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Regular Articles

Coexisting automated and human-driven vehicles: Well-posedness of a mixed nonlocal-local traffic model

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A R T I C L E I N F O

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A B S T R A C T

We present a macroscopic traffic flow model where standard vehicles coexist with vehicles informed on the traffic distribution. The resulting mixed nonlocal-local integro-differential PDEs is proved to generate a locally Lipschitz continuous semigroup whose orbits are uniquely characterized as solutions to the system, according to a natural definition of solution. The norms and function spaces adopted are intrinsic to the different nature of the equations.

1. Introduction

The forthcoming large scale introduction of autonomous vehicles (AVs) leads to consider vehicles that are *nonlocal*, in the sense that they are aware of traffic conditions also at a significant distance from their location, both in front and behind them. Analytically, this leads to nonlocal models, i.e., to models where integrals in the space variable allow a vehicle to react to suitable averages of the traffic density around — in front and behind — its position. The current literature offers a variety of models built on these premises, see [1,2,5,10,11,26].

At the same time, the presence of standard — or *local* — vehicles, i.e., aware of traffic conditions only at their position, can not be neglected. Indeed, the interplay between *local* and *nonlocal* vehicles (or individuals) is being considered in the traffic community, see [25,29,31,33,34], see also [23,24] for an analytical approach.

In the model we present here, these two kinds of drivers/vehicles are traveling simultaneously along the same road and, hence, they are interacting. The time and space dependent density of the *local* ones is $r = r(t,x)$, while the *nonlocal* ones are described by $\rho = \rho(t,x)$, t being time and x the one dimensional coordinate along the road. This results in the following natural extension of the classical LWR model [30,32]

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$$\begin{cases} \partial_t \rho + \partial_x \left(\rho\, v_{NL}\left((\rho + r) * \eta\right)\right) = 0 \\ \partial_t r + \partial_x \left(r\, v_L\left(\rho + r\right)\right) = 0. \end{cases} \tag{1.1}$$

The functions v_{NL} and v_L are the *speed laws* of the two populations, while η is a general averaging kernel. We underline that η is required neither to have compact support, nor to enjoy any sort of symmetry. In particular, the forward horizon can very well be far larger than the backward one, see for instance [16, § 3]. On the other hand, η may not be monotone, so that no *a priori* constant bound on the nonlocal density ρ is in general available for all times, though $\mathbf{L}^\infty$ bounds are available on all compact time intervals. On the contrary, in the present setting, the local density r is proved to remain for all times within the interval of the physical densities.

From the modeling point of view, (1.1) provides an environment not only for traffic descriptions, but also for investigating the interplay between AVs and standard *"short sighted"* (bulk) vehicles. According to a common approach, see [25,29], v_{NL} acts as the strategy assigned to the AVs in order to control the evolution of the whole traffic. It is then necessary to allow for the coexistence of AVs following different behavioral rules. We are thus led to extend (1.1) to the case of different classes, say k, of AVs:

$$\begin{cases} \partial_t \rho^i + \partial_x \left(\rho^i\, v^i_{NL}\left(\left(\sum_{j=1}^k \rho^j + r\right) * \eta^i\right)\right) = 0 & i = 1, \ldots, k \\ \partial_t r + \partial_x \left(r\, v_L\left(\sum_{j=1}^k \rho^j + r\right)\right) = 0. \end{cases} \tag{1.2}$$

Here we focus on the analytic properties of (1.2), providing its global in time well-posedness and a full set of *a priori* estimates. The different nature of the equations in (1.2) imposes the use of different techniques, relying on different assumptions. In particular, the *natural* regularities of $x \mapsto \rho(t,x)$ and of $x \mapsto r(t,x)$ turn out to be different, as also the norms that allow well-posedness and stability estimates. More precisely, the initial data (ρ_o, r_o) to be assigned to (1.2) as well as the solutions $x \mapsto \rho(t,x)$ and $x \mapsto r(t,x)$ find their natural environment in the sets

$$\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) := \left\{\rho \in \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \colon \rho' \in \mathbf{BV}(\mathbb{R};\mathbb{R}^k)\right\} \text{ and } \mathcal{BV}(\mathbb{R};\mathbb{R}) := (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};\mathbb{R}) \tag{1.3}$$

whereas stability estimates hold in the $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$ and $\mathbf{L^1}(\mathbb{R};\mathbb{R})$ norms. Indeed, the local Lipschitz continuous dependence of solutions with respect to the initial datum and with respect to time holds in the $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$ and $\mathbf{L^1}(\mathbb{R};\mathbb{R})$ topologies, provided the norms

$$\|\rho\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)} := \|\rho\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \mathrm{TV}(\rho') \quad \text{and} \quad \|r\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})} := \|r\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} + \mathrm{TV}(r) \tag{1.4}$$

are bounded. In particular, Example 2.4 below shows that the use of the $\mathbf{L^1}$ norm on ρ may not ensure the continuous dependence of the solutions to (1.2) on the initial datum.

The motivation for these differences lies in the need to use Kružkov [28] definition of solution for the local equation, since (1.2) obviously comprises the case of the scalar conservation law $\partial_t r + \partial_x\left(r\, v_L(r)\right) = 0$. In turn, this imposes $x \mapsto \rho(t,x)$ and $x \mapsto \rho_o(x)$ to be at least in $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$. Once this regularity is achieved, Kružkov theory [28], as well as the stability estimates [17], can be exploited to deal with the local equation in the proof of the well-posedness of (1.2). On the other hand, to deal with the nonlocal equation, our starting points are the well-posedness result and the stability estimates in [16], which we here improve to suit to the present setting. All this leads to the asymmetry between the assumptions required on the local and nonlocal equations and data — see **(vNL)** vs. **(vL)** and $\rho_o \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)$ vs. $r_o \in \mathcal{BV}(\mathbb{R};\mathbb{R})$. In particular, the present lack of a well-posedness and stability theory for general systems of conservation laws prevents us from allowing r to be vector valued.

In this connection, we underline a natural problem arising from the present result. The so-called *nonlocal-to-local limit* recently attracted a relevant attention in the literature, see [8,12–15,20,27] and the references therein. An analogous result in the case of (1.1), i.e., letting η tend to Dirac delta, would yield the existence of solutions to the LWR model with 2 populations introduced in [4], whose well-posedness is, to our knowledge, still an open problem. The case (1.2) of more than 2 populations might then follow.

The proof below relies on careful estimates, separately, on the local and on the nonlocal equations constituting (1.2). As a first step, more regular initial data are considered. Adapting and, where necessary, improving the available well-posedness and stability estimates in [16,17] allow to ignite a recursive procedure that yields a Cauchy sequence converging to a solution to (1.2), first locally in time and then globally. Careful *a priori* estimates ensure the (local) Lipschitz continuous dependence of solutions on time and on the initial data. A final bootstrap procedure allows to relax the assumptions on the initial data and leads to the existence result presented in Theorem 2.3. Theorem 2.5 proves uniqueness of solutions to (1.2), also making use of the basic estimates on the 2 separate problems.

The next section presents the main results while all proofs are deferred to Section 3.

2. Main results

Throughout, $T > 0$ is fixed and $k \in \mathbb{N} \setminus \{0\}$. We also shorten the notation for the sum $\rho^1 + \cdots + \rho^k$ setting $\Sigma\rho \coloneqq \sum_{j=1}^k \rho^j$. We denote by $\mathcal{C}(\cdots)$ a continuous and positive function non decreasing in each of its arguments, the exact value of which is not relevant.

We defer to Lemma 3.2 a characterization of $\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)$, as defined in (1.3). Here, we note that clearly $\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k) \subsetneq \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) \subsetneq \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$. Throughout, we use the norm

$$\|(\rho, r)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\times\mathbf{L^1}(\mathbb{R};\mathbb{R})} \coloneqq \|\rho\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \|r\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} . \tag{2.1}$$

Definition 2.1. By *solution* to (1.2) with initial datum $\rho_o \in \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$, $r_o \in \mathbf{L^1}(\mathbb{R};\mathbb{R})$ on the interval $[0,T]$, we mean $(\rho, r) \in \mathbf{C^0}\left([0,T]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$ such that setting

$$\begin{aligned} v^i(t,x) &\coloneqq v^i_{NL}\left(\left(\Sigma\rho(t) * \eta^i\right)(x) + \left(r(t) * \eta^i\right)(x)\right) , \\ w(t,x,r) &\coloneqq v_L\left(\Sigma\rho(t,x) + r\right) , \end{aligned}$$

ρ^i, for $i = 1, \ldots, k$, is a distributional solution [6, Definition 4.2] to

$$\begin{cases} \partial_t \rho^i + \partial_x\left(\rho^i\, v^i(t,x)\right) = 0 \\ \rho^i(0,x) = \rho^i_o(x) \end{cases} \tag{2.2}$$

and r is a Kružkov solution [28, Definition 1] to

$$\begin{cases} \partial_t r + \partial_x\left(r\, w(t,x,r)\right) = 0 \\ r(0,x) = r_o(x) . \end{cases} \tag{2.3}$$

Remark that the necessity for a higher regularity in ρ stems from the requirement that r be a Kružkov solution. Indeed, the standard definition of solution to (2.3), see [28, Definition 1], requires that the flux $(t,x,r) \mapsto r\, w(t,x,r)$ be differentiable with respect to the space variable, hence that $x \mapsto \rho(t,x)$ be in $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$. A further motivation of this regularity is in **Step 6** in the proof of Theorem 2.3, see Section 3.4.

The higher regularity of the solution to the nonlocal problem justifies a regularity of the nonlocal velocity v_{NL} higher than that of the local velocity v_L. Below, we assume that the speed laws v_{NL} and v_L in (1.2) and the averaging kernel η meet the following requirements:

(vNL) $v_{NL} \in (\mathbf{C^3} \cap \mathbf{W^{3,\infty}})(\mathbb{R};\mathbb{R}^k)$.

(vL) $v_L \in \mathbf{C^2}(\mathbb{R};\mathbb{R})$ and there exist $V_L, R_L > 0$ such that $v_L(r) = V_L$ for $r \leq 0$ and $v_L(r) = 0$ for $r \geq R_L$.

($\boldsymbol{\eta}$) $\eta \in (\mathbf{C^3} \cap \mathbf{W^{3,\infty}})(\mathbb{R};\mathbb{R}^k)$.

We first present a stability property of Definition 2.1.

Proposition 2.2. *Assume* **(vNL)**, **(vL)** *and* **($\boldsymbol{\eta}$)**. *Fix sequences* $(\rho_{o,n}, r_{o,n}) \in \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})$, $(\varepsilon_n, e_n) \in \mathbf{C^0}\left([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$ *and a function* (ρ_∞, r_∞) *belonging to* $\mathbf{C^0}\big([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\big)$ *such that for all* $n \in \mathbb{N}$ *the problem*

$$\begin{cases} \partial_t \rho_n^i + \partial_x \left(\rho_n^i \, v_{NL}^i \left(\Sigma\rho_n * \eta^i + (r_n + e_n) * \eta^i\right)\right) = 0 & i = 1,\ldots,k \\ \partial_t r_n + \partial_x \left(r_n \, v_L \left(\Sigma(\rho_n + \varepsilon_n) + r_n\right)\right) = 0 \\ \rho_n(0,x) = \rho_{o,n}(x) \\ r_n(0,x) = r_{o,n}(x) \end{cases}$$

admits a solution (ρ_n, r_n) *in the sense of Definition* 2.1. *Moreover, assume that*

$$\begin{aligned} \lim_{n\to+\infty} (\rho_n, r_n) &= (\rho_\infty, r_\infty) && \text{in } \mathbf{C^0}\left([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\right); \\ \lim_{n\to+\infty} (\varepsilon_n, e_n) &= (0,0) && \text{in } \mathbf{C^0}\left([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\right). \end{aligned} \tag{2.4}$$

Then, (ρ_∞, r_∞) *solves, in the sense of Definition* 2.1, *the problem*

$$\begin{cases} \partial_t \rho^i + \partial_x \left(\rho^i \, v_{NL}^i \left(\Sigma\rho * \eta^i + r * \eta^i\right)\right) = 0 & i = 1,\ldots,k \\ \partial_t r + \partial_x \left(r \, v_L \left(\Sigma\rho + r\right)\right) = 0 \\ \rho(0,x) = \rho_\infty(0,x) \\ r(0,x) = r_\infty(0,x). \end{cases} \tag{2.5}$$

We are now ready to state the existence and stability result for solutions to (1.2).

Theorem 2.3. *Let* **(vNL)**, **(vL)** *and* **($\boldsymbol{\eta}$)** *hold. Then, problem* (1.2) *generates a unique map*

$$\mathcal{S} \colon \mathbb{R}_+ \times \left(\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R})\right) \to \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R}) \tag{2.6}$$

such that

($\mathcal{S}$1) *$\mathcal{S}$ is a global semigroup, i.e.,* $\mathcal{S}_o = \mathbf{Id}$ *and for all* $t, \hat{t} \in \mathbb{R}_+$, $\mathcal{S}_t \circ \mathcal{S}_{\hat{t}} = \mathcal{S}_{t+\hat{t}}$.

($\mathcal{S}$2) *For any* $(\rho_o, r_o) \in \left(\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R})\right)$ *and any* $\bar{T} > 0$, *the orbit* $t \mapsto \mathcal{S}_t(\rho_o, r_o)$ *solves* (1.2) *on* $[0, \bar{T}]$ *in the sense of Definition* 2.1.

($\mathcal{S}$3) *The semigroup $\mathcal{S}$ is locally Lipschitz continuous with respect to the norm* (2.1)*: for all* $M > 0$, *there exists a constant*

$$\mathcal{C}_1 := \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, M, T\right)$$

such that for all $t, \hat{t} \in [0,T]$, *for all* (ρ_o, r_o), $(\hat{\rho}_o, \hat{r}_o) \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R})$ *with*

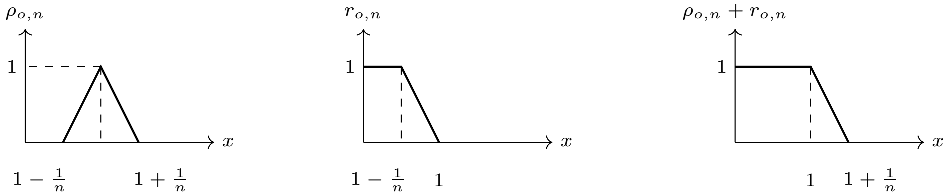

Fig. 2.1. The initial condition $(\rho_{o,n}, r_{o,n})$ and their sum in Example 2.4.

$$\begin{aligned} \|\rho_o\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)} &\le M\,, & \|r_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})} &\le M\,,\\ \|\widehat{\rho}_o\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)} &\le M\,, & \|\widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\le M\,, \end{aligned}$$

the following estimates hold:

$$\begin{aligned} \|\mathcal{S}_t(\rho_o,r_o)-\mathcal{S}_t(\widehat{\rho}_o,\widehat{r}_o)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\times\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\le (1+\mathcal{C}_1 t)\|(\rho_o-\widehat{\rho}_o, r_o-\widehat{r}_o)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\times\mathbf{L^1}(\mathbb{R};\mathbb{R})}\,,\\ \|\mathcal{S}_t(\rho_o,r_o)-\mathcal{S}_{\hat t}(\rho_o,r_o)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\times\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\le \mathcal{C}_1\left|t-\hat t\right|. \end{aligned}$$

($\mathcal{S}$4) *For all $t \in [0,T]$ and for all $(\rho_o,r_o) \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)\times\mathcal{BV}(\mathbb{R};\mathbb{R})$, call $(\rho(t),r(t)) := \mathcal{S}_t(\rho_o,r_o)$. Then, there exists a positive constant*

$$\mathcal{C}_2 := \mathcal{C}\left(\begin{array}{c}\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)},\\ \|\rho_o\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})}, T\end{array}\right)$$

such that

$$\begin{aligned} \|\rho(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} &= \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, & \|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\le \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})},\\ \|\rho(t)\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)} &\le \mathcal{C}_2, & \mathrm{TV}\,(r(t)) &\le \mathcal{C}_2. \end{aligned}$$

($\mathcal{S}$5) *Call $(\rho(t),r(t)) := \mathcal{S}_t(\rho_o,r_o)$, for $(\rho_o,r_o) \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)\times\mathcal{BV}(\mathbb{R};\mathbb{R})$. If $\rho_o^i \ge 0$ for $i = 1,\ldots,k$ and $r_o \in [0,R_L]$, then $\rho^i(t,x) \ge 0$ and $r(t,x) \in [0,R_L]$ for a.e. $(t,x) \in \mathbb{R}_+\times\mathbb{R}$ and $\|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} = \|r_o\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}$.*

The proof is deferred to § 3.4.

The asymmetry in **($\mathcal{S}$3)** in the local initial conditions r_o and $\widehat{r}_o$ is consequence of the application of [17, Theorem 2.6] about the stability of solutions to local problems with respect to the fluxes. We take advantage of this asymmetry in the uniqueness proof in Theorem 2.5.

Note that the constant $\mathcal{C}_1$ in **($\mathcal{S}$3)** actually depends also on $\|v_L\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}$, as is to be expected. Indeed, by **(vL)**, $\|v_L\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} \le \mathcal{C}\left(R_L, \|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\right)$.

The necessity of the requirement $\rho_o \in \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$ and the stability with respect to the $\mathbf{W^{1,1}}$-norm of the nonlocal initial datum are illustrated by the following example.

Example 2.4. Set $k = 1$ and choose the speed laws $v_{NL} \equiv 0$ and v_L satisfying **(vL)** such that $v_L(r) = 1 - r$ for $r \in [1/2, 1]$. Define for all $n \in \mathbb{N}$

$$\rho_{o,n} := \begin{cases} 0 & x < 1-1/n\\ 1+n(x-1) & x \in [1-1/n, 1]\\ 1-n(x-1) & x \in \,]1, 1+1/n]\\ 0 & x > 1+1/n \end{cases} \qquad r_{o,n} := \begin{cases} 0 & x < 0\\ 1 & x \in [0, 1-1/n]\\ n(1-x) & x \in \,]1-1/n, 1]\\ 0 & x > 1 \end{cases}$$

see Fig. 2.1.

Note that $\rho_{o,n}(x) + r_{o,n}(x) = 1$ for all $x \in [0,1]$. In this setting,

$$(\rho_{o,n}, r_{o,n}) \to (0, \chi_{[0,1]}) \text{ in } \mathbf{L^1}(\mathbb{R};\mathbb{R}^2) \quad \text{but for } t>0 \quad \mathcal{S}_t(\rho_{o,n}, r_{o,n}) \not\to \mathcal{S}_t(0,\chi_{[0,1]}) \text{ in } \mathbf{L^1}(\mathbb{R};\mathbb{R}^2)\,.$$

Indeed, $(\rho_{o,n}, r_{o,n})$ yields the stationary solution to (1.2), since $v_{NL} \equiv 0$ and $v_L\left(\rho_{o,n}(x) + r_{o,n}(x)\right) = v_L(1) = 0$ for all $x \in [0,1]$. Hence, for all $t > 0$, $\mathcal{S}_t(\rho_{o,n}, r_{o,n}) = (\rho_{o,n}, r_{o,n})$.

On the other hand, calling $(\rho(t), r(t)) = \mathcal{S}_t(0, \chi_{[0,1]})$, we have that $\rho(t) \equiv 0$ and r is the (non stationary) entropy solution to the Cauchy problem

$$\begin{cases} \partial_t r + \partial_x\left(r\, v_L(r)\right) = 0 \\ r(0,x) = \chi_{[0,1]}(x)\,. \end{cases}$$

Hence, the mere $\mathbf{L^1}$-convergence of the initial data does not guarantee the convergence in $\mathbf{L^1}(\mathbb{R};\mathbb{R}^2)$ of the solution at any positive time.

Note, for completeness, that $\rho_{o,n} \in \mathcal{BV}^1(\mathbb{R};\mathbb{R})$ and $r_{o,n} \in \mathcal{BV}(\mathbb{R};\mathbb{R})$ so that Theorem 2.3 applies and Item **($\mathcal{S}$3)** holds.

We now prove the uniqueness of solutions to (1.2) in the same class where existence is ensured by the semigroup trajectories exhibited in Theorem 2.3.

Theorem 2.5. *Let* **(vNL)**, **(vL)** *and* **(η)** *hold. Fix $(\rho_o, r_o) \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R})$ and assume that $(\rho_1, r_1), (\rho_2, r_2) \in \mathbf{C^0}\left([0,T]; \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R})\right)$ solve (1.2) with initial datum (ρ_o, r_o) in the sense of Definition 2.1. Then,*

$$(\rho_1, r_1) = (\rho_2, r_2)\,.$$

The proof is deferred to § 3.5.

As a consequence, the trajectories of the semigroup $\mathcal{S}$ constructed in Theorem 2.3 are uniquely characterized as the solutions to (1.2) in the sense of Definition 2.1.

3. Analytical proofs

Throughout, we use the following notation:

$$\begin{aligned} \|\rho\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} &:= \textstyle\sum_{i=1}^k \left\|\rho^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} & \|\rho\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} &:= \|\rho\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} + \|\rho'\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \\ \|\rho\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R}^k)} &:= \textstyle\sum_{i=1}^k \left\|\rho^i\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} & \|\rho\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)} &:= \|\rho\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R}^k)} + \|\rho'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R}^k)} \\ \mathrm{TV}(\rho) &:= \textstyle\sum_{i=1}^k \mathrm{TV}(\rho^i) \end{aligned}$$

and the vector convolution in the space variable is, for $i = 1, \ldots, k$,

$$\left(\rho(t) * \eta^i\right)(x) = \left(\left(\rho^1(t) * \eta^i\right)(x), \ldots, \left(\rho^k(t) * \eta^i\right)(x)\right). \tag{3.1}$$

The next two lemmas refer to general elementary functional analytic results of use below.

Lemma 3.1. *With the notation (1.3), if $q \in \mathcal{BV}(\mathbb{R};\mathbb{R}^k)$, then $\|q\|_{\mathbf{L^\infty}(\mathbb{R},\mathbb{R}^k)} \le \frac{1}{2}\,\mathrm{TV}(q)$.*

Proof of Lemma 3.1. We use below the left continuous representative of q. Since, $q \in \mathbf{L^1}(\mathbb{R};\mathbb{R}^k)$, then for every $\varepsilon > 0$, there exist a_ε and b_ε with $a_\varepsilon < 0$, $b_\varepsilon > 0$, $|q(a_\varepsilon)| < \varepsilon$ and $|q(b_\varepsilon)| < \varepsilon$. For any $x \in \mathbb{R}$ we thus have

$$|q(x)| \leq |q(a_\varepsilon)| + \mathrm{TV}\left(q;]-\infty, x[\right) \leq \varepsilon + \mathrm{TV}\left(q;]-\infty, x[\right) ;$$
$$|q(x)| \leq |q(b_\varepsilon)| + \mathrm{TV}\left(q; [x, +\infty[\right) \leq \varepsilon + \mathrm{TV}\left(q; [x, +\infty[\right) .$$

Summing the two latter results and passing to the infimum over ε completes the proof. □

Lemma 3.2 *(Characterization of $\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)$). With the notation (1.3)–(1.4), $\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)$ is the set of $\mathbf{W^{1,1}}$ limits of $\mathbf{W^{2,1}}$ bounded sequences in $(\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k)$. Moreover, if $\rho \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)$, then there exists $\rho_n \in (\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k)$ such that $\lim_{n\to+\infty} \rho_n = \rho$ in $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$ and*

$$\lim_{n\to+\infty} \|\rho_n\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)} = \|\rho\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)} . \tag{3.2}$$

Proof of Lemma 3.2. Fix ρ such that there exists a sequence ρ_n in $(\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k)$ bounded in $\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)$ and converging to ρ in $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$. Hence $\rho' = \lim_{n\to+\infty} \rho_n'$ in $\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)$, with ρ_n' bounded in $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$. By [3, Theorem 3.9], we have $\rho \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)$.

On the other hand, introduce a mollifier $\zeta \in \mathbf{C^\infty_c}(\mathbb{R};\mathbb{R}_+)$ with $\operatorname{spt}\zeta \subseteq [-1,1]$ and $\int_{\mathbb{R}} \zeta = 1$. For $n \in \mathbb{N}\setminus\{0\}$, define $\zeta_n(x) = n\,\zeta(n\,x)$ and $\rho_n = \rho * \zeta_n$. Clearly, $\rho_n \in (\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k)$, and by some classical results about convolution in [9,22] it holds that

$$\begin{aligned}
\|\rho_n\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\leq \|\rho\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \, \|\zeta\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} && \text{[By [9, Theorem 4.15]]}\\
\|\rho_n'\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\leq \|\rho'\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \, \|\zeta\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} && \text{[By [9, Theorem 4.15]]}\\
\|\rho_n''\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\leq \mathrm{TV}(\rho') \, \|\zeta\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} , && \text{[By [22, Proposition 8.49]]}
\end{aligned}$$

hence ρ is the $\mathbf{W^{1,1}}$ limit of a $\mathbf{W^{2,1}}$ bounded sequence in $(\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k)$. By the lower semicontinuity in $\mathbf{L^1}$ of the total variation, $\mathrm{TV}(\rho') \leq \liminf_{n\to\infty} \mathrm{TV}(\rho_n') = \liminf_{n\to\infty} \|\rho_n''\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}$, proving (3.2). □

3.1. Stability property of solutions

Proof of Proposition 2.2. Introduce for $i = 1, \ldots, k$ the maps

$$\begin{aligned}
v_n^i(t,x) &= v_{NL}^i \left(\left(\Sigma\rho_n(t) * \eta^i\right)(x) + \left((r_n(t) + e_n(t)) * \eta^i\right)(x) \right)\\
v_\infty^i(t,x) &= v_{NL}^i \left(\left(\Sigma\rho_\infty(t) * \eta^i\right)(x) + \left(r_\infty(t) * \eta^i\right)(x) \right)
\end{aligned}$$

and observe that by **(vNL)** and (2.4), we have the convergence $\lim_{n\to+\infty} v_n^i = v_\infty^i$ a.e. in $[0,T]\times\mathbb{R}$. Moreover, since

$$\left\| \left(\Sigma\rho_n * \eta^i\right) + (r_n + e_n) * \eta^i \right\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R})} \leq \|\Sigma\rho_n + r_n + e_n\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}))} \left\|\eta^i\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}$$

is uniformly bounded by (2.4), also $\sup_n \|v_n\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R}^k)} < +\infty$. By [9, Theorem 4.9] and (2.4), there exists a function $h \in \mathbf{L^1}([0,T]\times\mathbb{R};\mathbb{R})$ such that for a.e. $(t,x) \in [0,T]\times\mathbb{R}$, up to a subsequence,

$$\sup_{n\in\mathbb{N}} \|\rho_n(t,x)\| + \sup_{n\in\mathbb{N}} \|\partial_x\rho_n(t,x)\| + \sup_{n\in\mathbb{N}} \|\partial_x\varepsilon_n(t,x)\| + \sup_{n\in\mathbb{N}} |r_n(t,x)| \leq h(t,x). \tag{3.3}$$

This allows to apply the Dominated Convergence Theorem to the equality

$$\int_0^T \int_{\mathbb{R}} \left(\rho_n^i(t,x)\,\partial_t\varphi(t,x) + \rho_n^i(t,x)\,v_n^i(t,x)\,\partial_x\varphi(t,x) \right) \mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} (\rho_{o,n})^i(x)\,\varphi(0,x)\,\mathrm{d}x = 0$$

which holds by Definition 2.1, for all $i = 1, \ldots, k$, $n \in \mathbb{N} \setminus \{0\}$, $\varphi \in \mathbf{C_c^1}(]-\infty, T[\times \mathbb{R}; \mathbb{R})$. Hence, ρ_∞ solves the nonlocal equation in (2.5).

Concerning the local part of (2.5), we proceed similarly. Introduce

$$w_n(t,x,r) := v_L\left(\Sigma\left(\rho_n(t,x) + \varepsilon_n(t,x)\right) + r\right) \quad \text{and} \quad w_\infty(t,x,r) := v_L\left(\Sigma\rho_\infty(t,x) + r\right).$$

Clearly, since v_L satisfies **(vL)** and (2.4), $\lim_{n\to+\infty} w_n\left(t,x,r_n(t,x)\right) = w_\infty\left(t,x,r_\infty(t,x)\right)$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}$. We aim at applying again the Dominated Convergence Theorem to the integral inequality in the definition of Kružkov solution, i.e.

$$\int_0^T \int_{\mathbb{R}} |r_n(t,x) - k| \partial_t \varphi(t,x) \, \mathrm{d}x \, \mathrm{d}t \tag{3.4}$$

$$+ \int_0^T \int_{\mathbb{R}} \operatorname{sgn}(r_n(t,x) - k) \left[r_n(t,x) \, w_n\left(t,x,r_n(t,x)\right) - k \, w_n(t,x,k)\right] \partial_x \varphi(t,x) \, \mathrm{d}x \, \mathrm{d}t \tag{3.5}$$

$$- \int_0^T \int_{\mathbb{R}} \operatorname{sgn}(r_n(t,x) - k) \, k \, \partial_x w_n(t,x,k) \, \varphi(t,x) \, \mathrm{d}x \, \mathrm{d}t \geq 0 \tag{3.6}$$

for $k \in \mathbb{R}$, $n \in \mathbb{N}$ and $\varphi \in \mathbf{C_c^\infty}([0,T] \times \mathbb{R}; \mathbb{R}_+)$. To this aim, observe that the term in (3.4) is easily dominated. Concerning the integrands in (3.5) and (3.6), compute them for a.e. $(t,x) \in [0,T] \times \mathbb{R}$ and use h as in (3.3) to get that

$$\begin{aligned}
|(3.5)| &\leq \left(|r_n(t,x)| + |k|\right) \|v_L\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} |\partial_x \varphi(t,x)| \\
&\leq \left(h(t,x) + |k|\right) \|v_L\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} |\partial_x \varphi(t,x)|, \\
|(3.6)| &\leq |k| \, \|v_L'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \left|\Sigma\left(\partial_x \rho_n(t,x) + \partial_x \varepsilon_n(t,x)\right)\right| |\partial_x \varphi(t,x)| \\
&\leq |k| \, \|v_L'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \, h(t,x) \, |\partial_x \varphi(t,x)|.
\end{aligned}$$

Then, as functions of (t,x), the dominating quantities obtained above are in $\mathbf{L^1}([0,T] \times \mathbb{R}; \mathbb{R})$ and the Dominated Convergence Theorem allows to pass (3.4)–(3.5)–(3.6) to the limit $n \to +\infty$, proving that r_∞ solves (2.5). □

3.2. The nonlocal problem

We are interested in the system

$$\begin{cases} \partial_t \rho^i + \partial_x \left(\rho^i \, V^i\left(t,x,\rho * \eta^i\right)\right) = 0 & i = 1, \ldots, k \\ \rho(0,x) = \rho_o(x) \end{cases} \tag{3.7}$$

where $V = V(t,x,q)$ is such that $V \in \mathbf{C^0}([0,T]; \mathcal{V})$ and $\mathcal{V}$ is the space of speed laws

$$\begin{aligned}
\mathcal{V} &:= \left\{ V \in \mathbf{C^3}(\mathbb{R} \times \mathbb{R}^k; \mathbb{R}^k) \colon \begin{array}{l} V(\cdot, 0) \in \mathbf{L}^\infty(\mathbb{R}; \mathbb{R}^k), \\ \left[\partial_x V \quad \nabla_q V\right] \in \mathbf{W}^{2,\infty}(\mathbb{R} \times \mathbb{R}^k; \mathbb{R}^k \times \mathbb{R}^{k \times k}) \end{array} \right\} \\
\|V\|_{\mathcal{V}} &:= \|V(\cdot,0)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)} + \left\|\left[\partial_x V \quad \nabla_q V\right]\right\|_{\mathbf{W}^{2,\infty}(\mathbb{R} \times \mathbb{R}^k; \mathbb{R}^k \times \mathbb{R}^{k \times k})}.
\end{aligned} \tag{3.8}$$

Definition 3.3. By *solution to* (3.7) *on the time interval* I we mean $\rho \in \mathbf{C^0}\left(I; \mathbf{L^1}(\mathbb{R};\mathbb{R}^k)\right)$ that solves

$$\begin{cases} \partial_t \rho^i + \partial_x \left(\rho^i \; v^i(t,x)\right) = 0 & (t,x) \in I \times \mathbb{R} \\ \rho^i(0,x) = \rho_o^i(x) & x \in \mathbb{R} \end{cases} \quad \text{where } v^i(t,x) := V^i\left(t, x, \left(\rho(t) * \eta^i\right)(x)\right). \tag{3.9}$$

Above, a *solution to* (3.9) is a distributional solution [6, Definition 4.2], which is also a weak entropy solution in the sense of [28, Definition 1], see also [19, Corollary II.1].

For $i = 1, \ldots, k$, given $v^i \in \mathbf{C^0}\left([0,T]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R})\right)$, the solution to

$$\begin{cases} \partial_t \rho^i + \partial_x \left(\rho^i \; v^i(t,x)\right) = 0 \\ \rho^i(0,x) = \rho_o^i(x) \end{cases}$$

in the sense of Definition 3.3 is

$$\rho^i(t,x) = \rho_o^i\left(X^i(0;t,x)\right) \; \mathcal{E}^i(0,t,x) \qquad (t,x) \in [0,T] \times \mathbb{R}, \qquad \text{where} \tag{3.10}$$

$$\mathcal{E}^i(t,t_o,x_o) = \exp\left(\int_{t_o}^{t} \partial_x v^i\left(s, X^i(s;t_o,x_o)\right) \, \mathrm{d}s\right) \qquad (t,t_o,x_o) \in [0,T]^2 \times \mathbb{R} \tag{3.11}$$

and $s \mapsto X^i(s;t,x)$ is the solution to the Cauchy problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} X^i(s;t,x) = v^i\left(s, X^i(s;t,x)\right) \\ X^i(t;t,x) = x\,. \end{cases} \tag{3.12}$$

The change of variable

$$\xi = X^i(0;t_o,x_o) \tag{3.13}$$

and the following formulæ, from [7, Theorem 2.3.2, Theorem 2.3.3], are frequently used below:

$$\partial_{x_o} X^i(t;t_o,x_o) = \mathcal{E}^i(t,t_o,x_o) \quad \text{and} \quad \partial_{t_o} X^i(t;t_o,x_o) = -\mathcal{E}^i(t,t_o,x_o)\, v^i(t_o,x_o)\,. \tag{3.14}$$

For completeness, we recall below — without proof — properties concerning the problem (3.7) useful in the sequel.

Lemma 3.4 *([16, Theorem 2.3]). Assume* $\eta \in \mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)$, $V \in \mathbf{C^0}\left([0,T];\mathcal{V}\right)$ *and* $\rho_o \in \mathcal{BV}(\mathbb{R};\mathbb{R}^k)$ *Then,*

1. *There exists a unique solution* ρ *to* (3.7) *on* $[0,T]$ *in the sense of Definition 3.3.*
2. *If* $\widehat{\rho}_o \in \mathcal{BV}(\mathbb{R};\mathbb{R}^k)$ *and* $\widehat{\rho}$ *is the corresponding solution to* (3.7) *on* $[0,T]$, *then for all* $t \in [0,T]$

$$\begin{aligned} & \left\|\rho(t) - \widehat{\rho}(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \\ & \leq C\left(\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|V\|_{\mathbf{C^0}([0,t];\mathcal{V})}, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R}^k)}, t\right) \|\rho_o - \widehat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}. \end{aligned}$$

3. ρ *is locally Lipschitz continuous in* t*: for all* $t, \widehat{t} \in [0,T]$

$$\left\|\rho(t) - \rho(\widehat{t})\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \leq C\left(\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|V\|_{\mathbf{C^0}([0,T];\mathcal{V})}, \|\rho_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R}^k)}, T\right) \left|t - \widehat{t}\right|.$$

4. *If* $\widehat{V} \in \mathbf{C^0}([0,T];\mathcal{V})$ *and* $\widehat{\rho}$ *is the corresponding solution to* (3.7) *on* $[0,T]$*, then for all* $t \in [0,T]$*:*

$$\begin{aligned}&\|\rho(t)-\widehat{\rho}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\\&\le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)},\|V\|_{\mathbf{C^0}([0,t];\mathcal{V})},\|\rho_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R}^k)},t\right)\left\|V-\widehat{V}\right\|_{\mathbf{C^0}([0,t];\mathcal{V})} t.\end{aligned}$$

5. *For all* $t \in [0,T]$ *and for* $i = 1,\dots,k$*,* $\left\|\rho^i(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} = \left\|\rho^i_o\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}$.

6. *For* $i=1,\dots,k$*, if* $\rho^i_o \ge 0$*, then* $\rho^i(t,x)\ge 0$ *for a.e.* $(t,x)\in[0,T]\times\mathbb{R}$.

We now establish some qualitative and regularity properties of the solution to the nonlocal problem (3.7) under more restrictive hypotheses on the initial datum ρ_o, on the speed law V and on the kernel function η.

Lemma 3.5 *(Fine estimates on* ρ*). Suppose* $\eta \in (\mathbf{C^3}\cap\mathbf{W^{3,\infty}})(\mathbb{R};\mathbb{R}^k)$*,* $V \in \mathbf{C^0}([0,T];\mathcal{V})$ *and* $\rho_o \in (\mathbf{C^2}\cap\mathbf{W^{2,\infty}}\cap\mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k)$*. Then, the solution* ρ *to the nonlocal problem* (3.7)*, constructed in Lemma 3.4, satisfies the following properties.*

(N1) $\rho \in \mathbf{C^1}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$.

(N2) $\partial_x\rho \in \mathbf{L^1}\left([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)$*. In particular, for a.e.* $t\in[0,T]$

$$\|\partial_x\rho(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)},\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})},\|\rho_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)},t\right), \tag{3.15}$$

$$\left\|\partial^2_{xx}\rho(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)},\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})},\|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)},t\right), \tag{3.16}$$

$$\|\partial_x\rho\|_{\mathbf{L^1}([0,t]\times\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)},\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})},\|\rho_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)},T\right)t, \tag{3.17}$$

$$\left\|\partial^2_{xx}\rho\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)},\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})},\|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)},T\right)t. \tag{3.18}$$

(N3) ρ *is* $\mathbf{W^{1,1}}$ *locally Lipschitz continuous in* t*, i.e., for all* $t,\widehat{t}\in[0,T]$

$$\left\|\rho(t)-\rho(\widehat{t})\right\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)},\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})},\|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)},T\right)\left|t-\widehat{t}\right|.$$

(N4) $\rho \in \mathbf{L^\infty}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$ *with*

$$\|\rho\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)},\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})},\|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)},\|\rho_o\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R}^k)},T\right). \tag{3.19}$$

(N5) $\partial^2_{xx}\rho \in \mathbf{C^0}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$.

(N6) $\partial_x\rho, \partial_t\rho, \partial^2_{xx}\rho, \partial^2_{tx}\rho \in \mathbf{L^\infty}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$*. In particular,*

$$\|\partial_x\rho\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)},\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})},\|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)},\|\rho_o\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)},T\right). \tag{3.20}$$

Proof. Fix $i\in\{1,\dots,k\}$. Referring to (3.1), introduce $v^i(t,x) := V^i\left(t,x,(\rho(t)*\eta^i)(x)\right)$ and use the notation (3.10). We split the proof in different parts.

Claim 1. The maps $(t,x)\to v(t,x),\partial_x v(t,x)\in\mathbf{C^0}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$**. Moreover, for all** $t\in[0,T]$**:** $x\mapsto v(t,x)\in(\mathbf{C^3}\cap\mathbf{W^{3,\infty}})(\mathbb{R};\mathbb{R}^k)$ **and** $v\in\mathbf{L^\infty}\left([0,T];\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)\right)$**.** The continuity of the maps $(t,x)\to v(t,x),\partial_x v(t,x)$ is a consequence of the assumptions on V,η and the hypothesis $\rho\in\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}^k))$. Similarly, for all $t\in[0,T]$, $x\mapsto v(t,x)$ is of class $\mathbf{C^3}$ and allows the following estimates:

$$\begin{aligned}
\left|v^i(t,x)\right| &\le \|V\|_{\mathbf{C^0}([0,t];\mathcal{V})}\left(1+\left\|\eta^i\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\right)\\
\left|\partial_x v^i(t,x)\right| &\le \|V\|_{\mathbf{C^0}([0,t];\mathcal{V})}\left(1+\left\|(\eta^i)'\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\right)\\
\left|\partial^2_{xx} v^i(t,x)\right| &\le \|V\|_{\mathbf{C^0}([0,t];\mathcal{V})}\left(1{+}2\left\|(\eta^i)'\right\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}\|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}{+}\left\|(\eta^i)'\right\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\rho_o\|^2_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\right)\\
\left|\partial^3_{xxx} v^i(t,x)\right| &\le \|V\|_{\mathbf{C^0}([0,t];\mathcal{V})}\big(1+3\left\|\eta^i\right\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R})}\|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\\
&\qquad +3\left\|(\eta^i)'\right\|^2_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}\|\rho_o\|^2_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}+\left\|(\eta^i)'\right\|^3_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\rho_o\|^3_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\big)\,.
\end{aligned}$$

Finally, we also obtain

$$\begin{aligned}
\|v\|_{\mathbf{L}^\infty([0,T];\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}))} &\le Q \qquad \text{where}\\
Q \coloneqq 4\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})}&\big(1+\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}\|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\\
&+\|\eta'\|^2_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}\|\rho_o\|^2_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}+\|\eta'\|^3_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)}\|\rho_o\|^3_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\big)
\end{aligned} \tag{3.21}$$

completing the proof Claim 1.

Claim 2: The following regularities hold:

$$\begin{array}{ll}
\forall\, t\in[0,T] & [(t_o,x_o)\mapsto X(t;t_o,x_o)]\in\mathbf{C^1}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)\\
\forall\, t\in[0,T] & [(t_o,x_o)\mapsto \mathcal{E}(t,t_o,x_o)]\in\mathbf{C^1}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)
\end{array}$$

and moreover, for $i=1,\ldots,k$, $(t,t_o,x_o)\in[0,T]\times[0,T]\times\mathbb{R}$,

$$\left|\mathcal{E}^i(t,t_o,x_o)\right|\le e^{QT}\,. \tag{3.22}$$

Due to Claim 1, the continuity of X is an immediate consequence of $V\in\mathbf{C^0}([0,T];\mathcal{V})$ and $\rho\in\mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)\right)$. The continuity of $(t_o,x_o)\mapsto\mathcal{E}(t,t_o,x_o)$ follows from (3.11) and the $\mathbf{C^1}$ regularity of X holds, due to (3.14). An application of (3.14) yields that

$$\begin{aligned}
\partial_{x_o}\mathcal{E}^i(t,t_o,x_o) &= \partial_{x_o}\int_{t_o}^{t}\partial_x v^i\left(s,X^i(s;t_o,x_o)\right)\mathrm{d}s\\
&= \int_{t_o}^{t}\partial^2_{xx}v^i\left(s,X^i(s;t_o,x_o)\right)\,\mathcal{E}^i(s;t_o,x_o)\,\mathrm{d}s\\
\partial_{t_o}\mathcal{E}^i(t,t_o,x_o) &= \partial_{t_o}\int_{t_o}^{t}\partial_x v^i\left(s,X^i(s;t_o,x_o)\right)\mathrm{d}s\\
&= -\partial_x v^i(t_o,x_o)-\int_{t_o}^{t}\partial^2_{xx}v^i\left(s,X^i(s;t_o,x_o)\right)\mathcal{E}^i(s;t_o,x_o)\,v^i(t_o,x_o)\,\mathrm{d}s
\end{aligned} \tag{3.23}$$

proving the $\mathbf{C^1}$ regularity of $(t_o,x_o)\mapsto\mathcal{E}(t,t_o,x_o)$.

Moreover, referring to (3.21), the bound (3.22) follows, since for all $(t,t_o,x_o)\in[0,T]\times[0,T]\times\mathbb{R}$, $\left|\mathcal{E}^i(t,t_o,x_o)\right|\le e^{Q|t-t_o|}$.

Claim 3: (N1) holds. One obtains that $\rho\in\mathbf{C^0}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$ because of the continuity of ρ_o and Claim 2. Compute now

$$\begin{aligned}
\partial_x \rho^i(t,x) = {} & (\rho_o^i)'\left(X^i(0;t,x)\right)\,\mathcal{E}^i(0,t,x)^2 \\
& +\rho_o^i\left(X^i(0;t,x)\right)\int_t^0 \partial^2_{xx} v^i\left(s,X^i(s;t,x)\right)\,\mathcal{E}^i(s,t,x)\,\mathrm{d}s \\
\partial_t \rho^i(t,x) = {} & (\rho_o^i)'\left(X^i(0;t,x)\right)\left(-\mathcal{E}^i(0,t,x)\,v^i(t,x)\right)\mathcal{E}^i(0,t,x) \\
& +\rho_o^i\left(X^i(0;t,x)\right)\left(-\partial_x v^i(t,x)-\int_t^0 \partial^2_{xx} v^i\left(s,X^i(s;t,x)\right)\mathcal{E}^i(s;t,x)\,v^i(t,x)\,\mathrm{d}s\right),
\end{aligned} \tag{3.24}$$

proving that $\rho \in \mathbf{C^1}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$ due to $\rho_o \in \mathbf{C^1}(\mathbb{R};\mathbb{R}^k)$, Claim 1 and Claim 2.

Claim 4: (N2) holds. Use the change of variable (3.13) and (3.14) to obtain for a.e. $t \in [0,T]$

$$\begin{aligned}
\left\|\partial_x \rho^i(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \le {} & \int_{\mathbb{R}} \left|(\rho_o^i)'\left(X^i(0;t,x)\right)\,\mathcal{E}^i(0,t,x)^2\right|\mathrm{d}x \\
& + \int_{\mathbb{R}} \left|\rho_o^i\left(X^i(0;t,x)\right)\int_0^t \partial^2_{xx} v^i\left(s,X^i(s;t,x)\right)\mathcal{E}^i(s,t,x)\,\mathrm{d}s\right|\mathrm{d}x \\
\le {} & \int_{\mathbb{R}} \left|(\rho_o^i)'(\xi)\mathcal{E}^i\left(0,t,X^i(t;0,\xi)\right)\right|\mathrm{d}\xi + \int_{\mathbb{R}} \left|\rho_o^i(\xi)\right| Q\,t\,e^{Qt}\mathcal{E}^i(0;t,\xi)\,\mathrm{d}\xi \\
\le {} & \left\|\rho_o^i\right\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R})} e^{Qt}\left(1+Q\,t\,e^{Qt}\right)
\end{aligned}$$

proving (3.15). Since

$$\begin{aligned}
\partial^2_{xx}\rho^i(t,x) = {} & (\rho_o^i)''\left(X^i(0;t,x)\right)\,\mathcal{E}^i(0,t,x)^3 \\
& + 2(\rho_o^i)'\left(X(0;t,x)\right)\,\mathcal{E}^i(0,t,x)\partial_x\mathcal{E}^i(0,t,x) \\
& + (\rho_o^i)'\left(X^i(0;t,x)\right)\mathcal{E}^i(0,t,x)\int_t^0 \partial^2_{xx} v^i\left(s,X(s;t,x)\right)\mathcal{E}^i(s,t,x)\,\mathrm{d}s \\
& + \rho_o^i\left(X^i(0;t,x)\right)\int_t^0 \partial^3_{xxx} v^i\left(s,X^i(s;t,x)\right)\mathcal{E}^i(s,t,x)^2\,\mathrm{d}s \\
& + \rho_o^i\left(X^i(0;t,x)\right)\int_t^0 \partial^2_{xx} v^i\left(s,X^i(s;t,x)\right)\partial_x\mathcal{E}^i(s,t,x)\,\mathrm{d}s\,,
\end{aligned}$$

computations entirely similar to the ones used above ensure that

$$\left\|\partial^2_{xx}\rho(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \le \left\|\rho_o^i\right\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R})} e^{2Qt}\left(1+4e^{Qt}Qt+Q^2t^2\right).$$

proving (3.16). The bounds (3.17) and (3.18) readily follow.

Claim 5: (N3) and (N4) hold. Thanks to **3.** in Lemma 3.4, it is sufficient to prove the $\mathbf{L^1}$ local Lipschitz continuity in time of $\partial_x\rho$. To this aim, start from (3.24) and, using (3.22), compute:

$$
\begin{aligned}
&\left\|\partial_x \rho^i(t) - \partial_x \rho^i(\hat{t})\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\
\le & \int_{\mathbb{R}} \left|(\rho_o^i)'\left(X^i(0;t,x)\right)\mathcal{E}^i(0,t,x)^2 - (\rho_o^i)'\left(X^i(0;\hat{t},x)\right)\mathcal{E}^i(0,\hat{t},x)^2\right| \mathrm{d}x \\
& + \int_{\mathbb{R}} \left|\rho_o^i\left(X^i(0;t,x)\right)\int_0^t \partial^2_{xx} v^i\left(s, X^i(s;t,x)\right)\mathcal{E}^i(s,t,x)\,\mathrm{d}s\right. \\
& \left. - \rho_o^i\left(X^i(0;\hat{t},x)\right)\int_0^{\hat{t}} \partial^2_{xx} v^i\left(s, X^i(s;\hat{t},x)\right)\mathcal{E}^i(s,\hat{t},x)\,\mathrm{d}s\right| \mathrm{d}x \\
\le & 2e^{QT}\,\mathrm{TV}\left((\rho_o^i)'\right)\sup_{x\in\mathbb{R}}\left|X^i(0;t,x) - X^i(0;\hat{t},x)\right| && \text{[By [16, Lemma 4.1]]} \\
& + 2e^{2QT}\left\|(\rho_o^i)'\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\sup_{x\in\mathbb{R}}\left|\mathcal{E}^i(0,t,x) - \mathcal{E}^i(0,\hat{t},x)\right| && \text{[By (3.13), (3.14)]} \\
& + 2\,Q\,T\,e^{QT}\,\mathrm{TV}\left((\rho_o^i)'\right)\sup_{x\in\mathbb{R}}\left|X^i(0;t,x) - X^i(0;\hat{t},x)\right| && \text{[By [16, Lemma 4.1], (3.21)]} \\
& + Q\,e^{2QT}\left\|\rho_o^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\left|t-\hat{t}\right| && \text{[By (3.13), (3.14)]} \\
& + Qe^{2QT}\left\|\rho_o^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\int_0^{\hat{t}}\sup_{x\in\mathbb{R}}\left|X^i(s;t,x) - X^i(s;\hat{t},x)\right|\mathrm{d}s && \text{[By (3.13), (3.14), (3.21)]} \\
& + Qe^{QT}\left\|\rho_o^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\int_0^{\hat{t}}\sup_{x\in\mathbb{R}}\left|\mathcal{E}^i(s,t,x) - \mathcal{E}^i(s,\hat{t},x)\right|\mathrm{d}s\,. && \text{[By (3.13), (3.14), (3.21)]}
\end{aligned}
$$

To complete the above estimate, note that for all $s \in [0,T]$ and for all $x \in \mathbb{R}$,

$$
\begin{aligned}
\left|X^i(s;t,x) - X^i(s;\hat{t},x)\right| &\le Q\,e^{QT}\left|t-\hat{t}\right|; && \text{[By (3.12), (3.14) and (3.21)]} \\
\left|\mathcal{E}^i(s,t,x) - \mathcal{E}^i(s,\hat{t},x)\right| &\le \left(Q + Q^2Te^{QT}\right)\left|t-\hat{t}\right|. && \text{[By (3.21), (3.22) and (3.23)]}
\end{aligned}
$$

and **(N3)** follows. The standard embedding $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \hookrightarrow \mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)$ allows to prove **(N4)**.

Claim 6: (N5) holds. The computations in Claim 4 show that $\partial^2_{xx}\rho \in \mathbf{C^0}\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$ due to $\rho_o \in \mathbf{C^2}(\mathbb{R};\mathbb{R}^k)$, Claim 1 and Claim 2.

Claim 7: (N6) holds. For $(t,x) \in [0,T]\times\mathbb{R}$, by the computations in Claim 3, (3.21) and (3.22)

$$
\begin{aligned}
\left|\partial_x\rho^i(t,x)\right| &\le \left\|\rho_o^i\right\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})} e^{2QT} \\
\left|\partial_t\rho^i(t,x)\right| &\le \left\|(\rho_o^i)'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} e^{2QT}Q + \left\|(\rho_o^i)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left(Q + Q^2e^{QT}T\right) \\
\left|\partial^2_{xx}\rho^i(t,x)\right| &\le \left\|\rho_o^i\right\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R})}\left(e^{2QT} + 4e^{QT}QT + Q^2T^2\right)e^{QT} \\
\left|\partial^2_{tx}\rho^i(t,x)\right| &\le \left|\left((\rho_o^i)''\right)\left(X^i(0;t,x)\right)\partial_t X^i(0;t,x)\mathcal{E}^i(0,t,x)^2\right| \\
&\quad + \left|(\rho_o^i)'\left(X^i(0;t,x)\right)2\mathcal{E}^i(0,t,x)\partial_t\mathcal{E}^i(0,t,x)\right| \\
&\quad + \left|(\rho_o^i)'\left(X^i(0;t,x)\right)\partial_t X^i(0;t,x)\int_t^0 \partial^2_{xx}v^i\left(s,X^i(s;t,x)\right)\mathcal{E}^i(s,t,x)\,\mathrm{d}s\right| \\
&\quad + \left|\rho_o^i\left(X^i(0;t,x)\right)\partial^2_{xx}v^i(t,x)\right|
\end{aligned}
\tag{3.25}
$$

$$
\begin{aligned}
&+\left|\rho_o^i\left(X^i(0;t,x)\right)\int_t^0 \partial^3_{xxx}v^i\left(s,X^i(s;t,x)\right)\partial_t X^i(s;t,x)\mathcal{E}^i(s,t,x)\,\mathrm{d}s\right|\\
&+\left|\rho_o^i\left(X^i(0;t,x)\right)\int_t^0 \partial^2_{xx}v^i\left(s,X^i(s;t,x)\right)\partial_t \mathcal{E}^i(s,t,x)\,\mathrm{d}s\right|\\
&\le \left\|\rho_o^i\right\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R})} Q\left(1+TQ+e^{QT}\left(2+T^2Q^2\right)+e^{2QT}4QT+e^{3TQ}\right),
\end{aligned}
$$

completing the proof of Claim 7. □

Lemma 3.6 *(*A priori *estimate on V). Assume that $\eta \in \mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)$, v_{NL} satisfies* **(vNL)** *and $r \in \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$. Define for $i=1,\ldots,k$ and for $(t,x,q)\in[0,T]\times\mathbb{R}\times\mathbb{R}^k$*

$$
V^i(t,x,q) \coloneqq v^i_{NL}\left(\Sigma q + (r(t)*\eta^i)(x)\right).
$$

Then, V belongs to $\mathbf{C^0}\left([0,T];\mathcal{V}\right)$ and for all $t\in[0,T]$

$$
\|V\|_{\mathbf{C^0}([0,t];\mathcal{V})} \le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|r\|_{\mathbf{C^0}([0,t];\mathbf{L^1}(\mathbb{R};\mathbb{R}))}\right). \tag{3.26}
$$

Proof. Compute, for all $(t,x,q)\in[0,T]\times\mathbb{R}\times\mathbb{R}$ and $i,j,l,m=1,\ldots,k$

$$
\begin{aligned}
\left|V^i(t,x,0)\right| &\le \left\|v^i_{NL}\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\\
\left|\partial_x V^i(t,x,q)\right| &\le \left\|(\eta^i)'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
\left|\partial_{q_j} V^i(t,x,q)\right| &\le \left\|(v^i_{NL})'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\\
\left|\partial^2_{xx} V^i(t,x,q)\right| &\le \left\|(\eta^i)'\right\|^2_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|^2_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
&\quad+\left\|(\eta^i)''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
\left|\partial^2_{xq_j} V^i(t,x,q)\right| &\le \left\|(\eta^i)'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
\left|\partial^2_{q_jq_l} V^i(t,x,q)\right| &\le \left\|(v^i_{NL})''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\\
\left|\partial^3_{xxx} V^i(t,x,q)\right| &\le \left\|(\eta^i)'\right\|^3_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})'''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|^3_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
&\quad+3\left\|(\eta^i)'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(\eta^i)''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|^2_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
&\quad+\left\|(\eta^i)'''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
\left|\partial^3_{xxq_j} V^i(t,x,q)\right| &\le \left\|(\eta^i)'\right\|^2_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})'''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|^2_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
&\quad+\left\|(\eta^i)''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
\left|\partial^3_{xq_jq_l} V^i(t,x,q)\right| &\le \left\|(\eta^i)'\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|(v^i_{NL})'''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
\left|\partial^3_{q_jq_lq_m} V^i(t,x,q)\right| &\le \left\|(v^i_{NL})'''\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})},
\end{aligned}
$$

hence for all $t\in[0,T]$, $V(t)\in\mathcal{V}$. Moreover, since for all $t,t_o\in[0,T]$,

$$
\left\|r(t)*\eta^i - r(t_o)*\eta^i\right\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R})} \le \|r(t)-r(t_o)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\left\|\eta^i\right\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R})}
$$

we get that $V \in \mathbf{C^0}([0,T];\mathcal{V})$. The $\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)$ continuity of $t \mapsto V(t,\cdot,0)$ and of $t \mapsto [\partial_x V(t) \quad \nabla_q V(t)]$ in $\mathbf{W^{2,\infty}}(\mathbb{R}\times\mathbb{R}^k;\mathbb{R}^k\times\mathbb{R}^{k\times k})$ then follows by the regularity of v_{NL}. □

Lemma 3.7 *(Stability in $\mathbf{W^{1,1}}$). Fix $\mathcal{R} > 0$. Let $\eta \in \mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)$, v_{NL} satisfy* **(vNL)**. *For all $r, \hat r \in \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$ with $\|r\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}))} \le \mathcal{R}$, $\|\hat r\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}))} \le \mathcal{R}$, $\rho_o \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)$ and $\hat\rho_o \in \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)$, call $\rho, \hat\rho$ the solutions, in the sense of Definition 3.3, to*

$$\begin{cases} \partial_t \rho^i + \partial_x \left(\rho^i \, v^i_{NL}\left(\Sigma\rho * \eta^i + r * \eta^i\right)\right) = 0 & i = 1,\dots,k \\ \rho(0,x) = \rho_o(x) \end{cases} \tag{3.27}$$

$$\begin{cases} \partial_t \hat\rho^i + \partial_x \left(\hat\rho^i \, v^i_{NL}\left(\Sigma\hat\rho * \eta^i + \hat r * \eta^i\right)\right) = 0 & i = 1,\dots,k \\ \hat\rho(0,x) = \hat\rho_o(x) \end{cases} \tag{3.28}$$

Then, there exists a constant $\mathcal{C}_3$ such that for all $t \in [0,T]$

$$\|\rho(t) - \hat\rho(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \le (1 + \mathcal{C}_3\, t)\|\rho_o - \hat\rho_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \mathcal{C}_3 \int_0^t \|r(\tau) - \hat r(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\, \mathrm{d}\tau \tag{3.29}$$

where

$$\mathcal{C}_3 := \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \mathcal{R}, \|\rho_o\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)}, \|\hat\rho_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R}^k)}, t\right). \tag{3.30}$$

Using the particular form of (3.27)–(3.28), Lemma 3.7 improves the stability estimates **2.–4.** in Lemma 3.4.

Proof of Lemma 3.7. Throughout, fix $i = 1,\dots,k$. Lemma 3.6 ensures that we can apply Lemma 3.4 to (3.27) and (3.28). Hence, there exist unique $\rho, \hat\rho \in \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)\right)$ solutions to (3.27) and (3.28) in the sense of Definition 3.3. Introduce the velocities

$$\begin{aligned} v^i(t,x) &:= v^i_{NL}\left(\left(\Sigma\rho(t) * \eta^i\right)(x) + \left(r(t) * \eta^i\right)(x)\right), \\ \hat v^i(t,x) &:= v^i_{NL}\left(\left(\Sigma\hat\rho(t) * \eta^i\right)(x) + \left(\hat r(t) * \eta^i\right)(x)\right). \end{aligned}$$

The same computations as in Claim 1 in Lemma 3.5 and the estimate (3.26) in Lemma 3.6, yield that there exists a constant

$$Q := \mathcal{C}\left(\|\eta'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)}, \mathcal{R}, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\hat\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\right)$$

such that

$$\max\left\{\left\|\partial_x v^i\right\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}))}, \left\|\partial_x \hat v^i\right\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}))}\right\} \le Q. \tag{3.31}$$

With this choice, we have

$$\max\left\{\mathcal{E}^i(t,t_o,x_o), \hat{\mathcal{E}}^i(t,t_o,x_o)\right\} \le e^{Q|t-t_o|} \le e^{QT} \quad \forall t, t_o \in [0,T], x_o \in \mathbb{R}. \tag{3.32}$$

Moreover, for all $s,t \in [0,T]$, $s \le t$, the following estimate holds by an adaptation of [16, Proposition 4.2]:

$$\left|X^i(s;t,x) - \hat X^i(s;t,x)\right| \le \int_s^t \left\|v^i(\tau) - \hat v^i(\tau)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} e^{Q(t-\tau)}\, \mathrm{d}\tau \tag{3.33}$$

$$\le |t-s|\, e^{Q|t-s|} \left\|v^i - \hat v^i\right\|_{\mathbf{C^0}([0,t];\mathbf{L^\infty}(\mathbb{R};\mathbb{R}))}. \tag{3.34}$$

At last, taking advantage of the expression (3.11) and applying the estimates (3.31)–(3.34), one gets that for all $s, t \in [0, T]$, $s \le t$ $x \in \mathbb{R}$

$$\begin{aligned}
&\left|\mathcal{E}^i(s,t,x) - \widehat{\mathcal{E}}^i(s,t,x)\right| \\
\le& \max\left\{ e^{\|\partial_x \widehat{v}^i\|_{\mathbf{C^0}([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}|t-s|}, e^{\|\partial_x v^i\|_{\mathbf{C^0}([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}|t-s|} \right\} \\
&\times \left| \int_t^s \partial_x v^i\left(\tau, X^i(\tau;t,x)\right) - \partial_x \widehat{v}^i\left(\tau, \widehat{X}^i(\tau;t,x)\right) \mathrm{d}\tau \right| \\
\le& e^{Q|t-s|} \left| \int_t^s \partial_x v^i\left(\tau, X^i(\tau;t,x)\right) - \partial_x v^i\left(\tau, \widehat{X}^i(\tau;t,x)\right) \mathrm{d}\tau \right| \\
&+ e^{Q|t-s|} \left| \int_t^s \partial_x v^i\left(\tau, \widehat{X}^i(\tau;t,x)\right) - \partial_x \widehat{v}^i\left(\tau, \widehat{X}^i(\tau;t,x)\right) \mathrm{d}\tau \right| \\
\le& e^{Q|t-s|} \left(Q \int_s^t \left| X^i(\tau;t,x) - \widehat{X}^i(\tau;t,x) \right| \mathrm{d}\tau + \int_s^t \left\| \partial_x v^i(\tau) - \partial_x \widehat{v}^i(\tau) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}\tau \right) \\
\le& e^{Q|t-s|} \left(Q \int_s^t \int_\tau^t \left\| v^i(\tau') - \widehat{v}^i(\tau') \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} e^{Q(t-\tau')} \mathrm{d}\tau' \,\mathrm{d}\tau + \int_s^t \left\| \partial_x v^i(\tau) - \partial_x \widehat{v}^i(\tau) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}\tau \right) \\
\le& e^{Q|t-s|} \left(e^{Q|t-s|} Q|t-s| \int_s^t \left\| v^i(\tau) - \widehat{v}^i(\tau) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} \mathrm{d}\tau + \int_s^t \left\| \partial_x v^i(\tau) - \partial_x \widehat{v}^i(\tau) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}\tau \right) \\
\le& e^{2Q|t-s|} \left(Q|t-s| + 1 \right) \left\| v^i - \widehat{v}^i \right\|_{\mathbf{L^1}((s,t);\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}))} .
\end{aligned} \tag{3.35}$$

Now observe that for all $\tau \in [0, T]$

$$\begin{aligned}
&\left\| v^i(\tau,\cdot) - \widehat{v}^i(\tau,\cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \\
=& \sup_{x\in\mathbb{R}} \left| v^i_{NL}\left(\left(\Sigma\rho(\tau) * \eta^i\right)(x) + (r(\tau) * \eta^i)(x) \right) - v^i_{NL}\left(\left(\Sigma\widehat{\rho}(\tau) * \eta^i\right)(x) + (\widehat{r}(\tau) * \eta)(x) \right) \right| \\
\le& \|v'_{NL}\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)} \|\eta\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)} \left(\|\rho(\tau) - \widehat{\rho}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} + \|r(\tau) - \widehat{r}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \right),
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
&\left\| \partial_x v^i(\tau) - \partial_x \widehat{v}^i(\tau) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \\
\le& \left(\left\| (v^i_{NL})' \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} + \left\| (v^i_{NL})'' \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \left\| \eta^i \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \|\Sigma\widehat{\rho}(\tau) + \widehat{r}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \right) \\
&\times \left\| (\eta^i)' \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \left(\sum_{j=1}^k \left\| \rho^j(\tau) - \widehat{\rho}^j(\tau) \right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} + \|r(\tau) - \widehat{r}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \right) \\
\le& \left(\|v'_{NL}\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)} + \|v''_{NL}\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)} \|\eta\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)} \left(\|\widehat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} + \mathcal{R} \right) \right) \|\eta'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)} \\
&\times \left(\|\rho(\tau) - \widehat{\rho}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} + \|r(\tau) - \widehat{r}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \right)
\end{aligned} \tag{3.37}$$

by Item **5.** in Lemma 3.4 and

$$
\begin{aligned}
&\left\|\partial^2_{xx} v^i(\tau,\cdot) - \partial^2_{xx}\widehat{v}^i(\tau,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\\
\leq &\left(1+3\|\eta\|_{\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}^k)}\left(\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)}+\mathcal{R}\right)+\|\eta'\|^2_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)}\left(\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)}+\mathcal{R}\right)^2\right) \qquad (3.38)\\
&\times\|\eta\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^k)}\|v'_{NL}\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^k)}\left(\|\rho(\tau)-\widehat{\rho}(\tau)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)}+\|r(\tau)-\widehat{r}(\tau)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\right).
\end{aligned}
$$

Bound on the distance between $\rho^i(t)$ and $\widehat{\rho}^i(t)$ in $\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)$. Owing to (3.10), one obtains that for all $t\in[0,T]$

$$
\begin{aligned}
&\left\|\rho^i(t)-\widehat{\rho}^i(t)\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\\
\leq&\int_{\mathbb{R}}\left|\left(\rho^i_o\left(X^i(0;t,x)\right)-\widehat{\rho}^i_o\left(X^i(0;t,x)\right)\right)\mathcal{E}^i(0,t,x)\right|\mathrm{d}x\\
&+\int_{\mathbb{R}}\left|\left(\widehat{\rho}^i_o\left(X^i(0;t,x)\right)-\widehat{\rho}^i_o\left(\widehat{X}^i(0;t,x)\right)\right)\mathcal{E}^i(0,t,x)\right|\mathrm{d}x\\
&+\int_{\mathbb{R}}\left|\widehat{\rho}^i_o\left(\widehat{X}^i(0;t,x)\right)\left(\mathcal{E}^i(0,t,x)-\widehat{\mathcal{E}}^i(0;t,x)\right)\right|\mathrm{d}x\\
\leq&\left\|\rho^i_o-\widehat{\rho}^i_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}+2\,\mathrm{TV}(\widehat{\rho}^i_o)\sup_{x\in\mathbb{R}}\left|X^i(0;t,x)-\widehat{X}^i(0;t,x)\right| \qquad \text{[By [16, Lemma 4.1]]}\\
&+e^{3Qt}\left\|\widehat{\rho}^i_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}(Qt+1)\left\|v^i-\widehat{v}^i\right\|_{\mathbf{L}^1([0,t];\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}))} \qquad \text{[By (3.35)]}\\
\leq&\left\|\rho^i_o-\widehat{\rho}^i_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}+e^{Qt}\left(2\,\mathrm{TV}(\widehat{\rho}^i_o)+e^{2Qt}\left\|\widehat{\rho}^i_o\right\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}(Qt+1)\right)\left\|v^i-\widehat{v}^i\right\|_{\mathbf{L}^1([0,t];\mathbf{W}^{1,\infty}(\mathbb{R};\mathbb{R}))}. \qquad (3.39)
\end{aligned}
$$

where in the previous computations we exploit (3.33). Now, inserting (3.36) and (3.37) in (3.39) allows to prove

$$
\begin{aligned}
&\|\rho(t)-\widehat{\rho}(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)}\\
\leq&\|\rho_o-\widehat{\rho}_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)}\\
&+\mathcal{C}\left(\|\eta\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^k)},\|v_{NL}\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^k)},\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)},\|\widehat{\rho}_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R}^k)},\mathcal{R},t\right)\\
&\quad\times\int_0^t\left(\|\rho(\tau)-\widehat{\rho}(\tau)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)}+\|r(\tau)-\widehat{r}(\tau)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\right)\mathrm{d}\tau,
\end{aligned}
$$

and, by an application of Gronwall Lemma, it is possible to obtain

$$
\begin{aligned}
&\|\rho(t)-\widehat{\rho}(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)}\\
\leq&\left(1+\mathcal{C}\left(\|\eta\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^k)},\|v_{NL}\|_{\mathbf{W}^{2,\infty}(\mathbb{R};\mathbb{R}^k)},\|\rho_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)},\|\widehat{\rho}_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R}^k)},\mathcal{R},t\right)t\right)\\
&\times\left(\|\rho_o-\widehat{\rho}_o\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)}+\int_0^t\|r(\tau)-\widehat{r}(\tau)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})}\mathrm{d}\tau\right). \qquad (3.40)
\end{aligned}
$$

Bound on the distance between $\partial_x\rho$ and $\partial_x\widehat{\rho}$ in $\mathbf{C}^0\left([0,T];\mathbf{L}^1(\mathbb{R};\mathbb{R}^k)\right)$. An application of [16, Lemma 4.1], together with the estimates (3.31) and (3.32), leads to

$$
\begin{aligned}
&\left\|\partial_x \rho(t) - \partial_x \widehat{\rho}^i(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\
\leq &\int_{\mathbb{R}} \left| \left((\rho_o^i)'\left(X^i(0;t,x)\right) - (\rho_o^i)'\left(\widehat{X}^i(0;t,x)\right)\right) \mathcal{E}^i(0,t,x)^2 \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| (\rho_o^i)'\left(\widehat{X}^i(0;t,x)\right) \left(\mathcal{E}^i(0,t,x)^2 - \widehat{\mathcal{E}}^i(0;t,x)^2\right)\right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| (\rho_o^i)'\left(\widehat{X}^i(0;t,x)\right) - (\widehat{\rho}_o^i)'\left(\widehat{X}^i(0;t,x)\right)\right| \widehat{\mathcal{E}}^i(0;t,x)^2 \,\mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| \left(\rho_o^i\left(X^i(0;t,x)\right) - \rho_o^i\left(\widehat{X}^i(0;t,x)\right)\right) \int_0^t \partial^2_{xx} v^i\left(s, X^i(s;t,x)\right) \mathcal{E}^i(s,t,x)\,\mathrm{d}s \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| \rho_o^i\left(\widehat{X}^i(0;t,x)\right) \int_0^t \left(\partial^2_{xx} v^i\left(s, X^i(s;t,x)\right) - \partial^2_{xx}\widehat{v}^i\left(s, X^i(s;t,x)\right)\right) \mathcal{E}^i(s,t,x)\,\mathrm{d}s \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| \rho_o^i\left(\widehat{X}^i(0;t,x)\right) \int_0^t \left(\partial^2_{xx} \widehat{v}^i\left(s, X^i(s;t,x)\right) - \partial^2_{xx}\widehat{v}^i\left(s, \widehat{X}^i(s;t,x)\right)\right) \mathcal{E}^i(s,t,x)\,\mathrm{d}s \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| \rho_o^i\left(\widehat{X}^i(0;t,x)\right) \int_0^t \partial^2_{xx} \widehat{v}^i\left(s, \widehat{X}^i(s;t,x)\right) \left(\mathcal{E}^i(s,t,x) - \widehat{\mathcal{E}}^i(s;t,x)\right) \mathrm{d}s \right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \left| \left(\rho_o^i\left(\widehat{X}^i(0;t,x)\right) - \widehat{\rho}_o^i\left(\widehat{X}^i(0;t,x)\right)\right) \int_0^t \partial^2_{xx} \widehat{v}^i\left(s, \widehat{X}^i(s;t,x)\right) \widehat{\mathcal{E}}^i(s;t,x)\,\mathrm{d}s \right| \mathrm{d}x \\
\leq & 2e^{Qt}\,\mathrm{TV}((\rho_o^i)')\left\|X^i(0;t,\cdot) - \widehat{X}^i(0;t,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} \\
&+ 2e^{2Qt}\left\|(\rho_o^i)'\right\|_{\mathbf{L^1}(\mathbb{R},\mathbb{R})}\left\|\mathcal{E}^i(0,t,\cdot) - \widehat{\mathcal{E}}^i(0,t,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} \\
&+ e^{Qt}\left\|(\rho_o^i)' - (\widehat{\rho}_o^i)'\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} + 2Qte^{2Qt}\,\mathrm{TV}(\rho_o^i)\left\|X^i(0;t,\cdot) - \widehat{X}^i(0;t,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} \\
&+ e^{2Qt}\left\|\rho_o^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \int_0^t \left\|\partial^2_{xx} v^i(s) - \partial^2_{xx}\widehat{v}^i(s)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} \mathrm{d}s \\
&+ e^{2Qt}\left\|\rho_o^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\left\|\partial^3_{xxx}\widehat{v}^i\right\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R};\mathbb{R})} \int_0^t \left\|X^i(s;t,\cdot) - \widehat{X}^i(s;t,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} \mathrm{d}s \\
&+ Qe^{Qt}\left\|\rho_o^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \int_0^t \left\|\mathcal{E}^i(s;t,\cdot) - \widehat{\mathcal{E}}^i(s;t,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} \mathrm{d}s + Qte^{2Qt}\left\|\rho_o^i - \widehat{\rho}_o^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\
\leq &\left(1 + \mathcal{C}\left(\|\eta'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \mathcal{R}\right) t\right) \\
&\quad \times \left\|(\rho_o^i)' - (\widehat{\rho}_o^i)'\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\
&+ \mathcal{C}\left(\|\eta'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \mathcal{R}\right) t \left\|\rho_o^i - \widehat{\rho}_o^i\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\
&+ \mathcal{C}\left(\|\eta'\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \mathcal{R}, t\right)
\end{aligned}
$$

$$\times \int_0^t \left\| v^i(\tau;\cdot) - \widehat{v}^i(\tau,\cdot)\right\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}))} \mathrm{d}\tau$$

where in the last inequality we exploit (3.33), (3.35) and

$$\left\|\partial^3_{xxx}\widehat{v}\right\|_{\mathbf{C^0}([0,t];\mathbf{L^\infty}(\mathbb{R};\mathbb{R}^k))} \le \mathcal{C}\left(\|\eta'\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \mathcal{R}\right).$$

Owing to (3.36)–(3.38) and (3.40), we get

$$\begin{aligned}
&\|\partial_x\rho(t) - \partial_x\widehat{\rho}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \\
&\le \left(1 + \mathcal{C}\left(\|\eta'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{2,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \mathcal{R}\right) t\right) \|\rho_o' - \widehat{\rho}_o'\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \\
&\quad + \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R}^k)}, \mathcal{R}, t\right) t \\
&\quad \times \left(\|\rho_o - \widehat{\rho}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} + \int_0^t \|r(\tau) - \widehat{r}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \mathrm{d}\tau\right).
\end{aligned} \tag{3.41}$$

Finally, (3.29) and (3.30) follow from (3.40) and (3.41). □

3.3. The local problem

This paragraph is devoted to the Cauchy problem

$$\begin{cases} \partial_t r + \partial_x\left(r\, w(t,x,r)\right) = 0 \\ r(0,x) = r_o(x). \end{cases} \tag{3.42}$$

We refer to [28, Definition 1] for the definition of solution to (3.42). Uniqueness and Lipschitz continuous dependence on the initial data follow from [28, Theorem 1].

Lemma 3.8 *(Dependence on the initial datum and uniqueness). Fix $W > 0$. Assume*

(L1) $w \in \mathbf{C^1}([0,T] \times \mathbb{R} \times \mathbb{R}; [0,W])$.
(L2) $\partial^2_{tr} w, \partial^2_{xr} w \in \mathbf{L^\infty_{loc}}([0,T] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$.

If $r, \widehat{r} \in \mathbf{L^\infty}([0,T] \times \mathbb{R}; \mathbb{R})$ are Kružkov solutions to (3.42) with initial data $r_o, \widehat{r}_o \in \mathbf{L^\infty}(\mathbb{R};\mathbb{R})$, then, for a.e. $t \in [0,T]$,

$$\begin{gathered}
\|r(t) - \widehat{r}(t)\|_{\mathbf{L^1}([a+\sigma t, b-\sigma t];\mathbb{R})} \le \|r_o - \widehat{r}_o\|_{\mathbf{L^1}([a,b];\mathbb{R})} \qquad \text{where} \\
\sigma = \sup\left\{|w(\tau,\xi,q) + q\,\partial_r w(\tau,\xi,q)| \colon \tau \in [0,t]\,,\ \xi \in [a,b]\,,\ q \in [-M,M]\right\} \\
M = \max\left\{\|r_o\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}, \|\widehat{r}_o\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\right\}.
\end{gathered} \tag{3.43}$$

If moreover $\|r_o - \widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} < +\infty$ and

(L3) $\sup\left\{|\partial_r w(\tau,\xi,q)| \colon \tau \in [0,t]\,,\ \xi \in \mathbb{R}\,,\ q \in [-M,M]\right\} < +\infty$,

then

$$\|r(t) - \widehat{r}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \le \|r_o - \widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}. \tag{3.44}$$

Hence, (3.42) *admits at most a unique Kružkov solution.*

Proof of Lemma 3.8. We set $\varphi(t,x,r) = r\,w(t,x,r)$ and, with reference to the conservation law $\partial_t r + \partial_x \varphi(t,x,r) = 0$, we verify the assumptions in [28, § 3]. Indeed: $\varphi \in \mathbf{C^1}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$ due to **(L1)**. Furthermore, the maps $r \mapsto \partial_t\varphi(t,x,r)$ and $r \mapsto \partial_x\varphi(t,x,r)$ are Lipschitz continuous on every compact subset of $[0,T]\times\mathbb{R}\times\mathbb{R}$, due to **(L2)**. Hence, (3.43) follows from [28, Formula (3.1)] while [28, Theorem 2] ensures the uniqueness of the solution. Finally, (3.44) is obtained passing to the limits $a \to -\infty$ and $b \to +\infty$ in (3.43), thanks to **(L3)**. □

Corollary 3.9 *(Invariance of* $[0,R_L]$ *and mass conservation). Let* **(L1)** *and* **(L2)** *hold. Fix* $R_L > 0$ *and assume moreover that*

$$w(t,x,R_L) = 0 \quad \text{for all } (t,x) \in \mathbb{R}_+ \times \mathbb{R}\,. \tag{3.45}$$

If $r_o \in \mathbf{L^1}(\mathbb{R};[0,R_L])$*, then, for all* $t \geq 0$*, the Kružkov solution* r *to* (3.42) *satisfies*

$$r(t,x) \in [0,R_L] \quad \text{for a.e. } x \in \mathbb{R} \qquad \text{and} \qquad \|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} = \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\,. \tag{3.46}$$

Proof of Corollary 3.9. Clearly, $r_o \equiv 0$ yields the null solution by (3.45) while $\widehat{r}_o \equiv R_L$ yields the constant solution $\widehat{r}(t,x) = R_L$. The comparison principle [28, Theorem 3] ensures the invariance $r(t,x) \in [0,R_L]$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$. Then, the conservation of mass yields the latter equality. □

We now consider the existence of solutions to (3.42).

Lemma 3.10 *(Existence). Fix* $W > 0$*. Let* **(L1)** *and* **(L2)** *hold and assume also that*

- **(L4)** $\partial^2_{xx} w, \partial^2_{xr} w \in \mathbf{C^0}\left([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R}\right)$.
- **(L5)** $\partial_x w, \partial_r w \in \mathbf{L^\infty}\left([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R}\right)$.
- **(L6)** *There exists* $R \geq 0$ *such that for all* $r \leq -R$, $w(t,x,r) = W$ *and for all* $r \geq R$, $w(t,x,r) = 0$.
- **(L7)** $\partial^2_{xr} w \in \mathbf{L^\infty}\left([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R}\right)$.

Then, for any $r_o \in \mathbf{L^\infty}(\mathbb{R};\mathbb{R})$*, problem* (3.42) *admits a Kružkov solution.*

Proof of Lemma 3.10. Using the notation in [28] we set $\varphi(t,x,r) = r\,w(t,x,r)$ and we check that the assumptions in [28, 4° of § 5] are verified. Indeed: φ and $\partial_r\varphi$ are continuous by **(L1)**; $\partial^2_{xr}\varphi$ is continuous by **(L4)**; $\partial^2_{xx}\varphi$ is continuous by **(L4)**; $\partial_x\varphi$ and $\partial_r\varphi$ are bounded in $[0,T]\times\mathbb{R}\times[-M,M]$ for any M by **(L5)**; condition [28, (4.1)] directly follows; condition [28, (4.2)] holds by **(L5)**–**(L6)**–**(L7)**. Then, [28, Theorem 5] applies, completing the proof. □

Lemma 3.11 *(***BV** *estimate and stability). Fix* $r_o \in \mathbf{BV}(\mathbb{R};[0,R])$*. Consider the problems*

$$\begin{cases} \partial_t r + \partial_x\left(r\,w(t,x,r)\right) = 0 \\ r(0,x) = r_o(x) \end{cases} \qquad \text{and} \qquad \begin{cases} \partial_t \widehat{r} + \partial_x\left(\widehat{r}\,\widehat{w}(t,x,\widehat{r})\right) = 0 \\ \widehat{r}(0,x) = r_o(x) \end{cases} \tag{3.47}$$

with both $w, \widehat{w}$ *satisfying* **(L1)**, **(L5)**, **(L6)** *and* **(L7)***. Moreover,* w *also satisfies*

- **(L8)** *For all* $q \in [-R,R]$, $\int_0^T \int_{\mathbb{R}} \left|\partial^2_{xx} w(t,x,q)\right| \mathrm{d}x\,\mathrm{d}t < +\infty$.
- **(L9)** $\partial_t w, \partial^2_{tr} w, \partial^2_{tx} w \in \mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$.
- **(L10)** $\partial^2_{xx} w, \partial^2_{xr} w \in \mathbf{C^0}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$.

Call r the solution to (3.47), left. Then, for all $t \in [0,T]$, $r(t) \in \mathbf{BV}(\mathbb{R};\mathbb{R})$ and

$$\mathrm{TV}\,(r(t)) \leq \mathrm{TV}(r_o)e^{\kappa_o t} + R\int_0^t e^{\kappa_o(t-\tau)} \int_{\mathbb{R}} \left\|\partial^2_{xx} w(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x\,\mathrm{d}\tau \tag{3.48}$$

where

$$\kappa_o \leq 3\left(\|\partial_x w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} + R\left\|\partial^2_{xr} w\right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\right).$$

Additionally, calling $\widehat{r}$ the solution to (3.47), right, if $w, \widehat{w}$ satisfy

(L11) $\sup_{t\in[0,T]} \|\partial_x w(t,\cdot,\cdot)\|_{\mathbf{L}^1(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} + \sup_{t\in[0,T]} \|\partial_x \widehat{w}(t,\cdot,\cdot)\|_{\mathbf{L}^1(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} < +\infty,$

then

$$\|r(t)-\widehat{r}(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \leq t e^{\kappa_o t}\mathcal{C}\left(\mathrm{TV}(r_o), \int_0^t\int_{\mathbb{R}} \left\|\partial^2_{xx} w(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x\,\mathrm{d}\tau, R\right) \|w-\widehat{w}\|_{\mathcal{W}}. \tag{3.49}$$

In (3.49), we use the set

$$\mathcal{W} := \{w\colon [0,T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R} \text{ satisfying } \mathbf{(L1)}, \mathbf{(L5)}, \mathbf{(L6)}, \mathbf{(L11)}\} \tag{3.50}$$

with norm

$$\|w\|_{\mathcal{W}} := \|w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} + R\,\|\partial_r w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} + \sup_{t\in[0,T]} \|\partial_x w(t,\cdot,\cdot)\|_{\mathbf{L}^1(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} \tag{3.51}$$

where $\|\partial_x w(t,\cdot,\cdot)\|_{\mathbf{L}^1(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} = \int_{\mathbb{R}} \operatorname{ess\,sup}_{q\in\mathbb{R}} |\partial_x w(t,x,q)|\,\mathrm{d}x$.

Proof. Set $f(t,x,r) = r\,w(t,x,r)$ and $g(t,x,\widehat{r}) = \widehat{r}\,\widehat{w}(t,x,\widehat{r})$. We aim at applying [17, Theorem 2.6]. To this aim, we specialize to the present case the key assumptions **(H1)**, **(H2)** and **(H3)** therein, as relaxed in the remark before [17, Theorem 2.3].

(H1) $f, g \in \mathbf{C^1}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$;
$\partial^2_{xr} f,\ \partial^2_{xx} f,\ \partial^2_{x\widehat{r}} g,\ \partial^2_{xx} g \in \mathbf{C^0}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$;
$\partial_x f,\ \partial_r f,\ \partial^2_{xr} f,\ \partial_x g,\ \partial_{\widehat{r}} g,\ \partial^2_{x\widehat{r}} g \in \mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$.

(H2) $f \in \mathbf{C^1}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$;
$\partial^2_{xr} f,\ \partial^2_{tx} f,\ \partial^2_{tr} f \in \mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$;
$\int_0^T \int_{\mathbb{R}} \left\|\partial^2_{xx} f(t,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x\,\mathrm{d}t < +\infty$.

(H3) $f - g \in \mathbf{C^1}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$;
$\partial_r(f-g) \in \mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$;
$\int_0^T \int_{\mathbb{R}} \|\partial_x f(t,x,\cdot) - \partial_x g(t,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x\,\mathrm{d}t < +\infty$.

Requirement **(H1)** holds: f and g are of class $\mathbf{C^1}$ by **(L1)**; the existence and continuity of the second derivatives follow from **(L1)** and **(L10)**; $\partial_r f$ and $\partial_{\widehat{r}} g$ are in $\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$ by **(L1)**, **(L5)** and **(L6)**; $\partial^2_{xr} f$ and $\partial^2_{x\widehat{r}} g$ are in $\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$ due to **(L5)**, **(L6)** and **(L7)**; $\partial_x f$ and $\partial_x g$ are in $\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})$ due to **(L5)** and **(L6)**.

Requirement **(H2)** holds: $\partial^2_{tr} f$ and $\partial^2_{tx} f$ are of class $\mathbf{L}^\infty$ by **(L6)** and **(L9)**; finally

$$\int_0^T \int_{\mathbb{R}} \left\| \partial^2_{xx} f(t,x,\cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x \, \mathrm{d}t \le R \int_0^T \int_{\mathbb{R}} \left\| \partial^2_{xx} w(t,x,\cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x \, \mathrm{d}t < +\infty$$

by **(L6)**, **(L8)**.

Requirement **(H3)** holds: $f-g \in \mathbf{C}^1([0,T] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ by **(L1)** while $\partial_r(f-g)$ is in $\mathbf{L}^\infty([0,T] \times \mathbb{R} \times \mathbb{R}; \mathbb{R})$ due to **(L1)**, **(L5)**, **(L6)**; $\int_0^T \int_{\mathbb{R}} \|\partial_x f(t,x,\cdot) - \partial_x g(t,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x \, \mathrm{d}t < +\infty$ by **(L6)** and **(L11)**. With reference to the notation in [17, Formula (2.4)], introduce

$$\kappa_o := 3 \left\| \partial^2_{xr} f \right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} \le 3 \left(\|\partial_x w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} + R \left\| \partial^2_{xr} w \right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} \right).$$

Since f satisfies **(H1)** and **(H2)**, we apply [17, Theorem 2.5], which reads for all $t \in [0,T]$

$$\begin{aligned} \mathrm{TV}\,(r(t)) &\le \mathrm{TV}(r_o) e^{\kappa_o t} + \int_0^t e^{\kappa_o (t-\tau)} \int_{\mathbb{R}} \left\| \partial^2_{xx} f(\tau,x,\cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x \, \mathrm{d}\tau \\ &\le \mathrm{TV}(r_o) e^{\kappa_o t} + R \int_0^t e^{\kappa_o (t-\tau)} \int_{\mathbb{R}} \left\| \partial^2_{xx} w(\tau,x,\cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x \, \mathrm{d}\tau, \end{aligned}$$

proving (3.48). Recall for later use the estimate

$$e^a - 1 \le a\, e^a \qquad \forall\, a \ge 0\,. \tag{3.52}$$

With reference to the notation in [17, Theorem 2.6], we have

$$\begin{aligned} \kappa &= 2 \left\| \partial^2_{xr} f \right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} \\ \frac{e^{\kappa_o t} - e^{\kappa t}}{\kappa_o - \kappa} &= \frac{e^{\left\| \partial^2_{xr} f \right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} t} - 1}{\left\| \partial^2_{xr} f \right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}} \, e^{2\left\| \partial^2_{xr} f \right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} t} \\ &\le e^{3\left\| \partial^2_{xr} f \right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} t} t && \text{[By (3.52)]} \\ &= e^{\kappa_o t} t. \end{aligned}$$

We can now apply [17, Theorem 2.6] and obtain the estimate

$$\begin{aligned} & \|r(t) - \widehat{r}(t)\|_{\mathbf{L}^1(\mathbb{R};\mathbb{R})} \\ \le\; & \frac{e^{\kappa_o t} - e^{\kappa t}}{\kappa_o - \kappa} \mathrm{TV}(r_o) \|\partial_r (f-g)\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} \\ & + \left(\int_0^t \frac{e^{\kappa_o (t-\tau)} - e^{\kappa (t-\tau)}}{\kappa_o - \kappa} \int_{\mathbb{R}} \left\| \partial^2_{xx} f(\tau,x,\cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x \, \mathrm{d}\tau \right) \|\partial_r (f-g)\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} \\ & + \int_0^t e^{\kappa (t-\tau)} \int_{\mathbb{R}} \|\partial_x (f-g)(\tau,x,\cdot)\|_{\mathbb{R};\mathbb{R})} \mathrm{d}x \, \mathrm{d}\tau \\ \le\; & t\, e^{\kappa_o t}\, \mathrm{TV}(r_o) \left(\|w - \widehat{w}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} + R \|\partial_r w - \partial_r \widehat{w}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} \right) \end{aligned}$$

$$
\begin{aligned}
&+R\,t\,e^{\kappa_o t}\left(\int_0^t\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}\tau\right)\\
&\quad\times\left(\|w-\widehat{w}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}+R\|\partial_r w-\partial_r\widehat{w}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\right)\\
&+R\,e^{\kappa t}\int_0^t\int_{\mathbb{R}}\|\partial_x w(\tau,x,\cdot)-\partial_x\widehat{w}(\tau,x,\cdot)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}\tau\\
\le{}& t\,e^{\kappa_o t}\left(\mathrm{TV}(r_o)+R\left(\int_0^t\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}\tau\right)\right)\|w-\widehat{w}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\\
&+R\,t\,e^{\kappa_o t}\left(\mathrm{TV}(r_o)+R\left(\int_0^t\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}\tau\right)\right)\\
&\quad\times\|\partial_r w-\partial_r\widehat{w}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\\
&+Rt\,e^{\kappa t}\sup_{t\in[0,T]}\|\partial_x w(\tau,\cdot,\cdot)-\partial_x\widehat{w}(\tau,\cdot,\cdot)\|_{\mathbf{L}^1(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\\
\le{}& t\,e^{\kappa_o t}\left(\mathrm{TV}(r_o)+R\left(\int_0^t\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}\tau\right)\right)\\
&\times\left(\|w-\widehat{w}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}+R\,\|\partial_r w-\partial_r\widehat{w}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\right)\\
&+Rt\,e^{\kappa t}\sup_{t\in[0,T]}\|\partial_x w(\tau,\cdot,\cdot)-\partial_x\widehat{w}(\tau,\cdot,\cdot)\|_{\mathbf{L}^1(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\\
\le{}& t\,e^{\kappa_o t}\max\left\{\left(\mathrm{TV}(r_o)+R\left(\int_0^t\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}\tau\right)\right),R\right\}\|w-\widehat{w}\|_{\mathcal{W}}.
\end{aligned}
$$

The proof is completed. □

Remark that a straightforward consequence of Lemma 3.1, Lemma 3.11 and (3.48) is that for all $t\in[0,T]$

$$
\|r(t)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}+\mathrm{TV}\left(r(t)\right)\le\mathcal{C}\begin{pmatrix}\|\partial_x w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})},\left\|\partial^2_{xr}w\right\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})},\\ \int_0^T\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\mathrm{d}x\,\mathrm{d}\tau,R,\mathrm{TV}(r_o),T\end{pmatrix} \tag{3.53}
$$

where R is as in **(L6)**.

Lemma 3.12 *($\mathbf{L^1}$ continuity in time). Assume* **(L1)–(L11)***. Then, the solution r built in Lemma 3.8 and Lemma 3.10 belongs to $\mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$. Actually, r is locally $\mathbf{L^1}$ Lipschitz continuous in time, i.e., calling K the constant in the right hand side of (3.53), for all $t_1,t_2\in[0,T]$*

$$
\begin{aligned}
\|r(t_2)-r(t_1)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\le{}& K\left(\sup_{t\in[0,T]}\|\partial_x w(t,\cdot,\cdot)\|_{\mathbf{L^1}(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))}\right)|t_2-t_1| \\
&+K\left(+K\,\|\partial_r w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}+\|w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\right)|t_2-t_1|.
\end{aligned} \tag{3.54}
$$

Under conditions **(L5)–(L9)** and **(L11)**, the Kružkov solution to (3.42) belongs to the space $\mathbf{C^0}\big([0,T];\mathbf{L^1_{loc}}(\mathbb{R};\mathbb{R})\big)$ by [17, Remark 2.4]. We here prove that we also have the $\mathbf{L^1}$ Lipschitz continuity, locally in time.

Proof of Lemma 3.12. We follow the lines of [18, Theorem 4.3.1]. Since r is a distributional solution to (3.42), for any $t_1, t_2 \in [0,T]$ with $t_1 < t_2$, $\varphi \in \mathbf{C^1}([0,T];\mathbb{R})$ with $\varphi(t) = 1$ for all $t \in [0,t_2]$, $\psi \in \mathbf{C^1_c}(\mathbb{R};\mathbb{R})$ with $\|\psi\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \leq 1$, we have for $i = 1,2$

$$\int_{t_i}^{T} \int_{\mathbb{R}} r(t,x)\, \partial_t\varphi(t)\, \psi(x)\, \mathrm{d}x\, \mathrm{d}t + \int_{t_i}^{T} \int_{\mathbb{R}} r(t,x)\, w\left(t,x,r(t,x)\right)\, \varphi(t)\, \partial_x\psi(x)\, \mathrm{d}x\, \mathrm{d}t + \int_{\mathbb{R}} r(t_i,x)\, \varphi(t_i)\, \psi(x)\, \mathrm{d}x = 0\,.$$

Subtracting these expressions and using (L11), we get

$$\begin{aligned}
&\int_{\mathbb{R}} \left(r(t_2,x) - r(t_1,x)\right) \psi(x)\, \mathrm{d}x \\
&= \int_{t_1}^{t_2} \int_{\mathbb{R}} r(t,x)\, w\left(t,x,r(t,x)\right) \partial_x\psi(x)\, \mathrm{d}x\, \mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \mathrm{TV}\left(r(t,\cdot)\, w\left(t,\cdot,r(t,\cdot)\right)\right) \mathrm{d}t && [3,\text{ Def. } 3.4] \\
&\leq \int_{t_1}^{t_2} \Big(\|r(t)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\, \mathrm{TV}\left(w\left(t,\cdot,r(t,\cdot)\right)\right) + \mathrm{TV}\left(r(t)\right) \|w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\Big)\, \mathrm{d}t && [3,\text{ Ex. } 3.17] \\
&\leq \int_{t_1}^{t_2} K\left(\|\partial_x w(t,\cdot,\cdot)\|_{\mathbf{L^1}(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} + \|\partial_r w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R},\mathbb{R})}\, K\right) \mathrm{d}t && [\text{By } (3.53)] \\
&\quad + \int_{t_1}^{t_2} K\, \|w\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\, \mathrm{d}t\,. && [\text{By } (3.53)]
\end{aligned}$$

Passing to the supremum over all $\psi \in \mathbf{C^1_c}(\mathbb{R};\mathbb{R})$ with $|\psi(x)| \leq 1$ the proof is completed. □

Lemma 3.13. *Assume* **(vL)**. *Suppose*

(V1) $\rho \in \mathbf{C^1}([0,T]\times\mathbb{R};\mathbb{R}^k)$.
(V2) $\rho \in \mathbf{C^0}\left([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)$.
(V3) $\partial^2_{xx}\rho \in (\mathbf{C^0}\cap\mathbf{L^1})\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$.
(V4) $\partial_t\rho, \partial^2_{tx}\rho \in \mathbf{L}^\infty\left([0,T]\times\mathbb{R};\mathbb{R}^k\right)$.

Then, the map $w(t,x,r) \coloneqq v_L\left(\Sigma\rho(t,x) + r\right)$ *satisfies all the requirements* **(L1)** *to* **(L11)** *where the constant* R *in* **(L6)** *can be chosen bigger than* $R_L + \|\rho\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R}^k)}$.

Proof. We group the different requirements as follows.

(L1) and (L3): Compute

$$\begin{aligned}\partial_t w(t,x,r) &= v_L'\left(\Sigma\rho(t,x)+r\right)\ \Sigma\partial_t\rho(t,x)\\ \partial_x w(t,x,r) &= v_L'\left(\Sigma\rho(t,x)+r\right)\ \Sigma\partial_x\rho(t,x)\end{aligned} \quad \text{and} \quad \partial_r w(t,x,r) = v_L'\left(\Sigma\rho(t,x)+r\right),$$

so that **(vL)** and **(V1)** imply **(L1)**, while **(vL)** implies **(L3)**.

(L2): Direct computations show that $\partial^2_{tr} w$ and $\partial^2_{xr} w$ involve only first and second derivatives of v_L and first derivatives of ρ. Hence, **(vL)** and **(V1)** ensure that $\partial^2_{tr} w$ and $\partial^2_{xr} w$ are locally bounded.

(L4): Assumptions **(vL)**, **(V1)** and **(V3)** clearly imply **(L4)**.

(L5), (L7), and (L9): Note that as $\partial_x\rho \in \mathbf{L^1}\left([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)$ by **(V2)** and **(V3)**, **(V4)** implies that $\partial_x\rho \in \mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R}^k)$. Then, Assumptions **(vL)** and **(V4)** imply that $\partial_t w$, $\partial_x w$, $\partial_r w$, $\partial^2_{xr} w$, $\partial^2_{tr} w$, and $\partial^2_{tx} w$ belong to $\mathbf{L}^\infty\left([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R}\right)$.

(L6): By **(V2)** define $M = \|\rho\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R}^k)}$. If $r \geq R_L + M$, then $r + \Sigma\rho(t,x) \geq r - M \geq R_L$ and so, by **(vL)**, $w(t,x,r) = v_L\left(\Sigma\rho(t,x)+r\right) = 0$. If $r \leq -M$, then $r + \Sigma\rho(t,x) \leq r + M \leq 0$ and so, by **(vL)**, $w(t,x,r) = v_L\left(\Sigma\rho(t,x)+r\right) = W$. Then **(L6)** holds with $R = R_L + M$.

(L8): Assumptions **(vL)** and **(V3)** allow to compute

$$\partial^2_{xx} w(t,x,r) = v_L''\left(\Sigma\rho(t,x)+r\right)\left(\Sigma\partial_x\rho(t,x)\right)^2 + v_L'\left(\Sigma\rho(t,x)+r\right)\ \Sigma\partial^2_{xx}\rho(t,x).$$

Moreover, for every $r \in \mathbb{R}$, we have

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}} \left|\partial^2_{xx} w(t,x,r)\right| \mathrm{d}x\,\mathrm{d}t \qquad (3.55)\\ \leq&\|v_L''\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\int_0^T\int_{\mathbb{R}}\left|\Sigma\partial_x\rho(t,x)\right|^2\mathrm{d}x\,\mathrm{d}t + \|v_L'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\int_0^T\int_{\mathbb{R}}\left|\Sigma\partial^2_{xx}\rho(t,x)\right|\mathrm{d}x\,\mathrm{d}t\\ \leq&\|v_L''\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\partial_x\rho\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R}^k)}\|\partial_x\rho\|_{\mathbf{L^1}([0,T]\times\mathbb{R};\mathbb{R}^k)} + \|v_L'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\left\|\partial^2_{xx}\rho\right\|_{\mathbf{L^1}([0,T]\times\mathbb{R};\mathbb{R}^k)},\end{aligned}$$

proving **(L8)**.

(L10): Clearly, Assumptions **(vL)**, **(V1)**, and **(V3)** imply that the functions $\partial^2_{xx} w$ and $\partial^2_{xr} w$ belong to $\mathbf{C^0}\left([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R}\right)$, proving **(L10)**.

(L11): Assumptions **(vL)** and **(V2)** imply that, for every $M > 0$ and $t \in [0,T]$, it holds that

$$\begin{aligned}\sup_{t\in[0,T]}\|\partial_x w(t,\cdot,\cdot)\|_{\mathbf{L^1}(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} &= \sup_{t\in[0,T]}\int_{\mathbb{R}}\operatorname*{ess\,sup}_{q\in\mathbb{R}}\left|v_L'\left(\Sigma\rho(t,x)+q\right)\ \Sigma\partial_x\rho(t,x)\right|\mathrm{d}x\\ &\leq \|v_L'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\partial_x\rho\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}^k))} < +\infty,\end{aligned}$$

proving **(L11)**. □

Lemma 3.14. *Let* **(vL)** *hold. Moreover, suppose $r_o, \widehat{r}_o \in \mathcal{BV}(\mathbb{R};\mathbb{R})$ and $\rho, \widehat{\rho}$ satisfy* **(V1)**–**(V4)**. *Call $r, \widehat{r}$ the Kružkov solutions to*

$$\begin{cases}\partial_t r + \partial_x\left(r\, v_L\left(\Sigma\rho(t,x)+r\right)\right) = 0\\ r(0,x) = r_o(x)\end{cases} \quad \text{and} \quad \begin{cases}\partial_t \widehat{r} + \partial_x\left(\widehat{r}\, v_L\left(\Sigma\widehat{\rho}(t,x)+\widehat{r}\right)\right) = 0\\ \widehat{r}(0,x) = \widehat{r}_o(x).\end{cases} \qquad (3.56)$$

Then, for all $t \in [0,T]$

$$\begin{aligned}&\left\|r(t)-\widehat{r}(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\ \leq&\left\|r_o-\widehat{r}_o\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\ &+\mathcal{C}\left(\begin{array}{c}R_L,\|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})},\|\rho\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)},\\ \|\widehat{\rho}\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)},\|\partial_x\rho\|_{\mathbf{L^1}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))},\mathrm{TV}(\widehat{r}_o),t\end{array}\right)t\left\|\rho-\widehat{\rho}\right\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}.\end{aligned}$$

Proof. Call $w(t,x,q):=v_L\left(\Sigma\rho(t,x)+q\right)$, $\widehat{w}(t,x,q):=v_L\left(\Sigma\widehat{\rho}(t,x)+q\right)$ and note that, since **(vL)** holds and $\rho,\widehat{\rho}$ satisfy **(V1)**–**(V4)**, by Lemma 3.13, $w,\widehat{w}$ satisfy **(L1)**–**(L11)** and $w,\widehat{w}\in\mathcal{W}$ where $\mathcal{W}$ is defined in (3.50), with

$$R:=R_L+\max\left\{\|\rho\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R}^k)},\|\widehat{\rho}\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R}^k)}\right\}. \tag{3.57}$$

Hence, we may apply Lemma 3.8, Lemma 3.10 and Lemma 3.11 to problems (3.56). In particular, (3.44) and (3.49) yields that for all $t\in[0,T]$

$$\begin{aligned}\left\|r(t)-\widehat{r}(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\leq&\left\|r_o-\widehat{r}_o\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\ &+t\,e^{\kappa_o t}\,\mathcal{C}\left(\mathrm{TV}(\widehat{r}_o),\int_0^t\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\mathrm{d}x\,\mathrm{d}\tau,R\right)\|w-\widehat{w}\|_{\mathcal{W}}\end{aligned}$$

where

$$\begin{aligned}\kappa_o&\leq 3\left(\|\partial_x w\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}+R\left\|\partial^2_{xr}w\right\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\right)\\ &\leq 3\left(\|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}+R\,\|v_L''\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\right)\|\partial_x\rho\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R}^k)}.\end{aligned}$$

Furthermore, owing to (3.55),

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\mathrm{d}x\,\mathrm{d}\tau\\ \leq&\|v_L''\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|\partial_x\rho\|_{\mathbf{L^\infty}([0,t]\times\mathbb{R};\mathbb{R}^k)}\|\partial_x\rho\|_{\mathbf{L^1}([0,t]\times\mathbb{R};\mathbb{R}^k)}+\|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\left\|\partial^2_{xx}\rho\right\|_{\mathbf{L^1}([0,t]\times\mathbb{R};\mathbb{R}^k)}.\end{aligned} \tag{3.58}$$

We proceed now to estimate the term $\|w-\widehat{w}\|_{\mathcal{W}}$ by evaluating separately each of the summands in (3.51). It holds that

$$\begin{aligned}\|w-\widehat{w}\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}&=\sup_{t\in[0,T],x\in\mathbb{R},q\in\mathbb{R}}\left|v_L\left(\Sigma\rho(t,x)+q\right)-v_L\left(\Sigma\widehat{\rho}(t,x)+q\right)\right|\\ &\leq\|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\,\|\rho-\widehat{\rho}\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R}^k)}\\ &\leq\|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\,\|\rho-\widehat{\rho}\|_{\mathbf{C^0}([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}.\end{aligned}$$

Similarly, direct computations lead to

$$\begin{aligned}\|\partial_r w-\partial_r\widehat{w}\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}&=\sup_{t\in[0,T],x\in\mathbb{R},q\in\mathbb{R}}\left|v_L'\left(\Sigma\rho(t,x)+q\right)-v_L'\left(\Sigma\widehat{\rho}(t,x)+q\right)\right|\\ &\leq\|v_L''\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\|\rho-\widehat{\rho}\|_{\mathbf{C^0}([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}.\end{aligned}$$

At last, we get for all $t\in[0,T]$

$$
\begin{aligned}
&\|\partial_x w(t,\cdot,\cdot) - \partial_x \widehat{w}(t,\cdot,\cdot)\|_{\mathbf{L^1}(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} \\
= &\int_{\mathbb{R}} \sup_{q\in\mathbb{R}} \left| v_L'(\Sigma\rho(t,x)+q)\,\Sigma\partial_x\rho(t,x) - v_L'(\Sigma\widehat{\rho}(t,x)+q)\,\Sigma\partial_x\widehat{\rho}(t,x)\right| \mathrm{d}x \\
\le &\int_{\mathbb{R}} \sup_{q\in\mathbb{R}} \left| v_L'(\Sigma\rho(t,x)+q)\Sigma\left(\partial_x\rho(t,x) - \partial_x\widehat{\rho}(t,x)\right)\right| \mathrm{d}x \\
&+ \int_{\mathbb{R}} \sup_{q\in\mathbb{R}} \left| \left(v_L'(\Sigma\rho(t,x)+q) - v_L'(\Sigma\widehat{\rho}(t,x)+q)\right)\Sigma\partial_x\widehat{\rho}(t,x)\right| \mathrm{d}x \\
\le &\|v_L'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\partial_x\rho(t)-\partial_x\widehat{\rho}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \\
&+\|v_L''\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\partial_x\widehat{\rho}(t)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)}\,\|\rho(t)-\widehat{\rho}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\sup_{t\in[0,T]} \|\partial_x w(t,\cdot,\cdot) - \partial_x\widehat{w}(t,\cdot,\cdot)\|_{\mathbf{L^1}(\mathbb{R};\mathbf{L}^\infty(\mathbb{R};\mathbb{R}))} \\
\le &\left(\|v_L'\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} + \|v_L''\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}\|\partial_x\widehat{\rho}\|_{\mathbf{L}^\infty([0,T]\times\mathbb{R};\mathbb{R}^k)}\right)\|\rho-\widehat{\rho}\|_{\mathbf{C^0}([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}
\end{aligned}
$$

and, for all $t>0$, collecting the previous computations we get

$$
\begin{aligned}
&\|r(t)-\widehat{r}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\
\le &\|r_o-\widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\
&+\mathcal{C}\left(\begin{array}{c} R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \|\rho\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))}, \\ \|\widehat{\rho}\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))}, \|\partial_x\rho\|_{\mathbf{L^1}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}, \mathrm{TV}(\widehat{r}_o), t\end{array}\right)\|\rho-\widehat{\rho}\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}t.
\end{aligned}
$$

Note that as $\rho,\widehat{\rho}$ satisfy **(V1)**–**(V4)**, then $\rho,\widehat{\rho}$ belong to $\mathbf{C^0}\left([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)\right)$ and $\partial_x\rho\in\mathbf{L^1}\left([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)$. Indeed, $\rho\in\mathbf{C^0}\left([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)\right)$ due to **(V2)** and Sobolev embedding $\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\hookrightarrow\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)$, see [9, Theorem 8.8]. Moreover $\partial_x\rho\in\mathbf{C^0}\left([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)\right)$: first, $\partial_x\rho\in\mathbf{L^1}\left([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)\hookrightarrow\mathbf{L^1}\left([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)\right)$ owing to **(V2)** and **(V3)**; second, $\partial^2_{tx}\rho\in\mathbf{L}^\infty([0,t]\times\mathbb{R};\mathbb{R}^k)$ due to **(V4)**. Then, as $\partial_x\rho\in\mathbf{W^{1,1}}\left([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)\right)$, by [21, Theorem 2, Section 5.9] we get $\partial_x\rho\in\mathbf{C^0}\left([0,t];\mathbf{L}^\infty(\mathbb{R};\mathbb{R}^k)\right)$. The case of $\widehat{\rho}$ is analogous. □

3.4. Proof of Theorem 2.3

Fix a positive constant M. We consider first the case of an initial datum (ρ_o,r_o) with

$$
\begin{array}{c}
\rho_o\in\left(\mathbf{C^2}\cap\mathbf{W^{2,\infty}}\cap\mathbf{W^{2,1}}\right)(\mathbb{R};\mathbb{R}^k) \quad r_o\in\mathcal{BV}(\mathbb{R};\mathbb{R}) \\
\|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}+\|r_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})}\le M.
\end{array}
\tag{3.59}
$$

Under this condition, the proof is achieved by means of a sequence of approximate solutions (ρ_n,r_n) iteratively defined as follows.

$$
\begin{cases}
\partial_t\rho_1^i+\partial_x\left(\rho_1^i\,v_{NL}^i\left(\Sigma\rho_1*\eta^i+(r_o*\eta^i)(x)\right)\right)=0 & i=1,\ldots,k\\
\rho_1(0,x)=\rho_o(x)
\end{cases}
\tag{3.60}
$$

and

$$\begin{cases} \partial_t r_1 + \partial_x \left(r_1 \, v_L \left(\Sigma \rho_1(t,x) + r_1 \right) \right) = 0 \\ r_1(0,x) = r_o(x) \,. \end{cases} \tag{3.61}$$

For $n \geq 2$, define recursively the functions ρ_n and r_n as solutions to the problems

$$\begin{cases} \partial_t \rho_n^i + \partial_x \left(\rho_n^i \, v_{NL}^i \left(\Sigma \rho_n * \eta^i + (r_{n-1}(t) * \eta^i)(x) \right) \right) = 0 & i = 1, \ldots, k \\ \rho_n(0,x) = \rho_o(x) \end{cases} \tag{3.62}$$

and

$$\begin{cases} \partial_t r_n + \partial_x \left(r_n \, v_L \left(\Sigma \rho_n(t,x) + r_n \right) \right) = 0 \\ r_n(0,x) = r_o(x) \,. \end{cases} \tag{3.63}$$

Step 1. The functions ρ_n and r_n defined in (3.62) and (3.63) are well defined with $\rho_n \in \mathbf{C^0}\big([0,T]; \mathbf{W^{1,1}}(\mathbb{R}; \mathbb{R}^k)\big)$ and $r_n \in \mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$.

The inductive procedure on which the proof of the present step is based follows this scheme:

$n = 1$	**(vNL)** **(η)** (3.59)	$\Longrightarrow$	ρ_1 satisfies **(N1)**–**(N6)**
$n \geq 1$	ρ_n satisfies **(N1)**–**(N6)** and **(vL)**	$\Longrightarrow$	**(V1)**–**(V4)**
	(vL) and **(V1)**–**(V4)**	$\Longrightarrow$	**(L1)**–**(L11)**
	(L1)–**(L11)** and (3.59)	$\Longrightarrow$	$r_n \in \mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$
	(vNL) **(η)** (3.59) and $r_n \in \mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$	$\Longrightarrow$	ρ_{n+1} satisfies **(N1)**–**(N6)**

Fix an arbitrary $T > 0$. Under Assumptions **(vNL)**, **(η)** and with (ρ_o, r_o) as in (3.59), problem (3.60) fits into Lemma 3.4 and Lemma 3.5. Hence $\rho_1 \in \mathbf{C^0}\left([0,T]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)$ is well defined and ρ_1 satisfies **(V1)**–**(V4)** of Lemma 3.13.

Similarly, Assumption **(vL)**, (3.59) and the Properties **(V1)**–**(V4)** of ρ_1 ensure that Lemma 3.8 and Lemma 3.10 apply, so that r_1 is well defined and $r_1 \in \mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$.

Induction process. Fix $n \in \mathbb{N}, n \geq 2$ and suppose that $r_{n-1} \in \mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$. Then, under Assumptions **(vNL)**, **(η)** and (3.59), $\rho_n \in \mathbf{C^0}\left([0,T]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)$, solution to (3.62) is well defined by Lemma 3.4 and Lemma 3.5. Furthermore, ρ_n satisfies **(V1)**–**(V4)** of Lemma 3.13 by assumptions on r_{n-1}. Finally, Assumption **(vL)**, (3.59) and the Properties **(V1)**–**(V4)** of ρ_n ensure that Lemma 3.8 and Lemma 3.10 apply, so that r_n is well defined and $r_n \in \mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$.

Step 2. There exists a constant $\mathcal{C}_4 := \mathcal{C}(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R},\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R},\mathbb{R}^k)}, M, T)$ such that $\|\rho_n(t) - \rho_{n-1}(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \leq \mathcal{C}_4 \, \|r_{n-1} - r_{n-2}\|_{\mathbf{C^0}([0,t];\mathbf{L^1}(\mathbb{R};\mathbb{R}))} \, t$ for all $t \in [0,T]$.

Fix $n \in \mathbb{N} \setminus \{0,1\}$. We have $r_{n-2}, r_{n-1} \in \mathbf{C^0}\left([0,T]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$ by **Step 1**. Define $\mathcal{R} := \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}$. Lemma 3.8 ensures that $\max\{\|r_{n-2}\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}))}, \|r_{n-1}\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}))}\} \leq \mathcal{R}$. Apply Lemma 3.7 with $r := r_{n-2}$, $\widehat{r} := r_{n-1}$ obtaining that, for all $t \in [0,T]$,

$$\begin{aligned} & \|\rho_n(t) - \rho_{n-1}(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \\ \leq \; & \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, t \right) \\ & \times \int_0^t \|r_{n-1}(\tau) - r_{n-2}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \, \mathrm{d}\tau, \end{aligned}$$

concluding the step.

Step 3. There exists $\mathcal{C}_5 := \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v'_L\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, M, T\right)$ **such that** $\|r_n(t) - r_{n-1}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \leq \mathcal{C}_5 \|r_{n-1} - r_{n-2}\|_{\mathbf{C^0}([0,t];\mathbf{L^1}(\mathbb{R};\mathbb{R}))} t^2$ **for** $t \in [0,T]$**.**

Fix $n \in \mathbb{N} \setminus \{0,1\}$ and let ρ_{n-1}, ρ_n be as defined in **Step 1**. Then, as they satisfy **(V1)–(V4)**, we may apply Lemma 3.14, which reads for all $t \in [0,T]$

$$\begin{aligned}&\|r_n(t) - r_{n-1}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\ \leq&\mathcal{C}\begin{pmatrix} R_L, \|v'_L\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \|\rho_n\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, \\ \|\rho_{n-1}\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\partial_x\rho_n\|_{\mathbf{L^1}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}, \mathrm{TV}(r_o)\end{pmatrix} t\|\rho_n - \rho_{n-1}\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}.\end{aligned}$$

Recall now that $\rho_n = (\rho_n^1, \ldots, \rho_n^k)$ is such that for $i = 1, \ldots, k$, ρ_n^i solves

$$\begin{cases}\partial_t\rho_n^i + \partial_x\left(\rho_n^i V^i(t,x,\rho_n * \eta)\right) = 0\\ \rho_n^i(0,x) = \rho_o^i(x)\end{cases} \quad \text{for } V^i(t,x,q) := v_{NL}^i\left(\Sigma q + (r_{n-1}(t) * \eta^i)(x)\right), \tag{3.64}$$

with $V \in \mathbf{C^0}([0,T];\mathcal{V})$. Indeed,

$$\begin{aligned}\|V\|_{\mathbf{C^0}([0,T];\mathcal{V})} \leq&\mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|r_{n-1}\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}))}\right) && \text{[By Lemma 3.6]}\\ \leq&\mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\right). && \text{[By Lemma 3.8]}\end{aligned}$$

The latter bound, due to **(N2)**, **(N4)** and **(N6)** in Lemma 3.5, implies that

$$\begin{aligned}&\|\rho_n\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))}\\ \leq&\ \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, t\right) \qquad (3.65)\\ &\|\partial_x\rho_n\|_{\mathbf{L^1}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)}\\ \leq&\ \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, t\right). \qquad (3.66)\end{aligned}$$

Identical bounds are valid for ρ_{n-1}, too. Then, **Step 2** allows to conclude that

$$\begin{aligned}&\|r_n(t) - r_{n-1}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\ \leq&\mathcal{C}\begin{pmatrix}\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v'_L\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \\ \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})}, t\end{pmatrix} \mathcal{C}_4 \|r_{n-1} - r_{n-2}\|_{\mathbf{C^0}([0,t];\mathbf{L^1}(\mathbb{R};\mathbb{R}))} t^2,\end{aligned}$$

proving the claim of the present step.

Step 4. *A priori* estimates on (ρ, r) **in** $\mathbf{C^0}\left([0,T];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$**, solution to** (1.2) **in the sense of Definition 2.1.**

Once called $V^i(t,x,q) := v_{NL}^i\left(\Sigma q + \left(r(t) * \eta^i\right)(x)\right)$ and $w(t,x,r) := v_L\left(\Sigma\rho(t,x) + r\right)$, by Definition 2.1, ρ and r are solutions to (3.7) and (3.42) respectively. We claim that the function w satisfies the requirements **(L1)–(L11)**. Indeed, the following scheme holds:

$$\begin{array}{rcll} r\in \mathbf{C^0}\left([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R})\right),\ \textbf{(vNL)},\ \boldsymbol{(\eta)} & \Rightarrow & V\in\mathbf{C^0}([0,T];\mathcal{V}) & \text{by Lemma 3.6}\\ V\in\mathbf{C^0}([0,T];\mathcal{V}),\ (3.59),\ \boldsymbol{(\eta)} & \Rightarrow & \textbf{(N1)–(N6)}\text{ hold} & \text{by Lemma 3.5}\\ \textbf{(N1)–(N6)}\ ,\ \textbf{(vL)} & \Rightarrow & \textbf{(V1)–(V4)} & \\ \textbf{(vL)}\text{ and }\textbf{(V1)–(V4)} & \Rightarrow & w\text{ satisfies }\textbf{(L1)–(L11)} & \text{by Lemma 3.13.} \end{array}$$

Because of the fact w meets the requirements **(L1)–(L11)**, we apply Lemma 3.8: as $r \in \mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}))$ and the function $\widehat{r} \equiv 0$ are both Kružkov solutions to (3.42), then

$$\|r(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \le \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}. \tag{3.67}$$

By (3.26) in Lemma 3.6 and (3.67), one gets that

$$\begin{aligned} \|V\|_{\mathbf{C^0}([0,T];\mathcal{V})} &\le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|r\|_{\mathbf{C^0}([0,T];\mathbf{L^1}(\mathbb{R};\mathbb{R}))}\right)\\ &\le \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\right). \end{aligned} \tag{3.68}$$

Moreover by Lemma 3.5 we obtain the following *a priori* estimates:

$$\begin{array}{rcll} \|\rho(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} &=& \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} & [\text{By } \mathbf{5.} \text{ in Lemma 3.4}]\\ \|\partial_x\rho(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} &\le& \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|V\|_{\mathbf{C^0}([0,T];\mathcal{V})}, \|\rho_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)}, t\right) & [\text{By } \textbf{(N2)}]\\ \left\|\partial^2_{xx}\rho(t)\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} &\le& \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|V\|_{\mathbf{C^0}([0,T];\mathcal{V})}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, t\right) & [\text{By } \textbf{(N2)}]\\ \|\rho(t)\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R}^k)} &\le& \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|V\|_{\mathbf{C^0}([0,T];\mathcal{V})}, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R}^k)}, t\right) & [\text{By } \textbf{(N4)}]\\ \|\partial_x\rho(t)\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R}^k)} &\le& \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|V\|_{\mathbf{C^0}([0,T];\mathcal{V})}, \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, t\right) & [\text{By } \textbf{(N6)}] \end{array}$$

Hence, there exists a positive constant

$$\mathcal{C}_6 := \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, T\right) \tag{3.69}$$

such that

$$\|\rho(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} = \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \qquad \text{and} \qquad \|\rho(t)\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}_6\,. \tag{3.70}$$

We are now ready to prove the estimate on the total variation of r taking advantage of (3.48) in Lemma 3.11, with $R := R_L + \|\rho\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R};\mathbb{R}^k)}$. For all $t \in [0,T]$, $r(t)$ is in $\mathbf{BV}(\mathbb{R};\mathbb{R})$ and

$$\mathrm{TV}\left(r(t)\right) \le \mathrm{TV}(r_o)e^{\kappa_o t} + R\int_0^t e^{\kappa_o(t-\tau)}\int_{\mathbb{R}}\left\|\partial^2_{xx}w(\tau,x,\cdot)\right\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\,\mathrm{d}x\,\mathrm{d}\tau$$

where

$$\kappa_o \le 3\left(\|\partial_x w\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} + R\left\|\partial^2_{xr}w\right\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})}\right).$$

Computations analogous to the ones in the proof to Lemma 3.13 and the use of (3.70) yield

$$\kappa_o \le \mathcal{C}\left(\begin{array}{c}\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v'_L\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})},\\ \|\rho_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, T\end{array}\right)$$

and

$$\int_0^t \int_{\mathbb{R}} \left\| \partial^2_{xx} w(\tau, x, \cdot) \right\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \mathrm{d}x \, \mathrm{d}\tau$$
$$\leq \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, t \right).$$

So, one gets that for all $t \in [0,T]$

$$\mathrm{TV}\,(r(t)) \leq \mathcal{C}\left(\begin{array}{c} \|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \\ \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})}, T \end{array} \right)$$

and, thanks to Lemma 3.1, $\|r(t)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})}$ is bounded similarly. Finally, there exists a positive

$$\mathcal{C}_7 := \mathcal{C}\left(\begin{array}{c} \|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \\ \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})}, T \end{array} \right)$$

such that for all $t \in [0,T]$ it holds

$$\|r(t)\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})} + \|r(t)\|_{\mathbf{L}^\infty(\mathbb{R};\mathbb{R})} \leq \mathcal{C}_7 \,. \tag{3.71}$$

Step 5. There exists $T^* > 0$ such that the sequence $\{(\rho_n, r_n)\}$ constructed in Step 1, restricted to the time interval $[0,T^*]$, converges to a limit point (ρ, r) in $\mathbf{C^0}([0,T^*]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R}))$.

Let $\mathcal{C}_6$ and $\mathcal{C}_7$ be as in **Step 4**, so that (3.70) and (3.71) hold. Use $\mathcal{C}_4$ from **Step 2**, $\mathcal{C}_5$ from **Step 3** and recall that $\mathcal{C}(\cdot)$ is non decreasing in each argument. Then, there exists $\bar{\mathcal{C}}_5$ such that $\mathcal{C}_5 \leq \bar{\mathcal{C}}_5$, where

$$\begin{aligned} \mathcal{C}_5 &= \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, M, T \right) \qquad \text{and} \\ \bar{\mathcal{C}}_5 &:= \mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \mathcal{C}_6 + \mathcal{C}_7, T \right) \end{aligned}$$

so that, according to **Step 2** and **Step 3**, for all $t \in [0,T]$ and $n \in \mathbb{N} \setminus \{0,1\}$,

$$\begin{aligned} \|\rho_n(t) - \rho_{n-1}(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} &\leq \mathcal{C}_4 \, t \, \|r_{n-1} - r_{n-2}\|_{\mathbf{C^0}([0,T^*];\mathbf{L^1}(\mathbb{R};\mathbb{R}))}, \\ \|r_n(t) - r_{n-1}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\leq \bar{\mathcal{C}}_5 \, t^2 \, \|r_{n-1} - r_{n-2}\|_{\mathbf{C^0}([0,T^*];\mathbf{L^1}(\mathbb{R};\mathbb{R}))}. \end{aligned} \tag{3.72}$$

The choice $T^* := \min\left\{ 1/\sqrt{2\bar{\mathcal{C}}_5}, T \right\}$ implies that r_n is a Cauchy sequence in the space $\mathbf{C^0}\left([0,T^*]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$, hence it converges to a limit $r \in \mathbf{C^0}\left([0,T^*]; \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$. Then, (3.72) yields that ρ_n is a Cauchy sequence in $\mathbf{C^0}\left([0,T^*]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)$ and converges to $\rho \in \mathbf{C^0}\left([0,T^*]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)\right)$, completing this Step.

Step 6. The limit function $(\rho, r) \in \mathbf{C^0}\left([0,T^*]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$ solves (1.2) in the sense of Definition 2.1.

Apply Proposition 2.2 with $\rho_{o,n} = \rho_o$, $r_{o,n} = r_o$, $e_n = r_{n-1} - r_n$ and $\varepsilon_n = 0$.

Step 7. Existence of a global solution $(\rho, r) \in \mathbf{C^0}\left([0,T]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$.

Let T^* be as in the proof of **Step 5** and $(\rho, r) \in \mathbf{C^0}\left([0,T^*]; \mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k) \times \mathbf{L^1}(\mathbb{R};\mathbb{R})\right)$ be the solution obtained as limit of the Cauchy sequence $\{(\rho_n, r_n)\}$. We may apply Lemma 3.5, which ensures that $\rho(T^*) \in$

$(\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k)$ owing to **(N1)–(N5)** and the *a priori* estimate $\|\rho(T^*)\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)} \leq \mathcal{C}_6$ holds by **Step 4**. Furthermore, Lemma 3.8 and Lemma 3.11 yield that $r(T^*) \in \mathcal{BV}(\mathbb{R};\mathbb{R})$ and $\|r(T^*)\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})} \leq \mathcal{C}_7$. Hence, we can iterate the same construction in order to extend (ρ, r) to the time intervals $[T^*, 2T^*]$, $[2T^*, 3T^*]$, ... until we reach the final time T. Clearly, the extended (ρ, r) solves (1.2) on the whole $[0, T]$ in the sense of Definition 2.1.

Step 8. Locally Lipschitz dependence on the initial datum.

Fix $(\widehat{\rho}_o, \widehat{r}_o) \in (\mathbf{C^2} \cap \mathbf{W^{2,1}} \cap \mathbf{W^{2,\infty}})(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R})$. Consider the problems

$$\begin{cases} \partial_t \rho^i + \partial_x \left(\rho^i \, v_{NL}^i \left((\Sigma\rho + r) * \eta^i\right)\right) = 0 \\ \partial_t r + \partial_x \left(r \, v_L \left(\Sigma\rho + r\right)\right) = 0 \\ \rho(0) = \rho_o \\ r(0) = r_o \end{cases} \qquad \begin{cases} \partial_t \widehat{\rho}^i + \partial_x \left(\widehat{\rho}^i \, v_{NL}^i \left((\Sigma\widehat{\rho} + \widehat{r}) * \eta^i\right)\right) = 0 \\ \partial_t \widehat{r} + \partial_x \left(\widehat{r} \, v_L \left(\Sigma\widehat{\rho} + \widehat{r}\right)\right) = 0 \\ \widehat{\rho}(0) = \widehat{\rho}_o \\ \widehat{r}(0) = \widehat{r}_o \,. \end{cases}$$

To establish the distance between the solutions ρ and $\widehat{\rho}$ to the nonlocal problems, it is sufficient to apply (3.29) in Lemma 3.7. Indeed, call $\mathcal{R} := \max\left\{\|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, \|\widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\right\}$. By (3.29)–(3.30), for all $t \in [0, T]$

$$\|\rho(t) - \widehat{\rho}(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \leq (1 + \mathcal{C}_3\, t) \|\rho_o - \widehat{\rho}_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \mathcal{C}_3 \int_0^t \|r(\tau) - \widehat{r}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \, \mathrm{d}\tau. \tag{3.73}$$

On the other hand, in order to get the stability related to the local problem we take advantage of Lemma 3.14, which yields that for all $t \in [0, T]$

$$\begin{aligned} & \|r(t) - \widehat{r}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\ \leq & \|r_o - \widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \\ & + \mathcal{C}\begin{pmatrix} R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \|\rho\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))}, \\ \|\widehat{\rho}\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))}, \|\partial_x \rho\|_{\mathbf{L^1}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}, \mathrm{TV}(\widehat{r}_o) \end{pmatrix} t \, \|\rho - \widehat{\rho}\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}. \end{aligned}$$

Owing to **Step 4**, there exists a constant

$$\mathcal{C}_8 := \mathcal{C}\begin{pmatrix} \|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \\ \|\widehat{\rho}_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, \|\widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, t \end{pmatrix}$$

such that for all $t \in [0, T]$ one has

$$\|\rho\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))} + \|\partial_x \rho\|_{\mathbf{L^1}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))} + \|\widehat{\rho}\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))} \leq \mathcal{C}_8 \,.$$

So, collecting the previous estimates and (3.73), one obtains that there exists a constant

$$\mathcal{C}_9 := \mathcal{C}\begin{pmatrix} \|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \\ \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, \|\widehat{r}_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})}, t \end{pmatrix}$$

such that

$$\begin{aligned}
&\|r(t)-\widehat{r}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
&\le \|r_o-\widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} + \mathcal{C}_9\, t \left(\|\rho_o-\widehat{\rho}_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \int_0^t \|r(\tau)-\widehat{r}(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\,\mathrm{d}\tau \right).
\end{aligned}$$

Gronwall Lemma allows to conclude that for all $t \in [0,T]$

$$\|r(t)-\widehat{r}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \le \left(\|r_o-\widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} + \mathcal{C}_9\, t\, \|\rho_o-\widehat{\rho}_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \right) e^{\mathcal{C}_9 t^2}$$

and, completing (3.73),

$$\begin{aligned}
\|\rho(t)-\widehat{\rho}(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \le{}& (1+\mathcal{C}_3 t)\|\rho_o-\widehat{\rho}_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)}\\
&+\mathcal{C}_3 \left(\|r_o-\widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} t + \mathcal{C}_9 t^2 \|\rho_o-\widehat{\rho}_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \right) e^{\mathcal{C}_9 t^2}.
\end{aligned}$$

In conclusion, for all $t \in [0,T]$ there exists a constant

$$\mathcal{C}_{10} := \mathcal{C}\left(\begin{array}{c} \|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})},\\ \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|\widehat{\rho}_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, \|\widehat{r}_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})}, t \end{array} \right) \tag{3.74}$$

such that

$$\begin{aligned}
&\|\rho(t)-\widehat{\rho}(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \|r(t)-\widehat{r}(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
&\le (1+\mathcal{C}_{10}\, t) \left(\|\rho_o-\widehat{\rho}_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \|r_o-\widehat{r}_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \right).
\end{aligned} \tag{3.75}$$

Step 9. Local Lipschitz continuity in time.

Apply **(N3)** in Lemma 3.5 to obtain that for all $t, \widehat{t} \in [0,T]$,

$$\begin{aligned}
&\left\|\rho(t)-\rho(\widehat{t})\right\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)}\\
\le{}&\mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|V\|_{\mathbf{C^0}([0,T];\mathcal{V})}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, T \right) \left|t-\widehat{t}\right| && \text{[Use (3.68)]}\\
\le{}&\mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}, T \right) \left|t-\widehat{t}\right|.
\end{aligned} \tag{3.76}$$

Moreover, as we aim at applying Lemma 3.12, call $w(t,x,r) := v_L\left(\Sigma\rho(t,x)+r\right)$, so that by the proof to Lemma 3.13 one gets

$$\begin{aligned}
\sup_{q\in[-K,K]} \mathrm{TV}\left(w(t,\cdot,q)\right) &\le \|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} \|\partial_x \rho(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)}\\
\|\partial_r w\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} &\le \|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\\
\|w\|_{\mathbf{L^\infty}([0,T]\times\mathbb{R}\times\mathbb{R};\mathbb{R})} &\le V_L + \|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} R_L.
\end{aligned}$$

Hence, as **Step 4** implies $K = \mathcal{C}_7$ and $\|\partial_x\rho(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R}^k)} \le \mathcal{C}_6$, again by **Step 4**, one obtains

$$\begin{aligned}
&\left\|r(t)-r(\widehat{t})\right\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\\
&\le \mathcal{C}_7 \left(\|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})}\mathcal{C}_6 + \mathcal{C}_7 \|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} + V_L + \|v_L'\|_{\mathbf{L^\infty}(\mathbb{R};\mathbb{R})} R_L \right) \left|t-\widehat{t}\right| && \text{[By Lemma 3.12]}
\end{aligned}$$

$$\leq \mathcal{C}\left(\begin{array}{c}\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, V_L, \\ \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \|\rho_o\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)}, \|r_o\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})}, T\end{array}\right) \left|t-\hat{t}\right| \tag{3.77}$$

completing the proof to the local Lipschitz continuity in time.

Step 10. Definition and properties of $\mathcal{S}$.

By the arbitrariness of T, the above steps allow to define the map

$$\mathcal{S}\colon \mathbb{R}_+ \times (\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R}) \longrightarrow (\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R})$$

by $\mathcal{S}_t(\rho_o, r_o) := (\rho(t), r(t))$, where (ρ, r) is the solution constructed above.

By Definition 2.1 and the uniqueness of the solutions to (2.2) and to (2.3), for any $t_1 > 0$, the map $t \mapsto (\rho, r)(t_1 + t)$ solves

$$\begin{cases}\partial_t \tilde\rho^i + \partial_x \left(\tilde\rho^i\, \tilde v^i(t,x)\right) = 0 \\ \tilde\rho^i(0,x) = \rho^i(t_1, x)\end{cases} \qquad \tilde v^i(t,x) := v^i_{NL}\left(\left(\Sigma\tilde\rho(t_1+t) * \eta^i\right)(x) + \left(\tilde r(t_1+t) * \eta^i\right)(x)\right)$$
$$\begin{cases}\partial_t \tilde r + \partial_x \left(\tilde r\, \tilde w(t,x,\tilde r)\right) = 0 \\ \tilde r(0,x) = r(t_1, x)\end{cases} \qquad \tilde w(t,x,r) := v_L\left(\Sigma\tilde\rho(t_1+t,x) + \tilde r\right)$$

which ensures that $\mathcal{S}_t \circ \mathcal{S}_{t_1} = \mathcal{S}_{t+t_1}$, showing that $\mathcal{S}$ is a semigroup.

For fixed $M > 0$, call B_M the set of pairs (ρ_o, r_o) satisfying (3.59). The restriction of $\mathcal{S}$ to $[0,T] \times B_M$ is Lipschitz continuous with respect to the modulus in t and the norm (2.1) in (ρ_o, r_o) with a Lipschitz constant that, by (3.75), (3.76) and (3.77) results of the type

$$\mathcal{C}\left(\|\eta\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, \|v_{NL}\|_{\mathbf{W^{3,\infty}}(\mathbb{R};\mathbb{R}^k)}, R_L, V_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, M, T\right).$$

By uniform continuity, $\mathcal{S}$ can be extended to $[0,T] \times \overline{B}_M$, where $\overline{B}_M$ is the closure of B_M with respect to the norm (2.1), which is

$$\overline{B}_M = \left\{(\rho, r) \in \mathcal{BV}^1(\mathbb{R};\mathbb{R}^k) \times \mathcal{BV}(\mathbb{R};\mathbb{R})\colon \|\rho\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)} + \|r\|_{\mathcal{BV}(\mathbb{R};\mathbb{R})} \leq M\right\} \tag{3.78}$$

by Lemma 3.2. The arbitrariness of T and M allow to further extend $\mathcal{S}$ as in (2.6).

To prove the properties of $\mathcal{S}$, fix (ρ_o, r_o) in the domain of $\mathcal{S}$ and choose a sequence $(\rho_{o,n}, r_{o,n})$ in $(\mathbf{C^2} \cap \mathbf{W^{2,\infty}} \cap \mathbf{W^{2,1}})(\mathbb{R};\mathbb{R}^k) \times (\mathbf{L^1} \cap \mathbf{BV})(\mathbb{R};\mathbb{R})$ converging to (ρ_o, r_o) with respect to (2.1).

For all $t_1, t_2 \in \mathbb{R}_+$ we have $(\mathcal{S}_{t_1} \circ \mathcal{S}_{t_2})(\rho_{o,n}, r_{o,n}) = \mathcal{S}_{t_1+t_2}(\rho_{o,n}, r_{o,n})$, passing to the limit $n \to +\infty$ we prove **($\mathcal{S}$1)** using Proposition 2.2.

To prove **($\mathcal{S}$2)** apply Proposition 2.2 with $\varepsilon_n = e_n = 0$.

Property **($\mathcal{S}$3)** states the Lipschitz continuity of $\mathcal{S}$ on $[0,T] \times \overline{B}_M$ as defined in (3.78). This holds true by the very construction of $\mathcal{S}$, while the form of the Lipschitz constant follows from (3.75), (3.76) and (3.77). Similarly, **($\mathcal{S}$4)** follows from (3.67), (3.70), (3.71) and Lemma 3.2.

The positivity of ρ follows from **6.** in Lemma 3.4. Corollary 3.9 ensures both the invariance of $[0, R_L]$ and the conservation of the $\mathbf{L^1}$ norm, completing the proof of **($\mathcal{S}$5)**.

The proof of Theorem 2.3 is completed.

3.5. Proof of Theorem 2.5

Fix a positive ε and choose $\rho_o^\varepsilon \in \mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)$ such that

$$\|\rho_o^\varepsilon - \rho_o\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} < \varepsilon \quad \text{and} \quad \|\rho_o^\varepsilon\|_{\mathbf{W^{2,1}}(\mathbb{R};\mathbb{R}^k)} \leq \|\rho_o\|_{\mathcal{BV}^1(\mathbb{R};\mathbb{R}^k)} . \tag{3.79}$$

Call $(\rho^\varepsilon, r^\varepsilon)$ the solution to (1.2) with initial datum $(\rho_o^\varepsilon, r_o)$ constructed in Theorem 2.3.

Definition 2.1 allows to use Lemma 3.7 with $\rho = \rho_1$, $\widehat{\rho} = \rho_\varepsilon$ leading to the estimate

$$\|\rho_1(t) - \rho^\varepsilon(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \leq (1 + \mathcal{C}_3\, t)\|\rho_o - \rho_o^\varepsilon\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \mathcal{C}_3 \int_0^t \|r_1(\tau) - r^\varepsilon(\tau)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})}\, \mathrm{d}\tau \tag{3.80}$$

where $\mathcal{C}_3$ is defined in (3.30). Similarly, Definition 2.1 also allows to use Lemma 3.14 with $r = r^\varepsilon$ and $\widehat{r} = r_1$, leading to

$$\|r_1(t) - r^\varepsilon(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \leq \mathcal{C}_{11}\, t\, \|\rho_1 - \rho^\varepsilon\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))} . \tag{3.81}$$

where

$$\mathcal{C}_{11} := \mathcal{C}\begin{pmatrix} R_L, \|v_L'\|_{\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R})}, \|\rho^\varepsilon\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))}, \\ \|\rho_1\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))}, \|\partial_x \rho^\varepsilon\|_{\mathbf{L^1}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))}, \mathrm{TV}(r_o), t \end{pmatrix} .$$

Remark that the above expression allows to cope with the different regularities of ρ_1 and ρ^ε. Proceed as in **Step 4** in the proof of Theorem 2.3 to estimate the arguments in the right hand side, obtaining

$$\|\rho^\varepsilon\|_{\mathbf{C^0}([0,t];\mathbf{W^{1,\infty}}(\mathbb{R};\mathbb{R}^k))} \leq \mathcal{C}_6 \quad \text{and} \quad \|\partial_x \rho^\varepsilon\|_{\mathbf{L^1}([0,t];\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k))} \leq \mathcal{C}_6$$

where $\mathcal{C}_6$, defined in (3.69), is bounded uniformly in ε and, hence, the same holds for $\mathcal{C}_{11}$.

Insert (3.80) in (3.81) and apply Gronwall Lemma to obtain

$$\begin{aligned} \|r_1(t) - r^\varepsilon(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} &\leq (1 + \mathcal{C}_3\, t)\, \|\rho_o - \rho_o^\varepsilon\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \exp(\mathcal{C}_3\, \mathcal{C}_{11}\, t^2) \\ &\leq (1 + \mathcal{C}_3\, t)\, \exp(\mathcal{C}_3\, \mathcal{C}_{11}\, t^2)\, \varepsilon . \end{aligned}$$

Insert the latter estimate in (3.80) and use (3.79) to get

$$\|\rho_1(t) - \rho^\varepsilon(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} \leq (1 + \mathcal{C}_3\, t)\left(1 + \mathcal{C}_3\, t \exp(\mathcal{C}_3\, \mathcal{C}_{11}\, t^2)\right) \varepsilon .$$

Repeat the same procedure replacing ρ_1 with ρ_2 so that, by the triangle inequality, we get

$$\|\rho_1(t) - \rho_2(t)\|_{\mathbf{W^{1,1}}(\mathbb{R};\mathbb{R}^k)} + \|r_1(t) - r_2(t)\|_{\mathbf{L^1}(\mathbb{R};\mathbb{R})} \leq 2(1 + \mathcal{C}_3\, t)\left(1 + 2\,\mathcal{C}_3\, t \exp(\mathcal{C}_3\, \mathcal{C}_{11}\, t^2)\right) \varepsilon .$$

Since ε is arbitrary, the proof is completed.

Declaration of competing interest

The authors declare no conflicts of interest in this paper.

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