On the approximation of monotone variational inequalities in L^p spaces with probability measure

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Abstract In this paper we propose an approximation procedure for a class of monotone variational inequalities in probabilistic Lebesgue spaces. The implementation of the functional approximation in L^p, with $p > 2$, leads to a finite dimensional variational inequality whose structure is different from the one obtained in the case $p = 2$, already treated in the literature. The proposed computational scheme is applied to the random traffic equilibrium problem with polynomial cost functions.

1 Introduction

In many equilibrium problems arising in applied sciences, the data are often not known with precision and this uncertainty can be modeled by using some probability distributions. In this paper we are interested in the variational inequality approach to equilibrium problems which has been very fruitful in the last decades. Motivated by the need to cope with uncertain data, many authors have developed various approaches to the theory of random variational inequalities (the term stochastic variational inequalities is also used by numerous authors). Our contribution falls in the so called L^p approach to random variational inequalities introduced in [6, 7] and subsequently developed in a series of papers [4, 5, 8, 10, 11, 12]. A comparison of the rigorous L^p approach with a sample-path approach has been carried out in [9]. In this last paper, the authors also proposed a regularization method to deal with the case where the operator is monotone but not strictly monotone and applied their results to the traffic equilibrium problem with linear cost functions, which is mod-

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eled by a variational inequality in L^2. In this case, the regularization term is the identity operator, i.e., the Riesz isometry, and after a discretization procedure the original infinite dimensional variational inequality is transformed in a large number of *independent* finite dimensional variational inequalities. To the best of our knowledge, the above mentioned abstract regularization procedure has not yet been applied to random variational inequalities in L^p, with $p > 2$. In this paper, we show that when $p > 2$ the structure of the regularizing duality operator does not allow to split the L^p variational inequality into a large number of finite dimensional variational inequalities. Instead, it can be approximated by a single variational inequality whose operator $F : \mathbb{R}^L \to \mathbb{R}^L$ has a special structure such that all the summands in F, excepted the regularization term, depend on a number of variables which is much smaller than L. As an application of our results, we investigate the random traffic equilibrium problem with polynomial cost functions.

The paper is organized as follows. First, we give an overview of the L^p approach for random variational inequalities in Section 2.1. Then, in Section 2 we describe a functional approximation scheme combined with a regularization procedure to find approximated solutions of a random monotone variational inequality, while its implementation in L^p spaces, with $p > 2$, is analyzed in detail in Section 2.3. In Section 3 we apply the results illustrated in Section 2 to the random traffic network equilibrium problem with polynomial cost functions. The deterministic version of the problem and its variational inequality formulation are recall in Section 3.1. Section 3.2 is devoted to the stochastic version of the problem, where both the traffic demand and the travel cost functions may include random perturbations, and a stochastic variational inequality formulation is given. Finally, the regularization and approximation procedures described in Section 2 have been applied to some instances of the random traffic network equilibrium problem in order to show the impact of different probability distributions of the random data on the average cost at equilibrium.

2 Regularization of random variational inequalities

This Section is devoted to the regularization and approximation procedures for random monotone variational inequalities. In particular, Section 2.1 is an overview of the L^p approach for random variational inequalities; Section 2.2 describes a functional approximation scheme combined with a regularization procedure to find approximated solutions of a random monotone variational inequality, while in Section 2.3 we discuss in detail the implementation of the regularization and approximation procedures in L^p spaces with $p > 2$.

2.1 Random variational inequalities in probabilistic Lebesgue spaces

Let $(\Omega, \mathscr{A}, P)$ be a probability space, $A, B : \mathbb{R}^k \to \mathbb{R}^k$ two given mappings and $b, c \in \mathbb{R}^k$ two given vectors. Moreover, let R and S be two real-valued random variables defined on Ω, D a random vector in $\mathbb{R}^m$ and $G \in \mathbb{R}^{m\times k}$ a given matrix. For any $\omega \in \Omega$ we define a random set

$$M(\omega) := \{x \in \mathbb{R}^k : \; Gx \leq D(\omega)\}.$$

Consider the following random variational inequality: for almost every $\omega \in \Omega$, find $\hat{x} := \hat{x}(\omega) \in M(\omega)$ such that

$$(S(\omega)A(\hat{x}) + B(\hat{x}))^\top (z - \hat{x}) \geq (R(\omega)\, c + b)^\top (z - \hat{x}), \qquad \forall\, z \in M(\omega). \tag{1}$$

To facilitate the foregoing discussion, we set

$$T(\omega, x) := S(\omega)A(x) + B(x).$$

We assume that A, B and S are such that the map $T : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$ is a Carathéodory function, that is, for each fixed $x \in \mathbb{R}^k$ the function $T(\cdot, x)$ is measurable with respect to the σ-algebra $\mathscr{A}$, whereas for almost every $\omega \in \Omega$ the function $T(\omega, \cdot)$ is continuous. We also assume that $T(\omega, \cdot)$ is monotone for every $\omega \in \Omega$, i.e.,

$$(T(\omega, x) - T(\omega, y))^\top (x - y) \geq 0, \qquad \forall\, x, y \in \mathbb{R}^k, \; \forall\, \omega \in \Omega. \tag{2}$$

If (1) is uniquely solvable, then conditions can be given to ensure that the solution belongs to an L^p space for some $p \geq 2$. This allows us to compute statistical quantities such as mean values and variances of the solution. Since we are only interested in solutions with finite first- and second-order moments, another approach is to consider an integral variational inequality instead of the parametric variational inequality (1).

Thus, for a fixed $p \geq 2$, consider the Banach space $L^p(\Omega, P, \mathbb{R}^k)$ of random vectors V from Ω to $\mathbb{R}^k$ such that the expectation (p-moment) is given by

$$E^P(\|V\|^p) = \int_\Omega \|V(\omega)\|^p dP(\omega) < \infty.$$

For subsequent developments, we assume the following growth condition:

$$\|T(\omega, z)\| \leq \alpha(\omega) + \beta(\omega)\|z\|^{p-1}, \quad \forall\, z \in \mathbb{R}^k, \quad \text{for some } p \geq 2, \tag{3}$$

where $\alpha \in L^q(\Omega, P)$ and $\beta \in L^\infty(\Omega, P)$. Due to the above growth condition, the Nemytskii operator $\hat{T}$ associated to T acts from $L^p(\Omega, P, \mathbb{R}^k)$ to $L^q(\Omega, P, \mathbb{R}^k)$, where $p^{-1} + q^{-1} = 1$, and is defined by

$$\hat{T}(V)(\omega) := T(\omega, V(\omega)), \quad \omega \in \Omega. \tag{4}$$

It will be useful to notice that if $T(\omega,\cdot)$ is monotone for each ω, then $\hat{T}$ is monotone form $L^p(\Omega,P,\mathbb{R}^k)$ to $L^q(\Omega,P,\mathbb{R}^k)$, i.e.,

$$\int [T(\omega,V(\omega)) - T(\omega,U(\omega))]^\top \,(V(\omega)-U(\omega))dP(\omega) \geq 0$$

holds for all $U,V \in L^p(\Omega,P,\mathbb{R}^k)$. Assuming $D \in L^p_m(\Omega) := L^p(\Omega,P,\mathbb{R}^m)$, we introduce the following nonempty, closed and convex subset of $L^p_k(\Omega)$:

$$M^P := \{V \in L^p_k(\Omega) : GV(\omega) \leq D(\omega),\ P-a.s.\}.$$

Let $S(\omega) \in L^\infty$, $0 < \underline{s} < S(\omega) < \bar{s}$, and $R(\omega) \in L^q$. Equipped with these notations, we consider the following L^p formulation of (1): find $\hat{U} \in M^P$ such that for every $V \in M^P$ we have

$$\begin{aligned} &\int_\Omega (S(\omega)A[\hat{U}(\omega)] + B[\hat{U}(\omega))]^\top (V(\omega)-\hat{U}(\omega))\,dP(\omega) \\ &\qquad\qquad \geq \int_\Omega (b+R(\omega)\,c)^\top (V(\omega)-\hat{U}(\omega))dP(\omega). \end{aligned} \tag{5}$$

If problems (1) and (5) are uniquely solvable, then they are equivalent provided that the solution of (1) belongs to L^p.

To get rid of the abstract sample space Ω, we consider the joint distribution $\mathbb{P}$ of the random vector (R,S,D) and work with the special probability space $(\mathbb{R}^d,\mathscr{B}(\mathbb{R}^d),\mathbb{P})$, where $d := 2+m$ and $\mathscr{B}$ is the Borel σ-algebra on $\mathbb{R}^d$. For simplicity, we assume that R, S and D are independent random vectors and we set

$$r = R(\omega), \quad s = S(\omega), \quad t = D(\omega), \quad y = (r,s,t).$$

For each $y \in \mathbb{R}^d$, we define the set

$$M(y) := \{x \in \mathbb{R}^k :\ Gx \leq t\}.$$

The pointwise formulation of the variational inequality reads: find $\hat{x}$ such that $\hat{x}(y) \in M(y)$, $\mathbb{P}$-a.s., and for $\mathbb{P}$-almost every $y \in \mathbb{R}^d$ and for every $x \in M(y)$, we have

$$(sA[\hat{x}(y)] + B[\hat{x}(y)])^\top (x-\hat{x}(y)) \geq (rc+b)^\top (x-\hat{x}(y))\,. \tag{6}$$

In order to obtain the integral formulation of (6), consider the space $L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$ and introduce the closed and convex set

$$M_\mathbb{P} := \{v \in L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k) :\ \ Gv(r,s,t) \leq t,\ \ \mathbb{P}-a.s.\}.$$

Without any loss of generality, we assume that $R \in L^q(\Omega,P)$ and $D \in L^p(\Omega,P,\mathbb{R}^m)$ are nonnegative (otherwise we can use the standard decomposition in the positive part and the negative part). Moreover, we assume that the support (i.e., the set of possible outcomes) of $S \in L^\infty(\Omega,P)$ is the interval $[\underline{s},\bar{s}[\subset (0,\infty)$.

With these ingredients, we consider the variational inequality problem of finding $\hat{u} \in M_{\mathbb{P}}$ such that for every $v \in M_{\mathbb{P}}$ we have

$$\begin{aligned}\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} (sA[\hat{u}(y)] + B[\hat{u}(y)])^\top (v(y) - \hat{u}(y))\, d\mathbb{P}(y) \\ \geq \int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}^m_+} (b + r\,c)^\top (v(y) - \hat{u}(y))\, d\mathbb{P}(y).\end{aligned} \tag{7}$$

We conclude this section by recalling the following general result that ensures the solvability of an infinite dimensional variational inequality like (5) or (7) (see [13] for a recent survey on existence results for variational inequalities).

Theorem 1. *Let E be a reflexive Banach space and let E^* denote its topological dual space. We denote the duality pairing between E and E^* by $\langle \cdot, \cdot \rangle_{E,E^*}$. Let K be a nonempty, closed, and convex subset of E, and $A : K \to E^*$ be monotone and continuous on finite dimensional subspaces of K. Consider the variational inequality problem of finding $u \in K$ such that*

$$\langle Au, v - u \rangle_{E,E^*} \geq 0, \qquad \forall\, v \in K.$$

Then, a necessary and sufficient condition for the above problem to be solvable is the existence of $\delta > 0$ such that at least a solution of the variational inequality:

$$\textit{find } u_\delta \in K_\delta \textit{ such that } \langle Au_\delta, v - u_\delta \rangle_{E,E^*} \geq 0, \quad \forall\, v \in K_\delta$$

satisfies $\|u_\delta\| < \delta$, where $K_\delta = \{v \in K : \ \|v\| \leq \delta\}$.

In the next section, we show how the set $M_{\mathbb{P}}$ can be approximated by a sequence $\{M^n_{\mathbb{P}}\}$ of finite dimensional sets, and the functions r and s can be approximated by the sequences $\{\rho_n\}$ and $\{\sigma_n\}$ of step functions, with $\rho_n \to \rho$ in L^p and $\sigma_n \to \sigma$ in L^∞, respectively, where $\rho(r,s,t) = r$ and $\sigma(r,s,t) = s$. When the solution of (7) is unique, we can compute a sequence of step functions $\hat{u}_n$ which converges strongly to $\hat{u}$ under suitable hypotheses.

2.2 A functional approximation scheme for the random variational inequality

We start with a discretization of the space $X := L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$. We introduce a sequence $\{\pi_n\}$ of partitions of the support

$$\Upsilon := [0, \infty[\times [\underline{s}, \bar{s}[\times \mathbb{R}^m_+$$

of the probability measure $\mathbb{P}$ induced by the random elements R, S, and D. For this, we set

$$\pi_n = (\pi^R_n, \pi^S_n, \pi^D_n),$$

where

$$\begin{aligned}
&\pi_n^R := (r_n^0, \dots, r_n^{N_n^R}), \ \ \pi_n^S := (s_n^0, \dots, s_n^{N_n^S}), \ \ \pi_n^{D_i} := (t_{n,i}^0, \dots, t_{n,i}^{N_n^{D_i}}),\\
&0 = r_n^0 < r_n^1 < \dots r_n^{N_n^R} = n,\\
&\underline{s} = s_n^0 < s_n^1 < \dots s_n^{N_n^S} = \overline{s},\\
&0 = t_{n,i}^0 < t_{n,i}^1 < \dots t_{n,i}^{N_n^{D_i}} = n \quad (i = 1, \dots, m),\\
&|\pi_n^R| := \max\{r_n^j - r_n^{j-1} : j = 1, \dots, N_n^R\} \to 0 \ \ (n \to \infty),\\
&|\pi_n^S| := \max\{s_n^k - s_n^{k-1} : k = 1, \dots, N_n^S\} \to 0 \ \ (n \to \infty),\\
&|\pi_n^{D_i}| := \max\{t_{n,i}^{h_i} - t_{n,i}^{h_i-1} : h_i = 1, \dots, N_n^{D_i}\} \to 0 \ \ (i = 1, \dots, m; n \to \infty).
\end{aligned}$$

These partitions give rise to an exhausting sequence $\{\Upsilon_n\}$ of subsets of Υ, where each Υ_n is given by the finite disjoint union of the intervals:

$$I_{jkh}^n := [r_n^{j-1}, r_n^j[\times [s_n^{k-1}, s_n^k[\times I_h^n,$$

where we use the multi-index $h = (h_1, \cdots, h_m)$ and

$$I_h^n := \prod_{i=1}^m [t_{n,i}^{h_i-1}, t_{n,i}^{h_i}[.$$

For each $n \in \mathbb{N}$, we consider the space of the $\mathbb{R}^l$-valued step functions on Υ_n, extended by 0 outside of Υ_n:

$$X_n^l := \left\{ v_n : \ v_n(r,s,t) = \sum_j \sum_k \sum_h v_{jkh}^n 1_{I_{jkh}^n}(r,s,t), \quad v_{jkh}^n \in \mathbb{R}^l \right\},$$

where 1_I denotes the $\{0,1\}$-valued characteristic function of a subset I. To approximate an arbitrary function $w \in L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R})$, we employ the mean value truncation operator μ_0^n associated to the partition π_n given by

$$\mu_0^n w := \sum_{j=1}^{N_n^R} \sum_{k=1}^{N_n^S} \sum_h (\mu_{jkh}^n w)\, 1_{I_{jkh}^n}, \tag{8}$$

where

$$\mu_{jkh}^n w := \begin{cases} \dfrac{1}{\mathbb{P}(I_{jkh})} \displaystyle\int_{I_{jkh}^n} w(y)\, d\mathbb{P}(y), & \text{if } \mathbb{P}(I_{jkh}^n) > 0,\\ 0, & \text{otherwise.} \end{cases}$$

Analogously, for a L^p vector function $v = (v_1, \dots, v_l)$, we define

$$\mu_0^n v := (\mu_0^n v_1, \dots, \mu_0^n v_l),$$

for which one can prove that $\mu_0^n v$ converges to v in $L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^l)$.

To construct approximations for the set

$$M_{\mathbb{P}} = \left\{ v \in L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k) : \; Gv(r,s,t) \le t \,,\; \mathbb{P}-\text{a.s.} \right\},$$

we introduce the orthogonal projector $q : (r,s,t) \in \mathbb{R}^d \mapsto t \in \mathbb{R}^m$ and define, for each elementary cell I^n_{jkh}, the quantities

$$\overline{q}^n_{jkh} = (\mu^n_{jkh} q) \in \mathbb{R}^m \qquad \text{and} \qquad (\mu_0^n q) = \sum_{jkh} \overline{q}^n_{jkh} \, 1_{I^n_{jkh}} \in X_n^m.$$

This leads to the following sequence of polyhedra

$$M^n_{\mathbb{P}} := \{ v \in X_n^k : \;\; Gv^n_{jkh} \le \overline{q}^n_{jkh}, \quad \forall j,k,h \}.$$

Since our objective is to approximate the random variables R and S, we introduce

$$\rho_n = \sum_{j=1}^{N_n^R} r_n^{j-1} \, 1_{[r_n^{j-1}, r_n^j[} \in X_n \qquad \text{and} \qquad \sigma_n = \sum_{k=1}^{N_n^S} s_n^{k-1} \, 1_{[s_n^{k-1}, s_n^k[} \in X_n.$$

Notice that

$$\sigma_n(r,s,t) \to \sigma(r,s,t) = s \text{ in } L^\infty(\mathbb{R}^d,\mathbb{P}), \quad \rho_n(r,s,t) \to \rho(r,s,t) = r \text{ in } L^p(\mathbb{R}^d,\mathbb{P}).$$

Combining the above ingredients, for any $n \in \mathbb{N}$ we consider the following discretized variational inequality: find $\hat{u}_n := \hat{u}_n(y) \in M^n_{\mathbb{P}}$ such that, for every $v_n \in M^n_{\mathbb{P}}$, we have

$$\begin{aligned} &\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^d} [\sigma_n(y) A(\hat{u}_n) + B(\hat{u}_n)]^\top [v_n - \hat{u}_n] \, d\mathbb{P}(y) \\ &\qquad \ge \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}^d} [b + \rho_n(y) \, c]^\top [v_n - \hat{u}_n] \, d\mathbb{P}(y). \end{aligned} \tag{9}$$

We also assume that the probability measures P_R, P_S and P_{D_i} have the probability densities φ_R, φ_S and φ_{D_i}, with $i = 1, \ldots, m$, respectively. Therefore, for $i = 1, \ldots, m$, we have

$$dP_R(r) = \varphi_R(r) \, dr, \qquad dP_S(s) = \varphi_S(s) \, ds, \qquad dP_{D_i}(t_i) = \varphi_{D_i}(t_i) \, dt_i.$$

In absence of strict monotonicity, the solution of (5) and (7) is not unique. In this case (which often occurs in our application) the previous approximation procedure must be coupled with a *regularization* scheme as follows. We choose a sequence $\{\varepsilon_n\}$ of regularization parameters and choose the regularization map to be the duality map $J : L^p(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k) \to L^q(\mathbb{R}^d,\mathbb{P},\mathbb{R}^k)$. We assume that $\varepsilon_n > 0$ for every $n \in \mathbb{N}$ and that $\varepsilon_n \downarrow 0$ as $n \to \infty$.

We can then consider, for any $n \in \mathbb{N}$, the following regularized stochastic variational inequality: find $w_n = w_n^{\varepsilon_n}(y) \in M_{\mathbb{P}}^n$ such that, for every $v_n \in M_{\mathbb{P}}^n$, we have

$$\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} \left(\sigma_n(y) A[w_n(y)] + B[w_n(y)] + \varepsilon_n J(w_n(y))\right)^\top (v_n(y) - w_n(y))\, d\mathbb{P}(y)$$
$$\geq \int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} (b + \rho_n(y)\, c)^\top (v_n(y) - w_n(y))\, d\mathbb{P}(y). \tag{10}$$

As usual, the solution w_n will be referred to as the regularized solution. Weak and strong convergence of w_n to the minimal-norm solution of (7) can be proved under suitable hypotheses (see below). We also recall (see e.g. [1]) that in L^p we have

$$J(u) = \|u\|_{L^p}^{2-p} |u|^{p-2}\, u.$$

We recall the following convergence result (see [8]).

Theorem 2. *Assume that the growth condition* (3) *holds and* $T(\omega, \cdot)$ *is strongly monotone, uniformly with respect to* $\omega \in \Omega$*, that is there exists* $\tau > 0$ *such that*

$$(T(\omega, x) - T(\omega, y))^\top (x - y) \geq \tau \|x - y\|^2 \quad \forall\, x, y, \text{ a.e. } \omega \in \Omega.$$

Then the sequence $\{\hat{u}_n\}$*, where* $\hat{u}_n$ *is the unique solution of* (9)*, converges strongly in* $L^p(\mathbb{R}^d, \mathbb{P}, \mathbb{R}^k)$ *to the unique solution* $\hat{u}$ *of* (7)*.*

The following results (see [9]) highlight some of the features of the regularized solutions.

Theorem 3. *The following statements hold.*

1. *For every* $n \in \mathbb{N}$*, the regularized stochastic variational inequality* (10) *has the unique solution* w_n*.*
2. *Any weak limit of the sequence of regularized solutions* $\{w_n\}$ *is a solution of* (7)*.*
3. *The sequence of regularized solutions* $\{w_n\}$ *is bounded provided that the following coercivity condition holds: there exists a bounded sequence* $\{\delta_n\}$*, with* $\delta_n \in M_{\mathbb{P}}^n$*, such that*

$$\frac{\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m} [\sigma_n(y) A(u_n(y)) + B(u_n(y))]^\top (u_n(y) - \delta_n(y))\, d\mathbb{P}(y)}{\|u_n\|} \to \infty$$

as $\|u_n\| \to \infty$*.*

To obtain strong convergence, we need to use the concept of fast Mosco convergence [14], as given by the following definition.

Definition 1. Let X be a normed space, let $\{K_n\}$ be a sequence of closed and convex subsets of X and let $K \subset X$ be closed and convex. Let $\{\varepsilon_n\}$ be a a sequence of positive real numbers such that $\varepsilon_n \to 0$. The sequence $\{K_n\}$ is said to converge to K in the fast Mosco sense, related to ε_n, if

1. For each $x \in K$, $\exists \{x_n\} \in K_n$ such that $\varepsilon_n^{-1} \|x_n - x\| \to 0$;
2. For $\{x_m\} \subset X$, if $\{x_m\}$ weakly converges to $x \in K$, then $\exists \{z_m\} \in K$ such that $\varepsilon_m^{-1}(z_m - x_m)$ weakly converges to 0.

Theorem 4. *Assume that $M_{\mathbb{P}}^n$ converges to $M_{\mathbb{P}}$ in the fast Mosco sense related to ε_n. Moreover, assume that $\varepsilon_n^{-1}\|\sigma_n - \sigma\| \to 0$ and $\varepsilon_n^{-1}\|\rho_n - \rho\| \to 0$ as $n \to \infty$. Then the sequence of regularized solutions $\{w_n\}$ converges strongly to the minimal-norm solution of the stochastic variational inequality* (7)*, provided that $\{w_n\}$ is bounded.*

2.3 Implementation

In this section, we derive an equivalent form of the regularized stochastic variational inequality (10) suitable for being solved on a computer. We first rewrite (10) for the reader convenience: given any $n \in \mathbb{N}$, find $w_n = w_n^{\varepsilon_n}(y) \in M_{\mathbb{P}}^n$ such that, for every $v_n \in M_{\mathbb{P}}^n$, we have

$$\int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}_+^m} \big(\sigma_n(y) A[w_n] + B[w_n] + \varepsilon_n J(w_n)\big)^\top (v_n - w_n)\, d\mathbb{P}(y) \\ \geq \int_0^\infty \int_{\underline{s}}^{\bar{s}} \int_{\mathbb{R}_+^m} (b + \rho_n(y)\, c)^\top (v_n - w_n)\, d\mathbb{P}(y).$$

The solution of (10) is a step function which is determined by its constant (vector) values in each elementary cell I_{jlh}^n. Since for each partition of the support of $\mathbb{P}$ we have

$$[0,\infty[\times [\underline{s}, \bar{s}[\times \mathbb{R}_+^m = \bigcup_{j,l,h} I_{jlh}^n,$$

we can write (10) as

$$\sum_j \sum_l \sum_h \int_{I_{jlh}^n} \big(\sigma_n(y) A[w_n] + B[w_n] + \varepsilon_n J(w_n)\big)^\top (v_n - w_n)\, d\mathbb{P}(y) \\ \geq \sum_j \sum_l \sum_h \int_{I_{jlh}^n} (b + \rho_n(y)\, c)^\top (v_n - w_n)\, d\mathbb{P}(y). \tag{11}$$

Bearing in mind that the components of $A[w]$ and $B[w]$ are multivariate polynomials in $w_2, \ldots, w_n$, and that v_{jlh}^n denotes the constant vector value of $v_n(y)$ in the cell I_{jlh}^n, the value of $A[w]$ in I_{jlh}^n can be written as Av_{jlh}^n and, analougously, the value of $B[w]$ in I_{jlh}^n can be written as Bv_{jlh}^n.

For the subsequent development it is useful to notice that for a step function $w \in X_n^k$, we have:

$$\|w\|_{L^p} = \left[\sum_j \sum_l \sum_h \left(\sqrt{(w_{1jlh})^2 + \ldots + (w_{kjlh})^2} \right)^p \mathbb{P}(I_{jlh}^n) \right]^{1/p}.$$

Let us denote with L_n the total number of the cells I^n_{jlh} induced by the partition π_n and group all the values w^n_{jlh}, for any j,l,h, in a vector which, with abuse of notation, we denote $(w^n_1,\dots,w^n_{L_n}) \in \mathbb{R}^{k\times L_n}$, i.e., we use the same symbol for both a step function of X^k_n and its associated vector of $\mathbb{R}^{k\times L_n}$ which describes its constant values on each cell. Moreover, we make the position

$$\|w\|^{2-p}_{L^p} = f(w^n_1,\dots,w^n_{L_n}).$$

A way of ordering the elements w^n_{jlh} into a vector $(w^n_\alpha)_\alpha \in \mathbb{R}^{k\times L_n}$ will be specified later and is fundamental for the implementation of our approximation procedure.
We can thus associate to the set of step functions $M^n_{\mathbb{P}}$, the set

$$M^n = \{v^n \in \mathbb{R}^{k\times L_n} : \ v^n_{jlh} \in M^n_{jlh}, \quad \forall j,l,h\}$$

where

$$M^n_{jlh} = \{v^n_{jlh} \in \mathbb{R}^k : \ G v^n_{jlh} \le \bar{q}^n_{jlh}\}, \quad \forall\ j,l,h.$$

Equipped with these notations, (11) can be equivalently written as

$$\begin{aligned}
\sum_j\sum_l\sum_h s_n^{l-1} A[w^n_{jlh}]^\top (v^n_{jlh}-w^n_{jlh})\mathbb{P}(I^n_{jlh}) + \sum_j\sum_l\sum_h B[w^n_{jlh}]^\top (v^n_{jlh}-w^n_{jlh})\mathbb{P}(I^n_{jlh})\\
+\varepsilon_n \sum_j\sum_l\sum_h f(w^n_1,\dots,w^n_{L_n})|w^n_{jlh}|^{p-2}(w^n_{jlh})^\top (v^n_{jlh}-w^n_{jlh})\mathbb{P}(I^n_{jlh})\\
\ge \sum_j\sum_l\sum_h (b^\top + r_n^{j-1} c^\top)(v^n_{jlh}-w^n_{jlh})\mathbb{P}(I^n_{jlh}).
\end{aligned} \tag{12}$$

In (12) we can choose $v^n_{jlh} = w^n_{jlh}$ for all the cells excepted one, so as to simplify the factor $\mathbb{P}(I^n_{jlh})$. However, the resulting inequality cannot be interpreted as a variational inequality on a single cell, because the term f involves the variables of all the cells. We can then sum again the resulting expression over the indices j,l,h and obtain

$$\begin{aligned}
\sum_j\sum_l\sum_h s_n^{l-1} A[w^n_{jlh}]^\top (v^n_{jlh}-v^n_{jlh}) + \sum_j\sum_l\sum_h B[w^n_{jlh}]^\top (v^n_{jlh}-v^n_{jlh})\\
+\varepsilon_n \sum_j\sum_l\sum_h f(w^n_1,\dots,w^n_{L_n})|w^n_{jlh}|^{p-2}(w^n_{jlh})^\top (v^n_{jlh}-v^n_{jlh})\\
\ge \sum_j\sum_l\sum_h (b^\top + r_n^{j-1} c^\top)(v^n_{jlh}-v^n_{jlh}).
\end{aligned} \tag{13}$$

Let us notice that if $p=2$ the variational inequality above can be split into a large number of independent variational inequalities in $\mathbb{R}^k$, one for each elementary cell I_{jlh} (see e.g. [9]). This decomposition is not possible for $p>2$ but, in this case, the last expression represents a variational inequality in $\mathbb{R}^{k\times L_n}$ with a special structure. In order to specify the structure of the operator of (13), as well as the constant term in the right hand side, so as to obtain a computational scheme that can be implemented in a straightforward manner, we need to specify a way in which the two (scalar) indices j,l and the multi-index h are mapped into a single index α. Thus, remember that:

$$j=1,\dots,N_n^R, \quad l=1,\dots,N_n^S, \quad h_i=1,\dots,N_n^{D_i}, \quad i=1,\dots,m,$$

and define

$$\alpha = 1+(j-1)+(l-1)N_n^R+(h_1-1)N_n^R N_n^S+\cdots+(h_m-1)N_n^R N_n^S \prod_{i=1}^{m-1} N_n^{D_i}. \quad (14)$$

On the other hand, from any given value of $\alpha \in \{1,2,\dots,L_n\}$, we can derive the corresponding indices j,l,h. This can be done in various ways and here we describe a sequential algorithm. We recall that $\lfloor a/b \rfloor$ denotes the result of the integer division of a divided by b while $a \bmod b$ denotes the remainder. Define $\alpha_1 = \alpha - 1$ and compute

$$\begin{cases} j=(\alpha_1 \bmod N_n^R)+1, & \alpha_2 = \lfloor \alpha_1/N_n^R \rfloor, \\ l=(\alpha_2 \bmod N_n^S)+1, & \alpha_3 = \lfloor \alpha_2/N_n^S \rfloor, \\ h_1=(\alpha_3 \bmod N_n^{D_1})+1, & \alpha_4 = \left\lfloor \alpha_3/N_n^{D_1} \right\rfloor, \\ \vdots & \vdots \\ h_m=(\alpha_{m+2} \bmod N_n^m)+1. & \end{cases}$$

If we denote

$$T_l^n = s_n^{l-1}A+B \quad \text{and} \quad e_j^n = b+r_n^{j-1}c, \quad (15)$$

then (13) can be written as

$$\sum_\alpha [T_\alpha^n(w_\alpha^n)]^\top (v_\alpha^n - w_\alpha^n) + \varepsilon_n \sum_\alpha f(w_n)|w_\alpha^n|^{p-2}(w_\alpha^n)^\top (v_\alpha^n - w_\alpha^n) \\ \geq \sum_\alpha (e_\alpha^n)^\top (v_\alpha^n - w_\alpha^n). \quad (16)$$

Notice that the expressions for T_α^n and e_α^n can be derived from (15) by using the inversion of formula (14) given above. Finally, we remark that any of the numerous algorithms for finite dimensional variational inequalities can be exploited for solving (16).

3 Application to the traffic network equilibrium problem with random data

In this section we apply the results shown in Section 2 to the traffic network equilibrium problem with random data. First, we recall the deterministic version of the problem and its variational inequality formulation (Section 3.1). Section 3.2 deals with the problem where both the traffic demand and the travel cost functions include random perturbations and a stochastic variational inequality formulation is given. Moreover, we prove a convergence result for the average cost at equilibrium by exploiting the approximation and regularization procedure described in Section 2.2. Finally, Section 3.3 is devoted to some numerical experiments showing the impact of different probability distributions of the random data on the average cost at equilibrium.

3.1 An outline of the traffic network equilibrium problem

We now recall the basic definitions and the variational inequality formulation of a network equilibrium flow (see, e.g. [3, 17]). For a comprehensive treatment of all the mathematical aspects of the traffic network equilibrium problem, we refer the interested reader to the classical book of Patriksson [16]. A traffic network consists of a triple $G=(N,A,W)$, where $N=\{N_1,\dots,N_p\}$ is the set of nodes, $A=\{a_1,\dots,a_n\}$ represents the set of direct arcs (also called links) connecting pairs of nodes and $W=\{W_1,\dots,W_m\}\subseteq N\times N$ is the set of the origin-destination (O-D) pairs. The flow on the link a_i is denoted by f_i and we group all the link flows in a vector $f=(f_1,\dots,f_n)$. A path (or route) is defined as a set of consecutive links and we assume that each O-D pair W_j is connected by r_j paths whose set is denoted by P_j. All the paths in the network are grouped into a vector $(R_1,\dots,R_k)$. The link structure of the paths can be described by using the link-path incidence matrix $\Delta=(\delta_{ir})$, $i=1,\dots,n$, $r=1,\dots,k$, with entries $\delta_{ir}=1$, if $a_i\in R_r$, and 0 otherwise. To each path R_r it is associated a flow F_r. The path flows are grouped into a vector $(F_1,\dots,F_k)$ which is called the network path flow (or simply, the network flow if it is clear that we refer to paths). The flow f_i on the link a_i is equal to the sum of the flows on the paths containing a_i, so that $f=\Delta F$. The unit cost of traveling through a_i is a real function $c_i(f)\geq 0$ of the flows on the network, so that $c(f)=(c_1(f),\dots,c_n(f))$ denotes the link cost vector on the network. The meaning of the cost is usually that of travel time and, in the simplest case, the generic component c_i only depends on f_i. A very popular link cost function was introduced by the Bureau of Public Roads [2] and explicitly take into account the link capacities. More precisely, the travel cost for link a_i is given by

$$c_i(f_i)=t_i^0\left[1+\gamma\left(\frac{f_i}{u_i}\right)^{\beta}\right], \tag{17}$$

where u_i describes the capacity of link a_i, t_i^0 is the free flow travel time on link a_i, while β and γ are positive parameters. Analogously, one can define a cost on the paths as $C(F)=(C_1(F),\dots,C_k(F))$. Usually, $C_r(F)$ is just the sum of the costs on the links which build that path:

$$C_r(F)=\sum_{i=1}^{n}\delta_{ir}c_i(f),$$

or in compact form $C(F) = \Delta^\top c(\Delta F)$. For each pair W_j, there is a given traffic demand $D_j > 0$, so that $D = (D_1, \dots, D_m)$ is the demand vector of the network. Feasible path flows are nonnegative and satisfy the demands, i.e., belong to the set

$$K = \{F \in \mathbb{R}^k : \; F \geq 0 \;\text{ and }\; \Phi F = D\}, \tag{18}$$

where Φ is the pair-path incidence matrix whose entries, for $j = 1, \dots, m$, $r = 1, \dots, k$, are

$$\varphi_{jr} = \begin{cases} 1, & \text{if the path } R_r \text{ connects the pair } W_j, \\ 0, & \text{elsewhere.} \end{cases}$$

The notion of a user traffic equilibrium is given by the following definition.

Definition 2. A network flow $H \in K$ is a *Wardrop equilibrium* if, for each O-D pair W_j and for each pair of paths R_r, R_s which connect W_j, the following implication holds:

$$C_r(H) > C_s(H) \Longrightarrow H_r = 0;$$

that is, if traveling along the path R_r takes more time than traveling along R_s, then the flow along R_r vanishes.

Remark 1. Among the various paths which connect a given O-D pair W_j some will carry a positive flow and others zero flow. It follows from the previous definition that, for a given O-D pair, the travel cost is the same for all nonzero flow paths, otherwise users would choose a path with a lower cost. Hence, H is a Wardrop equilibrium if for each O-D pair W_j there exists $\lambda_j \in \mathbb{R}$ such that

$$C_r(H) \begin{cases} = \lambda_j, & \text{if } H_r > 0, \\ \geq \lambda_j, & \text{if } H_r = 0. \end{cases} \tag{19}$$

Hence, λ_j denotes the equilibrium cost shared by all the used paths connecting W_j. The variational inequality formulation of the Wardrop equilibrium is given by the following result (see, e.g., [3]).

Theorem 5. *A network flow $H \in K$ is a Wardrop equilibrium iff it satisfies the variational inequality*

$$C(H)^\top (F - H) \geq 0, \qquad \forall\, F \in K. \tag{20}$$

Sometimes it is useful to decompose the scalar product in (20) according to the various O-D pairs W_j:

$$\sum_{j=1}^{m} \sum_{r \in P_j} C_r(H)\,(F_r - H_r) \geq 0, \qquad \forall\, F \in K.$$

For the subsequent development the monotonicity properties of the cost operators will be exploited. We recall them in this section for the reader convenience.

Definition 3. A map $T:\mathbb{R}^k \to \mathbb{R}^k$ is said *monotone* if

$$(\,(T(x)-T(y)\,)^\top (x-y) \geq 0, \qquad \forall\, x,y \in \mathbb{R}^k,$$

and *strictly monotone* if the equality holds only for $x=y$. T is said *strongly monotone* if there exists $\tau > 0$ such that

$$(\,(T(x)-T(y)\,)^\top (x-y) \geq \tau \|x-y\|^2, \qquad \forall\, x,y \in \mathbb{R}^k.$$

The strict monotonicity assumption of the link cost functions is commonly used because it models the congestion effect. However, this does not necessarily implies the strict monotonicity of the path cost functions, as the following lemma shows.

Lemma 1.

1. *If c is monotone, then C is monotone.*
2. *If c is strictly monotone and Δ has full column rank, then C is strictly monotone.*
3. *If c is strongly monotone and Δ has full column rank, then C is strongly monotone.*

Proof.

1. If $F^1, F^2 \in K$, then

$$\begin{aligned}[F^1-F^2]^\top [C(F^1)-C(F^2)] &= [F^1-F^2]^\top \Delta^\top [c(\Delta F^1)-c(\Delta F^2)] \\ &= [\Delta F^1 - \Delta F^2]^\top [c(\Delta F^1)-c(\Delta F^2)] \\ &\geq 0.\end{aligned}$$

2. If $F^1 \neq F^2$, then $\Delta F^1 \neq \Delta F^2$ since Δ has full column rank, hence

$$\begin{aligned}[F^1-F^2]^\top [C(F^1)-C(F^2)] &= [F^1-F^2]^\top \Delta^\top [c(\Delta F^1)-c(\Delta F^2)] \\ &= [\Delta F^1 - \Delta F^2]^\top [c(\Delta F^1)-c(\Delta F^2)] \\ &> 0.\end{aligned}$$

3. Let $F^1, F^2 \in K$. The strong monotonicity of c implies that there exists $\tau > 0$ such that

$$\begin{aligned}[F^1-F^2]^\top [C(F^1)-C(F^2)] &= [F^1-F^2]^\top \Delta^\top [c(\Delta F^1)-(\Delta F^2)] \\ &= [\Delta F^1 - \Delta F^2]^\top [c(\Delta F^1)-c(\Delta F^2)] \\ &\geq \tau \|\Delta F^1 - \Delta F^2\|^2 \\ &= \tau (F^1-F^2)^\top \Delta^\top \Delta (F^1-F^2) \\ &\geq \tau \lambda_{min}(\Delta^\top \Delta) \|F^1-F^2\|^2,\end{aligned}$$

where $\lambda_{min}(\Delta^\top \Delta)$, which denotes the minimum eigenvalue of $\Delta^\top \Delta$, is positive since Δ has full column rank. $\square$

3.2 The stochastic VI formulation of the traffic network equilibrium problem

We now consider the traffic network equilibrium problem where both the demand and the costs are affected by random perturbations.

Let Ω be a sample space and P be a probability measure on Ω, and consider the following feasible set which takes into consideration random fluctuations of the demand:

$$K(\omega) = \{F \in \mathbb{R}^k : \; F \geq 0, \;\; \Phi F = D(\omega)\}, \qquad \omega \in \Omega.$$

Moreover, let $C : \Omega \times \mathbb{R}^k \mapsto \mathbb{R}^k$ be the random cost function. We can thus introduce ω as a random parameter in (20) and consider the problem of finding a vector $H(\omega) \in K(\omega)$ such that, $P-$a.s:

$$C(\omega, H(\omega))^\top (F - H(\omega)) \geq 0, \qquad \forall\, F \in K(\omega). \tag{21}$$

Definition 4. A random vector $H \in K(\omega)$ is a *random Wardrop equilibrium* if for P-almost every $\omega \in \Omega$, for each O-D pair W_j and for each pair of paths R_r, R_s which connect W_j, the following implication holds:

$$C_r(\omega, (H(\omega)) > C_s(\omega, (H(\omega))) \Longrightarrow H_r(\omega) = 0. \tag{22}$$

Consider then the set

$$\begin{aligned} K_P = \{F \in L^p(\Omega, P, \mathbb{R}^k) : \; & F_r(\omega) \geq 0, \; P.-\text{a.s.}, \quad \forall\, r = 1, \ldots, k, \\ & \Phi F(\omega) = D(\omega), \; P.-\text{a.s.}\}, \end{aligned}$$

which is convex, closed and bounded, hence weakly compact. Furthermore, assume that the cost function C satisfies the growth condition:

$$\|C(\omega, z)\| \leq \alpha(\omega) + \beta(\omega)\|z\|^{p-1}, \qquad \forall\, z \in \mathbb{R}^k, \; P.-\text{a.s.}, \tag{23}$$

for some $\alpha \in L^q(\Omega, P), \beta \in L^\infty(\Omega, P), p^{-1} + q^{-1} = 1$.

Remark 2. We note that polynomial cost functions are often used to model the network congestion, e.g., the BPR cost functions (17), hence condition (23) is naturally satisfied. In particular, with linear costs the functional setting is the Hilbert space L^2.

The Carathéodory function C gives rise to a Nemytskii map $\hat{C} : L^p(\Omega, P, \mathbb{R}^k) \to L^q(\Omega, P, \mathbb{R}^k)$ defined through the usual position

$$\hat{C}(F)(\omega) = C(\omega, F((\omega)), \tag{24}$$

and, with abuse of a notation, instead of $\hat{C}$, the same symbol C is often used for both the Carathéodory function and the Nemytskii map that it induces. We thus consider the following integral variational inequality: find $H \in K_P$ such that

$$\int_{\Omega} C(\omega, H(\omega))^{\top}(F - H(\omega))dP(\omega) \geq 0, \qquad \forall\, F \in K_P. \tag{25}$$

A solution of (25) satisfies the random Wardrop conditions in the sense shown by the following lemma (see [12] for the proof).

Lemma 2. *If $H \in K_P$ is a solution of* (25)*, then H is a random Wardrop equilibrium.*

As a consequence of the previous lemma, we get that there exists a vector function $\lambda \in L^p(\Omega, P, \mathbb{R}^m)$ such that

$$C_l(\omega, H(\omega)) = \lambda_j(\omega) \tag{26}$$

for any O-D pair W_j and any path R_l connecting W_j, with $H_l(\omega) > 0$, P-almost surely.

In order to better address the modeling and computational aspects, we specify how the deterministic and the random variables appear in the operator structure. More precisely, we assume that the operator is the sum of a purely deterministic term and of a random term, where randomness act as a modulation. With the specifying of the constant term in the operator explicitly, we have

$$C(\omega, H(\omega)) = S(\omega)A[H(\omega)] + B[H(\omega)] - b - R(\omega)c, \tag{27}$$

where $S \in L^{\infty}(\Omega, P), R \in L^q(\Omega),\ A, B : L^p(\Omega, P, \mathbb{R}^k) \to L^q(\Omega, P, \mathbb{R}^k), b, c \in \mathbb{R}^k$. The integral variational inequality (25) now reads

$$\begin{aligned} &\int_{\Omega} \left(S(\omega)(A[H(\omega)])^{\top} + (B[H(\omega)])^{\top}\right)(F - H(\omega))dP(\omega) \\ &\geq \int_{\Omega} (b^{\top} + R(\omega)c^{\top})(F - H(\omega))dP(\omega), \qquad \forall\, F \in K_P. \end{aligned} \tag{28}$$

The *average cost at equilibrium* is defined as

$$E_P[\lambda] = \int_{\Omega} \lambda(\omega)dP(\omega), \tag{29}$$

where $\lambda = \lambda(\omega) = (\lambda_1(\omega), \ldots, \lambda_m(\omega))$ is defined as in (26).

Remark 3. Let us note that the integral in (29) is different from zero under the natural assumption that in each path R_r there is a link where the cost is bounded from below by a positive number (uniformly in $\omega \in \Omega$). This hypothesis is fulfilled in real networks because the cost is positive for positive flows, but also the cost at zero flow (called the free flow time) is positive, because it represents the travel time without congestion.

As already explained in the previous Section, the random vector $t = D(\omega)$ and the two random variables $r = R(\omega)$ and $s = S(\omega)$ generate a probability $\mathbb{P}$ in the image space $\mathbb{R}^{2+m}$ of (r, s, t) from the probability P on the abstract sample space Ω. Hence, we can express the earlier defined quantities in terms of the image space variables, thus obtaining functions which can be approximated through a discretization procedure. The integration now runs over the image space variables, but to keep notation simple we just write $\int$ instead of $\int_0^\infty \int_{\underline{s}}^{\overline{s}} \int_{\mathbb{R}_+^m}$. The transformed expression read as follows:

$$E_{\mathbb{P}}[\lambda] = \int \lambda(r,s,t)d\mathbb{P}(r,s,t), \tag{30}$$

Let us recall that the solution $H = H(r,s,t)$ of the stochastic variational inequality which describes the network equilibrium can be approximated using the procedure explained in Section 2.2 by a sequence $\{H^n\}$ of step functions such that $H^n \to H$ in L^p, as $n \to \infty$. In the next result we give converging approximations for the mean values defined previously.

Theorem 6. *For any $n \in \mathbb{N}$, we denote*

$$C^n[\rho_n, \sigma_n, H^n(r,s,t)] = \sigma_n A[H^n(r,s,t)] + B[H(r,s,t)] - b - \rho_n c$$

and

$$\lambda^n(r,s,t) = (\lambda_1^n(r,s,t), \dots, \lambda_m^n(r,s,t)),$$

where $\lambda_j^n(r,s,t) = C_l^n[\rho_n, \sigma_n, H^n(r,s,t)]$ for all paths R_l connecting W_j, for which $H_l^n(r,s,t) > 0$, $\mathbb{P}$-a.s.. Let $\rho(r,s,t) = r$, $\sigma(r,s,t) = s$. If $\rho_n \to \rho$ strongly in L^q, $\sigma_n \to \sigma$ strongly in L^∞, and $H^n \to H$ strongly in L^p, then the sequence

$$\{E_{\mathbb{P}}[\lambda^n]\}_n = \left\{ \int \lambda^n(r,s,t)d\mathbb{P}(r,s,t) \right\}_n$$

converges to $E_{\mathbb{P}}[\lambda]$, as $n \to \infty$. Moreover, $Var(\lambda^n) \to Var(\lambda)$.

Proof. Since $H^n \to H$ strongly in L^p, it follows that $A[H^n] \to A[H]$ and $B[H^n] \to B[H]$, strongly in $L^q = L^{\frac{p}{p-1}}$ because of the continuity of the Nemytskii operators A and B. Moreover, $\rho_n \to \rho$ strongly in L^q and $\sigma_n \to \sigma$ strongly in L^∞. As a consequence,

$$\sigma_n A[H^n] + B[H^n] - b - \rho_n c \;\to\; \sigma A[H] + B[H] - b - \rho c$$

strongly in L^q, and also strongly in L^1 because $\mathbb{P}$ is a probability measure. Hence, for each $i = 1, \dots, k$, we get $C_i^n[\rho_n, \sigma_n, H^n] \to C_i[r,s,H]$ strongly in L^1. Moreover, since $q > 2$ strong convergence in L^q also implies convergence of variances and, by the definitions of λ and λ^n, the thesis is proved. $\square$

3.3 Numerical experiments

We now report some numerical tests obtained by implementing the approximation and regularization procedures described in the previous sections. We consider a stochastic framework where both the traffic demands and the cost functions are affected by random perturbations. In particular, we assume that the random Wardrop equilibria depend on random vectors $r = R(\omega)$ and $t = D(\omega)$. The numerical computation of random Wardrop equilibria has been performed by implementing in Matlab 2018a the approximation and regularization procedures described in Section 2.2 combined with the algorithm designed in [15] for deterministic Wardrop equilibria.

Example 1. We consider the network consisting of 5 nodes and 6 links shown in Fig. 1. We assume that (1,5) is the only O-D pair and the traffic demand is $D = 100 + \delta$, where δ is a random variable which varies in the interval $[-10, 10]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 2.5.

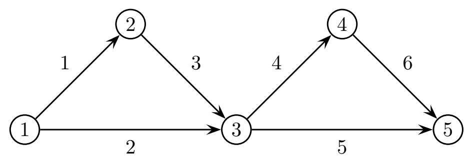

Fig. 1 Test network of Example 1.

The deterministic link cost functions are of the BPR form (17) defined as follows:

$$
\begin{aligned}
c_1 &= 0.5\left[1 + 0.15\,(f_1/5)^4\right], & c_2 &= 1 + 0.15\,(f_2/10)^4, \\
c_3 &= 0.5\left[1 + 0.15\,(f_3/5)^4\right], & c_4 &= 0.5\left[1 + 0.15\,(f_4/5)^4\right], \\
c_5 &= 1 + 0.15\,(f_5/10)^4, & c_6 &= 0.5\left[1 + 0.15\,(f_6/10)^4\right].
\end{aligned}
$$

The O-D pair is connected by four paths. We assume that the path cost operator is defined as in (27), where $S = 0$, $B(H) - b$ represents the deterministic path costs corresponding to the above link cost functions, while $c = -(1, \ldots, 1)$ and $r = R(\omega)$ is a random variable which varies in the interval $[0, 200]$ with either uniform distribution or truncated normal distribution with mean 100 and standard deviation 25.

Notice that in this case the link-path incidence matrix Δ has not full column rank and the path cost operator is monotone but not strongly monotone. Moreover, since the deterministic part of the path cost operator is polynomial with degree 4, the operator C satisfies the growth condition (23) with $p = 5$. Therefore, the approximated regularized variational inequality (13) cannot be decomposed into a large number of small size variational inequalities.

Both the intervals $[-10, 10]$ and $[0, 200]$ have been partitioned into N_I subintervals in the approximation procedure and the regularization parameter ε has been chosen equal to $1/(N_I)^6$.

Table 1 shows the convergence of the mean values and standard deviations of the cost at equilibrium λ for increasing values of N_I, assuming that the random variables

N_I	Cost at equilibrium Mean Value	Std Deviation
5	545.825	121.69
10	546.146	123.68
15	546.205	124.04
20	546.226	124.17
25	546.236	124.23
30	546.241	124.26

Table 1 Mean values and standard deviation of the cost at equilibrium when the random variables vary with uniform distribution.

N_I	Cost at equilibrium Mean Value	Std Deviation
5	537.218	48.54
10	537.469	52.12
15	537.524	52.87
20	537.544	53.15
25	537.553	53.27
30	537.559	53.34

Table 2 Mean values and standard deviation of the cost at equilibrium when the random variables vary with truncated normal distribution.

δ and r vary with uniform distribution. Similarly, Table 2 shows the mean values and standard deviations of λ, when δ and r vary with truncated normal distribution.

Example 2. We now consider the grid network shown in Fig. 2 consisting of 36 nodes and 60 links. We assume that there are three O-D pairs: (1,18), (13,30), (19,36) with traffic demands equal to $D = d + \delta(1,1,1)$, where $d = (150,100,200)$ and δ is a random variable which varies in the interval $[-50,50]$ with either uniform distribution or truncated normal distribution with mean 0 and standard deviation 10.

The deterministic link cost functions are of the BPR form (17) with $\gamma = 0.15$ and $\beta = 4$ for all the links, while $t_i^0 = 1$ and $u_i = 50$ for any $i = 1,\dots,30$, and $t_i^0 = 5$ and $u_i = 100$ for any $i = 31,\dots,60$.

We assume that the path cost operator is defined as in (27), where $S = 0$, $B(H) - b$ represents the deterministic path costs corresponding to the above link cost functions, while $c = -(1,\dots,1)$ and $r = R(\omega)$ is a random variable which varies in the interval $[0,20]$ with either uniform distribution or truncated normal distribution with mean 10 and standard deviation 2.

Notice that the link-path incidence matrix Δ has not full column rank since the total number of paths is greater than the number of links. Hence, the path cost operator is monotone but not strongly monotone. Moreover, similarly to Example 1, the cost operator satisfies the growth condition (23) with $p = 5$.

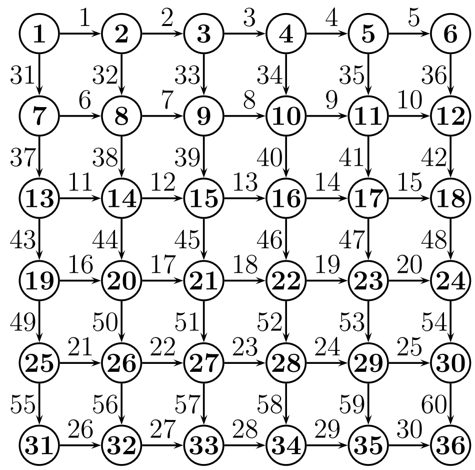

Fig. 2 Test network of Example 2.

Both the intervals $[-50,50]$ and $[0,20]$ have been partitioned into N_I subintervals in the approximation procedure and the regularization parameter ε has been chosen equal to $1/(N_I)^6$.

	Costs at equilibrium					
	Mean Values			Std Deviations		
N_I	(1,18)	(13,30)	(19,36)	(1,18)	(13,30)	(19,36)
5	19.860	22.011	22.735	3.366	4.869	5.080
10	19.940	22.044	22.833	3.456	4.942	5.229
15	19.970	22.073	22.867	3.476	4.997	5.251
20	19.974	22.078	22.871	3.481	5.012	5.256
25	19.976	22.080	22.873	3.483	5.018	5.259

Table 3 Mean values and standard deviations of the costs at equilibrium when the random variables vary with uniform distribution.

Tables 3 and 4 show the convergence of the mean values and standard deviations of the costs at equilibrium λ of the three O-D pairs, assuming that the random variables δ and r vary with uniform distribution or with truncated normal distribution, respectively.

	Costs at equilibrium					
	Mean Values			Std Deviations		
N_I	(1,18)	(13,30)	(19,36)	(1,18)	(13,30)	(19,36)
5	19.089	20.854	21.574	0.973	1.335	1.513
10	19.112	20.901	21.626	1.082	1.534	1.653
15	19.121	20.907	21.633	1.106	1.563	1.692
20	19.134	20.911	21.639	1.112	1.573	1.705
25	19.140	20.913	21.644	1.114	1.589	1.708

Table 4 Mean values and standard deviations of the costs at equilibrium when the random variables vary with truncated normal distribution.

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