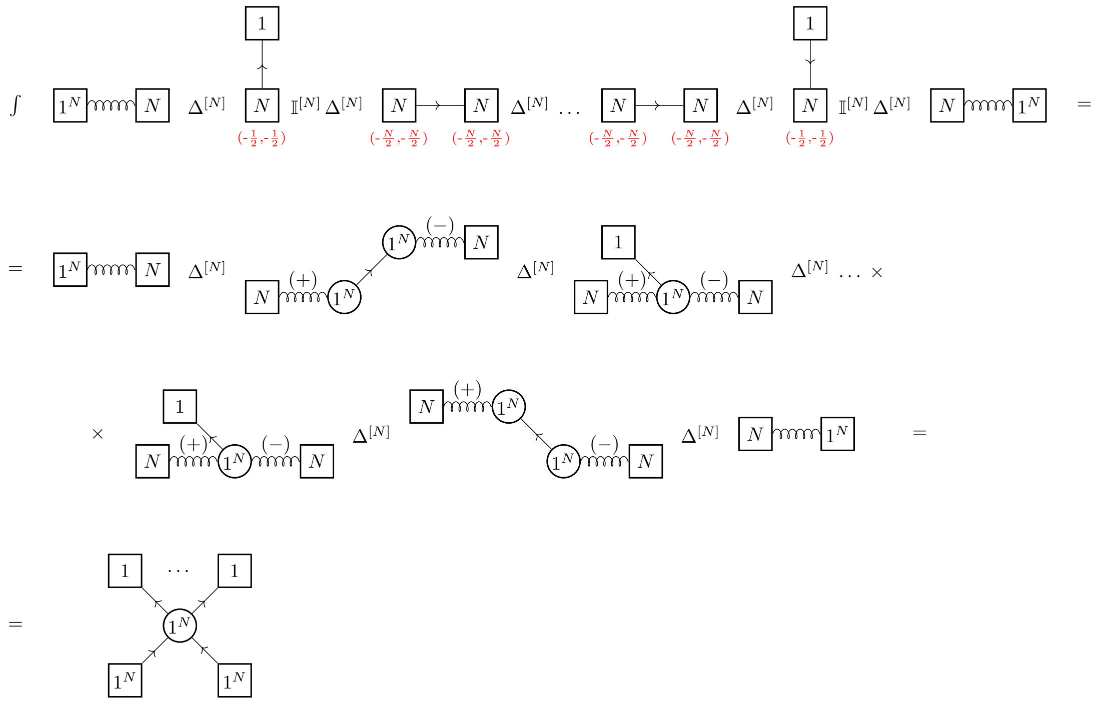
In Section 5.2, we started from the $\mathcal{N}=4$ mirror pair of the $U(N)$ SQCD and considered the real mass deformation leading to the $\mathcal{N}=2$ mirror duality relating the chiral $U(N)$ SQCD to its planar quiver dual pair, depicted in Figure 5.2.

It is interesting to consider the alternative situation where instead we take the chiral limit of the quiver on the dual side, which results in a planar limit for the SQCD. This can indeed be studied using the same strategies outlined in Section 5.2; however, here we will instead use the algorithm to generate the example. We begin with the chiral limit of the mirror of SQCD, namely:



(5.133)

We decompose this theory into fundamental blocks according to the rules described in the previous section, dualize each of them and then glue-back the result implementing the Identity-walls. These steps are depicted in the following figure:



(5.134)

Notice that the quiver considered in (5.133) actually does not follow the rule prescribed at the beginning of the section since it contains two tails built with asymmetric $U(N) \times U(M)$ bifundamentals. However we can bypass the problem of dualizing asymmetric bifundamental blocks by noticing that the two tails can be thought simply as a $\mathcal{S}$ and $\mathcal{S}^{-1}$-walls. These walls do not need to be dualized and in the last step they fuse to Identity-walls with two other walls coming from the dualization of the flavor blocks.

The resulting quiver can be then written explicitly in Lagrangian form, providing the planar mirror description for the original linear quiver gauge theory (5.133), as shown in Figure 5.21.

Interestingly, in the planar dual, the flavor symmetry is not the naive $U(1)^{F+2N}/U(1)$, but is instead enhanced to $S[U(F-1) \times U(2)^N]$, as shown in (5.133). This reveals an intriguing pattern of topological symmetry enhancement. The appearance of the $U(F-1)$ factor can be understood using the rule proposed in the previous section: the central part of the quiver consists of a sequence of $U(N)$ nodes, each with $[N, N]$ flavors and bifundamentals oriented in the same direction. However, the emergence of the multiple $U(2)$ factors is a new feature specific to this duality. Notably, this latter enhancement involves topological symmetries associated with non-adjacent nodes — that is, it is non-local — in contrast to the local structure underlying the $U(F-1)$ enhancement. Nonetheless, both enhancements are manifest in the planar dual.

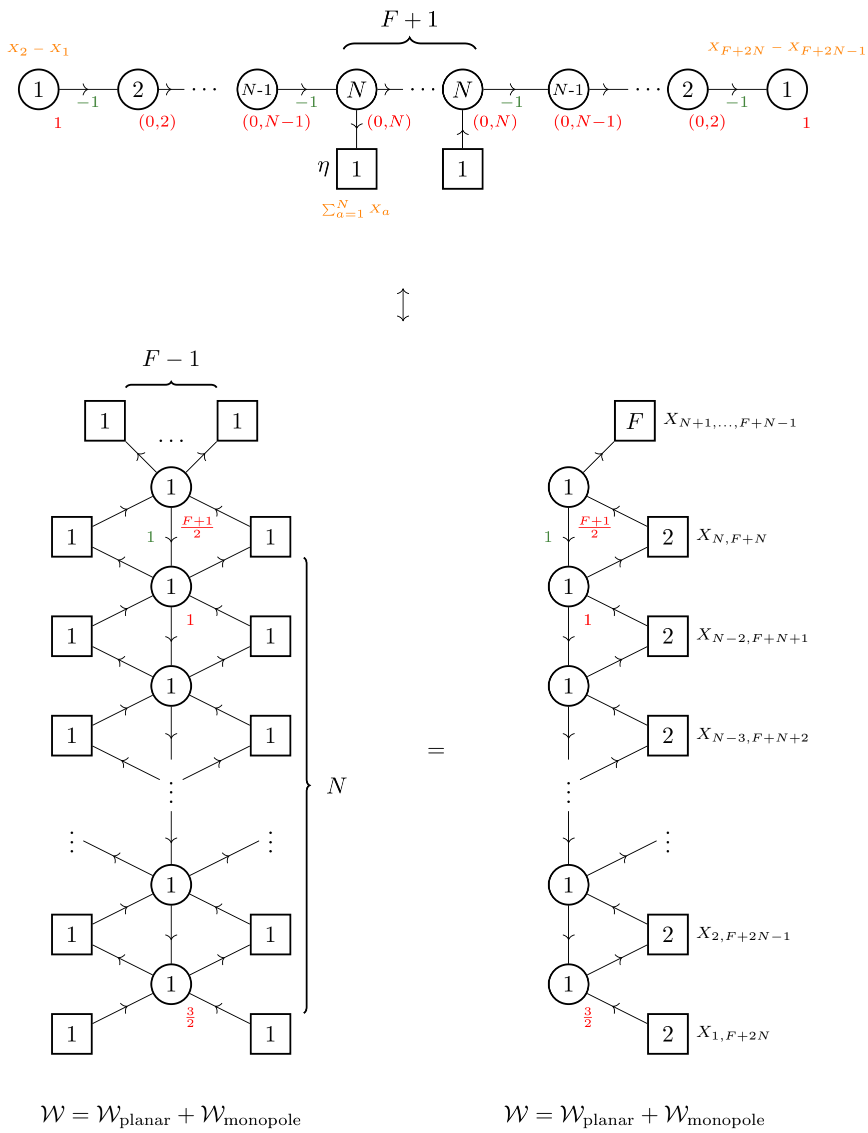


Figure 5.21: On the top, the mirror dual of the planar limit of the $\mathcal{N} = 4$ SQCD, reported from 5.133. On the bottom, the planar limit of the $\mathcal{N} = 4$ SQCD. We have suppressed the FI parameters and all the background terms for brevity. The FI terms for the gauge nodes are $\eta + \frac{iQ}{4}((F-1)\delta_{1,\alpha} - \delta_{N,\alpha})$. On the r.h.s. it is depicted the same theory but after recognizing the manifest flavor symmetry $S[U(F-1) \times U(2)^N]$ and writing it explicitly.

5.6 Conclusions

In this chapter we have presented a novel $\mathcal{N} = 2$ mirror duality that relates non-Abelian Chern-Simons theories and planar Abelian quiver theories. Such dualities can be obtained in two equivalent ways. We can start from any $\mathcal{N} = 4$ mirror pair and study the SUSY breaking deformation to $\mathcal{N} = 2$ using the strategies described in this chapter. Alternatively, we also have at disposal an algorithmic strategy that is derived from the $\mathcal{N} = 4$ dualization algorithm.

From the discussion presented in this chapter is it clear that it is possible to compute a planar Abelian mirror dual for a vast class of quiver gauge theories with $U(N)$ group,

however the Chern-Simons levels in these theories are constrained to be a functions of the rank and matter. In [44] more results are presented that are not reported in this thesis, which generalize this statement and we now briefly comment on them.
One possible generalization consists in taking real mass deformations, which in general have the effect of reducing the amount of matter, by giving them a non-zero real mass which generates a shift CS level. In this way it possible to relax the constraint on the CS level.
A second constraint concerns on the shape of the quiver, which in this chapter has been taken to be linear. However, by starting from more generic mirror dualities relating non-linear quivers, for example circular ones, we are able to straightforwardly generalize the result.
A final generalization consist in considering theories with different gauge groups. For example, it is possible to construct planar-Abelian duals of theories with $USp(2N)$ gauge groups, again by starting from suitable $\mathcal{N}=4$ mirror dualities.
Further generalizations, for example that involving ortho-symplectic gauge groups, are still missing and we plan to investigate further in the future.

Chapter 6

$\mathcal{N} = 0$: Further SUSY breaking deformations from $\mathcal{N} = 2$

Inspired from the results presented in Chapter 5, we propose $\mathcal{N} = 0$ mirror dualities for $SU(N)$ CS-QCD$_3$ with scalars and fermions. We will test this educated guesses by matching the global symmetries and the spectrum of the lightest operators.
The content of this chapter is based on [45].

6.1 Introduction

The first example of what can be considered a mirror-like duality between $\mathcal{N} = 0$ theories was proposed a very long time ago [38, 86] and goes by the name of particle-vortex duality. This duality relates a critical complex scalar field to a $U(1)$ gauge theory coupled to an Abelian Higgs model. A key feature of this duality is that it maps the non-gauged scalar to monopole operators (or vortices) of the gauge theory. More examples of similar dualities have been proposed later, for example a fermionic generalization of particle-vortex duality [87]. Other notable non-supersymmetric dualities are the so-called *bosonization* or *fermionization* dualities, in which gauge theories with Chern-Simons interactions coupled to fermions/bosons are dual to "free" theories of bosons/fermions. Interestingly, it was found in [88, 89] that it is possible to construct a web of dualities for Abelian Chern-Simons theories with bosons, that generalize particle-vortex duality, simply by assuming the bosonization duality, in a spirit not much different from [31] and from the dualization algorithm in Chapter 3.
More non-supersymmetric dualities, including those for non-Abelian gauge theories, have been established over the last decade [90, 91, 92, 93, 94]. These dualities have been checked mostly in two ways, namely the matching of the spectrum of the lightest operators and of the global symmetries of the two theories. A third highly non-trivial check is also the matching of the massive phases, that are typically described by TQFTs.

An intriguing observation about these dualities is that usually the non-supersymmetric dualities are very similar to their supersymmetric counterpart. The plan of this chapter is therefore to take inspiration from the mirror dualities of Chapter 5 to make educated guesses for non-supersymmetric ones that can be then tested. All the dualities that we will propose will be named mirror, as they will map operators constructed from elementary fields to dressed monopole operators. We will focus only on the case of the $SU(N)$ CS-SQCD$_3$ presented in Chapter 5 resulting, in fact, in planar Abelian dualities for the CS-QCD$_3$.
Interestingly, these dualities can be thought of as the result of an RG flow triggered by a

SUSY-breaking deformation. For example, sticking to the case of the $\mathcal{N}=2$ $SU(N)$ CS theory with F fundamental chirals, we imagine a mass deformation for all the fermions in the theory resulting in a $SU(N)$ CS theory coupled only to fundamental complex scalars. In the dual theory, we conjecture that this deformation corresponds also to a mild deformation. Specifically, we give a mass to either the fermion or the scalar composing each chiral field, resulting in an Abelian planar quiver theory constructed from bosons and fermions with CS interactions. We are thus assuming that the map of the SUSY breaking deformation is quite simple and that no phase transitions occur.

The proposed dualities will be tested through with the matching of global symmetries and the spectrum of the lightest operators, for which we compute the spin and the representation under the global symmetry. In the last section we also comment on the matching between massive phases of these theories, thus further motivating the dualities.

6.2 Linear Abelian Duals of CS-QED$_3$ with Bosons and Fermions

As a warm up, in this section we propose a duality for QED$_3$ with bosonic and fermionic matter and Chern-Simons interactions, which is a mild generalization of two *bosonization* dualities of [89]. This duality relates the $U(1)_{-N_s-N_f/2-k}$ QED theory with N_s scalars and N_f fermions to an Abelian linear quiver theory with N_s+N_f+k-1 $U(1)$ nodes. This duality is the $N=1$ case of the general duality for $U(N)$ or $SU(N)$ QCD with bosons and fermions discussed in this paper.

The relevant matter content are complex fermions, usually denoted as ψ, complex scalars, denoted as ϕ and real scalars denoted as σ.

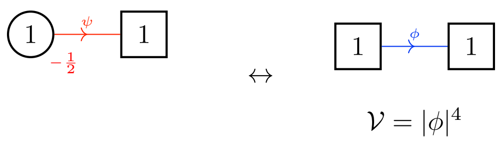
The simplest instances of the duality are the well known bosonization and fermionization dualities, which can be schematically written down as follows:[1]

$$U(1)_{-\frac{1}{2}} \;+\; 1\psi \qquad \leftrightarrow \qquad 1\phi \tag{6.3}$$

$$U(1)_{-1} \;+\; 1\phi \qquad \leftrightarrow \qquad 1\psi \tag{6.4}$$

It is convenient to write the dualities (6.3), (6.4) in a quiver notation in view of the generalization discussed below:

bosonization:



(6.5)

[1]In this paper we normalize a fermion charged under a $U(1)$ gauge field A as:

$$Z_{fermion} = |\det \not{D}_A|\, e^{-\frac{i\pi}{2}\eta(A)}. \tag{6.1}$$

therefore a mass deformation m for a fermion produces:

$$i\bar{\psi}\not{D}_A\psi \quad \xrightarrow{m\bar{\psi}\psi} \quad \begin{cases} 0 & m>0 \\ -\frac{1}{4\pi}AdA - 2\mathrm{CS}_g & m<0 \end{cases} \tag{6.2}$$

When we present dualities in compact notation, such as in (6.3), (6.4) and in the quiver-like notation described below we use the slightly imprecise but convenient notation where $\frac{i\pi}{2}\eta(A)$ is identified with a $\frac{1}{2}$ shift in the CS level of A, and we do not keep track of background gravitational CS terms.

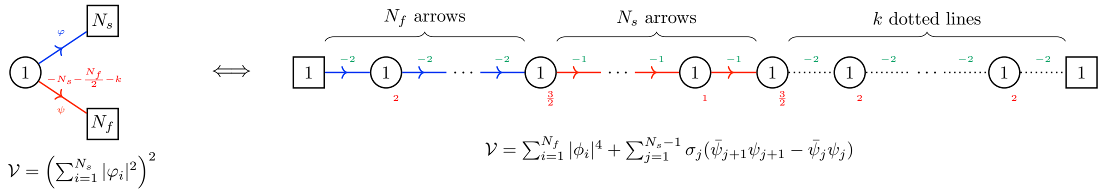
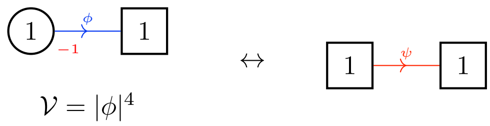


Figure 6.1: The proposed dual for QED$_3$ with N_s scalars, N_f fermions and CS level $-N_s-\frac{N_f}{2}-k$ with $k>0$. Round nodes denote $U(1)$ gauge groups, square node denote flavor groups. Red and blue arrows denote fermions and scalars, respectively, charged under the nodes that they connect. Red labels denote CS levels and green labels denote mixed CS levels.

fermionization:



(6.6)

We propose a generalization for the case of QED with N_f fermions ψ, N_s complex scalars ϕ and CS level $-N_s-\frac{N_f}{2}-k$ with $k\geqslant 0$, shown in Figure 6.1. This proposal is a refinement of the dualities already considered in [88, 89]. The dual is a $U(1)^{N_s+N_f+k-1}$ gauge theory with N_f complex scalars and N_s complex fermions charged under two $U(1)$ gauge groups with opposite charge. Furthermore for every $U(1)$ gauge group with at least two charged fermions there is a real scalar σ_j that interacts with the two fermions via the Yukawa coupling.

The off-diagonal mesons of the QED theory are mapped to monopole operators in the quiver theory. In particular, consider thes spin-0 and spin-1 mesons $\bar\psi_i\psi_j$ with $i<j$. The corresponding operator is a monopole with GNO flux $+1$ under all the gauge groups from the i-th to the $(j-1)$-th gauge group. As reviewed in Appendix B.1, this monopole is not gauge invariant and has gauge charges:

$$Q\left[\mathfrak{M}^{(0,\dots,0,\overset{i}{+},\dots,\overset{j-1}{+},0\dots)}\right]=(0,\dots,-1,\overset{i}{+1},0\dots,0,\overset{j-1}{+1},-1,0,\dots)\tag{6.7}$$

and can be made gauge invariant by dressing with the scalar ϕ_{i-1} connecting the $(i-1)$-th and the i-th node and the complex conjugate scalar ϕ_j^*. Therefore we have the mapping:

$$\bar\psi_i\psi_j\quad\leftrightarrow\quad\mathfrak{M}^{(0\ \dots\ 0\rightarrow+\ \dots\ +\leftarrow0\ \dots)}\quad,\qquad 1\le i<j\le N_f\tag{6.8}$$

where the two arrows denote the dressing with the corresponding scalars and the left-pointing arrow indicates that the dressing is performed with the complex conjugate scalar. Each boson in the monopole background "sees" a GNO flux $+1$ and therefore has spin-$\frac{1}{2}$, resulting in a spin-0 and a spin-1 operator. Similarly, the mesons $\varphi_i^*\varphi_j$ and $\bar\psi_i\varphi_j$ are mapped to:

$$\varphi_i^*\varphi_j\quad\leftrightarrow\quad\mathfrak{M}^{(0\ \dots\ 0\rightarrow+\ \dots\ +\leftarrow0\ \dots)}\quad,\qquad 1\le i<j\le N_s\tag{6.9}$$

$$\bar\psi_i\varphi_j\quad\leftrightarrow\quad\mathfrak{M}^{(0\ \dots\ 0\rightarrow+\ \dots\ +\leftarrow0\ \dots)}\tag{6.10}$$

which are spin-0 and spin-$\frac{1}{2}$ operators, respectively.

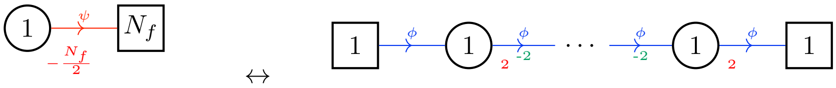
For generic k the monopole operators in the QED transform in non-trivial representations of the flavor symmetry $S(U(N_s) \times U(N_f))$. These monopole operators are expected to map to monopole operators of the dual theory. We will not discuss in detail the mapping of such operators.

When $k = 0$ the QED theory has some monopole operators which are neutral under the flavor symmetry $S(U(N_s) \times U(N_f))$. These are obtained by dressing the bare monopole $\mathfrak{M}^+$, which has gauge charge $-N_s$, with N_s (derivatives of) the scalars φ. The scalars are antisymmetrized in flavor indices, resulting in an operator that is charged under the topological symmetry but is neutral under $S(U(N_s) \times U(N_f))$.
On the mirror side, only for $k = 0$, the quiver has long mesonic operators connecting the two flavor nodes, built out of (derivatives of) all the fields. The long mesons are mapped to the monopoles described above:

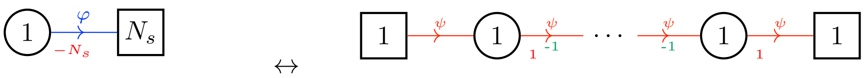
$$\mathfrak{M}^+ \overbrace{\partial_\bullet \varphi_{\{i_1} \cdots \partial_\bullet \varphi_{i_{N_s}\}}}^{N_s} \qquad \leftrightarrow \qquad \partial_\star \phi_1 \cdots \partial_\star \phi_{N_f} \partial_\star \psi_1 \cdots \partial_\star \psi_{N_s} \tag{6.11}$$

where $\partial_\bullet$ and $\partial_\star$ indicate generic sets of derivatives. On the l.h.s. Lorentz indices are contracted to obtain a non-vanishing operator. It would be interesting to precisely understand the mapping between the $\partial_\bullet$ derivatives on the right and the $\partial_\star$ derivatives on the left, but we will not attempt to do so in the present paper. While the spin of these operators depend on the sets of derivatives $\partial_\bullet$ and $\partial_\star$, respectively, the integrality of the spin is given by $\frac{N_s}{2}$ mod 1 on both sides.

For particular values of the three parameters N_f, N_s and k, which are all non-negative, our proposal 6.1 reproduces known dualities. For $k = 0$ and a single matter species we recover the Abelian dualities for fermionic or bosonic QED discussed in [89]:



$$V = \sum_{i=1}^{N_f} |\phi_i|^4 \tag{6.12}$$



$$\mathcal{V} = \left(\sum_{i=1}^{N_s} |\varphi_i|^2\right)^2 \qquad\qquad V = \sum_{i=1}^{N_s-1} \sigma_i(\bar{\psi}_{i+1}\psi_{i+1} - \bar{\psi}_i\psi_i) \tag{6.13}$$

In particular, for $k = 0$ and $N_f + N_s = 1$ we reproduce the original bosonization dualities (6.3), (6.4). For $k = 0$ and $N_f + N_s = 2$ the duality in Figure 6.1 reduces to dualities between QED theories. For $N_f = N_s = 1$ we recover a duality analyzed in [93] for $U(1)_{-\frac{3}{2}}$ with one scalar and one fermion. For $N_f = 2$, $N_s = 0$ or $N_s = 2$, $N_f = 2$ we recover dualities between scalar and fermionic QED. In the bosonic QED in (6.13) one can turn on a negative mass for all the scalars that preserves the $SU(N_s)$ global symmetry. The scalars condense, Higgsing the gauge group and the theory reduces to a $\mathbb{CP}^{N_s-1}$ NLSM. On the dual side the fermions take negative mass, resulting in a $(U(1)_0)^{N_s-1}$ gauge theory. We conjecture that the $N_s - 1$ dual photons together with the $N_s - 1$ real scalars σ_i combine into a $N_s - 1$ complex dimensional NLSM, reproducing the dual side.

Let us discuss some features of the duality in Figure 6.1. The symmetry of the QED theory is, up to discrete factors, $U(1)_{top} \times S(U(N_s) \times U(N_f))$. On the quiver side for $k > 0$ there are $N_f + N_s$ topological symmetries, while for $k = 0$ there are $N_f + N_s - 1$ topological symmetries and a $U(1)$ flavor symmetry rotating the fields. In both cases the rank of the global symmetry matches between the two theories. The duality implies that the $U(1)^{N_f+N_f}$ global symmetry of the quiver theory exhibit non-Abelian enhancement in the IR, matching the global symmetry of the QED theory.

Mass deformations for the matter fields of the QED theory are mapped to masses of the quiver theory, schematically:

$$\text{l.h.s.} \quad \begin{array}{c} |\phi|^2 \\ \bar{\psi}\psi \end{array} \quad \leftrightarrow \quad \begin{array}{c} \bar{\psi}\psi \\ -|\phi|^2 \end{array} \quad \text{r.h.s.} \tag{6.14}$$

One can check that turning on masses for the fermions and positive masses for the scalars one recovers the same duality 6.1 with different values of k, N_f and N_s. This provides a non-trivial consistency check of our proposal. Turning on a negative mass for a scalar Higgses the QED theory. In the dual side one of the fermions is integrated out with a positive mass, triggering a sequential confinement that results in a theory of free fields, matching the electric theory.

6.3 Planar Abelian Duals of CS-QCD$_3$ with Bosons

In this section, we construct a quiver gauge theory dual to non-Abelian QCD$_3$ with fundamental matter. Our proposal builds upon recent results [43, 44, 84] that established a duality between $\mathcal{N} = 2$ supersymmetric non-Abelian linear CS quiver gauge theories and planar Abelian quiver gauge theories with mixed CS interactions. Motivated by these supersymmetric dualities, we conjecture an analogous correspondence in the non-supersymmetric case: non-Abelian CS QCD$_3$ with fundamental fermionic and bosonic matter should be dual to a planar Abelian quiver theory containing both bosons and fermions. This expectation follows from the observation that our proposed non-supersymmetric dualities should emerge as mass deformations of their supersymmetric counterparts.

6.3.1 General Proposal

We propose the planar Abelian dual for $SU(N)$ QCD$_3$ with N_s critical scalars at Chern-Simons level $2N - N_s - k$ ($k \geqslant 0$) shown in Figure 6.2. This duality maps flavor symmetries to topological symmetries and many baryonic and mesonic operators of one theory are mapped to gauge invariant disorder operators (monopoles) of its mirror dual. In this section, we discuss some simple examples explicitly to justify our proposal. The proposed duality is believed to hold in the following range of parameters:

$$N_s \geqslant 2N, \qquad k \geqslant 0.$$

In the planar quiver shown in Figure 6.2, the parameter k counts the number of "empty" columns — columns that are connected to the rest of the quiver solely through horizontal BF couplings.

We now comment on the potential $\mathcal{V}$ shown in Figure 6.2 for the planar mirror theory:

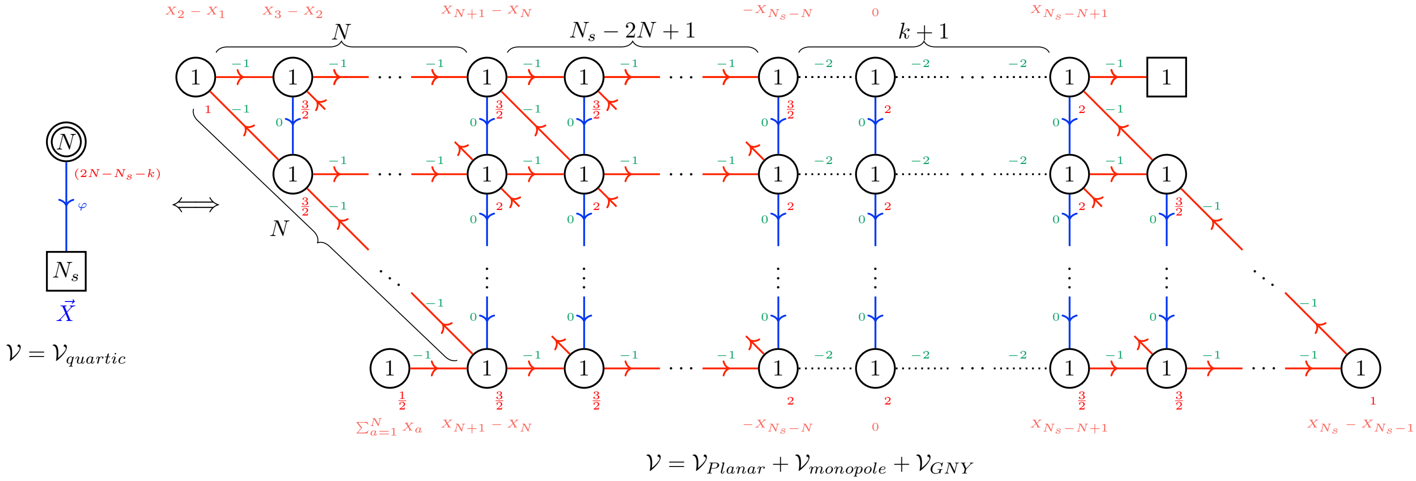
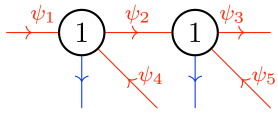


Figure 6.2: The proposed dual for $SU(N)$ QCD$_3$ with N_s scalars at Chern-Simons level $2N - N_s - k$ is shown here. Single/double circles (squares) correspond to U/SU gauge (flavor) symmetries. We describe scalar (fermion) bifundamental fields in blue (red). Chern-Simons levels of gauge nodes are indicated in maroon, while mixed CS couplings between gauge nodes are indicated in green. Dotted lines indicate the absence of any bifundamental matter fields and only the presence of BF interactions. BF couplings between adjacent gauge nodes connected by a fermionic field are -1, BF couplings between gauge nodes connected by a bosonic field are 0, while BF couplings between adjacent gauge nodes connected by dotted lines alone are -2. We also indicate the fugacities of the topological symmetries of each column in orange. The orientation of arrows indicates the representation of the field under the symmetry group, i.e.: an arrow pointing out of (into) a node transforms in the (anti-)fundamental representation of the symmetry group.

- $\mathcal{V}_{Planar}$ contains a cubic Yukawa interaction ($\phi_i\psi_j\psi_k + c.c.$) for every closed planar triangle. There is a term with a coefficient of +1 (-1) for each triangle closed clockwise (anticlockwise).
- $\mathcal{V}_{monopole}$ captures the monopole terms in the potential. For each vertical bosonic bifundamental field, there is a monopole potential with GNO flux $+1$ and -1 under the nodes connected by the arrow, from top to bottom ($\mathfrak{M}^{\binom{+}{-}} + c.c.$). As explained in Appendix B.1 these monopoles can be dressed to obtain a gauge invariant spin-0 operator. This breaks the topological symmetry of the gauge nodes in a column of the quiver to a single, diagonal combination. We label the unbroken $U(1)$ symmetry with the orange label at the top (or bottom) of the corresponding column. The label encodes the mapping of the $U(1)$ group to a subgroup $U(1) \subset U(N_s)$ of the electric flavor symmetry, as explained below.
- $\mathcal{V}_{GNY}$ captures additional cubic interactions between a real scalar field associated with the α^{th} gauge node in the I^{th} column, and the fermions charged under this gauge group (an arrow pointing out of (into) a node has charge +1 (-1) under the gauge symmetry). For example, consider a gauge theory represented by the quiver diagram in (6.15).



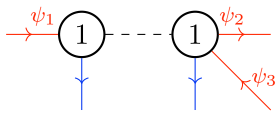
(6.15)

$\mathcal{V}_{GNY}$ for this theory has the form:

$$\mathcal{V}_{GNY} = \sigma_1(\bar{\psi}_2\psi_2 - \bar{\psi}_1\psi_1 - \bar{\psi}_4\psi_4) + \sigma_2(\bar{\psi}_3\psi_3 - \bar{\psi}_2\psi_2 - \bar{\psi}_5\psi_5). \tag{6.16}$$

Each gauge node **except** for the bottom-most gauge node has a GNY potential accompanying the Yukawa interactions.

By contrast, if we consider the following gauge theory (the dashed line indicates the absence of any fermionic bifundamental field, and the two nodes may share a bosonic bifundamental field or only mixed CS interactions):



(6.17)

we expect the identification of σ_1 and σ_2 so the resulting $\mathcal{V}_{GNY}$ has the form:

$$\mathcal{V}_{GNY} = \sigma(\bar{\psi}_2\psi_2 - \bar{\psi}_1\psi_1 - \bar{\psi}_3\psi_3). \tag{6.18}$$

In the quiver theory in Figure 6.2 there are $N(N_s - N)$ reals scalars σ.

In the QCD$_3$ theory, the quartic scalar potential is $\mathcal{V}_{quartic}$[2]:

$$\mathcal{V}_{quartic} = \sum_{\alpha,\beta=1}^{N}\sum_{I,J=1}^{N_s}\left[\varphi^{\alpha I}\bar{\varphi}_{\alpha J}\varphi^{\beta J}\bar{\varphi}_{\beta I} + (\varphi^{\alpha I}\bar{\varphi}_{\alpha I})(\varphi^{\beta J}\bar{\varphi}_{\beta J})\right] \tag{6.19}$$

which preserves the full $U(N_s)$ global symmetry.

In the planar mirror theory the presence of the interactions $\mathcal{V}_{monopole}$ and $\mathcal{V}_{Planar}$ are essential to match the rank of the global symmetry on the electric side. In the presence of these terms, the UV symmetry of the quiver theory is $U(1)_{top}^{N_s}$, where each $U(1)$ factor corresponds to the topological symmetries of a column of the quiver, broken to a diagonal combination by $\mathcal{V}_{monopole}$[3]. We further claim that the symmetry enhances in the IR to:

$$U(1)_{top}^{N_s} \to SU(N_s) \times U(1) \tag{6.20}$$

reproducing the $SU(N_s) \times U(1)$ flavor and baryonic symmetries of the electric theory respectively. This is most transparent in our choice of parametrization of the fugacities of the topological symmetries of each column in Figure 6.2. Explicitly, the diagonal

[2]N.B.: Latin indices $i, j = 1, \ldots, F$ run over flavor indices, and Greek indices $\alpha, \beta = 1, \ldots, N$ run over gauge indices.

[3]There are $N_s - 1$ columns which contribute $N_s - 1$ $U(1)$ factors, and the additional $U(1)$ factor is given by the topological symmetry of the bottom-most gauge node. The topological symmetries associated to the columns within the "empty" region of the quiver can be reabsorbed by a gauge transformation and do not contribute to the global symmetry.

combination of the topological symmetries in the i-th column, associated to $X_{i+1} - X_i$, is mapped to the combination $U(1)_{X_{i+1}} - U(1)_{X_i}$ in the maximal torus of $U(N_s)$. Similarly, the topological symmetry of the bottom left node, associated to $\sum_{a=1}^{N} X_a$, is mapped to the linear combination $\sum_{i=1}^{N} U(1)_{X_i}$. Therefore the fugacities denoted with orange labels in Figure 6.2 encode the explicit embedding of $U(1)_{top}^{N_s}$ in the enhanced $U(N_s)$ global symmetry group.

Moreover, the faithful symmetry of the QCD theory is [94, 93]:

$$\frac{U(N_s)}{\mathbb{Z}_N} \rtimes \mathbb{Z}_2^{\mathcal{C}} \tag{6.21}$$

where $\mathbb{Z}_2^{\mathcal{C}}$ is charge conjugation, acting as complex conjugation on the scalars for $N \geqslant 2$[4]. $\mathbb{Z}_N$ is generated by the following element of the maximal torus of $U(N_s)$:

$$\left\{ e^{\frac{2\pi i}{N}}, \dots, e^{\frac{2\pi i}{N}} \right\} \in \{U(1)_{X_1}, \dots, U(1)_{X_N}\} \tag{6.23}$$

All the gauge invariant dressed monopoles in the dual quiver transform trivially under this element, and are therefore compatible with being components of representations of the enhanced $\frac{U(N_s)}{\mathbb{Z}_N}$.

The fact that there are no mesonic $U(1)$ symmetries in the planar theory is a non-trivial statement and can be checked as follows by looking at the planar quiver:

$$\begin{aligned} \#\text{mesonic } \; U(1)s = \; & \#\text{lines} - \#\text{gauge nodes} - \#\text{triangles} = \\ = \; & [3N(N_s - N) - 2N_s + 3] - [N(N_s - N) + 1] \\ & - [2N(N_s - N) - 2(N_s - 1)] = \\ = \; & 0 \end{aligned} \tag{6.24}$$

where we assume that there is a mesonic $U(1)$ rotating each boson/fermion in the mirror, and a cubic potential term associated to each triangle appearing in the quiver[5].

The presence of the Gross-Neveu-Yukawa potential $\mathcal{V}_{GNY}$ also plays an important role in the duality and can be motivated as follows. On adding a negative mass deformation for the N_s scalars in the electric theory, the gauge group is completely Higgsed and the IR phase of the theory is a sigma model with the target manifold identified as the coset space [95, 96]:

$$\frac{SU(N_s)}{SU(N_s - N) \times SU(N)} \tag{6.25}$$

which has $2NN_s - 2N^2 + 1$ real dimensions. Upon mapping the same deformation in the mirror theory, we expect to flow to $N(N_s - N) + 1$ copies of $U(1)_0$, whose dual photons parameterize $(\mathbf{S}^1)^{N(N_s - N)+1}$ in the deep IR. This can be achieved by giving a negative mass to all the fermions and a positive mass to all the scalars in the quiver. The dual photons are expected to combine with the $N(N_s - N)$ real scalars σ, resulting in a NLSM with real dimension $2N(N_s - N) + 1$, consistent with the QCD theory.

[4]For $N = 2$, as $SU(2) \cong USp(2)$, the global symmetry of the electric $SU(2)$ QCD$_3$ theory is:

$$\frac{USp(2N_s)}{\mathbb{Z}_2}; \tag{6.22}$$

which is broken by $\mathcal{V}_{quartic}$ to $\frac{U(N_s)}{\mathbb{Z}_2} \rtimes \mathbb{Z}_2^{\mathcal{C}}$ [93].

[5]Note that this straightforward computation holds when the parameter $k = 0$. For $k > 0$, matching the $U(1)$ symmetries becomes more subtle. Nonetheless, the matching can be verified in this case as well.

In the rest of this section, we will make these ideas more precise by considering some explicit examples. Through these examples, we illustrate the effects of increasing the Chern-Simons level k and the rank of the gauge symmetries on the respective planar duals.

6.3.2 Example: $SU(2)_{-1}$ QCD$_3$ with $N_s = 5$ scalars

Let us begin with the example of $SU(2)_{-1}$ with 5 scalars and its planar dual description. Since this is our first example, we are pedantic and give many details to illustrate our proposal. We propose the duality shown in Figure 6.3.

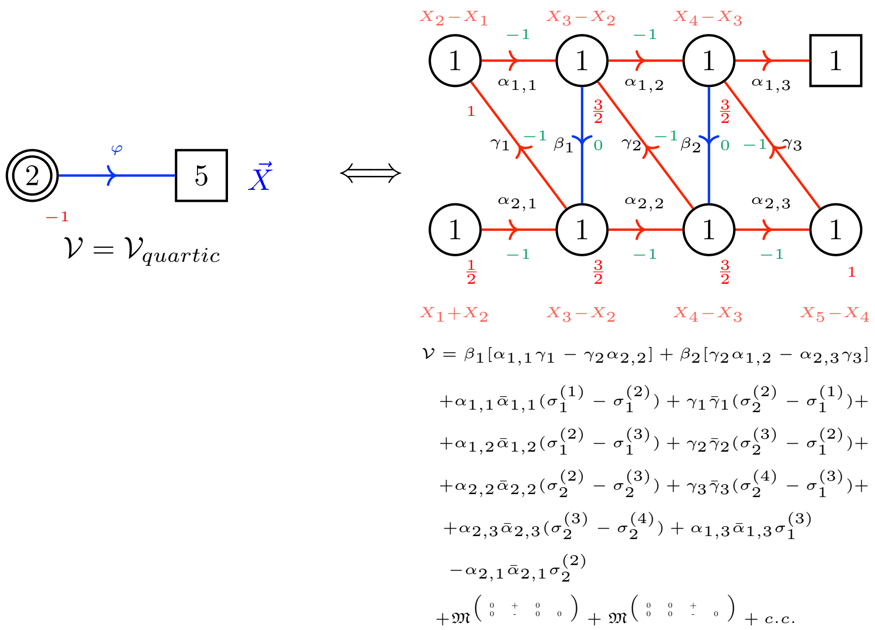


Figure 6.3: The dual for $SU(2)_{-1}$ QCD$_3$ with 5 scalars is shown here. All labelling conventions follow those of Figure 6.2. The superscript (subscript) for the real scalar σ indicates its column (position in the column), e.g.: $\sigma_1^{(3)}$ corresponds to the 1$^{\text{st}}$ gauge node in the 3$^{\text{rd}}$ column.

Henceforth, we refer to the SU$(2)_{-1}$ QCD$_3$ with 5 scalars as the "electric" theory and the planar theory as its "mirror". We are mainly concerned with the mapping of the following gauge-invariant operators in the electric theory:

1. 25 **mesons** of the form $\varphi_i\bar{\varphi}_j$ transforming in the adjoint representation of the flavor symmetry group $U(5)$. Hence, with respect to the maximal torus $U(1)^5_{X_i} \subset U(5)$, they are charged under $U(1)_{X_i} - U(1)_{X_j}$. The off-diagonal mesons (those with $i \neq j$) should then be mapped to monopole operators of the dual quiver theory. The proposed mapping for the mesonic operators is shown in Table 6.1. The diagonal mesons $|\varphi_i|^2$ are neutral under the $U(1)_{X_i}$ and are expected to map to mesons of the dual theory.

2. $25-5 = 20$ **conserved currents** associated to the global symmetry enhancement $U(1)^5 \to U(5)$. These are as spin-1 operators charged under $U(1)_{X_i} - U(1)_{X_j}$ with $i \neq j$ and are expected to be mapped to monopole operators, as illustrated below.

3. $\binom{5}{2} = 10$ **baryons** constructed by anti-symmetrizing the color indices with the help of the Levi-Civita tensor $\epsilon_{\alpha\beta}\varphi_i^\alpha\varphi_j^\beta$. Together, these transform in the 2-index antisymmetric representation of the flavor symmetry group $SU(5)$ and are charged under $U(1)_{X_i} + U(1)_{X_j}$, with $i \neq j$. The proposed mapping for the baryonic operators is shown in Table 6.2.

The off-diagonal mesons, off-diagonal currents and the baryons are charged under the maximal torus $U(1)^5_{X_i} \subset U(5)$, therefore they should be mapped to monopole operators in the dual quiver theory. In what follows, we propose a mapping for these operators, starting from the off-diagonal mesons and the baryons. Thus, the task at hand is to determine gauge invariant monopole operators that are **Lorentz scalars** and are charged under $U(1)_{X_i} \mp U(1)_{X_j}$, which are mapped to the topological symmetries of the quiver theory as described above. We make this statement precise in the next paragraphs by considering some examples. The techniques needed to compute the gauge charges of monopole operators, the dressing needed to construct gauge invariant monopoles and the spin of the resulting operator are reviewed in Appendix B.1.

"Mesonic" Monopole Operators We study monopole operators with the right quantum numbers to be mapped to the mesons of the electric theory here. As an example, we consider the following monopole operator:

$$\mathfrak{M}^{\left(\begin{smallmatrix} & + & & + & & 0 & \\ 0 & & 0 & & 0 & & 0 \end{smallmatrix}\right)} \tag{6.26}$$

This operator has charge $+1$ under the topological symmetries of the first two gauge nodes; hence, it is charged under $U(1)_{X_3} - U(1)_{X_1}$, and transforms as a component of the adjoint representation of $SU(5)$. We can calculate the gauge charge of this operator:

$$\begin{pmatrix} 2 & -1 & 0 & \\ 0 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 3 & -1 & \\ 0 & 0 & -1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 & 2 & -1 & \\ 0 & -1 & -1 & 0 \end{pmatrix}. \tag{6.27}$$

Clearly, it is not gauge invariant and must be dressed appropriately. We choose the minimal dressing[6]:

$$\begin{pmatrix} & + & & 2 & & - & \\ 0 & & - & & - & & 0 \end{pmatrix} \quad , \tag{6.28}$$

hence, we see that the operator:

$$\mathfrak{M}^{\left(\begin{smallmatrix} & + & & + & & 0 & \\ 0 & & 0 & & 0 & & 0 \end{smallmatrix}\right)}_{\gamma_1\gamma_2\bar{\alpha}_{1,2}} \tag{6.29}$$

is gauge invariant. We denote this operator by combining the GNO flux assignment and the dressing in the following way:

$$\mathfrak{M}^{\left(\begin{smallmatrix} & + & & + & & 0 & \\ 0 & & 0 & & 0 & & 0 \end{smallmatrix}\right)}_{\gamma_1\gamma_2\bar{\alpha}_{1,2}} \equiv \begin{pmatrix} & + & & + & & 0 & \\ 0 & & 0 & & 0 & & 0 \end{pmatrix} \quad . \tag{6.30}$$

[6]By this, we mean that we use the minimal number of fields to dress the bare monopole and construct a gauge-invariant operator. A non-minimal dressing—i.e., using additional fields—would yield an operator with a higher scaling dimension. In this work, we focus exclusively on mapping the operators with the lowest scaling dimensions.

As each of the three fermionic fields is quantized in a background of total magnetic flux 1, their spin statistics changes to that of spin-0 bosons[7]. We conclude that the dressed monopole operator has Lorentz spin 0, and is the right operator to be mapped to $\varphi_3\bar{\varphi}_1$.

We can readily identify the off-diagonal mesonic operators and summarize their dual descriptions—including necessary dressings—in Table 6.1. The mapping of diagonal mesonic operators $|\varphi_i|^2$ is less straightforward because these operators are neutral under the maximal torus of $U(5)$, visible in the quiver UV Lagrangian. We expect that these operators are mapped to suitable linear combinations of mesonic operators of the quiver, compatible with the mass deformations discussed in Section 6.6.1. We defer the study of the mapping of diagonal mesonic operators to future work.

Table 6.1: Dressed monopole operators of the planar mirror (Figure 6.3) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Gauge Charge	Spin	"Electric" Meson
$\begin{pmatrix} + & 0 & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 2 & - & 0 & \\ 0 & - & 0 & 0 \end{pmatrix}$	0	$\varphi_2\bar{\varphi}_1$
$\begin{pmatrix} + & + & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} + & 2 & - & \\ 0 & - & - & 0 \end{pmatrix}$	0	$\varphi_3\bar{\varphi}_1$
$\begin{pmatrix} + & + & + & \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} + & + & 2 & \\ 0 & - & - & - \end{pmatrix}$	0	$\varphi_4\bar{\varphi}_1$
$\begin{pmatrix} + & + & + & \\ 0 & 0 & 0 & + \end{pmatrix}$	$\begin{pmatrix} + & + & + & \\ 0 & - & -2 & + \end{pmatrix}$	0	$\varphi_5\bar{\varphi}_1$
$\begin{pmatrix} 0 & + & 0 & \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} - & 3 & - & \\ 0 & 0 & - & 0 \end{pmatrix}$	0	$\varphi_3\bar{\varphi}_2$
$\begin{pmatrix} 0 & + & + & \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} - & 2 & 2 & \\ 0 & 0 & - & - \end{pmatrix}$	0	$\varphi_4\bar{\varphi}_2$
$\begin{pmatrix} 0 & + & + & \\ 0 & 0 & 0 & + \end{pmatrix}$	$\begin{pmatrix} - & 2 & + & \\ 0 & 0 & -2 & + \end{pmatrix}$	0	$\varphi_5\bar{\varphi}_2$
$\begin{pmatrix} 0 & 0 & + & \\ 0 & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & - & 3 & \\ 0 & 0 & 0 & - \end{pmatrix}$	0	$\varphi_4\bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & + & \\ 0 & 0 & 0 & + \end{pmatrix}$	$\begin{pmatrix} 0 & - & 2 & \\ 0 & 0 & - & + \end{pmatrix}$	0	$\varphi_5\bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & 0 & + \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & - & \\ 0 & 0 & - & 2 \end{pmatrix}$	0	$\varphi_5\bar{\varphi}_4$

[7]In principle, they also have a spin-1 channel. Unless otherwise stated we dress the monopoles with the zero modes of the fermion fields, resulting in a spin-0 operator [32, 97].

Let us comment on a particular feature of the monopole operators in the quiver theories considered in this paper, which is a consequence of the presence of the monopole interaction $\mathcal{V}_{monopole}$. When looking for a condidate for the dual of an electric meson, say $\varphi_4\bar{\varphi}_3$, we may also consider the monopole:

$$\begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & + & 0 \end{pmatrix}. \tag{6.31}$$

Indeed, this is a spin-0 gauge invariant operator with the right charges to be mapped to $\varphi_4\bar{\varphi}_3$, similarly to the monopole listed in Table 6.1. This is possible due to the presence of the monopole potential $\mathcal{V}_{monopole}$: even though these two monopoles have different GNO flux assignments they have the same charges under the linear combination of topological symmetries preserved by $\mathcal{V}_{monopole}$. We expect these two monopoles to mix, therefore the electric meson $\varphi_4\bar{\varphi}_3$ will be mapped to some linear combination of the two:

$$\alpha \begin{pmatrix} 0 & 0 & 0 & \\ 0 & 0 & + & 0 \end{pmatrix} + \beta \begin{pmatrix} 0 & 0 & + & + & \\ 0 & 0 & 0 & & 0 \end{pmatrix} \leftrightarrow \varphi_4\bar{\varphi}_3, \quad (|\alpha|^2 + |\beta|^2 = 1). \tag{6.32}$$

This phenomenon is ubiquitous in the quiver theories studied in this paper and the amount of monopoles that can mix between each other increases as N increases due to the higher number of monopole interactions in $\mathcal{V}_{monopole}$. In Table 6.1 and in the rest of this paper we only report one dressed monopole that has the correct charges to be mapped to a given operator in the QCD theory[8]. It is understood that, in most cases, there are other monopole operators with the same global charges but different GNO flux assignment and dressing, and the mapping will involve a suitable linear combination of these operators. It would be interesting to analyze this mixing systematically, for example by extending the techniques of [97, 32, 98, 99] in the presence of monopole interactions, but we leave this to future work.

Conserved Currents We now examine how the $U(5)$ flavor symmetry currents - including the baryonic $U(1)$ subgroup - map between the electric and dual theories. These spin-1 conserved currents take the form:

$$J^\mu_{jk} = i\left(\bar{\varphi}_j \partial^\mu \varphi_k - \partial^\mu \bar{\varphi}_j \varphi_k\right) \tag{6.33}$$

where $j, k = 1, ..., 5$, transforming in the adjoint representation of $U(5)$. Note that under charge conjugation: $(J^\mu_{jk})^* = J^\mu_{kj}$.

The off-diagonal currents map to dressed spin-1 monopole operators in the dual theory. For example, J^μ_{21} corresponds to:

$$J^\mu_{12} \qquad \leftrightarrow \qquad \mathfrak{M}\begin{pmatrix} + & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.34}$$

where we dress with the spin-1 mode of one fermion and the spin-0 mode of the other fermion, resulting in a spin-1 operator. Under charge conjugation, this maps to J^μ_{21}:

$$J^\mu_{21} \qquad \leftrightarrow \qquad \mathfrak{M}\begin{pmatrix} - & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.35}$$

[8]Notice that also on the QCD side there are in general many operators with the same spin and global charges. For example any operator can be multiplied by a spin-0 flavor singlet or it can be acted on by contracted derivatives.

where, again, we dress with a spin-0 and a spin-1 mode.

The other off-diagonal spin 1 currents map in a similar way. In particular, the (off-diagonal) current J^{μ}_{jk} is mapped to the same bare monopole as the meson $\varphi_j \bar{\varphi}_k$, the difference being that the former is dressed such that it has spin-1 while the latter has spin-0.

The five diagonal currents J^{μ}_{jj} map to topological $U(1)$ currents in the dual theory:

$$J^{\mu}_{jj} \longleftrightarrow \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_{\nu}A^{(j)}_{\rho} \tag{6.36}$$

with the gauge field combinations:

$$A^{(j)}_{\mu} = \begin{cases} A^{(1,1)}_{\mu} & j=1 \\ A^{(2,1)}_{\mu} + A^{(2,2)}_{\mu} & j=2 \\ A^{(3,1)}_{\mu} + A^{(3,2)}_{\mu} & j=3 \\ A^{(4,1)}_{\mu} & j=4 \\ A^{(2,1)}_{\mu} & j=5 \end{cases} \tag{6.37}$$

Here $A^{(j,k)}_{\mu}$ denotes the gauge field for the $U(1)$ node at row j and column k in quiver (6.3). The mapping (6.36) reproduces the mapping of the maximal torus of the $U(5)$ global symmetry across the duality.

"Baryonic" Monopole Operators We follow the same line of reasoning as before to study monopole operators with the right quantum numbers to be mapped to the baryons of the electric theory here. As an example, we consider the following *bare* monopole operator:

$$\mathfrak{M}^{\left(\begin{smallmatrix} 0 & 0 & 0 & \\ + & + & 0 & 0 \end{smallmatrix}\right)} \tag{6.38}$$

This operator has charge $+1$ under the topological symmetries of the bottom two gauge nodes; hence, it is charged under $U(1)_{X_3} + U(1)_{X_1}$, and therefore can be identified with a component of the two-index antisymmetric representation of $SU(5)$. We can calculate the gauge charge of this operator:

$$\begin{pmatrix} 0 & 0 & 0 & \\ 1 & -1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 & \\ -1 & 3 & -1 & 0 \end{pmatrix} \equiv \begin{pmatrix} -1 & 0 & 0 & \\ 0 & 2 & -1 & 0 \end{pmatrix}. \tag{6.39}$$

Clearly, it is not gauge invariant and must be dressed appropriately. We again choose the minimal dressing that cancels the gauge charges of the bare monopole:

$$\begin{pmatrix} - & 0 & 0 & \\ 0 & 2 & - & 0 \end{pmatrix}, \tag{6.40}$$

Therefore the dressed monopole operator:

$$\mathfrak{M}^{\left(\begin{smallmatrix} 0 & 0 & 0 & \\ + & + & 0 & 0 \end{smallmatrix}\right)}\bar{\gamma}_1\bar{\alpha}_{2,2} \equiv \begin{pmatrix} 0 & 0 & 0 & \\ + & + & 0 & 0 \end{pmatrix}; \tag{6.41}$$

is a gauge invariant Lorentz scalar and has the correct charges to be mapped to $\varphi_3\varphi_2$ (color indices have been suppressed). Similarly, we can identify the 10 baryonic operators, reported with the appropriate dressing in Table 6.2.

Table 6.2: Dressed monopole operators of the planar mirror (Figure 6.3) and their putative mapping to baryonic operators of the electric theory (color indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Gauge Charge	Spin	"Electric" Baryon
$\begin{pmatrix} 0 & 0 & 0 & \\ + \leftarrow & 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 & \\ + \leftarrow & - & 0 & 0 \end{pmatrix}$	0	$\varphi_2\varphi_1$
$\begin{pmatrix} 0 \searrow & 0 & 0 & \\ + & + \leftarrow & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} - \searrow & 0 & 0 & \\ 0 & 2 \leftarrow & - & 0 \end{pmatrix}$	0	$\varphi_3\varphi_1$
$\begin{pmatrix} 0 \searrow & 0 \searrow & 0 & \\ + & + & + \leftarrow & 0 \end{pmatrix}$	$\begin{pmatrix} - \searrow & - \searrow & 0 & \\ 0 & + & 2 \leftarrow & - \end{pmatrix}$	0	$\varphi_4\varphi_1$
$\begin{pmatrix} 0 \searrow & 0 \searrow & 0 \searrow & \\ + & + & + & + \end{pmatrix}$	$\begin{pmatrix} - \searrow & - \searrow & - \searrow & \\ 0 & + & + & + \end{pmatrix}$	0	$\varphi_5\varphi_1$
$\begin{pmatrix} + \leftarrow & 0 & 0 & \\ + & + \leftarrow & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} + \leftarrow & - & 0 & \\ 0 & + \leftarrow & - & 0 \end{pmatrix}$	0	$\varphi_3\varphi_2$
$\begin{pmatrix} + \leftarrow & 0 \searrow & 0 & \\ + & + & + \leftarrow & 0 \end{pmatrix}$	$\begin{pmatrix} + \leftarrow & -2 \searrow & 0 & \\ 0 & 0 & 2 \leftarrow & - \end{pmatrix}$	0	$\varphi_4\varphi_2$
$\begin{pmatrix} + \leftarrow & 0 \searrow & 0 \searrow & \\ + & + & + & + \end{pmatrix}$	$\begin{pmatrix} + \leftarrow & -2 \searrow & - \searrow & \\ 0 & 0 & + & + \end{pmatrix}$	0	$\varphi_5\varphi_2$
$\begin{pmatrix} + & + \leftarrow & 0 \searrow & \\ + & + & + & + \end{pmatrix}$	$\begin{pmatrix} 0 & + \leftarrow & -2 \searrow & \\ 0 & 0 & 0 & + \end{pmatrix}$	0	$\varphi_5\varphi_3$
$\begin{pmatrix} + & + & + \leftarrow & \\ + & + & + & + \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & + \leftarrow & \\ 0 & 0 & 0 & 0 \end{pmatrix}$	0	$\varphi_5\varphi_4$

We reiterate that we only consider dressed minimal flux operators in the mapping.

6.3.3 Example: $SU(2)_{-2}$ QCD$_3$ with $N_s = 5$ scalars

We now consider the example of $SU(2)_{-2}$ with 5 scalars. Notice that it has a lower Chern-Simons level than that of the theory considered in Section 6.3.2; this minor difference will greatly alter the operator map. We propose the duality shown in Figure 6.4.

As before, we are mainly concerned with determining the monopole operators that are dual to the 24 **mesons** and 10 **baryons**. We do not show the explicit calculation and state the results in Tables 6.3 and 6.4 respectively.

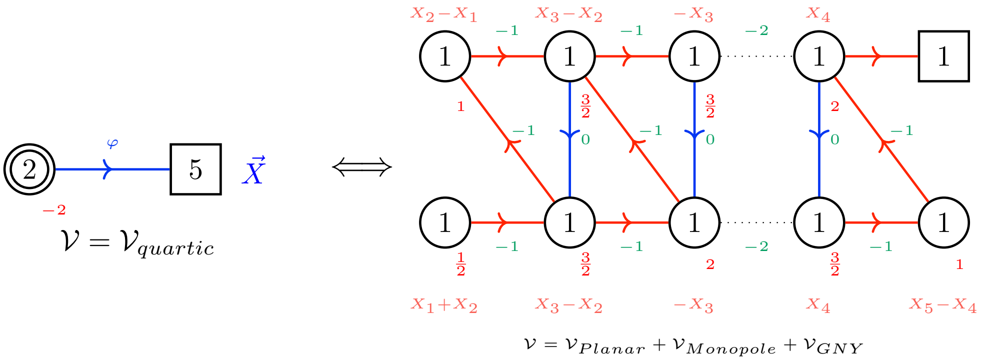


Figure 6.4: The dual for $SU(2)_{-2}$ QCD$_3$ with 5 scalars is shown here. All labelling conventions follow those of Figure 6.2.

Table 6.3: Dressed monopole operators of the planar mirror (Figure 6.4) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} + & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2\bar{\varphi}_1$	$\begin{pmatrix} + & + & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3\bar{\varphi}_1$
$\begin{pmatrix} + & + & + & + & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4\bar{\varphi}_1$	$\begin{pmatrix} + & + & + & + & \\ 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_5\bar{\varphi}_1$
$\begin{pmatrix} 0 & + & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3\bar{\varphi}_2$	$\begin{pmatrix} 0 & + & + & + & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4\bar{\varphi}_2$
$\begin{pmatrix} 0 & + & + & + & \\ 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_5\bar{\varphi}_2$	$\begin{pmatrix} 0 & 0 & + & + & \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4\bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & + & + & \\ 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_5\bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_5\bar{\varphi}_4$

Table 6.4: Dressed monopole operators of the planar mirror (Figure 6.4) and their putative mapping to baryonic operators of the electric theory (color indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ + & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2\varphi_1$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ + & + & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3\varphi_1$
$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ + & + & + & + & 0 \end{pmatrix}$	$\varphi_4\varphi_1$	$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ + & + & + & + & + \end{pmatrix}$	$\varphi_5\varphi_1$

Continued on next page

Table 6.4 – continued from previous page

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} + & 0 & 0 & 0 & \\ + & + & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3\varphi_2$	$\begin{pmatrix} + & 0 & 0 & 0 & \\ + & + & + & + & 0 \end{pmatrix}$	$\varphi_4\varphi_2$
$\begin{pmatrix} + & 0 & 0 & 0 & \\ + & + & + & + & + \end{pmatrix}$	$\varphi_5\varphi_2$	$\begin{pmatrix} + & + & 0 & 0 & \\ + & + & + & + & 0 \end{pmatrix}$	$\varphi_4\varphi_3$
$\begin{pmatrix} + & + & 0 & 0 & \\ + & + & + & + & + \end{pmatrix}$	$\varphi_5\varphi_3$	$\begin{pmatrix} + & + & + & + & \\ + & + & + & + & + \end{pmatrix}$	$\varphi_5\varphi_4$

6.3.4 Example: $SU(3)_{-1}$ QCD$_3$ with $N_s = 7$ scalars

We consider the case of $SU(3)_{-1}$ with 7 scalars and its planar dual as our final example of this section. We propose the duality shown in Figure 6.5.

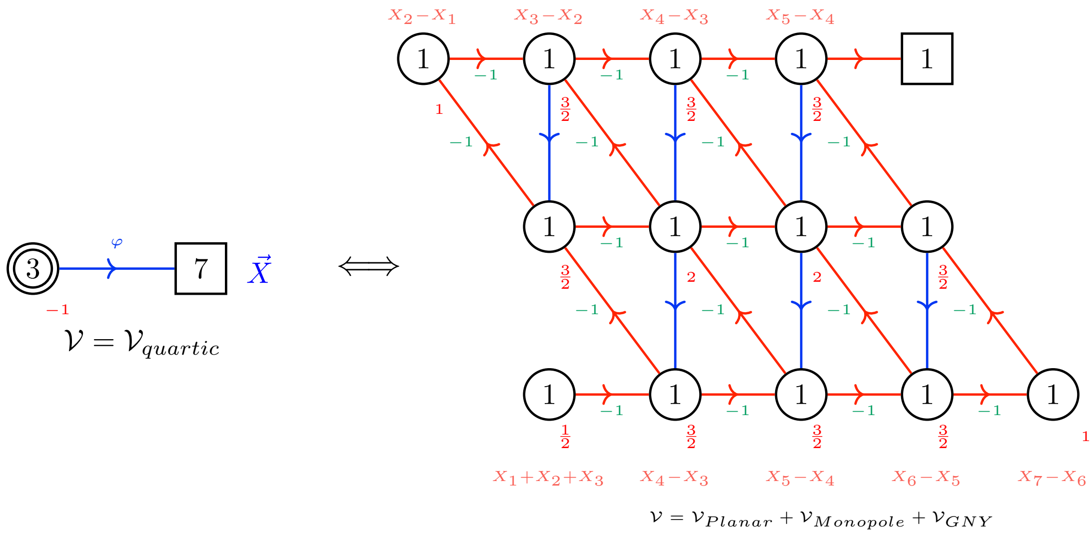


Figure 6.5: The dual for $SU(3)_{-1}$ QCD$_3$ with 7 scalars is shown here. All labelling conventions follow those of Figure 6.2. Mixed BF couplings between gauge nodes connected by vertical scalar fields have been suppressed for brevity.

We are concerned with the mapping of 48 **mesonic operators** (identified by being charged under $U(1)_{X_i} - U(1)_{X_j}$) and $\binom{7}{3} = 35$ **baryonic operators** (identified by being charged under $U(1)_{X_i} + U(1)_{X_j} + U(1)_{X_k}$), all of which are Lorentz scalars. Since the reasoning we follow is the same as before, we leave the complete operator map to the reader and only provide the GNO fluxes of the bare monopoles as a representative example of the general trends of construction and identification of these operators.

We report the monopoles mapped to mesons of the electric theory in Equation (6.42). Note that we only specify the off-diagonal elements of the meson matrix in Equation (6.42) and the remaining monopoles can be constructed by charge conjugation).

$$
\begin{aligned}
\mathbf{M} \leftrightarrow \Bigg\{ & \mathfrak{M}^{\begin{pmatrix} + & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & + & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & + & + & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \\
& \mathfrak{M}^{\begin{pmatrix} + & + & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & + & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & + & + & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \\
& \mathfrak{M}^{\begin{pmatrix} 0 & + & + & + & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & + & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & + & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \\
& \mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & + & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & + & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \\
& \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}}, \\
& \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}} \Bigg\}
\end{aligned}
\tag{6.42}
$$

In order for these monopole to be gauge invariant they need to be appropriately dressed as explained in Appendix B.1. The fermions used to dress the monopoles, in the chosen monopole background, have a spin-0 mode. Using this mode we dress the monopoles to obtain spin-0 operators that therefore have the correct spin to map to mesons of the QCD_3.

We report the monopoles mapped to baryons of the electric theory in Equation (6.43). Note that monopoles mapped to anti-baryons can be constructed by charge conjugation.

$$
\begin{gathered}
\mathbf{B} \leftrightarrow \Big\{ \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&0&0&0&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&+&0&0&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&0&0&0& \\ &+&+&0&0&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&0&0&0& \\ &+&+&0&0&0 \end{pmatrix}}, \\
\mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&+&+&0&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&0&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&0&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&0&0 \end{pmatrix}}, \\
\mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&0&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&+&0&0&& \\ &+&+&0&0& \\ &+&+&+&0&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&+&+&+&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&+&0 \end{pmatrix}}, \\
\mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&+&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&+&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&0 \end{pmatrix}}, \\
\mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&+&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&+&+&0&& \\ &+&+&+&0& \\ &+&+&+&+&0 \end{pmatrix}}, \\
\mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&+ \end{pmatrix}}, \\
\mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&+&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&+ \end{pmatrix}}, \\
\mathfrak{M}^{\begin{pmatrix} +&+&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&+&+&0&& \\ &+&+&+&0& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0&0&0&0&& \\ &+&+&+&+& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&0&0&0&& \\ &+&+&+&+& \\ &+&+&+&+&+ \end{pmatrix}}, \\
\mathfrak{M}^{\begin{pmatrix} +&+&0&0&& \\ &+&+&+&+& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&+&+&0&& \\ &+&+&+&+& \\ &+&+&+&+&+ \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} +&+&+&+&& \\ &+&+&+&+& \\ &+&+&+&+&+ \end{pmatrix}} \Big\}
\end{gathered}
\tag{6.43}
$$

These monopoles are not gauge invariant and hence they should be dressed. The dressing can be done analogously as in Table 6.2 (or 6.4). The resulting operators have Lorentz spin 0, matching that of the baryons.

6.3.5 Mapping Gauge Invariant Operators: The General Pattern

Before we conclude, we briefly comment on the general trend of mapping gauge invariant operators between the dual theories. We remind the reader that the proposed duality exchanges flavor and topological symmetries; hence, mesons, conserved currents and baryons of the electric theory will be mapped to suitably dressed disorder operators (monopoles) of the mirror theory. We ask the reader to keep Figure 6.2 in mind for the rest of the discussion. We also take this opportunity to remind the reader that a specific choice of fugacities of the topological symmetry parametrizes each column.

Mesons and Conserved Currents

Mesons and conserved currents are Lorentz scalars and vectors, respectively, that transform in the adjoint representation of the flavor symmetry group $SU(N_s)$. We now specify the general pattern of mapping these operators to monopoles of the mirror theory (Figure 6.2). We focus on the mapping of the $\frac{1}{2}N_s(N_s-1)$ "off-diagonal" mesons $\varphi_I\bar{\varphi}_J$ with $J < I$. The case $J > I$ can be obtained by taking the Hermitian conjugate of these operators, which map to monopoles with negative GNO fluxes.

A generic minimal flux mesonic operator **cannot** be charged under the topological symmetry of the first $U(1)$ gauge node in the last row of the quiver diagram. This is due to the parameterization of topological fugacities and representation-theoretic constraints. Modulo a subtlety arising from the presence of a monopole interaction in the dual quiver—already discussed in Section 6.3.1—we propose that *off-diagonal* mesonic operators are mapped to monopoles carrying GNO flux under the gauge nodes in the top row and/or along the rightmost diagonal of the quiver.

To describe this mapping systematically, we find it convenient to organize the mesons $\varphi_I\bar{\varphi}_J$ into three distinct classes, based on the values of the flavor indices I and J:

- First, we consider the case where both I and J are less than $N_s - N$.
- Next, we study the regime where $J < N_s - N \leqslant I$.
- Finally, we address the case in which both I and J are greater than or equal to $N_s - N$.

$\varphi_I\bar{\varphi}_J$, where $J < I < N_s - N$: The simplest case is that of the meson $\varphi_{I+1}\bar{\varphi}_I$ with $(1 \leqslant I \leqslant N_s - N - 1)$. In the mirror theory, this is mapped to a monopole with $+1$ GNO flux under the top node of the I^{th} column:

$$\varphi_{I+1}\bar{\varphi}_I \qquad \leftrightarrow \qquad \left(\begin{array}{ccc} \overbrace{0 \ \dots \ \boxed{+} \ \dots \ 0 \ 0}^{N_s-N} \ \overbrace{0 \ \dots \ 0 \ 0}^{k} & & \\ \uparrow & 0 & \\ I \quad 0 \text{ GNO Flux} & & \ddots \\ & & \quad 0 \end{array} \right)^{\!N} \tag{6.44}$$

Thereafter, $+1$ charges must propagate contiguously from left to right. A meson of the form $\varphi_{I+1+A}\bar{\varphi}_I$ $(A \neq 0)$ is dual to a monopole with the following charge assignment:

$$\varphi_{I+1+A}\bar{\varphi}_I \qquad \leftrightarrow \qquad \left(\begin{array}{ccc} \overbrace{0 \ \dots \ \underbrace{+ \ \dots \ +}_{A} \ 0}^{N_s-N} \ \overbrace{0 \ \dots \ 0 \ 0}^{k} & & \\ & 0 & \\ & & \ddots \\ 0 \text{ GNO Flux} & & \quad 0 \end{array} \right)^{\!N} \tag{6.45}$$

$\varphi_I\bar{\varphi}_J$**, where** $J < N_s - N \leqslant I$**:** In this case, the dual monopole has a contiguous string of $+1$ GNO fluxes starting from the J^{th} column and ending on the $(I+k)^{\text{th}}$ column. In particular, each of the k nodes inside the region of the quiver connected by BF coupling has GNO flux $+1$. The first case of this type is the meson $\varphi_{N_s-N+1}\bar{\varphi}_I$, which is mapped to a monopole of the form:

$$\varphi_{N_s-N+1}\bar{\varphi}_I \quad \leftrightarrow \quad \left(\begin{array}{cccc} \overbrace{0 \;\dots\; + \;\dots\; +}^{N_s-N}\;\overbrace{+ \; + \;\dots\; + \; +}^{k} & & & \\ \hline & \multicolumn{1}{|c}{0} & & \\ \text{0 GNO Flux} & & \ddots & \\ & & & \multicolumn{1}{|c}{0} \end{array}\right)_{\!N} \tag{6.46}$$

Once a charge has been assigned to a column, no additional charges can be assigned to the same column in lower rows. The propagation of fluxes for lower rows follows a strict "staircase" pattern, i.e.: charges in the lower rows can only be assigned to the rightmost column. Thus, a generic mesonic monopole mapped to $\varphi_{N_s-N+1+J}\bar{\varphi}_I$ $(1 \leqslant J \leqslant N)$ will have the following flux assignment:

$$\varphi_{N_s-N+1+J}\bar{\varphi}_I \quad \leftrightarrow \quad \left(\begin{array}{ccccc} \overbrace{0 \;\dots\; + \; +}^{N_s-N}\;\overbrace{+ \;\dots\; + \; +}^{k} & & & & \\ \hline & \multicolumn{1}{|c}{+} & & & \\ \text{0 GNO Flux} & & \ddots & & \\ & & & \multicolumn{1}{|c}{+} & \\ & & & & \multicolumn{1}{|c}{0} \end{array}\right)_{\!J} \tag{6.47}$$

$\varphi_I\bar{\varphi}_J$**, where** $N_s - N \leqslant J < I$**:** Mesons corresponding to values of $I \geqslant N_s - N + 1$ will instead have their $+1$ flux assigned to the rightmost column of the lower rows, and subsequent charge assignment will follow the usual staircase assignment. Thus, the meson $\varphi_I\bar{\varphi}_J$ $(N_s - N + 1 \leqslant J < I \leqslant N_s)$ will be mapped to a monopole of the form:

$$\varphi_I\bar{\varphi}_J \quad \leftrightarrow \quad \left(\begin{array}{ccccc} \overbrace{0 \;\dots\; 0 \; 0}^{N_s-N}\;\overbrace{0 \;\dots\; 0 \; 0}^{k} & & & & \\ \hline & \multicolumn{1}{|c}{+} & & & \\ \text{0 GNO Flux} & & \ddots & & \\ & & & \multicolumn{1}{|c}{+} & \\ & & & & \multicolumn{1}{|c}{0} \end{array}\right)_{\!I-J} \tag{6.48}$$

Generically, all the monopoles considered above will have spin-0 and spin-1 channels after they have been dressed - the spin-0 operators correspond to the off-diagonal elements of the meson matrix, while the spin-1 operators correspond to conserved currents in the original electric theory. We remind the reader that the diagonal elements $|\varphi_i|^2$ are mapped to mass deformations (refer Section 6.6.1 for a more detailed discussion on mass deformations) in the dual theory, while the diagonal currents map to topological $U(1)$ currents (refer Section 6.3.2 for an example of the dressing) in the mirror.

Baryons

Baryons are Lorentz scalars that transform in the rank-N_s antisymmetric representation of the flavor symmetry group $SU(N_s)$. We describe how these operators map to minimal flux operators in the planar mirror theory. We focus on the mapping of $\binom{N_s}{N}$ baryons to monopoles in the dual theory. Anti-baryons can be obtained by taking the Hermitian conjugate of these operators, which would map to monopoles with negative GNO fluxes.

$(\prod_{j=1}^{N-1}\varphi_j)\varphi_{N+I}$, where $0 \leqslant I \leqslant N_s + k - N - 1$: A generic minimal flux baryonic monopole operator **must** carry charge $+1$ under the topological symmetry of the first $U(1)$ gauge node in the last row of the quiver diagram[9].

Accordingly, the baryon $\prod_{j=1}^{N}\varphi_j = \varphi_1 \dots \varphi_N$ is dual to the minimal flux configuration:

$$\prod_{j=1}^{N}\varphi_j \quad \leftrightarrow \quad \left(\begin{array}{l} \quad \text{0 GNO Flux} \\ \\ \overline{+\,}| \end{array}\right) \tag{6.49}$$

Thereafter, the charge assignment must propagate contiguously from left to right in a row. A baryon of the form $\left(\prod_{j=1}^{N-1}\varphi_j\right)\varphi_{N+I} = \varphi_1 \dots \varphi_{N-1}\varphi_{N+I}$ is dual to the flux configuration:

$$\left(\prod_{j=1}^{N-1}\varphi_j\right)\varphi_{N+I} \quad \leftrightarrow \quad \left(\begin{array}{l} \qquad\qquad \text{0 GNO Flux} \\ \\ \overline{\underbrace{+\ +\ \dots\ +}_{I}}| \end{array}\right) \tag{6.50}$$

As before, care must be taken if there is a BF-coupled region (of length k), and each of the k nodes inside the region of the quiver connected only by BF couplings must have GNO flux $+1$. For instance, the baryon $\varphi_1 \dots \varphi_{N-1}\varphi_{N_s-N+1}$ is dual to the flux configuration:

$$\left(\prod_{j=1}^{N-1}\varphi_j\right)\varphi_{N_s-N+1} \quad \leftrightarrow \quad \left(\begin{array}{l} \qquad\qquad\qquad\qquad \text{0 GNO Flux} \\ \\ \overline{+\ \underbrace{+\ \dots\ +}_{N_s-N+1}\ \underbrace{+\ \dots\ +}_{k}\ +}| \end{array}\right) \tag{6.51}$$

$(\prod_{k=1}^{N-J}\varphi_j)(\prod_{k=N-J+2}^{N+1}\varphi_k)$, where $1 \leqslant J \leqslant N-1$: The flux assignment in higher rows follows a "staircase" pattern. For instance, the baryon $\left(\prod_{j=1}^{N-2}\varphi_j\right)\varphi_N\varphi_{N+1} = \varphi_1 \dots \varphi_{N-2}\varphi_N\varphi_{N+1}$ is dual to the flux configuration:

[9]The topological symmetry associated to this node is parametrized by $\sum_{k=1}^{N} X_k$, whereas all other nodes have topological symmetries parametrized by $X_i - X_{i+1}$. Thus, the group theoretic properties of a monopole acquiring charge $+1$ under this specific node follow from representation theory.

$$\left(\prod_{j=1}^{N-2}\varphi_j\right)\varphi_N\varphi_{N+1} \qquad \leftrightarrow \qquad \left(\begin{array}{cccccccccc} & & & & & & & & & 0\text{ GNO Flux}\\ + & & & & & & & & & \\ + & \underbrace{+ \;|\; \dots \quad 0}_{N_s-N+1} & \underbrace{0 \quad \dots \quad 0}_{k} & \underbrace{0 \quad \dots \quad 0}_{N} & & & & & & \end{array}\right)$$

(6.52)

Thus, the baryon $\left(\prod_{k=1}^{N-J}\varphi_k\right)\left(\prod_{k=N-J+2}^{N+1}\varphi_k\right) = \varphi_1\dots\varphi_{N-J}\varphi_{N-J+2}\dots\varphi_{N+1}$ is dual to the flux configuration:

$$\left(\prod_{k=1}^{N-J}\varphi_k\right)\left(\prod_{k=N-J+2}^{N+1}\varphi_k\right) \qquad \leftrightarrow \qquad \left(\begin{array}{ccccc} 0 & & & & 0\text{ GNO Flux}\\ & \ddots & & & \\ & & \left.\begin{array}{c} + \\ \ddots \\ + \end{array}\right\}_{J} & & \\ & & + & \underbrace{+ \;|\; \dots \quad 0}_{N_s-N+1}\;\underbrace{0 \quad \dots \quad 0}_{k}\;\underbrace{0 \quad \dots \quad 0}_{N} & \end{array}\right)$$

(6.53)

Mapping a generic baryon: Unlike mesons, baryons allow column reuse: a single column may host fluxes in multiple rows, provided the staircase rule is respected. Thus, a generic baryonic monopole will have the following flux assignment:

$$\left(\prod_{m=1}^{N-J-1}\varphi_m\right)\left(\prod_{l=1}^{J}\varphi_{N+b_l-l}\right)\varphi_{N+I} \qquad \leftrightarrow \qquad \left(\begin{array}{cccc} 0 & & & \\ & \ddots & & \\ & & \left.\begin{array}{l} \overbrace{+ \;\dots\; +}^{b_J}\; 0 \;\dots \\ \quad\ddots \\ \quad\quad \overbrace{+ \;+\; \dots \;+}^{b_1}\; 0 \;\dots \end{array}\right\}_{J} & \\ & & & + \;\underbrace{+ \;\dots\; \dots\; + \;\dots\; +}_{I}\; 0 \;\dots \end{array}\right)$$

(6.54)

We note that $k \geqslant 0$ does not change the counting of the mesonic and baryonic operators.

6.4 Planar Abelian Duals of CS-QCD$_3$ with Bosons and Fermions

In this section, we generalize the proposal of the previous section to non-Abelian theories with fundamental scalars and fermions. We anticipate that adding a fermionic flavor in the electric theory corresponds to "bosonizing" a column in the planar theory. We provide several examples of this here.

6.4.1 General Proposal

We propose the following planar, Abelian dual for $SU(N)$ QCD$_3$ with N_s scalars and N_f fermions at Chern-Simons level $2N - N_s - \frac{N_f}{2} - k$ in Figure 6.6. The proposed duality is believed to hold in the following range of parameters[10]:

$$N_s + N_f \geqslant 2N, \qquad N_s \geqslant 2N, \qquad k \geqslant 0 \tag{6.55}$$

This proposal should be seen as a generalization of the one considered for scalar QCD$_3$ obtained by replacing N_f scalars with N_f fermions and replacing the corresponding fermionic columns in the planar mirror with purely bosonic columns.

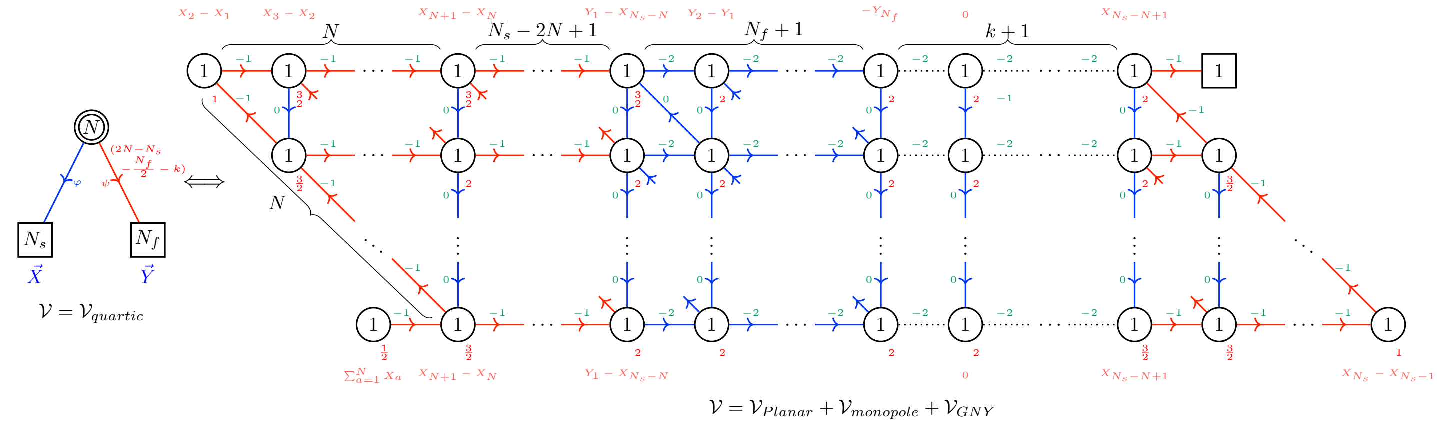


Figure 6.6: The proposed dual for $SU(N)$ QCD$_3$ with N_s scalars and N_f fermions at Chern-Simons level $2N - N_s - \frac{N_f}{2} - k$ is shown here. As before, single/double circles (squares) correspond to U/SU gauge (flavor) symmetries, scalar (fermion) bifundamental fields are indicated in blue (red), Chern-Simons levels of gauge nodes are indicated in maroon, while mixed BF couplings between gauge nodes are indicated in green. Dotted lines indicate the absence of any bifundamental matter fields and only the presence of BF interactions. Note that the Chern-Simons level of each gauge node is equal to minus half the sum of all BF couplings with the surrounding nodes. We also indicate the fugacities of the topological symmetries of each column in orange.

We briefly comment on the potential $\mathcal{V}$ shown in Figure 6.6. In addition to quartic scalar interactions (Equation 6.19), mixed scalar-fermion interactions are also generated radiatively in the electric QCD$_3$ theory and are captured by the electric potential $\mathcal{V}_{quartic}$:

$$\mathcal{V}_{quartic} = \sum_{\alpha,\beta=1}^{N} \sum_{I,J=1}^{N_s} \sum_{A,B=1}^{N_f} \Big[\varphi^{\alpha I}\bar{\varphi}_{\alpha J}\varphi^{\beta J}\bar{\varphi}_{\beta I} + (\varphi^{\alpha I}\bar{\varphi}_{\alpha I})(\varphi^{\beta J}\bar{\varphi}_{\beta J}) + \varphi^{\alpha I}\bar{\varphi}_{\beta I}\bar{\psi}_{\alpha A}\psi^{\beta A}$$
$$+ (\varphi^{\alpha I}\bar{\varphi}_{\alpha I})(\psi^{\beta A}\bar{\psi}_{\beta A})\Big]. \tag{6.56}$$

On the other hand, in addition to the usual Yukawa interactions, the monopole potential, and the GNY potential, an extra contribution is captured by $\mathcal{V}_{Planar}$ in the mirror

[10] Examples with $N_s < 2N$ are provided in Section 7 of [45].

theory. $\mathcal{V}_{Planar}$ contains a cubic scalar interaction ($\phi_i\phi_j\phi_k$+$c.c.$) for every closed planar triangle. There is a term with a coefficient of $+1$ (-1) for each triangle closed clockwise (anticlockwise). $\mathcal{V}_{monopole}$ involves every monopole with GNO fluxes $+1/-1$ under two nodes connected by a vertical scalar, analogous to the case discussed in Section 6.3, and are dressed as explained in Appendix B.1. As before, we claim $\mathcal{V}_{monopole}$ and $\mathcal{V}_{Planar}$ are essential to match the rank of the global symmetry on the electric side. In the presence of these terms, the UV symmetry of the quiver theory is $U(1)^{N_s}_{top} \times U(1)^{N_f}_{top}$, where each $U(1)$ factor corresponds to the topological symmetry of the $N_s + N_f - 1$ columns of the quiver, and the additional $U(1)$ factor is the topological symmetry of the bottom-most gauge node. We also claim the symmetry enhances in the IR:

$$U(1)^{N_s+N_f}_{top} \to U(N_s) \times U(N_f) \tag{6.57}$$

which is mapped precisely to the flavor symmetries of the electric theory. Furthermore, we expect the faithful symmetry to be [93]:

$$\frac{U(N_s) \times U(N_f)}{\mathbb{Z}_N} \rtimes \mathbb{Z}_2^{\mathcal{C}} \tag{6.58}$$

where $\mathbb{Z}_N$ is the center of $SU(N)$ and $\mathbb{Z}_2^{\mathcal{C}}$ is a charge conjugation symmetry.

In the rest of this section, we make these ideas more precise by considering some examples.

A Note on Higher Spin Baryons and Dual Monopoles We take a moment to comment on higher spin baryonic operators and their dual flux configurations here. Consider a generic baryon of the electric theory:

$$(\chi_1)^{i_1}(\chi_2)^{i_2}\cdots(\chi_F)^{i_F} \tag{6.59}$$

where each χ_i is either a boson or fermion, and the powers satisfy $\sum_{k=1}^{F} i_k = N$ [11]. This baryon maps to the following monopole operator in the mirror theory:

$$\begin{array}{c} \begin{array}{cccccc} \overbrace{\quad}^{j_1} & \overbrace{\quad}^{j_2} & \cdots & & \overbrace{\quad}^{j_N} & \cdots \end{array} \\ \begin{pmatrix} + & + & \dots & + & + & & & & & \\ & + & \dots & + & + & + & & & & \\ & & & & & & & & & \\ & & & + & + & \dots & \dots & 0 & 0 & \\ & & & \boxed{+} & + & \dots & & \dots & 0 & \underbrace{0}_{j_{F-1}} \end{pmatrix} \end{array} \tag{6.60}$$

[11]N.B.: If multiple copies of the same scalar appear, derivatives must be inserted; otherwise, the antisymmetric contraction vanishes.

Here, j_k denotes the number of + in column k (excluding the base corner + which is always present). The j_k are related to the baryonic charges i_k via:

$$\begin{cases} j_1 & = 1 - i_1 \\ j_2 & = 2 - i_1 - i_2 \\ \vdots & \\ j_N & = N - \sum_{k=1}^{N} i_k \\ j_{N+1} & = N - \sum_{k=1}^{N+1} i_k \\ \vdots & \\ j_{F-1} & = N - \sum_{k=1}^{F-1} i_k \end{cases} \tag{6.61}$$

some j_k may be negative—for instance, if $i_1 > 1$, then $j_1 < 0$. In such cases, negative fluxes (denoted by "$-$" symbols) must appear in the corresponding column. The fluxes within each column should be distributed across rows in a way that ensures the resulting monopole is *minimally* dressed. We will see some explicit examples of this now.

Finally, we note that the number of spin-J baryons that can be built without derivatives in $SU(N)$ QCD$_3$ with N_s scalars and N_f fermions is $\binom{N_f+2J}{2J} \times \binom{N_s}{N-2J}$, constructed by (anti-)symmetrizing the (bosonic) fermionic fields. Consider in particular the spin-J baryons:

$$\psi_{\{i_1}\psi_{i_2}\dots\psi_{i_{2J}\}}\varphi_{[j_i}\varphi_{j_2}\dots\varphi_{j_{N-2J}]} \tag{6.62}$$

where the flavor indices of scalars j_n are antisymmetrized and the flavor indices of fermions i_n are symmetrized. The fermions are contracted to give a spin-J operator. This operator transform in the $2J$-index symmetric representation of $SU(N_f)$ and in the $(N-2J)$-index antisymmetric representation of $SU(N_s)$. The operator in the highest weight:

$$(\psi_1)^{2J}\varphi_1\varphi_2\dots\varphi_{N-2J} \tag{6.63}$$

is mapped to the following monopole on the mirror side:

$$\left(\begin{array}{cccccccccccc} 0 & 0 & \dots & 0 & 0 & & & & & & & \\ & 0 & \dots & & & & & & & & & \\ & & + & + & + & \dots & + & & & & & \\ & & & + & + & \dots & + & \Big\}_J & \dots & 0 & 0 & \\ & & & + & + & \dots & \underbrace{+}_{N_s - N} & & & \dots & 0 & 0 \end{array}\right) \tag{6.64}$$

6.4.2 Example: $SU(3)_{-\frac{1}{2}}$ QCD$_3$ with $N_s = 6$ scalars and $N_f = 1$ fermion

Let us begin with the example of $SU(3)_{-\frac{1}{2}}$ with 6 scalars and 1 fermion, and its planar dual description. We propose the duality shown in Figure 6.7.

We are mainly concerned with mapping the following gauge invariant operators of the electric theory:

1. There are 48 **mesonic operators** of which:

   (a) 35 spin-0 mesons of the form $\varphi_i\bar{\varphi}_j$, identified by $X_i - X_j$ (including their charge conjugates).

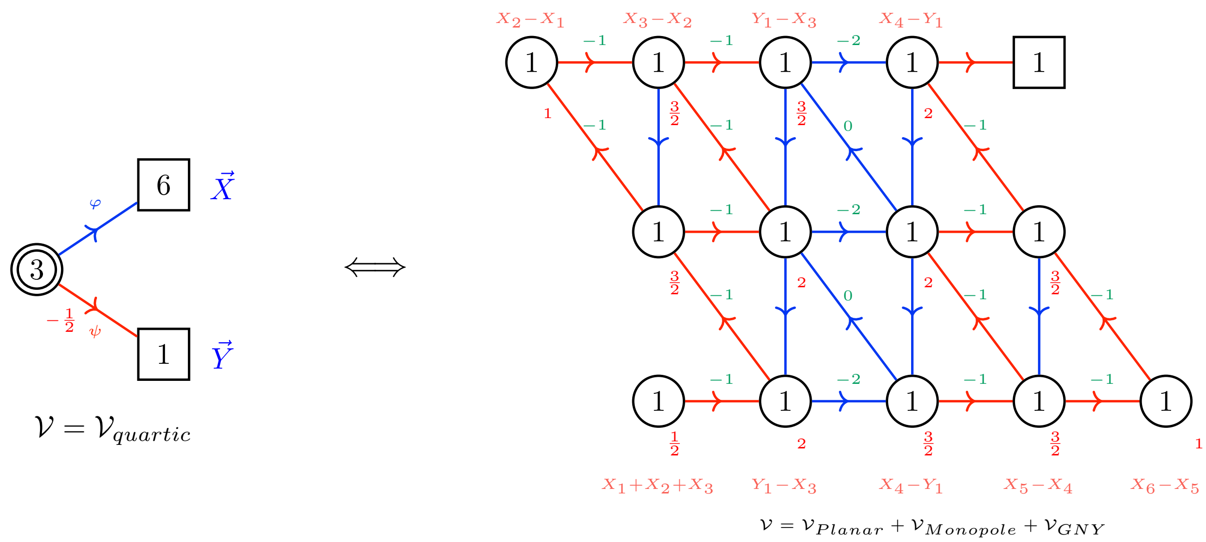


Figure 6.7: The dual for $SU(3)_{-\frac{1}{2}}$ QCD$_3$ with 6 scalars and 1 fermion is shown here. All labelling conventions are the same as Figure 6.6, and BF couplings between nodes connected by a vertical scalar field have been suppressed for brevity.

(b) 12 spin-$\frac{1}{2}$ mesons of the form $\varphi_i\bar{\psi}$, identified by $X_i - Y_1$ (including their charge conjugates).

(c) 1 meson with a spin-0 and spin-1 channel $\psi\bar{\psi}$.

As already discussed in the previous cases the mapping of diagonal mesons, which are neutral under the maximal torus of the global symmetry, is less straightforward, and we leave it to future work. The mapping of these off-diagonal mesons is given in Table 6.5.

2. There are many **baryonic operators**:

(a) $\binom{6}{3} = 20$ spin-0 baryons of the form $\varphi_i\varphi_j\varphi_k$, identified by $X_i + X_j + X_k$.

(b) 6 baryons with a spin-0 and spin-1 channel of the form $\psi\psi\phi_i$, identified by $2Y_1 + X_i$.

(c) $\binom{6}{2} = 15$ spin-$\frac{1}{2}$ baryons of the form $\psi\varphi_i\varphi_j$, identified by $Y_1 + X_i + X_j$.

(d) 1 baryon with spin-$\frac{3}{2}$ channels, identified by $3Y_1$, constructed by symmetrizing ψ in the flavor and spinor indices.

The mapping of baryons is given in Table 6.6.

Table 6.5: Dressed monopole operators of the planar mirror (Figure 6.7) and their putative mapping to mesonic operators of the electric theory. The remaining mesonic operators can be constructed by charge conjugation.

GNO Flux	Meson	GNO Flux	Meson
$\begin{pmatrix} + & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_2\bar{\varphi}_1$	$\begin{pmatrix} + & + & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3\bar{\varphi}_1$
$\begin{pmatrix} + & + & + & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi\bar{\varphi}_1$	$\begin{pmatrix} + & + & + & + & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4\bar{\varphi}_1$
$\begin{pmatrix} + & + & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5\bar{\varphi}_1$	$\begin{pmatrix} + & + & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_6\bar{\varphi}_1$
$\begin{pmatrix} 0 & + & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_3\bar{\varphi}_2$	$\begin{pmatrix} 0 & + & + & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi\bar{\varphi}_2$
$\begin{pmatrix} 0 & + & + & + & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4\bar{\varphi}_2$	$\begin{pmatrix} 0 & + & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5\bar{\varphi}_2$
$\begin{pmatrix} 0 & + & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_6\bar{\varphi}_2$	$\begin{pmatrix} 0 & 0 & + & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi\bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & + & + & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_4\bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5\bar{\varphi}_3$
$\begin{pmatrix} 0 & 0 & + & + & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_6\bar{\varphi}_3$	$\begin{pmatrix} 0 & 0 & 0 & - & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi\bar{\varphi}_4$
$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\varphi_5\bar{\varphi}_4$	$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & + & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_6\bar{\varphi}_4$
$\begin{pmatrix} 0 & 0 & 0 & - & & \\ & 0 & 0 & 0 & - & \\ & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\psi\bar{\varphi}_5$	$\begin{pmatrix} 0 & 0 & 0 & 0 & & \\ & 0 & 0 & 0 & 0 & \\ & 0 & 0 & 0 & 0 & + \end{pmatrix}$	$\varphi_6\bar{\varphi}_5$
$\begin{pmatrix} 0 & 0 & 0 & - & & \\ & 0 & 0 & 0 & - & \\ & 0 & 0 & 0 & 0 & - \end{pmatrix}$	$\psi\bar{\varphi}_6$		

Table 6.6: Dressed monopole operators of the planar mirror (Figure 6.7) and their putative mapping to baryonic operators of the electric theory (all indices have been suppressed). The remaining anti-baryonic operators can be constructed by charge conjugation.

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&0&0&0&0 \end{pmatrix}$	$\varphi_1\varphi_2\varphi_3$	$\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&+&0&0&0 \end{pmatrix}$	$\varphi_1\varphi_2\psi$
$\begin{pmatrix} 0&0&0&0&& \\ &+&0&0&0& \\ &+&+&0&0&0 \end{pmatrix}$	$\varphi_1\varphi_3\psi$	$\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&+&+&0&0 \end{pmatrix}$	$\varphi_1\varphi_2\varphi_4$
$\begin{pmatrix} 0&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&0&0 \end{pmatrix}$	$\varphi_1\varphi_3\varphi_4$	$\begin{pmatrix} 0&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&0&0 \end{pmatrix}$	$\varphi_1\varphi_4\psi$
$\begin{pmatrix} +&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&0&0 \end{pmatrix}$	$\varphi_2\varphi_3\varphi_4$	$\begin{pmatrix} +&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&0&0 \end{pmatrix}$	$\varphi_2\varphi_4\psi$
$\begin{pmatrix} +&+&0&0&& \\ &+&+&0&0& \\ &+&+&+&0&0 \end{pmatrix}$	$\varphi_3\varphi_4\psi$	$\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_1\varphi_2\varphi_5$
$\begin{pmatrix} 0&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_1\varphi_3\varphi_5$	$\begin{pmatrix} +&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_2\varphi_3\varphi_5$
$\begin{pmatrix} 0&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_1\varphi_5\psi$	$\begin{pmatrix} +&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_2\varphi_5\psi$
$\begin{pmatrix} +&+&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_3\varphi_5\psi$	$\begin{pmatrix} 0&0&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_1\varphi_4\varphi_5$
$\begin{pmatrix} +&0&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_2\varphi_4\varphi_5$	$\begin{pmatrix} +&+&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_3\varphi_4\varphi_5$
$\begin{pmatrix} +&+&+&0&& \\ &+&+&+&0& \\ &+&+&+&+&0 \end{pmatrix}$	$\varphi_4\varphi_5\psi$	$\begin{pmatrix} 0&0&0&0&& \\ &0&0&0&0& \\ &+&+&+&+&+ \end{pmatrix}$	$\varphi_1\varphi_2\varphi_6$
$\begin{pmatrix} 0&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&+&+ \end{pmatrix}$	$\varphi_1\varphi_3\varphi_6$	$\begin{pmatrix} +&0&0&0&& \\ &+&0&0&0& \\ &+&+&+&+&+ \end{pmatrix}$	$\varphi_2\varphi_3\varphi_6$
$\begin{pmatrix} 0&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&+ \end{pmatrix}$	$\varphi_1\varphi_6\psi$	$\begin{pmatrix} +&+&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&+ \end{pmatrix}$	$\varphi_3\varphi_6\psi$
$\begin{pmatrix} +&0&0&0&& \\ &+&+&0&0& \\ &+&+&+&+&+ \end{pmatrix}$	$\varphi_2\varphi_6\psi$	$\begin{pmatrix} 0&0&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&+ \end{pmatrix}$	$\varphi_1\varphi_4\varphi_6$
$\begin{pmatrix} +&0&0&0&& \\ &+&0&0&0& \\ &+&+&0&0&0 \end{pmatrix}$	$\varphi_2\varphi_3\psi$	$\begin{pmatrix} +&0&0&0&& \\ &+&+&+&0& \\ &+&+&+&+&+ \end{pmatrix}$	$\varphi_2\varphi_4\varphi_6$

Continued on next page

Table 6.6 – continued from previous page

GNO Flux	Baryon	GNO Flux	Baryon
$\begin{pmatrix} + & + & 0 & 0 & \\ & + & + & + & 0 \\ & + & + & + & + & + \end{pmatrix}$	$\varphi_3\varphi_4\varphi_6$	$\begin{pmatrix} + & + & + & 0 & \\ & + & + & + & 0 \\ & + & + & + & + & + \end{pmatrix}$	$\varphi_4\varphi_6\psi$
$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ & + & + & + & + \\ & + & + & + & + & + \end{pmatrix}$	$\varphi_1\varphi_5\varphi_6$	$\begin{pmatrix} + & 0 & 0 & 0 & \\ & + & + & + & + \\ & + & + & + & + & + \end{pmatrix}$	$\varphi_2\varphi_5\varphi_6$
$\begin{pmatrix} + & + & 0 & 0 & \\ & + & + & + & + \\ & + & + & + & + & + \end{pmatrix}$	$\varphi_3\varphi_5\varphi_6$	$\begin{pmatrix} + & + & + & 0 & \\ & + & + & + & + \\ & + & + & + & + & + \end{pmatrix}$	$\psi\varphi_5\varphi_6$
$\begin{pmatrix} + & + & + & + & \\ & + & + & + & + \\ & + & + & + & + & + \end{pmatrix}$	$\varphi_4\varphi_5\varphi_6$	$\begin{pmatrix} + & + & + & 0 & \\ & + & + & 0 & 0 \\ & + & + & 0 & 0 & 0 \end{pmatrix}$	ψ^3
$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ & + & + & 0 & 0 \\ & + & + & 0 & 0 & 0 \end{pmatrix}$	$\psi^2\varphi_1$	$\begin{pmatrix} + & 0 & 0 & 0 & \\ & + & + & 0 & 0 \\ & + & + & 0 & 0 & 0 \end{pmatrix}$	$\psi^2\varphi_2$
$\begin{pmatrix} + & + & 0 & 0 & \\ & + & + & 0 & 0 \\ & + & + & 0 & 0 & 0 \end{pmatrix}$	$\psi^2\varphi_3$	$\begin{pmatrix} + & + & + & 0 & \\ & + & + & 0 & 0 \\ & + & + & + & 0 & 0 \end{pmatrix}$	$\psi^2\varphi_4$
$\begin{pmatrix} + & + & + & 0 & \\ & + & + & 0 & 0 \\ & + & + & + & + & 0 \end{pmatrix}$	$\psi^2\varphi_5$	$\begin{pmatrix} + & + & + & 0 & \\ & + & + & 0 & 0 \\ & + & + & + & + & + \end{pmatrix}$	$\psi^2\varphi_6$

6.4.3 Example: $SU(2)_{-\frac{5}{2}}$ with $N_s = 5$ scalars and $N_f = 1$ fermion

We now consider the example of $SU(2)_{-\frac{5}{2}}$ with 5 scalars and 1 fermion. Notice that this theory has $k = 1$. We propose the duality shown in Figure 6.8.

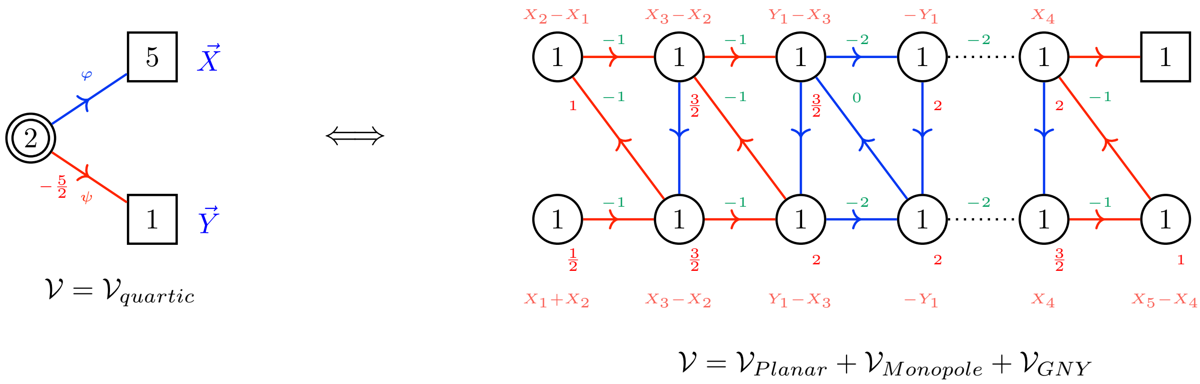


Figure 6.8: The dual for $SU(2)_{-\frac{5}{2}}$ QCD$_3$ with 5 scalars and 1 fermion is shown here. All labelling conventions are the same as Figure 6.6, and BF couplings between nodes connected by a vertical scalar field have been suppressed for brevity.

We are mainly concerned with the mapping of $\binom{5}{2} = 10$ spin-0 mesons, 5 spin-$\frac{1}{2}$ mesons, and $\binom{5}{2} = 10$ spin-0 baryons, 5 spin-$\frac{1}{2}$ baryons and 1 baryon with spin-1 channel. Since the details of the dressings are identical to the discussions preceding this section, we leave the complete operator map to the reader and provide the GNO fluxes of the bare monopoles are a representative example of the general trends of identifying these operators.

The mapping of the off-diagonal spin-0 mesons is provided in Equation 6.65 (the remaining mesons can be identified by charge conjugation):

$$
\begin{aligned}
\mathbf{M}|_{\text{spin-0}} \leftrightarrow \Big\{ & \mathfrak{M}^{\begin{pmatrix} + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}}, \\
& \mathfrak{M}^{\begin{pmatrix} 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & + & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \\
& \mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & + & + \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}} \Big\}.
\end{aligned}
\tag{6.65}
$$

A generic spin-0 mesonic monopole operator will be dressed as follows[12]:

$$
\mathfrak{M}^{\begin{pmatrix} 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \quad \mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & + \cdots & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}
\tag{6.66}
$$

The mapping of the off-diagonal spin-$\frac{1}{2}$ mesons is provided in Equation 6.67 (the remaining mesons can be identified by charge conjugation):

$$
\begin{aligned}
\mathbf{M}|_{\text{spin-}\frac{1}{2}} \leftrightarrow \Big\{ & \mathfrak{M}^{\begin{pmatrix} + & + & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & + & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \\
& \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & 0 & + \end{pmatrix}} \Big\}.
\end{aligned}
\tag{6.67}
$$

A generic spin-$\frac{1}{2}$ meson will be dressed as follows:

$$
\mathfrak{M}^{\begin{pmatrix} 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \quad \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & + \cdots & + \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}
\tag{6.68}
$$

The mapping of the spin-0 baryons is provided in Equation 6.69 (the remaining antibaryons can be identified by charge conjugation):

[12] We indicate the presence of only BF couplings in the second example by keeping a dotted line explicit between the two nodes.

$$\mathbf{B}|_{\text{spin-}0} \leftrightarrow \Bigg\{ \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & + & + & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & + & + & + \end{pmatrix}},$$
$$\mathfrak{M}^{\begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & + & + & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & + & + & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & + & + & + \\ + & + & + & + & + & + \end{pmatrix}},$$
$$\mathfrak{M}^{\begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & + & + & + \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & + & + & + \end{pmatrix}} \Bigg\}. \tag{6.69}$$

A generic spin-0 baryon will be dressed as follows:

$$\mathfrak{M}^{\begin{pmatrix} +\leftarrow 0 & 0 & 0 & 0 & \\ + & +\leftarrow 0 & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \\ + & + & + & + \cdots + \leftarrow 0 \end{pmatrix}} \tag{6.70}$$

The mapping of the spin-$\frac{1}{2}$ baryons is provided in Equation 6.71 (the remaining antibaryons can be identified by charge conjugation):

$$\mathbf{B}|_{\text{spin-}\frac{1}{2}} \leftrightarrow \Bigg\{ \mathfrak{M}^{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & 0 & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & + & 0 & 0 \\ + & + & + & + & + & 0 \end{pmatrix}},$$
$$\mathfrak{M}^{\begin{pmatrix} + & + & + & 0 & 0 \\ + & + & + & + & + & + \end{pmatrix}} \Bigg\}. \tag{6.71}$$

A generic spin-$\frac{1}{2}$ baryon will be dressed as follows:

$$\mathfrak{M}^{\begin{pmatrix} +\leftarrow 0 & 0 & 0 & 0 & \\ + & + & +\leftarrow 0 & 0 & 0 \end{pmatrix}}, \mathfrak{M}^{\begin{pmatrix} + & + & +\leftarrow 0 & 0 & \\ + & + & + & + \cdots + \leftarrow 0 \end{pmatrix}} \tag{6.72}$$

Finally, the baryon ψ^2 obtained by symmetrizing the flavor and spin indices is mapped to the dressed monopole:

$$\mathfrak{M}^{\begin{pmatrix} + & + & +\leftarrow 0 & 0 & \\ + & + & +\leftarrow 0 & 0 & 0 \end{pmatrix}} \tag{6.73}$$

6.5 Planar Abelian Duals of Unitary CS-QCD$_3$ with Bosons and Fermions

We have so far focused on constructing duals of $SU(N)$ QCD$_3$ with mixed matter content. These results, however, extend naturally to the case of *unitary* QCD$_3$ by gauging the baryonic symmetry. From the discussion in Section 6.3, we know that the baryonic symmetry of the $SU(N)$ QCD$_3$ maps to the topological symmetry of the bottom-left gauge node of the planar-abelian quiver (Figure 6.2). Gauging of this topological symmetry with zero CS level introduces a $U(1)_0$ sector with no charged matter whose path integral results in a delta-function, which effectively ungauges the $U(1)$ bottom-left gauge symmetry. While we did not keep track of background CS levels when presenting the duality in Figure 6.2, we expect the duality to involve a non-zero CS level for the baryonic symmetry. In analogy with the SUSY result of [43, 44], we expect a background CS at level -2 for the baryonic symmetry on the QCD side of 6.2. Therefore we propose that the dual description for $U(N)$ QCD$_3$ with N_s critical scalars and N_f fermions, where the theory is defined at Chern-Simons level $(2N-N_s-\frac{N_f}{2}-k, -N_s-\frac{N_f}{2}-k)$ [13] is the planar-abelian quiver depicted in Figure 6.9.

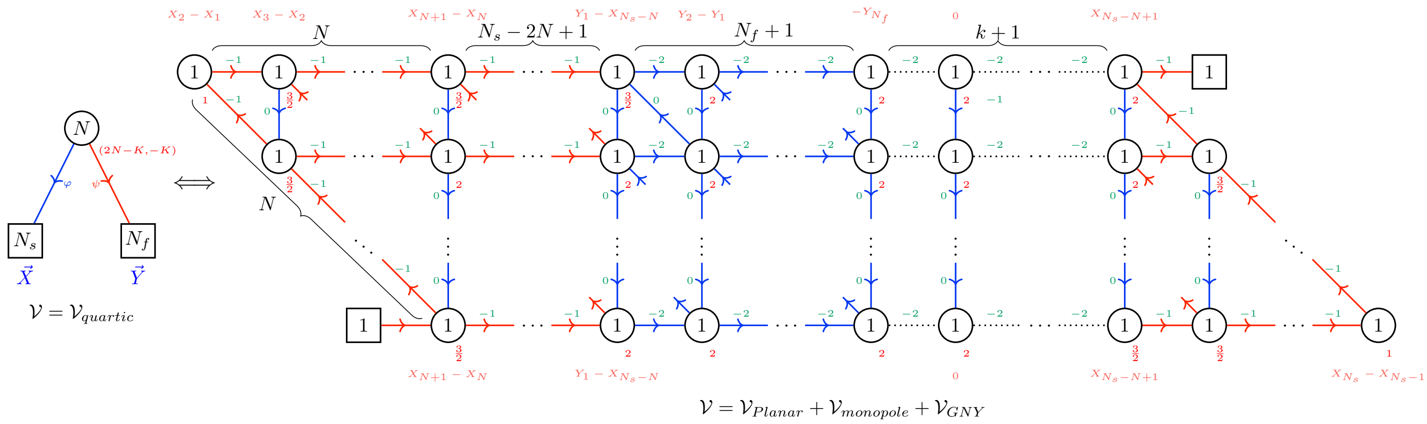


Figure 6.9: The dual of $U(N)$ QCD$_3$ with N_s scalars and N_f fermions at Chern-Simons level $(2N-K,\ -K)$, where for clarity we define $K = N_s + \frac{N_f}{2} + k$, is shown here. All labelling conventions are the same as Figure 6.6.

A particularly interesting special case arises when $k = 0$, shown in Figure 6.10. In this setting, the planar mirror theory contains a gauge-invariant mesonic operator built from bifundamental fields that connect the two $U(1)$ flavor nodes. This meson must map to a monopole operator in the electric $U(N)$ QCD$_3$ theory that is neutral under the $S(U(N_s) \times U(N_f))$ global symmetry. Consider the bare monopole $\mathfrak{M}$ with GNO fluxes $(+, 0, \ldots, 0)$ in the $U(N)_{2N-N_s-\frac{N_f}{2},\,-N_s-\frac{N_f}{2}}$ QCD$_3$ that transforms as the highest weight of the (N_s-2N)-index symmetric conjugate representation of $SU(N)$ and has charge N_s under the diagonal $U(1)$. This monopole can be made gauge invariant by dressing with N_s modes of the scalars φ, where two sets of N scalars are antisymmetrized in gauge indices and the other $N_s - 2N$ are symmetrized in gauge indices. By further symmetrizing all the scalars in $SU(N_s)$ indices we obtain operators that are neutral under $S(U(N_s) \times U(N_f))$ and have charge -1 under the topological symmetry, which are mapped to the long mesons described above. Therefore, we have

[13] We adopt the standard notation:

$$U(N)_{(k,\,k+\ell N)} \cong \frac{SU(N)_k \times U(1)_{N(k+\ell N)}}{\mathbb{Z}_N},$$

which corresponds to the Chern-Simons action

$$-\frac{ik}{4\pi}\int \mathrm{tr}(A \wedge dA) - \frac{i\ell}{4\pi}\int \mathrm{tr}(A) \wedge \mathrm{tr}(dA).$$

the mapping:

$$\mathfrak{M}^+ \overbrace{\partial_\bullet\varphi_1^{[1}\partial_\bullet\varphi_1^2\ldots\partial_\bullet\varphi_1^{N_s-2N}}^{N_s-2N}\ \overbrace{\varphi_1^{N_s-2N+1}\varphi_2^{N_s-2N+2}\ldots\varphi_N^{N_s-N}}^{N}\ \overbrace{\varphi_1^{N_s-N+1}\varphi_2^{N_s-N+2}\ldots\varphi_N^{N_s]}}^{N} \leftrightarrow \text{long mesons} \tag{6.74}$$

where lower indices are $SU(N)$ gauge indices and upper indices are $SU(N_s)$ flavor indices, which are completely antisymmetrized. The modes with gauge index 1 "see" the GNO flux and have half-integer spin, while the other modes have spin-0. Then the spin of this operator mod 1 is given by $\frac{1}{2}(N_s - 2N + 2)$ mod 1. This matches the integrality of the spin of the long mesons, because each meson contains an even (odd) number of fermions if N_s is even (odd).

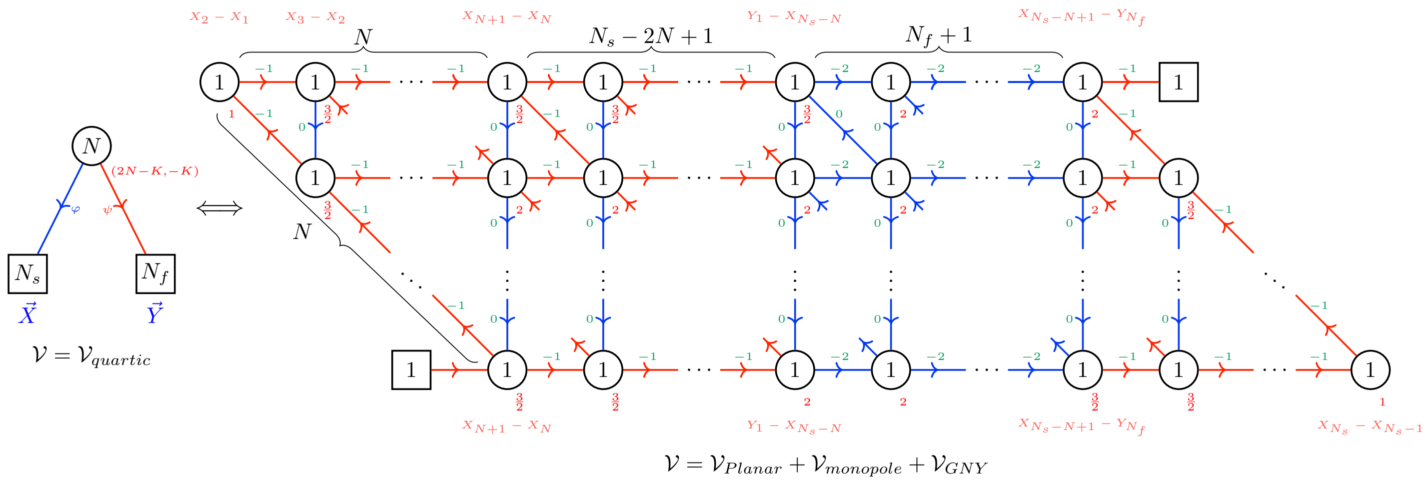


Figure 6.10: The dual of $U(N)$ QCD$_3$ with N_s scalars and N_f fermions at Chern-Simons level $(2N-K,\ -K)$, with $K = N_s + \frac{N_f}{2}$, is shown here. All labeling conventions follow those of Figure 6.6.

Prior to gauging the baryonic $U(1)$ symmetry, one can also introduce a Chern-Simons term for it at level Δ_ℓ, following the discussion in [82]. This leads to a $U(N)$ gauge theory with CS level $(2N - N_s - \frac{N_f}{2} - k,\ \Delta_\ell N - N_s - \frac{N_f}{2} - k)$. In the mirror dual, this corresponds to the addition of a new $U(1)_{\Delta_\ell}$ gauge node, coupled to the rest of the quiver via BF interactions.

6.6 Mass Deformations and RG Flow Across Dualities

In this section, we focus on studying the mapping of mass deformations across dualities, which serves as an important consistency check for our proposed dualities. We consider mass deformations for the fundamental fields in the QCD theory and propose a corresponding deformation in the Abelian dual that is partly inspired by the structure of the SUSY dualities of [44] and is a natural generalization of the Abelian case.

We also investigate the topological phases obtained by giving mass to all the charged fields in the QCD_3 theory and provide a consistent mapping of these deformations for some values of N, k and the number of charged fields.

6.6.1 Mass for a Single Field

A simple deformation that can be studied in the QCD_3 theory on the l.h.s. of Figure 6.6, consists of giving a mass to a boson or to a fermion that can be either positive or negative. One could think of it as a linear potential term for a mesonic operator of the form $|\varphi|^2$ or $\bar{\psi}\psi$. In principle, knowing the map of these operators across the duality 6.6 gives the knowledge of the deformation that is turned on in the dual theory. However, as we partially comment in Subsection 6.3.5, the map for these operators is not trivial[14]. Partially inspired by the SUSY ancestor of the duality, for which the deformations will be analyzed in [84], we consider examples to discuss massive deformations of the duality 6.6.

An Electric Scalar with Positive Mass

Concretely, consider $SU(2)$ at CS level $4-F$ with F scalars and its planar mirror dual (top row of Figure 6.11). Suppose we turn on a positive mass term $m^2|\phi_3|^2$ in the electric QCD_3 theory for one of the scalar fields (ϕ_3). We claim that this deformation maps to real mass terms for the fermions in one of the columns of the dual theory as:

$$m^2|\phi_3|^2 \quad \leftrightarrow \quad m(\bar{\psi}_1\psi_1 - \bar{\psi}_2\psi_2 + \bar{\psi}_3\psi_3)\,, \qquad m > 0 \tag{6.75}$$

where ψ_1 and ψ_3 are the horizontal fermions in the second column and ψ_2 is the diagonal fermion in the second column. Upon integrating out the massive scalar field, the electric theory flows to $SU(2)$ QCD_3 with $F-1$ scalars at CS level $4-F$. On the mirror side, we integrate out the fermions in the second column, leaving behind only BF couplings between the horizontal gauge nodes, as shown in the bottom row of Figure 6.11. The resulting duality is exactly the one we propose for $SU(N)$ with $F-1$ scalars at CS level $2N-k-(F-1)$, where $N=2$ and $k=1$ in this case. Notice that this is a natural generalization of the Abelian case in Figure 6.1, where giving a positive mass to a boson on the QED side corresponds to giving a positive mass to a fermion in the quiver side. The choice of signs for the fermionic masses in (6.75) reproduces the BF couplings expected from the general proposal 6.2.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric scalar with a positive mass results in the integration of a column of fermions in the bulk of its planar mirror, leaving behind only horizontal BF couplings between gauge nodes.

[14]We recall that the reason behind the difficulty of this map is that these operators are not charged under the Cartan subgroup of the global symmetry. Indeed, in the dual theory, there are a large number of operators, carrying the correct spin, that are not charged under the global symmetry, and in principle any linear combination of them is a valid candidate as the target of the map.

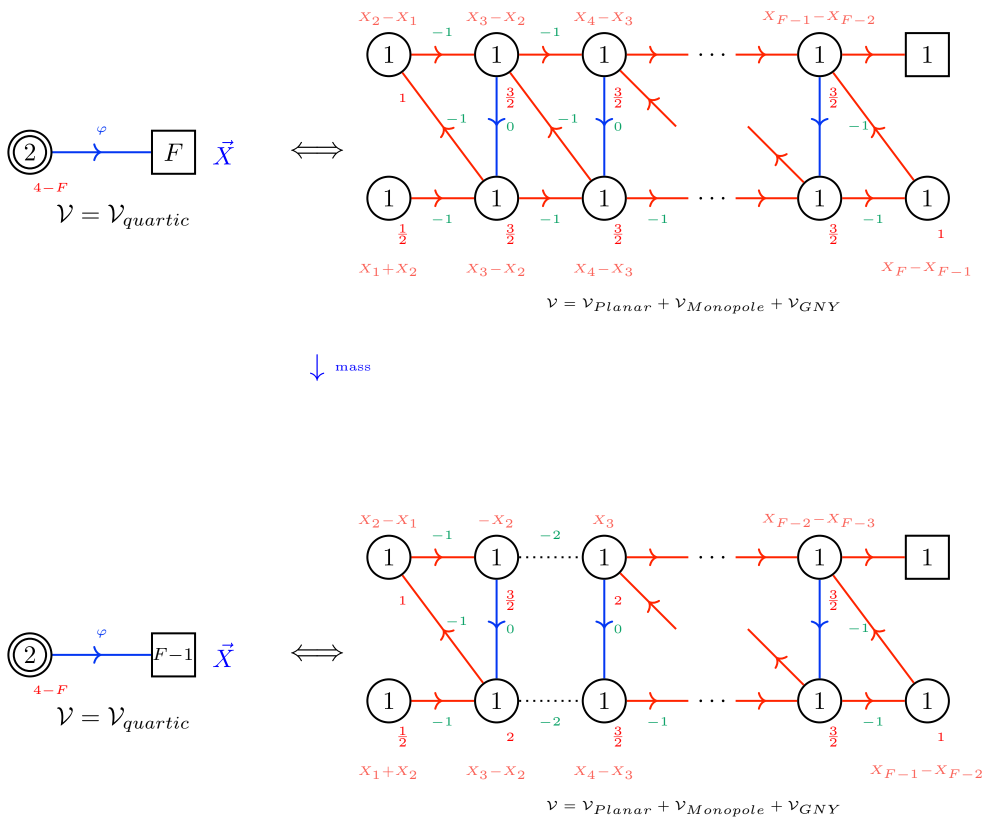


Figure 6.11: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single scalar field in $SU(2)_{4-F}$ QCD$_3$ with F scalar fields and the corresponding fermionic column in its planar dual (top row). Notice that CS and BF interactions have been generated according to the sign of the corresponding fermion masses. The resulting duality is exactly what we propose for $SU(2)$ with $F-1$ scalars at CS level $4-F$.

One can immediately realize that the number of columns composed of fermions in the bulk of the the dual quiver is $N_s - 2N$, where N_s is the number of scalars. Therefore we can use this strategy to integrate out only $N_s - 2N$ bosons out of N_s. Therefore we can not use this strategy to flow to QCD$_3$ with $N_s < 2N$, which indeed is beyond the range of validity of the duality 6.6. In principle one could study the effect of further massive deformations to go beyond the range $N_s \geqslant 2N$, this is a quite non-trivial exercise and we leave it for a future work.

One could also consider a negative mass for an electric scalar, which would condense and Higgs the gauge group $SU(N)_{2N-N_s-k} \to SU(N-1)_{2(N-1)-(N_s-1)-k'}$, where $k' = k-1$. We stress that the map of the deformation in (6.75) works only for $m > 0$. It would be interesting to see how this may be seen in the planar mirror theory, but it is beyond the scope of this paper.

An Electric Fermion with Positive Mass

Concretely, consider $SU(3)$ at CS level $6 - N_s - \frac{N_f}{2}$ with N_s scalars and N_f fermions, and its planar mirror dual (top row of Figure 6.12). Suppose we turn on a positive mass term for a fermionic field $m\bar{\psi}_{N_f}\psi_{N_f}$. We claim this deformation maps to a negative mass term for the horizontal bosonic fields in a column of the mirror theory:

$$m\bar{\psi}_{N_f}\psi_{N_F} \quad \leftrightarrow \quad -m^2(|\phi_H^{(1)}|^2 + |\phi_H^{(2)}|^2 + |\phi_H^{(3)}|^2)\,. \tag{6.76}$$

Upon integrating out the massive fermion, the electric theory flows to $SU(3)$ QCD$_3$ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} + \frac{1}{2}$. On the mirror side, the deformation cause the horizontal scalars to take a negative mass and thus a VEV, Higgsing pairs of $U(1)$ gauge groups connected by an horizontal scalar to the diagonal $U(1)$. This effectively identifies the two columns of $U(1)$ gauge groups to the left and right of the column of scalars. Furthermore, some of the cubic interactions for the scalars turn into effective masses, which allow one to integrate out the 2 diagonal scalars and 2 of the vertical scalars. The resulting quiver is shown in the bottom row of Figure 6.12. The resulting duality coincides with our proposed dual for $SU(N)$ with N_s scalars and $N_f - 1$ fermions at CS level $2N - N_s - \frac{(N_f-1)}{2}$, where $N = 3$ in this case.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric fermion with a positive mass results in Higgsing the horizontal bosons in a column in the bulk of its planar mirror, thereby shortening the quiver. One can check that CS and BF coupling of the resulting theory agree with our proposal (Figure 6.6), providing a consistency check of our results.

An Electric Fermion with Negative Mass

Concretely, consider $SU(3)$ at CS level $6 - N_s - \frac{N_f}{2}$ with N_s scalars and N_f fermions, and its planar mirror dual (top row of Figure 6.13). Suppose we turn on a negative mass term for a fermionic field $m\bar{\psi}_{N_f}\psi_{N_f}$. We claim this deformation maps to a positive mass term for the bosonic fields in a column of the mirror theory:

$$-m\bar{\psi}_{N_f}\psi_{N_F} \quad \leftrightarrow \quad m^2(|\phi_{diag}^{(1)}|^2 + |\phi_{diag}^{(2)}|^2 + |\phi_H^{(1)}|^2 + |\phi_H^{(2)}|^2 + |\phi_H^{(3)}|^2)\,. \tag{6.77}$$

Upon integrating out the massive fermion, the electric theory flows to $SU(3)$ QCD$_3$ with N_s scalars and $N_f - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} - \frac{1}{2}$. On the mirror side, the same deformations leads to the integration of a column of bosons, leaving behind only BF couplings between horizontal gauge nodes, as shown in the bottom row of Figure 6.13. The resulting duality is exactly the we propose for $SU(N)$ with N_s scalars and $N_f - 1$ fermions at CS level $2N - N_s - \frac{(N_f-1)}{2} - k$, where $N = 3$ and $k = 1$ in this case.

Immediate generalizations to electric gauge groups with higher rank follow from this observation. We reiterate that integrating out an electric fermion with a negative mass results in the integration of a column of bosons in the bulk of its planar mirror, leaving behind only BF couplings between horizontal gauge nodes.

We conclude by mentioning that in the duality 6.6, each fermion in the QCD$_3$ corresponds to a column of bosons in the dual theory. Thereforem the results summarized in Figure 6.12 and 6.13 can be used to study the effect on the dual theory of a mass for any number of fermions in the QCD$_3$.

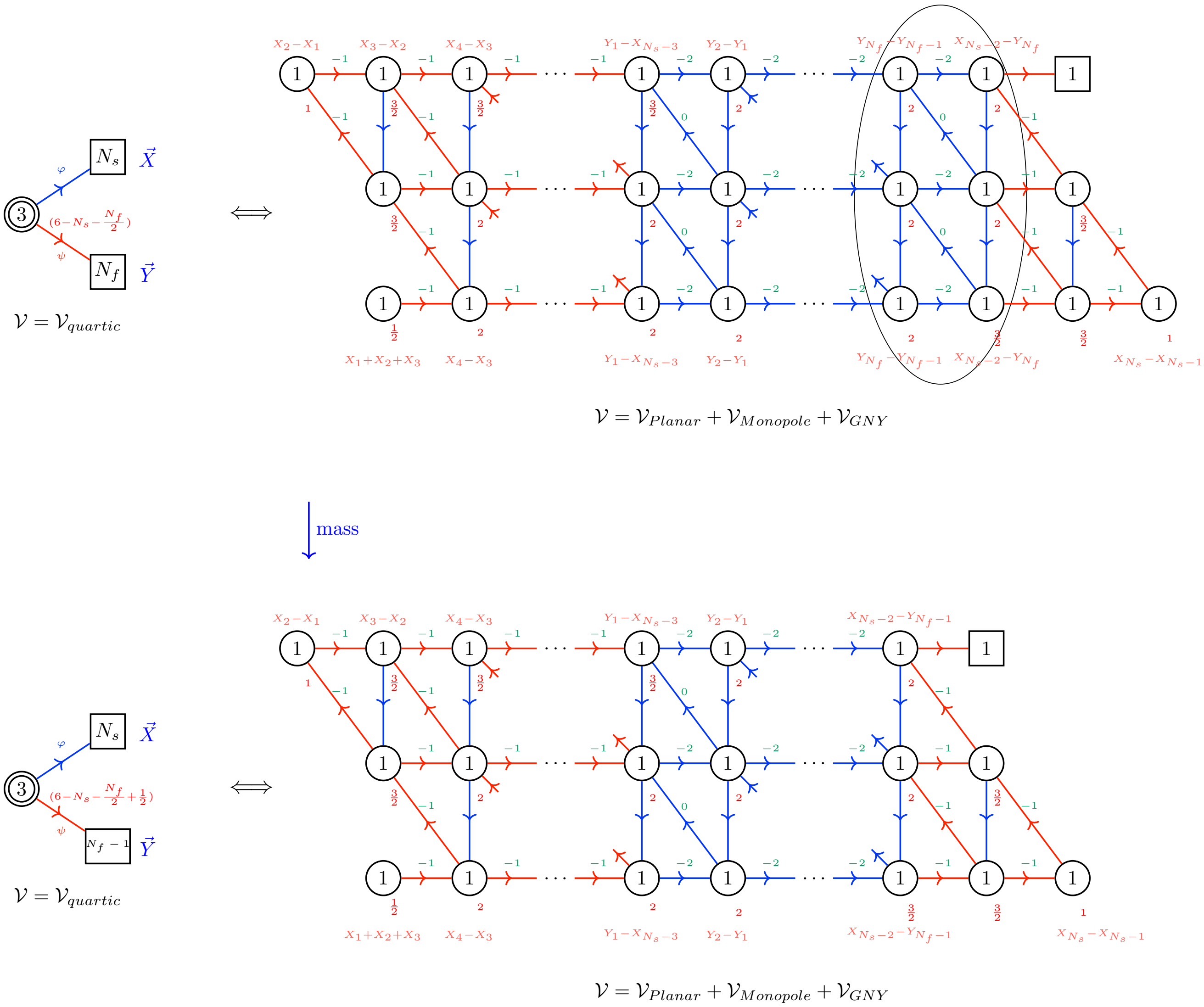


Figure 6.12: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single fermion with a positive mass in $SU(3)_{6-N_s-\frac{N_f}{2}}$ QCD$_3$ with N_s scalar fields and N_f fermionic fields and the corresponding bosonic column in its planar dual (top row). The resulting duality is exactly what we propose for $SU(3)$ with N_s scalars and $N_F - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} + \frac{1}{2}$.

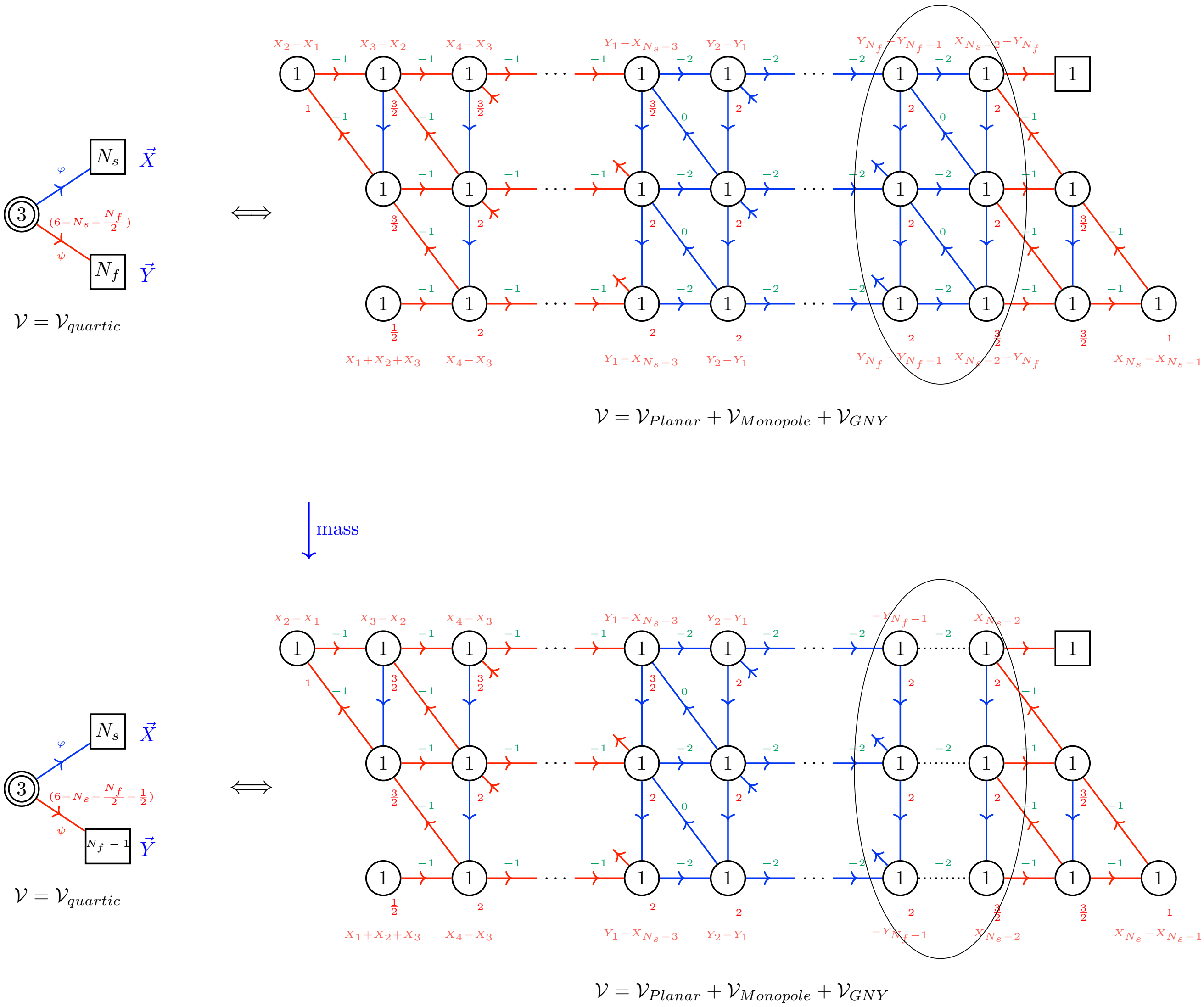


Figure 6.13: Schematic depiction of the duality (bottom row) resulting from the RG flow induced by integrating out a single fermion with negative mass in $SU(3)_{6-N_s-\frac{N_f}{2}}$ QCD$_3$ with N_s scalar fields and N_f fermionic fields and the corresponding bosonic column in its planar dual (top row). The resulting duality is exactly what we propose for $SU(3)$ with N_s scalars and $N_F - 1$ fermions at CS level $6 - N_s - \frac{N_f}{2} - \frac{1}{2}$.

6.6.2 A consistency check via particle-vortex

Consider $SU(2)$ at CS level 0 with 4 scalars and its planar mirror dual (top row of Figure 6.14). Suppose we turn on a positive mass term $m^2|\phi_1|^2$ in the electric QCD$_3$ theory for the scalar field ϕ_1. We claim that this deformation maps to the mirror theory, resulting in the duality shown in the middle row of Figure 6.14.

We can further deform the resulting QCD$_3$ by giving a mass also the ϕ_3 scalar field, resulting in a $SU(2)_0$ QCD$_3$ with only two scalars. We claim that this deformation corresponds to giving mass to the remaining fermions, leading to the duality in the last row of Figure 6.14. Notice that in this last step we actually remain with one scalar and two $U(1)$ gauge nodes, the diagonal combination decouples as a $U(1)_1$ which is an almost trivial theory.

At this point the $U(1)_0$ theory with one scalar can be also dualized using particle-vortex [38], resulting in a $O(2)$ Wilson-Fisher scalar. This in turn is also related, via bosonization, to a $U(1)_{\frac{1}{2}}$ theory with a fundamental fermion (6.3). The duality between $SU(2)_0$ with two scalars and $U(1)_{\frac{1}{2}}$ with one fermion, is consistent with a duality proposed in [93].

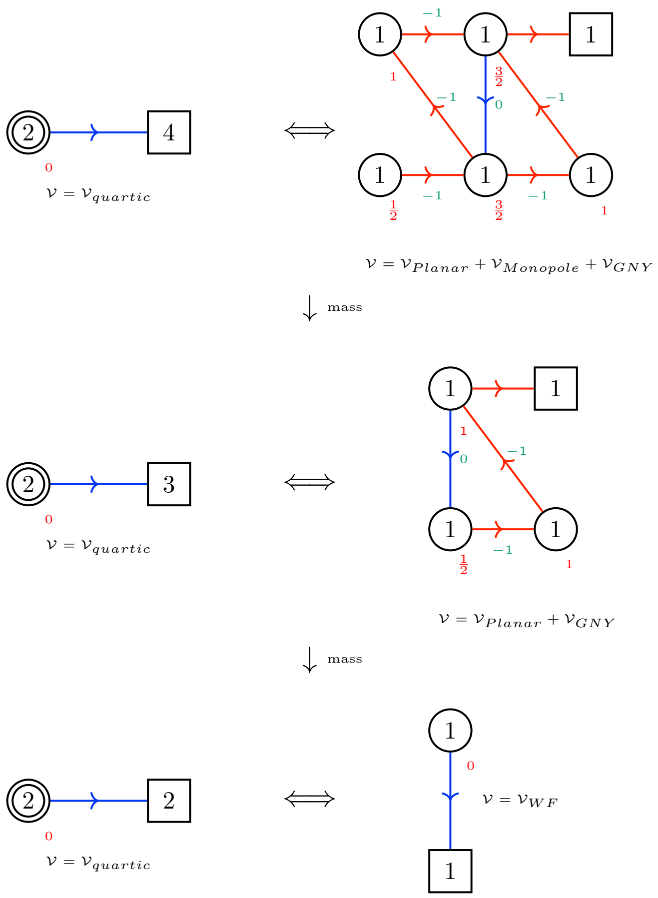


Figure 6.14: Schematic depiction of the duality conjectured for $SU(2)_0$ with 2 scalars. We omit writing topological fugacities for brevity. As per our conventions, fermionic (bosonic) fields are shown in red (blue), and (mixed) Chern-Simons interactions in maroon (green).

We can generalize this result further to $SU(N)_0$ with N scalars by starting from $SU(N)_0$ with $2N$ scalars and integrating out N of them, and find that $SU(N)_0$ with N scalars is also dual to an $O(2)$ Wilson-Fisher scalar.

6.6.3 TQFT Dualities from Mass Deformations

Instead of considering mass deformations for a single boson or fermion, we can consider a mass deformation for all of them. In particular we could consider a positive mass for all the scalars and a homogeneous mass for the fermions in the QCD$_3$, which always leads to a TQFT, and track the effect in the dual theory that should flow to a dual TQFT to match the IR phase. A general analysis shows that the resulting TQFT duality obtained is quite complicated. A possible strategy is to consider gapped phases of the $\mathcal{N}=2$ theories involved in the ancestor duality [44]. In particular one can study the flow from the ancestor $\mathcal{N}=2$ duality to a TQFT duality which we believe is equivalent to the TQFT duality we would reach starting from our proposed dual pair. The problem of studying RG flows in the SUSY case will be addressed in detail in [84], here we anticipate the result in the following two cases, consisting in the SQCD theories whose Abelian planar dual contains one or zero "bulk" columns.

We start by discussing the TQFT duality obtained starting from the duality for $SU(N)_{-1}$ with $N_s = 2N$ scalars and zero fermions 6.2. Turning on positive masses for all the scalars results in $SU(N)_{-1}$ CS theory. The planar Abelian dual theory has a single empty bulk column, see Figure 6.2[15].

As explained above we can consider a chiral-planar SUSY duality [44] where the electric theory flows to the same $SU(N)_{-1}$ TQFT. There the mass deformation can be followed on the planar Abelian side, resulting in a $U(1)^2$ CS thery with CS matrix:

$$\mathbf{K} = \begin{pmatrix} N & -N \\ -N & N+1 \end{pmatrix}; \tag{6.78}$$

This is the TQFT duality reported in quiver notation in Figure 6.15. We expect that this is the same TQFT duality one would obtain by deforming the non-SUSY duality for $SU(N)_{-1}$ with $N_s = 2N$ scalars and zero fermions. In the remainder of this Section we show that this duality can be proved via known TQFT dualities, providing a check for the flows discussed above.

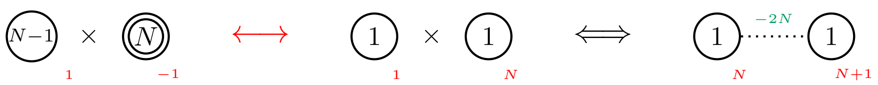


Figure 6.15: Duality between $SU(N)_{-1} \times U(N-1)_1$ CS theory and the Abelian CS theory associated to the CS matrix (6.78). (Mixed) Chern-Simons interactions are indicated in (green) red. The red arrow indicates level/rank duality for $SU(N)_{-1}$, while the double arrow indicates a duality between Abelian TQFTs described in the main body.

The 1-form symmetry group of the Abelian CS theory associated to $\mathbf{K}$ is $\mathbb{Z}_N^{(1)}$, as can be deduced from the Smith normal form of $\mathbf{K}$ [100], matching the $SU(N)_{-1}$ on the electric side. The duality between the Abelian CS theory associated to $\mathbf{K}$ and $SU(N)_{-1}$ can be proved by first applying level/rank duality (red arrow in 6.15), and then relating the resulting Abelian theories (double arrow in 6.15). The last step requires us to find

[15]The analysis discussed in this Section extends naturally to other QCD theories whose Abelian dual has a single "bulk" column.

two 2×2 matrices $\mathbf{Q}$ and $\mathbf{P}$ with integer coefficients, as described in [100], satisfying the condition

$$\mathbf{Q}^t \mathbf{K}_L^{-1} \mathbf{Q} - \mathbf{K}^{-1} = \mathbf{P}, \tag{6.79}$$

where $\mathbf{K}_L^{-1} = \mathrm{diag}(1, \frac{1}{N})$ and $\mathbf{K}^{-1}$ are the inverse Chern-Simons matrices of the two dual theories, respectively. The following matrices satisfy these requirements:

$$\mathbf{Q} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \tag{6.80}$$

where the diagonal elements of $\mathbf{P}$ are even, as required for spin preservation. The matrix $\mathbf{Q}$ is the (left-)invertible matrix specifying the bijective mapping of the $2|\det(\mathbf{K})|$ independent Wilson lines, from the dual side to the electric side, i.e., $\vec{\alpha}_L = \mathbf{Q}\vec{\alpha}$, where $\vec{\alpha}_L$ and $\vec{\alpha}$ are the charge vectors of the Wilson line on the electric and dual side, respectively. Notice that the sequence of dualities depicted in 6.15 requires an additional SPT $U(N-1)_1$ on the l.h.s. It would be interesting to keep track of the invertible backgrounds generated by the deformation of the quiver theory considered here in order to determine the precise STP factors involved in the duality for $SU(N)_{-1}$ with $2N+1$ scalars, but we leave this analsis to future work.

Similarly one can consider the case of $SU(N)_0$ QCD with $N_s = 2N$ scalars. By giving positive masses to all the scalars the theory flows to $SU(N)_0$ YM. The analysis of the flow to the TQFT from the corresponding SUSY theory, together with its planar Abelian dual, predicts that the dual flows to $U(1)_1$, which is an almost trivial TQFT matching the electric side.

6.7 Conclusions

In this chapter we have proposed a novel class of dualities for the CS QCD_3. Interestingly, this duality displays different features compared to to those proposed in [90, 101, 92, 93], namely those that we propose relate a non-Abelian gauge theory to an Abelian CS theory that has the shape of a planar quiver.

The duality we propose is valid only for large value of the number of bosons with respect to the CS level of the theory. In particular the number of scalars N_s must be greater than or equal to $2N$, where the gauge group is $U/SU(N)$. In [45] we also present dualities for $N_s = 2N-1, 2N-2$, that must satisfy $N_f \geqslant 1$ or $N_f \geqslant 2$, respectively. These dualities are not discussed in this thesis. It would be interesting to extend these dualities beyond the range currently available.
It would also be interesting to generalize these dualities to other gauge groups, such as $USp(2N)$ or $O(N)$.

Another interesting future perspective consists in further studying the massive phases of these theories and how they match across the duality. This is intriguing since in the cases discussed in Section 6.6 the matching is always done using level-rank duality for the TQFT at play. However, in cases beyond those discussed in Section 6.6 the resulting TQFT duality is much more intricate and does not appear as any level-rank or similar duality to the best of our knowledge. To do this we aim to exploit the fact that the massive phases of the SUSY version of these dualities, those that are discussed in Chapter 5, are equal to those of the non-SUSY case. We can then use the tools available in presence of SUSY to study these TQFT dualities and gain a better insight of the massive phases of non-SUSY dualities.

Appendix A

S_b^3 partition function

The partition function of 3d $\mathcal{N}=2$ theories on the three-sphere was computed thorugh localization techniques first in [18, 80, 19] on the round S^3 and later generalized to the case of the squashed three-sphere S_b^3 in [20][1]. We now focus on the latter.

The partition function of a generic 3d $\mathcal{N}=2$ generically depends on the real mass parameters $\vec{m}$ and set of Fayet-Iliopoulos parameters $\vec{\eta}$. To explain the generic expression of the partition function let us take as an example a generic gauge theory with a generic gauge group G, the generalization to a quiver theory is straightforward. The partition function generically reads:

$$Z(\vec{m},\vec{\eta}) = \frac{1}{|\mathcal{W}_G|}\int_{\mathbb{R}^{\mathrm{rk}\,G}}\prod_{i=1}^{\mathrm{rk}\,G}\mathrm{d}z_i\; Z^{(k)}_{\text{classical}}(\vec{\eta},\vec{m},\vec{z})Z_{\text{gauge}}(\vec{z})Z_{\text{matter}}(\vec{m},\vec{z})\,. \tag{A.1}$$

where G is the gauge group whose rank is rk$G$ and $|\mathcal{W}_G|$ is the dimension of the Weyl group. The integration is performed over the set $\vec{z}$ parameterizing the Cartan subgroup of G.
The partition function is written as a product of double-sine functions defined as:

$$s_b(x) = \prod_{m,n}\frac{(m+\frac{1}{2})b+(n+\frac{1}{2})b^{-1}-ix}{(m+\frac{1}{2})b+(n+\frac{1}{2})b^{-1}+ix}\,. \tag{A.2}$$

Satisfying the property $s_b(x)s_b(-x)=1$. b is the squashing parameter and it is useful to define the following contribution $Q=b+b^{-1}$.

To define the various contribution let us consider

- The classical part encodes the Chern-Simons and the Background Field interactions that can be effectively written by introducing the vector $\vec{w}$, which joins together the gauge $\vec{z}$ parameters together with the real $\vec{m}$ and $\vec{\eta}$ real parameters. We can then write:

  $$Z^{(k)}_{\text{classical}}(\vec{\eta},\vec{m},\vec{z}) = \mathrm{e}^{-i\pi\sum_{a,b}w_a k_{ab} w_b}\,, \tag{A.3}$$

  where k_{ab} is a matrix encoding the level of CS and BF couplings, satisfying suitable quantization conditions and symmetry preserving properties.

  When we turn off all the non-dynamical contributions, the classical part depends only on the FI parameters $\vec{\eta}$ and the gauge parameters $\vec{z}$, as well as only the dynamical CS terms.

[1] See also [102] for a comprehensive review of the partition function of 3d theories on various three-manifolds.

- The gauge part encodes the contribution of the vector multiplet:

$$Z_{\text{gauge}}(\vec{z}) = \frac{1}{\prod_\alpha s_b\left(\frac{iQ}{2} - \alpha(\vec{z})\right)}\,, \tag{A.4}$$

  where α are the roots of G.

- The matter part is obtained as the product of the contributions of chiral multiplets. The contribution of a chiral multiplet in the representation R of G, rotated by a $U(1)$ global symmetry with real parameter x and with charge r under the $\mathcal{N}=2$ $U(1)_R$ R-symmetry group, it is:

$$Z^{(r)}_{\text{chir}}(\vec{z},x) = \prod_{\omega\in R} s_b\left(\tfrac{iQ}{2}(1-r) - \omega(\vec{z}) - x\right)\,. \tag{A.5}$$

  Where ω are the weights of the representation R. The collection of all the independent x parameters appearing in the final contribution, gives the set $\vec{m}$.

  Notice that it is possible that there is matter charged under topological symmetries, for example monopole flipping fields. In this case the matter part depends not only on the real masses $\vec{m}$ but also on the FI parameters $\vec{\eta}$.

As already mentioned, the R-symmetry group of 3d $\mathcal{N}=2$ theories is abelian and thus it is possible that it mixes with abelian global symmetries along the RG flow. The R-charges at the IR fixed point can be determined by the so-called "F-extremization" procedure [80]. The statement is that the value of the R-charges at the SCFT point maximizes the free energy defined as to be the logarithm of the partition function. In general, given a set of $U(1)_a$ abelian global symmetry and a trial UV R-symemtry $U(1)_{R_0}$, we introduce mixing coefficients $\vec{t}$ and define:

$$R(\vec{t}) = R_0 + \sum_a t_a q_a\,, \tag{A.6}$$

where R_0 and Q_a are the UV R-charge and the charge under the $U(1)_a$ global symmetry, respectively. To define a function $Z(\vec{q})$ depending on mixing parameters we take the real parameter m_a associated to the $U(1)_a$ symmetry and make it complex. The imaginary part of m_a is proportional to the mixing parameter $q_a \sim \text{Im}(m_a)$ (up to a fixed normalization). The superconformal R-symmetry is then the one obtained from (A.6) with $\vec{q} = \vec{q}_{sc}$ which minimizes $Z(\vec{q})$, thus satisfying:[2]

$$\begin{cases} \left.\dfrac{\partial Z(\vec{q})}{\partial q_a}\right|_{\vec{q}=\vec{q}_{sc}} = 0 & \forall\, a \\ \left.\dfrac{\partial^2 Z(\vec{q})}{\partial q_a \partial q_b}\right|_{\vec{q}=\vec{q}_{sc}} > 0 & \forall\, a,b \end{cases} \tag{A.7}$$

[2]The second derivative of the partition function can not be zero due to an identity relating it to the central charge of the global symmetry, which can not be zero if the symmetry is faithful [61].

Appendix B

Monopole charges

B.1 Generalities of monopole operators

In this appendix we review various properties of monopole operators in $3d$ gauge theories.

B.1.1 Chern-Simons Fields Coupled to Fermions

Consider a gauge theory with N_f massless fermions ψ, described by the Lagrangian

$$\sum_{j=1}^{N_f} i\bar{\psi}_j \not{D}_A \psi_j + \frac{k_0}{4\pi} AdA, \tag{B.1}$$

where A is a $U(1)$ gauge field, and k_0 is the *bare* Chern-Simons (CS) level, quantized as an integer. The presence of fermions and the CS term implies an anomaly in the time-reversal symmetry $\mathcal{T}$. When quantizing the theory, a regulator for the fermion determinant that explicitly breaks time-reversal invariance (e.g., Pauli-Villars regularization) introduces a phase $\exp\left(-\frac{i\pi}{2}\eta(A)\right)$ in the partition function, where η is the APS η-invariant [103]. For our purposes, this phase can be interpreted as a shift in the bare CS level of $U(1)_A$ by $-\frac{N_f}{2}$, accompanied by the addition of a mixed gravitational CS term $-N_f\, CS_g$, which avoids Dirac string singularities. We denote this theory as $U(1)_{k_0-N_f/2}$, where the subscript $k = k_0 - N_f/2$ represents the shifted CS level, which can be a half-integer.

Introducing a real mass m for all fermions and integrating them out generates a CS interaction. At low energies, the effective CS level becomes

$$k_{IR} = k_0 - \frac{N_f}{2} + \frac{N_f}{2}\mathrm{sign}(m), \tag{B.2}$$

which is always an integer. Additionally, a gravitational CS term is induced:

$$N_f\, CS_g\, \mathrm{sign}(m). \tag{B.3}$$

Of particular interest is the case where $k = 0$. When a positive mass ($m > 0$) is turned on for all fermions, the flow leads to a trivial theory with a vanishing Lagrangian. Conversely, a negative mass ($m < 0$) results in:

$$-\frac{N_f}{4\pi} AdA - 2N_f\, CS_g. \tag{B.4}$$

The gravitational term $-2N_f\, CS_g$ is sometimes denoted as $U(N_f)_1$ in the literature. Time reversal acts on a fermion as:

$$\mathcal{T} : i\bar{\psi}\not{D}_A\psi \quad \rightarrow \quad i\bar{\psi}\not{D}_A\psi + \frac{1}{4\pi}AdA + 2CS_g \tag{B.5}$$

In the present paper a major role is played by quiver-like theories, where fermions can be charged under multiple gauge groups. Consider then a $U(1)^{(a)} \times U(1)^{(b)}$ gauge theory with a fermion ψ of charge $+1$ and -1 under the two gauge groups, bare CS levels $k_{0,a}$ and $k_{0,b}$ and mixed CS level $k_{0,ab}$:

$$\mathcal{L} = i\bar{\psi}\not{D}_{a-b}\psi + \frac{k_{0,a}}{4\pi}ada + \frac{k_{0,b}}{4\pi}bdb + \frac{k_{0,ab}}{4\pi}adb, \tag{B.6}$$

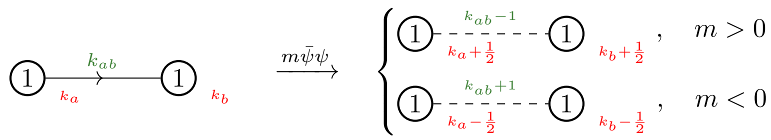
the regularization of the fermion determinant produces a $-\frac{1}{2}$ shift for the gauge field $a-b$:

$$-\frac{1}{8\pi}(a-b)d(a-b) = -\frac{1}{8\pi}ada - \frac{1}{8\pi}bdb + \frac{1}{4\pi}adb \tag{B.7}$$

resulting in the shifted levels:

$$k_a = k_{0,a} - \frac{1}{2}\,, \quad k_b = k_{0,b} - \frac{1}{2}\,, \quad k_{ab} = k_{0,ab} + 1\,. \tag{B.8}$$

Furthermore a mass deformation for the fermion produces an additional $\pm\frac{1}{2}$ shift in the CS levels k_a, k_b and a ∓ 1 shift in the mixed CS level k_{ab}. In the quiver notation used throughout this paper the theory (B.6) and its mass deformations are denoted as follows[1]:



(B.9)

B.1.2 Quantum Numbers of Monopole Operators

In three dimensions, local disorder operators can be defined in the presence of a $U(1)$ gauge group [97, 104]. To begin, note that in $2+1$ dimensions, there exists a conserved current

$$J = *F, \tag{B.10}$$

where F is the $U(1)$ field strength. This current is identically conserved due to the Bianchi identity, $d * J = dF = 0$. The corresponding charge is referred to as the vortex charge or topological charge.

A monopole operator of topological charge 1 is defined by integrating over field configurations with a Dirac singularity at a point $p \in \mathbb{R}^3$, such that a magnetic flux of 1 passes through any $\mathbf{S}^2$ surrounding p. This results in an operator $O(x)$ with operator product expansion (OPE) with J^μ:

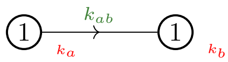
$$J^\mu(x)O(0) \sim \frac{1}{4\pi}\frac{x^\mu}{|x|^3}O(0) + \text{less singular terms.} \tag{B.11}$$

The quantum numbers of these monopole operators can be determined by analyzing the fermionic zero modes in the radially quantized theory with flux along S^2 [97, 99].

[1] In the quiver notation used in this paper we do not keep track of background gravitational CS levels CS_g.

The contribution from fermionic zero modes to the gauge charge of the monopole can be encoded in an effective shift of $\frac{1}{2}$ to the CS level for each charged fermion. Therefore in a $U(1)_k$ theory with N_f fermions the gauge charge of a monopole with GNO flux $+1$ is $k + \frac{N_f}{2}$, which is equal to the *bare* CS level k_0. Gauge invariant operators can then be constructed by dressing this bare monopole with (derivatives of) fundamental charged fields.

This analysis can be extended to theories with multiple gauge groups, as an example consider the theory with two gauge nodes (B.6):



(B.12)

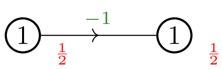
The fermion is charged under $a-b$ and its zero modes contributes as a $\frac{1}{2}$ shift to the CS level of $a-b$, which is equivalent to a $\frac{1}{2}$ shift for the CS levels k_a, k_b and a -1 shift to the mixed CS k_{ab}. Consider a generic bare monopole $\mathfrak{M}$ with fluxes n under $U(1)_a$ and m under $U(1)_b$, denoted as $\mathfrak{M}^{(n,m)}$. Gauss' law dictates that the monopole carries gauge charges $(n(k_a+\frac{1}{2}) + m\frac{(k_{ab}-1)}{2}, m(k_b+\frac{1}{2}) + n\frac{(k_{ab}-1)}{2})$ under $U(1)_a \times U(1)_b$. We write:

$$\begin{aligned} Q\left[\mathfrak{M}^{(n,m)}\right] =& nQ\left[\mathfrak{M}^{(+,0)}\right] + mQ\left[\mathfrak{M}^{(0,+)}\right] \\ =& n\left(k_a + \frac{1}{2}, \frac{k_{ab}-1}{2}\right) + m\left(\frac{k_{ab}-1}{2}, k_b + \frac{1}{2}\right) \\ =& \left(n(k_a+\frac{1}{2}) + m\frac{(k_{ab}-1)}{2}, m(k_b+\frac{1}{2}) + n\frac{(k_{ab}-1)}{2}\right) \end{aligned} \tag{B.13}$$

Gauge-invariant states can be constructed by dressing with modes of ψ (or $\bar{\psi}$) in very specific cases. Since ψ has charges $(+1,-1)$, a monopole dressed p times with ψ and q times with $\bar{\psi}$ has charge:

$$Q\left[\mathfrak{M}^{(n,m)}\psi^p\bar{\psi}^q\right] = \left(n(k_a+\frac{1}{2}) + m\frac{(k_{ab}-1)}{2} + p - q,\ m(k_b+\frac{1}{2}) + n\frac{(k_{ab}-1)}{2} + q - p\right) \tag{B.14}$$

As an example, for $k_a = k_b = \frac{1}{2}$ and $k_{ab} = -1$ the monopoles $\mathfrak{M}^{(+,0)}\partial_\bullet\bar{\psi}$ and $\mathfrak{M}^{(0,+)}\partial_\bullet\psi$ are gauge invariant[2]:



$$\rightarrow \quad \begin{aligned} Q\left[\mathfrak{M}^{(+,0)}\partial_\bullet\bar{\psi}\right] &= (0,0), \\ Q\left[\mathfrak{M}^{(0,+)}\partial_\bullet\psi\right] &= (0,0) \end{aligned} \tag{B.15}$$

B.1.3 Statistical Transmutation via Flux Attachment

In the presence of a monopole background, the spin of fundamental fields in the ground state undergoes a shift — a phenomenon known as *statistical transmutation*. This shift depends on the GNO charge (or flux) of the monopole, the intrinsic spin of the particle, and its gauge charge.

In what follows, we summarize the effects of statistical transmutation for a particle of spin s, and examine how this behavior is modified in the presence of multiple $U(1)$ gauge factors, particularly in the context of the quiver gauge theories discussed in the main text.

[2]Here $\partial_\bullet$ denotes a generic set of derivatives.

1. Single Gauge Group

Consider a spin s field φ with charge q under a $U(1)$ gauge group. On the background of a monopole with flux m the ground state of the field φ has the following spin channels:

$$|\frac{qm}{2} - s|, |\frac{qm}{2} - s| + 1, \ldots, \frac{qm}{2} + s,$$

For example in the background of a monopole with flux $m = 1$ a fermion has a spin-0 and a spin-1 channel and a scalar has a spin-$\frac{1}{2}$ channel.

2. Bifundamental Matter

Consider a spin-s field φ with charge $(+1, -1)q$ under a $U(1) \times U(1)$ gauge symmetry, shown explicitly in Equation (B.16):

$$\textcircled{1} \xrightarrow{\varphi} \textcircled{1} \tag{B.16}$$

The dressed operator $\mathfrak{M}^{(m,n)}\varphi$ will have the following spin channels:

$$|\frac{q}{2}(m-n) - s|, |\frac{q}{2}(m-n) - s| + 1, \ldots, \frac{q}{2}(m-n) + s,$$

which depends on the difference of the GNO fluxes of the two gauge groups. In particular in the background of a monopole $\mathfrak{M}^{(+,+)}$ the bifundamental field φ "sees" a vanishing effective flux, and its spin is not transmuted.

3. Absence of Transmutation from Distant Fluxes

Finally, we consider the quiver gauge theory shown in Equation (B.17):

$$\textcircled{1} \xrightarrow{\varphi_1} \textcircled{1} \xrightarrow{\varphi_2} \textcircled{1} \tag{B.17}$$

where φ_i are bifundamental spin-s_i fields. On the background of a monopole $\mathfrak{M}^{[0,0,m]}$ the field φ_1 does not "see" the magnetic flux, and its spin remains s_1. On the other hand the field φ_2 "sees" an effective flux $-m$, and has spin channels shifted by $-m$:

$$\begin{array}{ll} \varphi_1 : \ |-s_1|, \ |-s_1| + 1, \ \ldots, \ |s_1| & \\ \varphi_2 : \ \left|-s_2 - \frac{m}{2}\right|, \ \left|-s_2 - \frac{m}{2}\right| + 1, \ \ldots, \ \left|s_2 - \frac{m}{2}\right| & \end{array} \qquad \text{on } \mathfrak{M}^{[0,0,m]} \tag{B.18}$$

Generally, consider a field φ of spin-s and charges q_i under some $U(1)_i$ gauge groups, with $i = 1, \ldots, M$. Then in the background of a monopole with GNO fluxes m_i the field φ has the same spin channels as a spin-s field with charge 1 in the background of an "effective" GNO flux m_{eff}:

$$m_{eff} = \sum_{i=1}^{M} q_i m_i \tag{B.19}$$

B.2 BPS monopole operators

We now restrict our attention to monopole operators in supersymmetric theories. In these theories certain configurations define half-BPS monopole operators [78], in this section we review some of their properties such as how to compute their R-charge and the charges under global symmetries.

Let us consider as an example a theory with gauge group G coupled to matter in various representations. As already reviewed, a monopole operator is defined by a GNO, or magnetic, flux [105] which is a vector taking values over the Cartan subgroup of G and thus is a (rk G)-dimensional vector $\vec{m}$.
The R-charge of this monopole, as well as any charge under an abelian global symmetry is obtained as [34, 78]:

$$q[\mathfrak{M}^{\vec{m}}] = \Lambda \sum_{i=1}^{\mathrm{rk}G} m_i - \frac{1}{2} \sum_{f \in \text{fermions}} \sum_{\omega \in R_f} |\omega(\vec{m})| q_f \tag{B.20}$$

This formula is often referred to as the "monopole formula" The first sum runs over all the fermions in the theory. R_f is the representation of the fermion under the gauge symmetry, ω are the weights of the representation and q_f is the charge of the fermion under the R-symmetry or the global $U(1)$ symmetry. Moreover, the factor Λ is the mixing between the topological symmetry and the R-symmetry, or equivalently the level of the BF coupling involving the topological symmetry and the abelian global symmetry in interest.

The monopole operator can also be gauge variant. It is possible to determine the highest weight of the representation of the gauge group G under which it transforms by computing the charge of the fermion under each $U(1)_i$ factor in the Cartan subgroup of G:

$$k_i \sum_{i=1}^{\mathrm{rk}G} m_i - \frac{1}{2} \sum_{f \in \text{fermions}} \sum_{\omega \in R_f} |\omega(\vec{m})| q_f^{(i)} \tag{B.21}$$

where in this case $q_f^{(i)}$ is the charge of the fermion under $U(1)_i$. These operators hence require a dressing which might change properties of the starting monopole, such as the charges under global symmetries as well as properties under SUSY transformation. Referred to the latter, it is possible for example to obtain a quarter-BPS operator by starting from an half-BPS one after the dressing.

We recall that these monopole operators, by construction, can not carry charges under manifest simple global symmetry groups. However, they can still transform in representation under simple global symmetries due to various possible phenomena. For example, if the monopole operator is not gauge invariant, after the dressing it may transform in non-trivial representations due to the extra matter fields. Also, the restriction applies only to manifest simple global symmetries. Many monopoles can in fact recollect together to form non-trivial representations of global symmetries that are emergent in the IR.

Appendix C

Special 3d $\mathcal{N} = 2, 4$ theories

C.1 $FT[U(N)]$ theory: the $\mathcal{S}$-wall

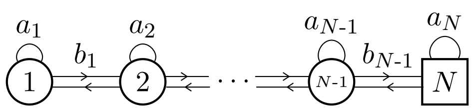
The $FT[U(N)]$ SCFT is a variation of the $T[SU(N)]$ $\mathcal{N} = 4$ theory of [28] in which we introduce an extra BF coupling and a chiral field fliping one moment map, as we will explain momentarily.

The $FT[U(N)]$ theory is characterized by a $SU(N) \times SU(N)$ global symmetry on top of the $\mathcal{N} = 4$ R-symmetry. We will however work *off-shell* by considering the global symmetry effectively as $U(N) \times U(N)$ which will be useful to perform the gauging of these symmetries. We will denote this theory in short with the following symbol:

$$\boxed{N} \text{- - - - - -} \boxed{N} \tag{C.1}$$

to highlight the presence of the two $U(N)$ global symmetries.

The $FT[U(N)]$ theory has the following quiver description:



$$\mathcal{W} = \sum_{i=1}^{N-1} b_i (a_i + a_{i+1}) \tilde{b}_i \tag{C.2}$$

We will consider an $\mathcal{N} = 2^*$ parameterization in which the bifundamentals have $U(1)_R$ charge 1 and the adjoint chirals 0. Also we have a global symmetry, which is the commutanto of $U(1)_R$ in the $\mathcal{N} = 4$ R-symmetry, that is $U(1)_A$ with associated real mass m_A. The bifundamentals have $U(1)_A$ charge -1 and the adjoint chirals 2[1].

The UV global symmetry is $SU(N) \times U(1)^{N-1} \times U(1)_\tau$. The $U(1)^{N-1}$ symmetry is given by the collection of topological symmetries associated to each gauge group and in the IR they enhance to $SU(N)$.

The IR spectrum of the theory is given by the moment map on the Coulomb branch and the chiral adjoint a_N. Both have R-charge 0 and $U(1)_A$ charge 2. Notice that the moment map on the Higgs branch is set to zero in the chiral ring due to the presence of the flipping field.

[1]This choice of R-charge is not the superconformal one. Indeed $U(1)_A$ can mix with $U(1)_R$ along the RG-flow to give the superconformal R-symmetry.

The partition function of the $FT[U(N)]$ theory can be defined recursively as:

$$
\begin{aligned}
\mathcal{Z}_{FT}^{(N)}(\vec{X};\vec{Y};m_A) &= \mathrm{e}^{2\pi i Y_N \sum_{i=1}^{N} X_i} \prod_{i,j=1}^{N} s_b\left(i\frac{Q}{2} + X_i - X_j - 2m_A\right) \\
&\times \int \mathrm{d}\vec{Z}_{N-1}^{(N-1)}\, \Delta_{N-1}(\vec{Z}^{(N-1)}) \mathrm{e}^{-2\pi i Y_N \sum_{a=1}^{N-1} Z_a^{(N-1)}} \prod_{a=1}^{N-1}\prod_{i=1}^{N} s_b\left(\pm(Z_a^{(N-1)} - X_i) + m_A\right) \\
&\times \mathcal{Z}_{FT}^{(N-1)}\left(Z_1^{(N-1)}, \cdots, Z_{N-1}^{(N-1)}; Y_1, \cdots, Y_{N-1}; m_A\right) ,
\end{aligned} \tag{C.3}
$$

with the basis for the recursion given by:

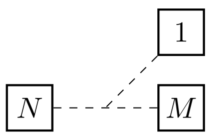
$$
\mathcal{Z}_{FT}^{(1)}(X;Y;m_A) = \mathrm{e}^{2\pi i XY} \prod_{i,j=1}^{N} s_b\left(i\frac{Q}{2} - 2m_A\right) . \tag{C.4}
$$

Asymmetric $\mathcal{S}$-wall Starting from the $FT[U(N)]$ theory it is possible to perform a deformation with the effect of breaking the two $U(N)$ global symmetries. To do this we give a VEV to the moment maps in form of Jordan-block matrices. This VEVs are uniquely specified by two partitions (ρ, σ) of N. In this work we are interested only in the case where one of the two partitions is trivial and the other is such that the $U(N)$ symmetry is broken down to $U(M) \times U(1)$, with $M < N$. Let us consider $\vec{X}$ to be the set of mass parameters for the unbroken $U(N)$ group and $\vec{Y}$ those of $U(M)$ and v for $U(1)$, then this deformation is implemented at the level of the partition function by the specialization:

$$
X_j = \left(\frac{N-M+1-2j}{2}\right)(iQ - 2m_A) + v \quad \text{for} \quad j = 1, \ldots, N-M\,, n \tag{C.5}
$$

$$
X_j = Y_{j-N+M} \quad \text{for} \quad j = N-M+1, \ldots, N\,. \tag{C.6}
$$

The resulting theory is depicted as an asymmetric $FT[U(N)]$ theory as:



(C.7)

Web of self-dualities The $FT[U(N)]$ theories satisfies to important self-dualities.

The first self-duality is obtained applying mirror duality and it is exact up to flipping field.

$$
\begin{aligned}
Z_{FT}^{(N)}(\vec{X},\vec{Y};m_A) =& Z_{FT}^{(N)}(\vec{Y},\vec{X};\tfrac{iQ}{2} - m_A)\times \\
&\times \prod_{i,j=1}^{N} s_b(\tfrac{iQ}{2} + X_i - X_j - 2m_A) s_b(\tfrac{iQ}{2} + Y_i - Y_j - 2m_A)\,.
\end{aligned} \tag{C.8}
$$

Note that in the dual frame the set of parameters $\vec{X}$ and $\vec{Y}$ are exchanged, reflecting the swap of Higgs and Coulomb branches. Due to the presence of a flipper in the definition of the $FT[U(N)]$ theory we also have extra chiral fields flipping the adjoint operators charged under the $U(N)$ global symmetries.

The second self-duality is again exact up to flipping fields. This second duality is called flip-flip [57] and it relates the $FT[U(N)]$ theory to itself with two flipping fields:

$$\begin{aligned} Z_{FT}^{(N)}(\vec{X},\vec{Y};m_A) =& Z_{FT}^{(N)}(\vec{X},\vec{Y};\tfrac{iQ}{2}-m_A)\times \\ & \times\prod_{i,j=1}^{N} s_b(\tfrac{iQ}{2}+X_i-X_j-2m_A)s_b(\tfrac{iQ}{2}+Y_i-Y_j-2m_A)\,. \end{aligned} \tag{C.9}$$

Notice that this second duality is very similar to the previous one, however the $\vec{X}$ and $\vec{Y}$ are not swapped.

It follows almost trivially that combining the two dualities previously introduced we obtain an exact self-duality:

$$Z_{FT}^{(N)}(\vec{X},\vec{Y};m_A) = Z_{FT}^{(N)}(\vec{Y},\vec{X};m_A) \tag{C.10}$$

that acts only by exchanging $\vec{X}\leftrightarrow\vec{Y}$, thus the Higgs and the Coulomb branch. Notice that this duality does not act on the m_A parameter.

These dualities can be proved by assuming the self-mirror duality for the $T[SU(N)]$ theory and also by iterating Aharony duality [55].

C.2 $FM[U(N)]$ theory: the improved bifundamental

The $FM[U(N)]$ theory is a $3d$ =2 SCFT that was introduced in [67].

The $FM[U(N)]$ theory admits a UV Lagrangian description as a quiver of $N-1$ unitary gauge nodes given in figure C.1, see also table C.1 for the charges and representation of all the fields. The FM theory has the UV global symmetry group:

$$S[U(N)\times U(1)^N]\times U(1)_\tau\times U(1)_\Delta\,, \tag{C.11}$$

in addition to the $U(1)_R$ symmetry. At the IR fixed point, the SCFT is characterized by the enhanced global symmetry:

$$S[U(N)\times U(N)]\times U(1)_\tau\times U(1)_\Delta\,. \tag{C.12}$$

The gauge invariant operators indeed reorganize into representations of the IR symmetry group. The list of the chiral ring generators of the $FM[U(N)]$ SCFT, along with their charges and representations, is given in table C.2.

We will denote the $FM[U(N)]$ by the following symbol:

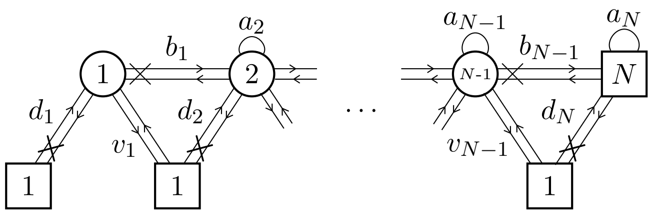
$$\boxed{N}\sim\!\sim\!\sim\!\sim\boxed{N} \tag{C.13}$$

which makes manifest the two $U(N)$ global symmetries.

The S_b^3 partition function of the FM theory can be defined recursively as:[2]

$$\begin{aligned} Z_{FM}^{(N)}(\vec{X},\vec{Y},\tau,\Delta) =& s_b\big(-i\frac{Q}{2}+2\Delta\big)\prod_{j=1}^{N}s_b\big(i\frac{Q}{2}-\Delta\pm(Y_N-X_j)\big)s_b(-\frac{iQ}{2}+\tau) \\ & s_b(-\frac{iQ}{2}+\tau)^{N-1}\prod_{j<k=1}^{N}s_b\big(-i\frac{Q}{2}+\tau\pm(X_j-X_k)\big)\int d\vec{Z}_{N-1}\Delta_{N-1}(\vec{Z}) \\ & \prod_{j=1}^{N}\prod_{k=1}^{N-1}s_b\big(\frac{iQ}{2}-\frac{\tau}{2}\pm(X_j-Z_k)\big)\prod_{j=1}^{N-1}s_b\big(-\frac{iQ}{2}+\frac{\tau}{2}+\Delta\pm(Z_j-Y_N)\big) \\ & Z_{FM}^{(N-1)}\big(\vec{Z},\{Y_1,\dots,Y_{N-1}\},\tau,\frac{\tau}{2}+\Delta\big)\,, \end{aligned} \tag{C.14}$$

[2]Note that w.r.t. [67] we adopt a different notation and cosider the adjoint chiral fields to be traceless.



$$\mathcal{W} = \sum_{j=1}^{N-1}\left[b_j(a_j + a_{j+1})\tilde{b}_j + Flip[b_j\tilde{b}_j]\right] + \\ + \sum_{j=1}^{N}(\mathfrak{M}_j^+ + \mathfrak{M}_j^-) + \sum_{j=1}^{N-1}(\tilde{v}_j b_j \tilde{d}_{j+1} + v_j \tilde{b}_j d_{j+1})$$

Figure C.1: Quiver representation of the UV completion of the $FM[U(N)]$ SCFT. Each node, square or round, labeled with a number n, represents a gauge or flavor $U(n)$ group, respectively. Each line is a $\mathcal{N} = 2$ chiral in the fundamental/antifundamental representation of the nodes to whom is attached, depending whether the arrow is outgoing or ingoing. Arches denote fields in the traceless adjoint representation. Crosses denote flipping fields. In the superpotential we also have monopoles, we denote by $\mathfrak{M}_i^\pm$ the monopole with charge ± 1 under the topological symmetry associated to the i-th gauge node.

	$U(1)_{R_0}$	$U(1)_\tau$	$U(1)_\Delta$	$U(1)_{Y_j}$	$U(N)$
$b_i, \tilde{b}_i$	0	1/2	0	0	$\mathbf{1}$
$b_{N-1}, \tilde{b}_{N-1}$	0	1/2	0	0	$\mathbf{N}, \bar{\mathbf{N}}$
a_i	2	-1	0	0	$\mathbf{1}$
a_N	2	-1	0	0	$\mathbf{N^2} - \mathbf{1}$
$v_i, \tilde{v}_i$	2	$\frac{N-i-2}{2}$	-1	$\mp\delta_{i,j+1}$	$\mathbf{1}$
$d_i, \tilde{d}_i$	0	$\frac{i-N}{2}$	$+1$	$\pm\delta_{i,j}$	$\mathbf{1}$
$d_N, \tilde{d}_N$	0	0	$+1$	$\pm\delta_{N,j}$	$\bar{\mathbf{N}}, \mathbf{N}$

Table C.1: List of abelian charges and representation under the global symmetries of all the fields of the $FM[U(N)]$ theory in Figure C.1. The $U(1)_{Y_j}$ symmetries are the boxes forming the saw in Figure C.1 numbered from left to right.

	$U(N)_X$	$U(N)_Y$	R charge
A_X	$\mathbf{N^2-1}$	$\mathbf{1}$	$2-\tau$
A_Y	$\mathbf{1}$	$\mathbf{N^2-1}$	$2-\tau$
Π	$\mathbf{N}$	$\bar{\mathbf{N}}$	Δ
$\tilde{\Pi}$	$\bar{\mathbf{N}}$	$\mathbf{N}$	Δ
$\mathsf{B}_{n,m}$	$\mathbf{1}$	$\mathbf{1}$	$2n-2\Delta+(m-n)\tau$

Table C.2: List of all gauge invariant operators that generate the holomorphic spectrum of the $FM[U(N)]$ SCFT. The R-charge is given as a trial value mixed with the other two abelian symmetries of the theory, $U(1)_\tau$ and $U(1)_\Delta$, whose mixing values are given by the two real variables τ and Δ (to avoid clutter we denote the real mass and the mixing coefficient by the same letter). The $\mathsf{B}_{n,m}$ are $U(N)\times U(N)$ singlets, for $n=1,\dots,N$ and $m=1,\dots,N+1-n$, defined by $\mathsf{B}_{1,m}=\mathcal{F}[d_{N+1-m}\tilde{d}_{N+1-m}]$, $\mathsf{B}_{n>1,m}=v_{N-m}a^{n-2}_{N-m}\tilde{v}_{N-m}$. To distinguish the two symmetries we named $U(N)_X$ the one manifest in Figure C.1 and $U(N)_Y$ the one that is emergent. A similar logic distinguish the two operators $\mathsf{A}_{X,Y}$.

with the basis of the recursion given by:

$$Z^{(1)}_{FM}(X,Y,\tau,\Delta)=s_b\big(-\frac{iQ}{2}+2\Delta\big)s_b\big(\frac{iQ}{2}-\Delta\pm(X-Y)\big)\,. \tag{C.15}$$

The vectors $\vec{X}$ and $\vec{Y}$ are the parameters for the manifest and emergent $U(N)$ symmetries respectively, and $\vec{Z}$ is the set of parameters for the gauge group $U(N-1)$.

Mirror self-duality The $FE[U(N)]$ theory enjoys an exact self-duality similar to that of the $FT[U(N)]$ theory that reads:

$$Z^{(N)}_{FM}(\vec{X},\vec{Y},\tau,\Delta)=Z^{(N)}_{FM}(\vec{Y},\vec{X},\tau,\Delta)\,. \tag{C.16}$$

Where the two non-abelian global symmetries are swapped, meaning that we have exchanged the manifest and emergent $U(N)$ symmetries in the UV representation C.1.

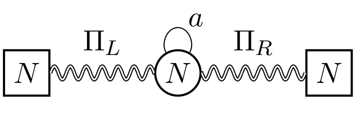
Fusion to identity Similarly as for the $FT[U(N)]$ theory, also the $FM[U(N)]$ theory satisfies a fusion to identity property. As a partition function identity it reads:

$$\int d\vec{Z}_N\Delta_N(\vec{Z},\tau)Z^{(N)}_{FM}(\vec{X},\vec{Z},\tau,\Delta)Z^{(N)}_{FM}(\vec{Z},\vec{Y},\tau,-\Delta)={}_{\vec{X}}\mathbb{I}_{\vec{Y}}(\tau)\,, \tag{C.17}$$

where the identity operator is defined as:

$${}_{\vec{X}}\mathbb{I}_{\vec{Y}}(\tau)=\frac{1}{\Delta_N(\vec{X},\tau)}\sum_{\in S_N}\prod_{j=1}^{N}\delta(X_j-Y_{(j)})\,. \tag{C.18}$$

Which consist in two $FM[U(N)]$ theories glued together with the addition of a monopole superpotential $\mathcal{W}=\mathfrak{M}^++\mathfrak{M}^-$ being dual to an identity operator. We depict this relation as:



$\Longleftrightarrow$ $\mathbb{I}$-wall

Π_L	Δ
Π_R	$-\Delta$
a	τ

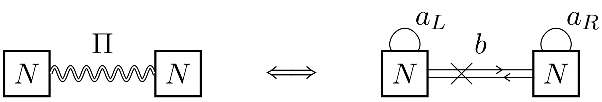
$$\mathcal{W}=\mathcal{W}_{\text{gluing}}+\mathfrak{M}^++\mathfrak{M}^- \tag{C.19}$$

The superpotential $\mathcal{W}_{\text{gluing}}$ contains the coupling between the adjoint a and the two adjoint operators A_L and A_R of the left and right $FM[U(N)]$ theories.

Interesting deformations Along the lines traced for the $FE[USp(2N)]$ theory, we present two types of deformation also for the $FM[U(N)]$ theory. The first deformation is realized by adding the linear superpotential term $\delta\mathcal{W} = \mathsf{B}_{1,1}$, which has the effect of breaking completely $U(1)_\Delta$ while it preserves $U(1)_\tau$. This consist in the specialization $\Delta = 0$ which reduces the $FM[U(N)]$ theory to the $\mathbb{I}$-wall theory:

$$\mathcal{I}_{FM}^{(N)}(\vec{x}, \vec{y}, \tau, \Delta = 0) = {}_{\vec{X}}\mathbb{I}_{\vec{Y}}(\tau)\,, \tag{C.20}$$

Another interesting case is the specialization obtained by the linear superpotential $\delta\mathcal{W} = \mathsf{B}_{1,2}$. This has the effect of breaking $U(1)_\Delta \times U(1)_\tau$ down to a $U(1)$ subgroup defined by the constraint $\Delta = 1 - \tau/2$. The $FM[U(N)]$ theory is deformed into a bifundamental hyper multiplet coupled to adjoint singlets:



Π	$\tau/2$
$a_{L,R}$	$2-\tau$
b	$\tau/2$

$\mathcal{W} = \mathsf{B}_{1,2}$ $\qquad\qquad$ $\mathcal{W} = b(a_L + a_R)\tilde{b} +$
$+Flip[b\tilde{b}]$ $\qquad$ (C.21)

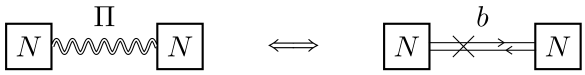
As a partition function identity this translate into:

$$\begin{aligned} Z_{FM}^{(N)}\big(\vec{X}, \vec{Y}, \tau, \Delta = \frac{\tau}{2}\big) =& s_b\big(-\frac{iQ}{2} + \tau\big)^{2N-1} \prod_{j<k=1}^{N} \Big[s_b\big(-\frac{iQ}{2} + \tau \pm (X_j - X_k)\big) \\ & s_b\big(-\frac{iQ}{2} + \tau \pm (Y_j - Y_k)\big) s_b\big(\frac{iQ}{2} - \frac{\tau}{2} \pm (X_j - Y_k)\big)\Big]\,. \end{aligned} \tag{C.22}$$

One can also iron a FM theory into a bifundamental hypermultiplet also using the deformation $\delta\mathcal{W} = \mathsf{B}_{2,1}$. This has the effect of breaking $U(1)_\Delta \times U(1)_\tau$ down to a $U(1)$ subgroup defined by the constraint $\Delta = \frac{iQ-\tau}{2}$. We have the following property:

$$Z_{FM}^{(N)}\big(\vec{X}, \vec{Y}, \tau, \Delta = \frac{iQ - \tau}{2}\big) = s_b\big(\frac{iQ}{2} - \tau\big) \prod_{j,k=1}^{N} s_b\big(\tau \pm (X_j - Y_k)\big)\,. \tag{C.23}$$

Graphically we have:



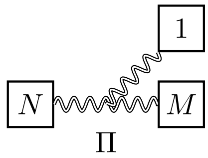
Π	$1-\tau/2$
b	$1-\tau/2$

$\mathcal{W} = \mathsf{B}_{2,1}$ $\qquad\qquad$ $\mathcal{W} = Flip[b\tilde{b}]$ $\qquad$ (C.24)

The second category of deformations is given by $U(N)$ breaking superpotential terms. This can be obtained by giving VEVs to any of the two adjoint operators. We consider the case of a VEV such that it breaks one of the global $U(N)$ symmetries down to $U(M) \times U(1)$. Suppose that $\vec{X}, \vec{Y}$ are the set of mass parameters respectively for $U(N)$ and $U(M)$ and V is that of $U(1)$, the specialization is as follows:

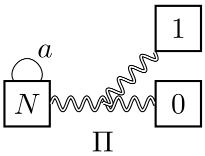
$$\begin{aligned} X_i &= \frac{N - M + 1 - 2j}{2}\tau + V \qquad \text{for} \quad i = 1, \dots, N - M\,, \\ X_i &= Y_{i-N+M} \qquad \text{for} \quad i = N - M + 1, \dots, N\,. \end{aligned} \tag{C.25}$$

We depict the resulting theory as an "asymmetric" improved bifundamental:



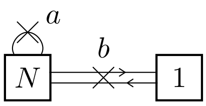
(C.26)

The case $M = 0$ enjoys a duality with a flipped fundamental flavor as:



$\mathcal{W} = 0$

$\Longleftrightarrow$



$\mathcal{W} = \sum_{j=0}^{N-1} Flip[ba^j\tilde{b}] +$
$+ \sum_{j=2}^{N} Flip[Tr a^j]$

$$\begin{array}{c|c} a & \tau \\ \Pi & \Delta \\ b & \frac{(1-N)}{2}\tau + \Delta \end{array}$$

(C.27)

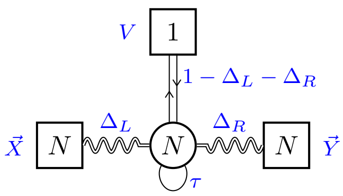
As a identity between partition function we have:

$$Z_{FM}^{(N)}\big(\vec{X}, \{\frac{N-1}{2}\tau + V, \dots, \frac{1-N}{2}\tau + V\}, \tau, \Delta\big) = n \tag{C.28}$$

$$= \prod_{j=2}^{N} s_b\big(-\frac{iQ}{2} + j\tau\big) \prod_{j=1}^{N} \big[s_b\big(\frac{iQ}{2} - \frac{1-N}{2}\tau - \Delta \pm (X_j - V)\big) s_b\big(-\frac{iQ}{2} + (j-N)\tau + 2\Delta\big)\big]\,. \tag{C.29}$$

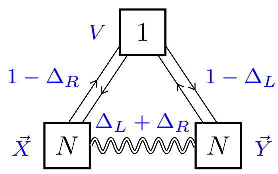
Star-triangle and star-star dualities The $FM[U(N)]$ theory is involved in interesting dualities that are named "star-triangle" and "star-star" dualities. For a complete review of them one may look at [42], here we only list them.

The first duality is the so-called star-triangle duality, often also called braid due to its connection to the 4d duality introduced in [64] via dimensional reduction and real mass deformations. The duality related two $FM[U(N)]$ theories glued together with the addition of a flavor and a generalized WZ-model, that is a theory composed of a $FM[U(N)]$ theory and chiral multiplets with cubic superpotential terms:



$\mathcal{W} = \mathcal{W}_{\text{gluing}} + \mathfrak{M}^+ + \mathfrak{M}^-$

$\longleftrightarrow$

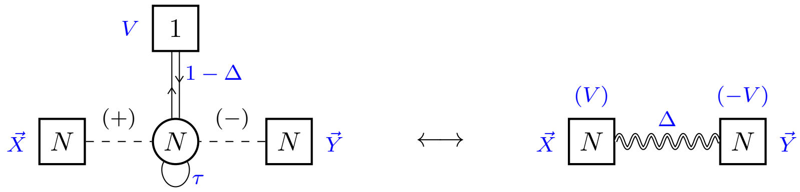


$\mathcal{W} = \mathcal{W}_{\text{cubic}}$

(C.30)

Notice that to avoid cluttering we now give in blue the parameterization of the global symmetries such that each polynomial can be seen as the trial R-charge plus a combination of real mass parameters weighted by the charges under the corresponding global symmetry. On the l.h.s. the gluing is performed with the usual coupling between the adjoint chiral and the adjoint operators in the $FM[U(N)]$ theories. In addition we also have a monopole superpotential. On the r.h.s. the cubic superpotential couples the bifundamental operator of the $FM[U(N)]$ theory to the chirals forming a triangular loop.

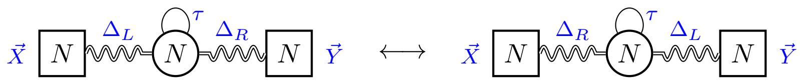
The monopole superpotential on the r.h.s. can be turned off by performing a suitable real mass deformation and land to the following duality:



$$\mathcal{W}=\mathcal{W}_{\text{gluing}} \qquad\qquad \mathcal{W}=0 \tag{C.31}$$

which we interpret as a basic $\mathcal{N}=2$ mirror duality in Chapter 4. We refer to [41, 42] for more details.

The last interesting duality is in the form of a star-star duality, which related two theories obtained from the gluing of two $FM[U(N)]$ theories.



$$\mathcal{W}=\mathcal{W}_{\text{gluing}} \qquad\qquad \mathcal{W}=\mathcal{W}_{\text{gluing}} \tag{C.32}$$

which in Chapter 4 we interpret as a swapping duality between two improved bifundamentals. We refer to [41, 42] for more details. Notice in particular that this duality swaps the two symmetries $U(1)_{\Delta_L} \leftrightarrow U(1)_{\Delta_R}$ while mantaining the order of the $U(N)_X$ and $U(N)_Y$ symmetries.

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