ADAPTING TO CLIMATE CHANGE: A TWO-STAGE NASH EQUILIBRIUM MODEL OF COALITION FORMATION

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Abstract. We propose a two-stage noncooperative game theoretic model to describe the coalition formation of countries which decide to jointly invest in research and developments projects to mitigate the damages induced by climate changes. The first-stage game is a finite game where each country has only two strategies: sign or not an international agreement with other countries, while the second-stage game is a generalized Nash equilibrium problem, where each country aims to find the optimal levels of pollutant emission and effort in research and development projects. The variational equilibrium of the second-stage game is reformulated as an equivalent variational inequality with a reduced number of variables and the monotonicity of the corresponding operator is investigated. Finally, the impact of the coalition on the reduction of the environmental damage is numerically investigated.

Keywords. Two-stage noncooperative game; Nash equilibrium; Variational inequality; Monotone operator.

1. INTRODUCTION

The well justified need to establish international agreements to jointly fight the adverse effects of climate change led in 1992 to the United Nations Framework Convention on Climate Change (UNFCCC), which is a multilateral treaty to stabilize greenhouse gas (GHG) concentrations “at a level that would prevent dangerous anthropogenic (human-induced) interference with the climate system”. The subsequent Kyoto Protocol, in 1997, aimed to commit the signatory countries (Parties) to reduce the GHG emissions, starting from 2005, but was not implemented by most of the Parties. A new, legally binding, international treaty was then signed in Paris in 2015 by 196 countries, and every year countries who have joined the UNFCCC meet to measure progress and negotiate multilateral responses to climate change, with the last conference of the parties (COP) being held at the end of 2023 in Dubai. COPs are crucial in bringing governments together while also mobilizing the private sector, civil society, industry and individuals to tackle the climate crisis. While reducing GHG emissions is considered a priority, in the last years the COPs also focused on accelerating a global energy transition, and helping countries adapt and build resilience climate damages. The new key-word launched by the COPs was thus *adaptation*, in the broad sense of adjustments in ecological, economic or social systems to counterbalance the already occurred, or potential, impact of climate change. It was thus suggested

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that international agreements also include the possibility that the Parties cooperate in research aimed at developing new strategies and technologies to implement adaptation.

Mathematical models of international agreements on climate change have then been designed to incorporate adaptation [3, 4, 7, 9, 15]. Our treatment is along the same lines as in [14, 16, 17] and further develop the Game Theory model put forward in [21]. Specifically, we assume that each country controls two variables: the level of pollutant emissions (related to its industrial production) and the amount of financial resources spent on research and development (R&D) to find out new adaptation technologies. Each country aims to optimize a utility function including the revenue due to the industrial production, the investment costs in R&D and the environmental impact. We consider the possibility that some of the countries jointly invest in some research project to increase the reduction of the environmental impact of pollutant emissions. Since the environmental impact measured by each country depends on the pollutant emissions of all the countries, we formulate the problem of finding the optimal levels of pollutant emissions and investments in R&D of each country as a generalized Nash equilibrium problem (GNEP) (see, e.g., [5, 19, 25]), where the countries that jointly invest in R&D projects share a budget constraint. We focus on the variational equilibria of such GNEP and prove that, under mild assumptions, they can be found by solving a reduced variational inequality which only contains the variables which describe the investments in R&D. Moreover, we provide some conditions on the model's parameters that guarantee the strong monotonicity of the map of the reduced variational inequality and hence the uniqueness of the variational equilibrium of the GNEP. The analysis above holds for a fixed coalition and does not investigate the problem of the coalition formation, i.e., whether a country should find convenient to be a member of a research team with other countries. For this, we consider a two-stage game model, to be solved backwards, that is, we first look for the variational equilibrium corresponding to all possible coalitions and then model the membership problem as a finite Nash game, where each country has a binary variable describing its membership in the international coalition. Our numerical experiments show that the *grand coalition*, with all countries counted in, is a Nash equilibrium of the membership game. Moreover, we provide a sensitivity analysis of the total pollutant emission and total environmental impact at equilibrium with respect to some model parameters.

The contribution of the paper is twofold. From a modeling point of view, we propose a two-stage game model, where the first-stage finite game formulates the membership problem, while the second-stage game is a GNEP with shared constraints. Our model generalizes both the approach in [21], where only the second-stage game is considered, and in [16, 17], where the model parameters do not depend on the specific country and a GNEP model is not considered. From a methodological point of view, we reformulate the variational equilibrium of the second-stage game as an equivalent variational inequality with a reduced number of variables and investigate the monotonicity of the corresponding operator.

The paper is structured as follows. In the subsequent Section 2 we introduce the notation and formulate the two-stage game model. In Section 3, we assume that the emissions of all countries are strictly positive and, by suitably exploiting the KKT conditions associated to our problem, show that the second-stage GNEP can be reformulated as one which involves the investment variables only. The case where no coalition is formed is also investigated and we show how it can be cast in the framework of the so called Network Games (see, e.g., [11]); this section also contains a brief recall on the variational inequality approach to be used. Section 4 is devoted to investigate the monotonicity properties of the operator of our variational inequality (the so called *pseudogradient* of the game). In Section 5 we discuss some numerical examples of our model, while Section 6 outlines some future research directions.

2. TWO-STAGE GAME MODEL

Throughout the paper, vectors in $\mathbb{R}^m$ are understood as columns, when involved in matrix operations, $a^\top$ denotes the transpose of vector a and $a^\top b$ the standard scalar product in $\mathbb{R}^m$. Furthermore, as is usual in Game Theory, the notation $x_{-i} = (x_1,\dots,x_{i-1},x_{i+1},\dots,x_m)$ and $x = (x_i,x_{-i}) = (x_1,\dots,x_m)$ will be used when we want to distinguish the role of x_i from all the other components of vector x. We are then given a set of M countries, whose industrial production generates pollution, e.g., GHG emissions. A standard assumption in economic models is that the industrial production P_i of country i generates a pollutant emission e_i, which is an increasing function of the production, such that zero production implies zero emissions; accordingly the revenue of a country can be expressed as a function of its emissions. A standard functional form of the revenue R_i is (see, e.g., [14, 16, 17])

$$R_i(e_i) = \alpha_i\, e_i - \frac{1}{2}e_i^2, \qquad \alpha_i > 0.$$

The environmental impact D_i for each country i can depend from the emissions of all the countries. Specifically, we assume that, in absence of mitigation strategies, the impact is given by

$$D_i(e) = \beta_i \sum_{j=1}^{M} e_j, \qquad \beta_i > 0. \tag{2.1}$$

Let us now denote by k_i the effort (money investment) in R&D of country i, which entails a cost given by

$$C_i(k_i) = \frac{c}{2}k_i^2, \qquad c > 0,$$

and assume that each country assigns to this activity a maximum budget of $\bar{k}_i$. The outcome of the research activity of a given country, will allow a reduction of the environmental impact (2.1), which can thus be updated to

$$D_i(e,k_i) = (\beta_i - \theta\, k_i) \sum_{j=1}^{M} e_j, \tag{2.2}$$

where the parameter $\theta > 0$ defines the adaptability (see [16]). We assume $\beta_i > \theta k_i$ to ensure that the environmental damage cannot be transformed to a benefit, and define $K_i := \min\{\bar{k}_i, \beta_i/\theta\}$ for each $i = 1,\dots,M$.

In the case where a group I of S countries decide to jointly develop research and strategies to mitigate the environmental damages, and sign a binding agreement, the utility function of each signatory country $i \in I$ is given by

$$u_i^{\mathscr{S}}(e,k) = \alpha_i e_i - \frac{1}{2}e_i^2 - \frac{c}{2}k_i^2 - \left(\beta_i - \theta \sum_{j\in I} k_j\right) \sum_{j=1}^{M} e_j, \tag{2.3}$$

that is the difference between the revenue and the sum of the investment cost and the environmental impact. Notice that the reduction of the environmental impact depends on the joint investment of all the signatory countries.

On the other hand, the utility function of each non-signatory country $i \notin I$ is

$$u_i(e,k) = \alpha_i e_i - \frac{1}{2}e_i^2 - \frac{c}{2}k_i^2 - (\beta_i - \theta k_i)\sum_{j=1}^{M} e_j, \tag{2.4}$$

where the reduction of the environmental impact depends on only its own investment k_i.

Clearly, the case where no coalition is formed, or the case of all countries allied (*grand coalition*) can be viewed as a special case of the formulation above. We assume $c > \theta^2$ so that the standard concavity assumption on both $u_i(e,k)$ and $u_i^{\mathscr{S}}(e,k)$, with respect to (e_i,k_i) for each fixed value of (e_{-i},k_{-i}), is satisfied.

Each country must make two types of decisions: first, it must decide whether to sign a binding agreement with other countries to be part of an international coalition; then, once the coalition is formed, it must decide on the optimal level of pollutant emissions and investment in R&D. Since the utility of each country depends on the choices made by other countries, both in terms of the coalition to be formed and the levels of pollutant emissions and investment in research and development, we formulate this problem as a two-stage non-cooperative game where the players are the countries considered. In the first-stage game we endow each player i with a binary variable s_i such that:

$$s_i = \begin{cases} 1 & \text{if player } i \text{ signs the agreement,} \\ 0 & \text{otherwise.} \end{cases}$$

Once the coalition represented by the vector s is fixed, we define the second-stage game as a GNEP, where each signatory player i, i.e., with $s_i = 1$, aims to solve the problem

$$\max_{e_i,k_i} u_i^{\mathscr{S}}(e,k) \tag{2.5a}$$

$$\text{s.t. } e_i \geq 0, \tag{2.5b}$$

$$k_i \geq 0, \tag{2.5c}$$

$$\sum_{j\in I} k_j \leq K_I, \tag{2.5d}$$

where $I = \{i: \ s_i = 1\}$ and $K_I = \sum_{j\in I} K_j$, while each non-signatory player i, i.e., with $s_i = 0$, aims to solve the problem

$$\max_{e_i,k_i} u_i(e,k) \tag{2.6a}$$

$$\text{s.t. } e_i \geq 0, \tag{2.6b}$$

$$0 \leq k_i \leq K_i. \tag{2.6c}$$

Notice that the second-stage game is a GNEP because constraints (2.5d) are shared between the players joining the coalition. Without loss of generality, we assume that the set I of players joining the coalition is $I = \{1,\dots,S\}$. Let $z_i = (e_i,k_i)$ for any $i = 1,\dots,M$, $z = (z_1,\dots,z_M)$, and, with a slight abuse of a notation, write $u_i(e,k) = u_i(z)$ and $u_i^{\mathscr{S}}(e,k) = u_i^{\mathscr{S}}(z)$. Moreover, let

$$Z = \left\{ z \in \mathbb{R}_+^{2M} : \ \sum_{j\in I} k_j \leq K_I, \quad k_i \leq K_i, \ \forall\, i \notin I \right\}.$$

A simultaneous solution of the optimization problems (2.5)–(2.6) is thus a generalized Nash equilibrium specified as follows.

Definition 2.1 (Generalized Nash equilibrium of the second-stage game)**.** A point $z^* \in Z$ is a generalized Nash equilibrium of the second-stage game iff

$$u_i^{\mathscr{S}}(z_i^*, z_{-i}^*) \geq u_i^{\mathscr{S}}(z_i, z_{-i}^*), \qquad \forall\, z_i \text{ such that } (z_i, z_{-i}^*) \in Z, \qquad \forall\, i \in I, \tag{2.7}$$

$$u_i(z_i^*, z_{-i}^*) \geq u_i(z_i, z_{-i}^*), \qquad \forall\, z_i \text{ such that } (z_i, z_{-i}^*) \in Z, \qquad \forall\, i \notin I. \tag{2.8}$$

It is well known that GNEPs may have infinite solutions (see, e.g., [25]) among which the so called variational ones are of particular interest from the socio-economic point of view. Indeed, variational solutions are characterized by the fact that the Lagrange multipliers corresponding to the shared constraints are the same for all players. In our problem, if, at equilibrium, the investments are positive for all members of the coalition, then the marginal utilities, $\partial u_i^{\mathscr{S}}/\partial k_i$ of all the players in the coalition are the same, which can be considered as a desirable fairness requirement. A variational equilibrium z^* of the second-stage game is any solution of the variational inequality $VI(Z,T)$: find $z^* \in Z$ such that

$$T(z^*)^\top (z - z^*) \geq 0, \qquad \forall\, z \in Z, \tag{2.9}$$

where $T : \mathbb{R}^{2M} \to \mathbb{R}^{2M}$ is the pseudo-gradient of the game, i.e.,

$$T(z) = -\left(\nabla_{z_1} u_1^{\mathscr{S}}(z), \ldots, \nabla_{z_S} u_S^{\mathscr{S}}(z), \nabla_{z_{S+1}} u_{S+1}(z), \ldots, \nabla_{z_M} u_M(z)\right).$$

If for each coalition, identified by the vector s, the second-stage game yields a unique variational equilibrium $z^*(s) = (e^*(s), k^*(s))$ (Theorem 4.1 below provides sufficient conditions for the uniqueness of the variational equilibrium), then we can define the utility function of each player i in the first-stage game as

$$U_i(s) = \begin{cases} u_i^{\mathscr{S}}(z^*(s)) & \text{if } s_i = 1, \\ u_i(z^*(s)) & \text{if } s_i = 0, \end{cases}$$

that is the utility of player i evaluated at the variational equilibrium $z^*(s)$ of the second-stage game. Thus, the first-stage game (or membership problem) is a M-person finite game where each player i aims to solve the problem

$$\max_{s_i} \; U_i(s) \tag{2.10a}$$

$$\text{s.t. } \; s_i \in \{0,1\}. \tag{2.10b}$$

A vector s^* is therefore a Nash equilibrium of the first-stage game if every player who is in the coalition has no incentive in getting out, and every player who is out has no incentive in getting in.

Definition 2.2 (Nash equilibrium of the first-stage game)**.** s^* is a Nash equilibrium of the first-stage game iff

$$U_i(s_i^*, s_{-i}^*) \geq U_i(1 - s_i^*, s_{-i}^*) \tag{2.11}$$

holds for each player $i = 1, \ldots, M$.

We remark that a trivial Nash equilibrium of the first-stage game is $s^* = (0, \ldots, 0)$, that is, the case where no country signs the international agreement. For the reader's convenience we summarize the main features of the first-stage and second-stage games in Table 1.

	Game	Variables	Player i's problem
First-stage	Finite	$s_i \in \{0,1\}$	(2.10)
Second-stage	GNEP	$z_i = (e_i, k_i)$	(2.5) if $s_i = 1$ (2.6) if $s_i = 0$

TABLE 1. Two-stage game model.

3. Reduced Second-Stage Game and Variational Inequality formulation

We now show that under suitable assumptions on the parameters, the variational equilibria of the second-stage game correspond to the ones of a game involving the k_i variables only.

Theorem 3.1. *If $\alpha_i > \beta_i$ for any $i = 1,\dots,M$, then $z^* = (e^*, k^*)$ is a variational equilibrium of the second-stage game if and only if k^* solves the variational inequality $VI(\Omega, F)$: find $k^* \in \Omega$ such that*

$$F(k^*)^\top (k - k^*) \geq 0, \qquad \forall\, k \in \Omega, \tag{3.1}$$

where

$$\Omega = \left\{ k \in \mathbb{R}^M_+ : \quad \sum_{i=1}^{S} k_i \leq K_I, \quad k_i \leq K_i, \;\; \forall\, i = S+1,\dots,M \right\}, \tag{3.2}$$

$$F(k) = (cI + A)k + q, \tag{3.3}$$

with

$$A = -\theta^2 \begin{bmatrix} S & \dots & S & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ S & \dots & S & 1 & \dots & 1 \end{bmatrix}, \qquad q = \left(-\theta \sum_{i=1}^{M} (\alpha_i - \beta_i), \dots, -\theta \sum_{i=1}^{M} (\alpha_i - \beta_i) \right)^\top,$$

and

$$e_i^* = \begin{cases} \alpha_i - \beta_i + \theta \sum\limits_{j \in I} k_j^* & \forall\, i \in I, \\ \alpha_i - \beta_i + \theta k_i^* & \forall\, i \notin I. \end{cases}$$

Moreover, if $c \geq S\theta^2$, then the solutions of $VI(\Omega, F)$ are the variational equilibria of the reduced GNEP where each signatory country ($i \in I$) aims to solve

$$\max_{k_i} \; v_i^{\mathscr{S}}(k) = -\frac{1}{2}(c - S\theta^2)k_i^2 + S\theta^2 k_i \sum_{\substack{j \in I \\ j \neq i}} k_j + \theta^2 k_i \sum_{j \notin I} k_j + \theta k_i \sum_{j=1}^{M} (\alpha_j - \beta_j) \tag{3.4a}$$

$$\text{s.t.} \;\; k_i \geq 0, \tag{3.4b}$$

$$\sum_{j \in I} k_j \leq K_I, \tag{3.4c}$$

and each non signatory country ($i \notin I$) aims to solve

$$\max_{k_i} v_i(k) = -\frac{1}{2}(c-\theta^2)k_i^2 + S\theta^2 k_i \sum_{j\in I} k_j + \theta^2 k_i \sum_{\substack{j\notin I\\ j\neq i}} k_j + \theta k_i \sum_{j=1}^{M}(\alpha_j - \beta_j) \tag{3.5a}$$

$$s.t.\quad 0 \le k_i \le K_i. \tag{3.5b}$$

Proof. Solving $VI(Z,T)$ is equivalent to solve the corresponding KKT system:

$$e_i - \alpha_i + \beta_i - \theta \sum_{j\in I} k_j - \lambda_i = 0, \qquad \forall\, i \in I, \tag{3.6a}$$

$$ck_i - \theta e_i - \theta \sum_{\substack{i=1\\ j\neq i}}^{M} e_j - \mu_i + \rho = 0, \qquad \forall\, i \in I, \tag{3.6b}$$

$$e_i - \alpha_i + \beta_i - \theta k_i - \lambda_i = 0, \qquad \forall\, i \notin I, \tag{3.6c}$$

$$ck_i - \theta e_i - \theta \sum_{\substack{j=1\\ j\neq i}}^{M} e_j - \mu_i + \nu_i = 0, \qquad \forall\, i \notin I, \tag{3.6d}$$

$$e_i\lambda_i = 0, \qquad \lambda_i \ge 0, \quad e_i \ge 0, \qquad \forall\, i = 1,\dots,M, \tag{3.6e}$$

$$\mu_i k_i = 0, \qquad \mu_i \ge 0, \quad k_i \ge 0, \qquad \forall\, i = 1,\dots,M, \tag{3.6f}$$

$$\rho\Big(\sum_{j\in I} k_j - K_I\Big) = 0, \qquad \rho \ge 0, \quad \sum_{j\in I} k_j \le K_I, \tag{3.6g}$$

$$\nu_i(k_i - K_i) = 0, \qquad \nu_i \ge 0, \quad k_i \le K_i, \qquad \forall\, i \notin I, \tag{3.6h}$$

where λ_i is the multiplier associated with the nonnegativity of emission e_i, μ_i is the multiplier associated with the nonnegativity of investment k_i, ν_i is the multiplier associated with the upper bound for the investment $k_i \le K_i$ of the players outside the coalition, and ρ is the multiplier associated with coalition's investment upper bound $\sum_{j\in I} k_j \le K_I$. Let us notice that all the players $i \in I$ share the same Lagrange multiplier associated to the shared constraint (see, e.g., the discussion in [5, 19, 25]).

If $\alpha_i > \beta_i$ for any $i = 1,\dots,M$, then we get:

$$e_i = \begin{cases} \alpha_i - \beta_i + \theta \sum_{j\in I} k_j + \lambda_i \ge \alpha_i - \beta_i > 0 & \forall\, i \in I,\\ \alpha_i - \beta_i + \theta k_i + \lambda_i \ge \alpha_i - \beta_i > 0 & \forall\, i \notin I. \end{cases}$$

Hence, the complementarity conditions imply $\lambda_i = 0$ for any $i = 1,\dots,M$. We then have:

$$e_i = \begin{cases} \alpha_i - \beta_i + \theta \sum_{j\in I} k_j, & \forall\, i \in I\\ \alpha_i - \beta_i + \theta k_i, & \forall\, i \notin I, \end{cases} \tag{3.7}$$

that is, the emissions can be expressed as a function of the investments. Let us further notice that $k_i > 0$ for any $i \notin I$. In fact, assume by contradiction that $k_i = 0$, then (3.6h) implies $\nu_i = 0$ and (3.6d) gives $\mu_i = -\theta \sum_{j=1}^{M} e_j < 0$ which is a contradiction. Then, it must be $k_i > 0$ and accordingly $\mu_i = 0$ holds for any $i \notin I$. We then use (3.7) to express the sum of all emissions as:

$$\sum_{i=1}^{M} e_i = \sum_{i=1}^{M} (\alpha_i - \beta_i) + \theta S \sum_{j \in I} k_j + \theta \sum_{j \notin I} k_j. \tag{3.8}$$

We then use the above expression in the KKT conditions (3.6b) and (3.6d)-(3.6h) to get rid of the emission variables:

$$ck_i - \theta^2 S k_i - \theta^2 S \sum_{j \neq i, j \in I} k_j - \theta^2 \sum_{j \notin I} k_j - \theta \sum_{i=1}^{M} (\alpha_i - \beta_i) - \mu_i + \rho = 0, \qquad \forall\, i \in I, \tag{3.9a}$$

$$ck_i - \theta^2 S \sum_{j \in I} k_j - \theta^2 k_i - \theta^2 \sum_{\substack{j \notin I \\ j \neq i}} k_j - \theta \sum_{i=1}^{M} (\alpha_i - \beta_i) - \mu_i + \nu_i = 0, \qquad \forall\, i \notin I, \tag{3.9b}$$

$$\mu_i k_i = 0, \qquad \mu_i \geq 0, \qquad k_i \geq 0, \qquad \forall\, i = 1, \ldots, M, \tag{3.9c}$$

$$\rho\Big(\sum_{j \in I} k_j - K_I\Big) = 0, \qquad \rho \geq 0, \qquad \sum_{j \in I} k_j \leq K_I, \tag{3.9d}$$

$$\nu_i (k_i - K_i) = 0, \qquad \nu_i \geq 0, \qquad k_i \leq K_i, \qquad \forall\, i \notin I. \tag{3.9e}$$

It is easy to check that (3.9) is the KKT system corresponding to the variational inequality $VI(\Omega, F)$.

Finally, if $c \geq \theta^2 S$, then the objective functions $v_i^{\mathscr{S}}$ and v_i are concave with respect to k_i and the operator F is the pseudo-gradient of the reduced game. Thus, the variational equilibria of the reduced game are the solutions of $VI(\Omega, F)$. □

Remark 3.1. The special case of the reduced game where no international agreement is signed, that is, each country invests independently in adaptation technologies, can be framed in the class of Network Games [11, 22, 24]. Indeed, when $I = \emptyset$, instead of formulas (3.4a) and (3.5a), we only have, for any $i = 1, \ldots, M$:

$$\max_{0 \leq k_i \leq K_i} v_i(k) = -\frac{1}{2}(c - \theta^2) k_i^2 + \theta^2 k_i \sum_{\substack{j=1 \\ j \neq i}}^{M} k_j + \theta k_i \sum_{j=1}^{M} (\alpha_j - \beta_j). \tag{3.10}$$

Let us now put: $\gamma := \frac{\theta}{c - \theta^2} \sum_{i=1}^{M} (\alpha_j - \beta_j)$ and $\delta := \frac{\theta^2}{c - \theta^2}$. Then, the minimization problems (3.10) can be written as

$$\max_{0 \leq k_i \leq K_i} -\frac{1}{2} k_i^2 + \gamma k_i + \delta k_i \sum_{\substack{j=1 \\ j \neq i}}^{M} k_j. \tag{3.11}$$

It can be easily proven that the simultaneous solution of the optimization problems (3.11) amounts to solve the system

$$k_i = \min \left\{ K_i,\ \gamma + \delta \sum_{\substack{j=1 \\ j \neq i}}^{M} k_j \right\}, \qquad i = 1, \ldots, M. \tag{3.12}$$

The latter system can be solved in closed form in the case where $K_i = +\infty$ for any $i = 1, \ldots, M$. To that end, let us put $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^M$, denote with G the matrix with entries are $g_{ij} = 1$ if

$i \neq j$ and $g_{ii} = 0$ and observe that G can be interpreted as the adjacency matrix of an undirected complete graph. Then, system (3.12) reads as

$$(I - \delta G)k = \gamma \mathbf{1}.$$

If the spectral radius $r(G)$ of G is such that $r(G)\delta < 1$, the matrix $(I - \delta G)$ is nonsingular and the solution can be represented as the following series expansion:

$$k = \gamma (I - \delta G)^{-1} \mathbf{1} = \gamma \sum_{p=0}^{\infty} \delta^p G^p \mathbf{1}.$$

The series expansion above, involving the powers of the adjacency matrix of a graph, has been introduced in the seminal paper [2] in a network model of social relationships. Let us notice that with G given above, the spectral radius is $M - 1$.

Since our approach to Nash equilibrium problems is based on variational inequalities, we now recall some concepts which will be used to ensure the existence and the uniqueness of solution to a finite-dimensional variational inequality problem (see, e.g., [6, 13, 20]).

Definition 3.1. A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on $K \subset \mathbb{R}^n$ iff:

$$[T(x) - T(y)]^\top (x - y) \geq 0, \qquad \forall\, x, y \in K.$$

If the equality holds only when $x = y$, T is said to be strictly monotone.
T is said to be β-strongly monotone on K iff there exists $\beta > 0$ such that

$$[T(x) - T(y)]^\top (x - y) \geq \beta \|x - y\|^2, \qquad \forall\, x, y \in K.$$

For linear operators on $\mathbb{R}^n$ the two concepts of strict and strong monotonicity coincide and are equivalent to the positive definiteness of the corresponding matrix. Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g., [6, 20]).

Theorem 3.2. *If $K \subset \mathbb{R}^n$ is a compact convex set and $T : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on K, then the variational inequality problem $VI(K, T)$ admits at least one solution. In the case K is unbounded, existence of a solution may be established under the following coercivity condition:*

$$\lim_{\|x\| \to +\infty} \frac{[T(x) - T(x_0)]^\top (x - x_0)}{\|x - x_0\|} = +\infty,$$

for $x \in K$ and some $x_0 \in K$. Furthermore, if T is strictly monotone on K the solution is unique.

4. Monotonicity Properties

In this section we give conditions which guarantee the strong monotonicity of the map F defined in (3.3).

Theorem 4.1. *If*

$$c > \frac{S^2 + M - S + \sqrt{M(S^3 + M - S)}}{2}\, \theta^2, \tag{4.1}$$

then the map F defined in (3.3) is strongly monotone on $\mathbb{R}^M$. Moreover, if $c > M^2\theta^2$, then F is strongly monotone on $\mathbb{R}^M$ for every coalition $I \subseteq \{1, \ldots, M\}$.

Proof. To study the strong monotonicity of the affine map F, we analyze the positive definiteness of its Jacobian matrix $\nabla F = cI + A$. We start with considering the symmetric part of the matrix $\frac{A}{-\theta^2}$:

$$B := \frac{A + A^\top}{-2\theta^2} = \begin{array}{c|ccc|ccc|} & 1 & \dots & S & S+1 & \dots & M \\ \hline 1 & S & \dots & S & \frac{S+1}{2} & \dots & \frac{S+1}{2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ S & S & \dots & S & \frac{S+1}{2} & \dots & \frac{S+1}{2} \\ \hline S+1 & \frac{S+1}{2} & \dots & \frac{S+1}{2} & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M & \frac{S+1}{2} & \dots & \frac{S+1}{2} & 1 & \dots & 1 \end{array} \tag{4.2}$$

To find the eigenvalues of B, let us group the components of the eigenvector v as $(v_1, \dots, v_S, v_{S+1}, \dots, v_M)$ and write $Bv = \lambda v$ accordingly:

$$\begin{cases} Sv_1 + \dots + Sv_S + \frac{S+1}{2} v_{S+1} + \dots + \frac{S+1}{2} v_M = \lambda v_1, \\ \quad\vdots \\ Sv_1 + \dots + Sv_S + \frac{S+1}{2} v_{S+1} + \dots + \frac{S+1}{2} v_M = \lambda v_S, \\ \frac{S+1}{2} v_1 + \dots + \frac{S+1}{2} v_S + v_{S+1} + \dots + v_M = \lambda v_{S+1}, \\ \quad\vdots \\ \frac{S+1}{2} v_1 + \dots + \frac{S+1}{2} v_S + v_{S+1} + \dots + v_M = \lambda v_M. \end{cases} \tag{4.3}$$

Notice that 0 is an eigenvalue of B with multiplicity $M-2$ since $\text{rank}(B) = 2$. In order to find the remaining eigenvalues, we sum up the first S equations in the system above, and then the remaining $M-S$ equations, to obtain the following system of two equations:

$$\begin{cases} S\left(S\sum_{i=1}^{S} v_i + \frac{S+1}{2} \sum_{i=S+1}^{M} v_i\right) = \lambda \sum_{i=1}^{S} v_i \\ (M-S)\left(\frac{S+1}{2} \sum_{i=1}^{S} v_i + \sum_{i=S+1}^{M} v_i\right) = \lambda \sum_{i=S+1}^{M} v_i. \end{cases} \tag{4.4}$$

Let us now set: $\sigma = \sum_{i=1}^{S} v_i$ and $\tau = \sum_{i=S+1}^{M} v_i$, and get the system:

$$\begin{cases} (S^2 - \lambda)\sigma + \frac{S(S+1)}{2}\tau = 0 \\ \frac{(M-S)(S+1)}{2}\sigma + (M-S-\lambda)\tau = 0. \end{cases} \tag{4.5}$$

The condition that the determinant Δ of the system above is zero reads

$$\lambda^2 - (S^2 + M - S)\lambda + S^2(M-S) - \frac{S(M-S)(S+1)^2}{4} = 0,$$

which yields to

$$\begin{aligned}\lambda_{1,2} &= \frac{S^2+M-S\pm\sqrt{(M-S+S^2)^2+S(M-S)(S+1)^2-4S^2(M-S)}}{2}\\ &= \frac{S^2+M-S\pm\sqrt{M(S^3+M-S)}}{2}.\end{aligned}$$

Therefore, the eigenvalues of B are

$$\begin{cases}\lambda_1 = \frac{S^2+M-S-\sqrt{M(S^3+M-S)}}{2} < 0 \text{ with multiplicity } 1,\\ \lambda_2 = \frac{S^2+M-S+\sqrt{M(S^3+M-S)}}{2} > 0 \text{ with multiplicity } 1,\\ \lambda_3 = 0 \text{ with multiplicity } M-2.\end{cases}$$

Finally, the eigenvalues of $\frac{1}{2}(\nabla F + \nabla F^\top)$ are

$$\begin{cases}c-\theta^2\lambda_1 > 0 \text{ with multiplicity } 1,\\ c-\theta^2\lambda_2 > 0 \text{ with multiplicity } 1,\\ c > 0 \text{ with multiplicity } M-2,\end{cases}$$

where the second inequality follows from assumption (4.1). Hence, ∇F is positive definite and F is strongly monotone on $\mathbb{R}^M$.

Finally, notice that

$$\lambda_2(S) = \frac{S^2+M-S+\sqrt{M(S^3+M-S)}}{2}$$

is a strictly increasing function of S. Hence, if $c > M^2\theta^2$, then

$$c > M^2\theta^2 = \theta^2\lambda_2(M) \geq \theta^2\lambda_2(S), \qquad \forall\, S = 1,\dots,M,$$

thus we get $c-\theta^2\lambda_2(S) > 0$ for any S, that is F is strongly monotone on $\mathbb{R}^M$ for every coalition. □

5. NUMERICAL EXPERIMENTS

In this section, we show some numerical results to illustrate the two-stage game model described in the previous Sections.

We consider a set of $M = 10$ countries with different values of parameters α_i, β_i and $\bar{k}_i$ defined as follows:

$$\begin{aligned}\alpha &= (10,\ 9.5,\ 9,\ 8.5,\ 8,\ 7.5,\ 7,\ 6.5,\ 6,\ 5),\\ \beta &= (0.1,\ 0.11,\ 0.115,\ 0.12,\ 0.125,\ 0.13,\ 0.135,\ 0.14,\ 0.145,\ 0.15),\\ \bar{k} &= (2,\ 1.9,\ 1.8,\ 1.7,\ 1.6,\ 1.5,\ 1.4,\ 1.3,\ 1.2,\ 1),\end{aligned}$$

and we assume $c = 0.05$ and $\theta = 0.001$. The countries are considered in descending order of wealth, so that the richer a country i is, the greater the value of α_i and $\bar{k}_i$. On the other hand, we assume that the environmental impact of global emissions is greater for poor countries than for rich countries, that is, the poorer a country i is, the greater the value of β_i. The choice of parameters was partially inspired by [16].

In this setting, the condition $c > M^2\theta^2$ holds, thus Theorem 4.1 guarantees the affine map F defined in (3.3) is strongly monotone for every coalition $I \subseteq \{1,\dots,M\}$. Hence, we reformulated $VI(\Omega,F)$ as an equivalent convex quadratic optimization problem (see [1]) and solved by means of the MATLAB function `quadprog` from the optimization toolbox. Computations were implemented in MATLAB R2024a and tested on an Apple M1 Max system with 64 GB of RAM running under macOS 14.4.

In order to find all the Nash equilibria of the first-stage game, we first computed the variational equilibrium $z^*(s)$ of the second-stage game for all possible binary vectors $s \in \{0,1\}^{10}$, and then evaluated the utility function of each player at $z^*(s)$. The results show that the first-stage game has exactly two Nash equilibria: $s^* = (0,\dots,0)$, i.e., no countries sign the agreement, and $s^* = (1,\dots,1)$, i.e., all countries sign the agreement. All other binary vectors s are not Nash equilibria, i.e., they do not represent stable coalitions between countries. For instance, the vector $s = (1,1,1,1,1,0,0,0,0,0)$, representing the coalition formed by the first five countries, is not a Nash equilibrium of the first-stage game because the following relations hold:

$$
\begin{aligned}
U_6(1,1,1,1,1,0,0,0,0,0) &= 18.3235 < 18.8965 = U_6(1,1,1,1,1,1,0,0,0,0),\\
U_7(1,1,1,1,1,0,0,0,0,0) &= 14.3186 < 14.8919 = U_7(1,1,1,1,1,0,1,0,0,0),\\
U_8(1,1,1,1,1,0,0,0,0,0) &= 10.5632 < 11.1373 = U_8(1,1,1,1,1,0,0,1,0,0),\\
U_9(1,1,1,1,1,0,0,0,0,0) &= 7.0573 < 7.6327 = U_9(1,1,1,1,1,0,0,0,1,0),\\
U_{10}(1,1,1,1,1,0,0,0,0,0) &= 1.1735 < 1.7530 = U_{10}(1,1,1,1,1,0,0,0,0,1),
\end{aligned}
$$

i.e., every country that is not part of the coalition would have an advantage in joining it.

Table 2 shows, for each of the two equilibria of the first-stage game, the variational equilibrium of the second-stage game, i.e., the emissions and investments at the equilibrium for each country, together with the corresponding utilities and environmental impacts. The results highlight that, for all countries, the equilibrium where the agreement is signed is more beneficial than the one where it is not signed. In fact, in that case each country increases its utility at the equilibrium and decreases its environmental impact. Moreover, this benefit is greater for poor countries than for rich countries. From the global point of view, the total investment in research and development in the case where the agreement is signed is about 8% higher than in the case where the agreement is not signed, but this increase allows the total utility to be increased by about 5% and the environmental impact to be reduced by about 11%, compared with small increase in total emissions of about 0.2%.

In the following, we analyze the sensitivity of the Nash equilibria with respect to changes of parameters c, θ and $\bar{k}$. Specifically, we modify one parameter at a time by assuming c varies in the interval $[0.01,\ 0.1]$, θ varies in $[0.0005,\ 0.005]$, while

$$
\bar{k} = \varepsilon\,(2,\ 1.9,\ 1.8,\ 1.7,\ 1.6,\ 1.5,\ 1.4,\ 1.3,\ 1.2,\ 1),
$$

where $\varepsilon \in [0.5,\ 1.5]$. Even in this setting, the first-stage game has exactly two Nash equilibria: $s^* = (0,\dots,0)$ and $s^* = (1,\dots,1)$.

Figure 1 shows the total pollutant emission, total effort in R&D, total utility, and total environmental impact at equilibrium as a function of parameter c, for each of the two Nash equilibria at first-stage. The results suggest that, for any value of c in the range considered, signing the agreement by all countries allows for an increase in total utility and a decrease in total environmental impact for a small increase in total pollutant emission and total investment in R&D.

	$s^* = (0,\dots,0)$				$s^* = (1,\dots,1)$			
Country	$e_i^*(s^*)$	$k_i^*(s^*)$	u_i	D_i	$e_i^*(s^*)$	$k_i^*(s^*)$	$u_i^{\mathscr{S}}$	D_i
1	9.9015	1.5149	42.4781	7.4597	9.9151	1.5146	43.5002	6.4388
2	9.3915	1.5149	36.8446	8.2171	9.4051	1.5146	37.8655	7.1977
3	8.8865	1.5149	31.8404	8.5958	8.9001	1.5146	32.8606	7.5771
4	8.3815	1.5149	27.0861	8.9745	8.3951	1.5146	28.1057	7.9565
5	7.8765	1.5149	22.5818	9.3533	7.8901	1.5146	23.6007	8.3359
6	7.3715	1.5000	18.3274	9.7331	7.3851	1.5146	19.3458	8.7153
7	6.8664	1.4000	14.3227	10.1194	6.8801	1.5146	15.3408	9.0947
8	6.3613	1.3000	10.5674	10.5057	6.3751	1.5146	11.5858	9.4741
9	5.8562	1.2000	7.0617	10.8920	5.8701	1.5146	8.0807	9.8535
10	4.8510	1.0000	1.1780	11.2859	4.8651	1.5146	2.2006	10.2329
Total	75.74	13.97	212.29	95.14	75.88	15.15	222.49	84.88

TABLE 2. Values of pollutant emission (e_i), effort in R&D (k_i), utility (u_i) and environmental impact (D_i) of each country at the variational equilibrium of the second-stage GNEP, for each of the two Nash equilibria at first-stage: no country signs the agreement ($s^* = (0,\dots,0)$) or all countries sign the agreement ($s^* = (1,\dots,1)$).

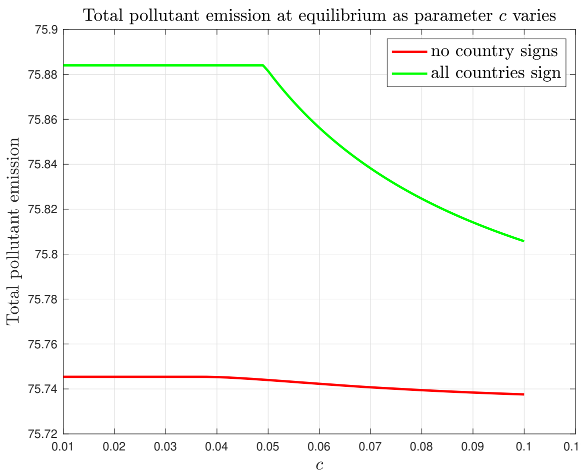
Figures 2 and 3 show the same quantities of Figure 1 as parameters θ and ε vary, respectively. Even in these cases, the results suggest that signing the agreement by all countries allow for an increase in total utility and a decrease in total environmental impact. Moreover, the quantities analyzed, particularly total utility and total environmental impact, seem to be more sensitive to the adaptability parameter θ than to the other two parameters.

To sum up, all the numerical results of the considered example show that there are only two stable coalitions in the first-stage membership problem: no country signs the agreement or all countries sign it. However, the equilibrium in which countries cooperate is globally preferable to the other because it represents a win-win situation for countries and the environment. In fact, for a small increase in total emissions, the total utility of countries is greater and the environmental impact less than the equilibrium in which countries do not cooperate.

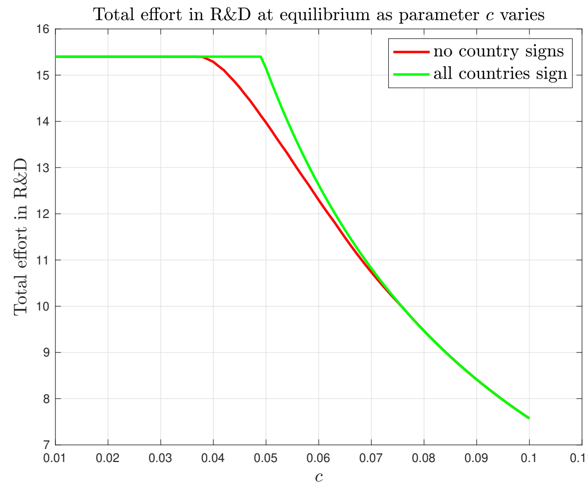
6. Conclusions and further research perspectives

In this paper, we investigated a game-theoretical model of international agreement stipulated to mitigate the environmental damage due to climate change, by investing in new technologies. While the ecological issues are of the utmost importance in the agenda of the European Commission, there is no unanimous agreement on how countries with different levels of industrialization and social welfare should contribute to mitigation strategies. In this respect, our model provides a solution which entails the equality of the marginal utilities (with respect to investments) of the cooperating countries, which can be viewed as a social equity requirement.

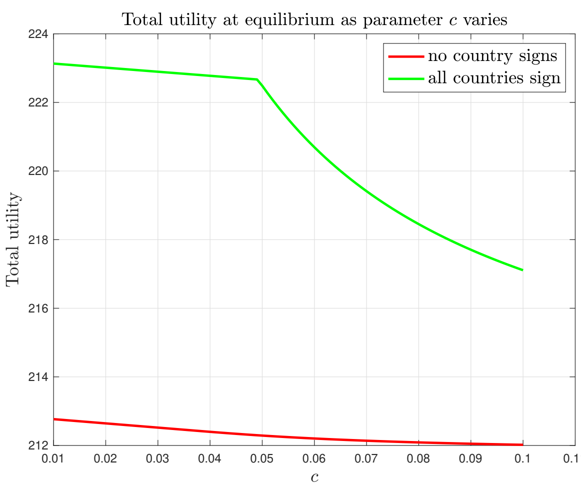
In future work, we plan to investigate models where countries can form coalitions not only to invest in *R&S* but also to jointly bound emissions, in the spirit of Kyoto Protocol (see, e.g.



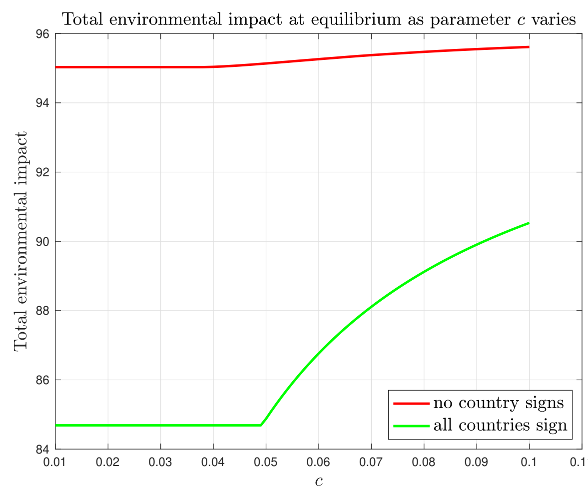
(A) Total pollutant emission at equilibrium.



(B) Total effort in R&D at equilibrium.



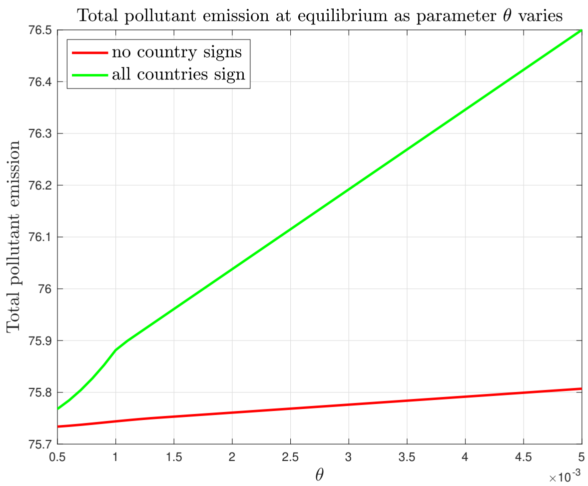
(C) Total utility at equilibrium.



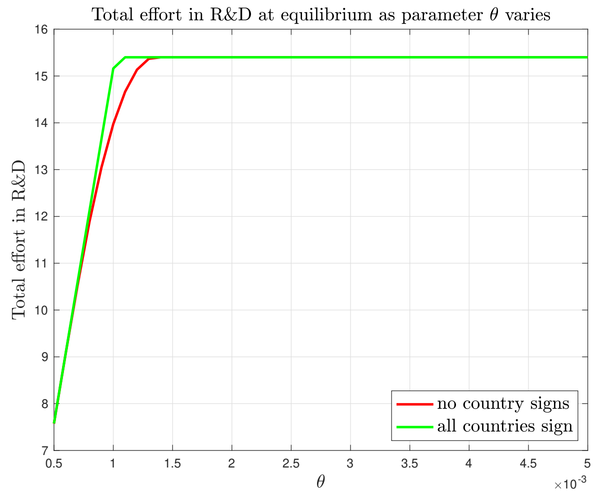
(D) Total environmental impact at equilibrium.

FIGURE 1. Total pollutant emission, total effort in R&D, total utility, total environmental impact at equilibrium, for each of the two Nash equilibria at first-stage, as a function of parameter c.

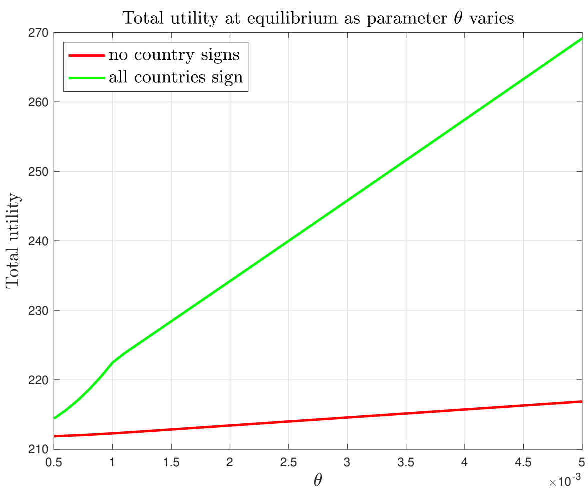
[10]). Moreover, it would be interesting considering the *spillover* of the research results of the signatory countries on the other countries. Indeed, while it is assumed that the countries in the coalition share the outcomes of their research efforts, one cannot rule out that part of the knowledge thus acquired becomes accessible also to the non-signatory countries. Another challenging research direction regards the computation of the first-stage Nash equilibrium for large coalitions. Indeed, following our approach based on the complete enumeration of all possible coalitions it is impossible to treat the case of more than around twenty signatory countries. This problem was circumvented by the authors in [16] by treating all the countries on the same footing from the point of view of their GHG emission potential, which is not realistic. As a consequence, what matters in their computations is not the number of possible coalitions, but only their size. New efficient numerical methods will have to be developed to tackle this problem.



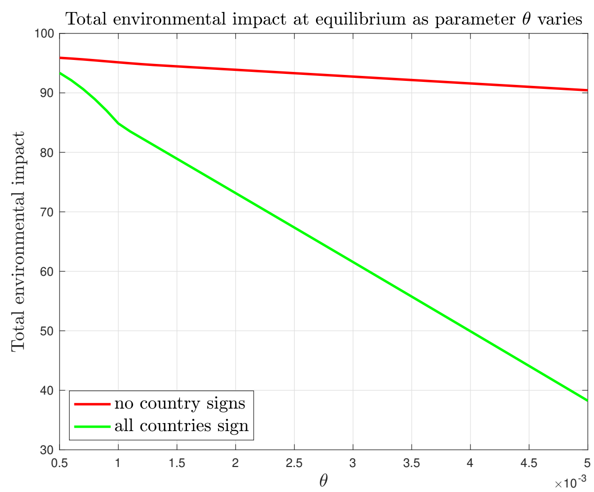
(A) Total pollutant emission at equilibrium.



(B) Total effort in R&D at equilibrium.



(C) Total utility at equilibrium.



(D) Total environmental impact at equilibrium.

FIGURE 2. Total pollutant emission, total effort in R&D, total utility, total environmental impact at equilibrium, for each of the two Nash equilibria at first-stage, as a function of parameter θ.

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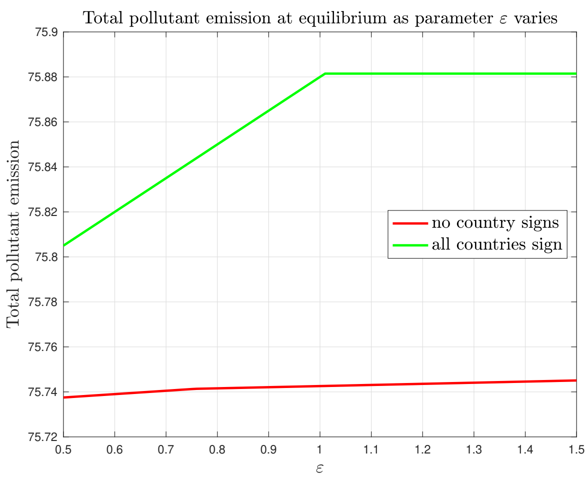
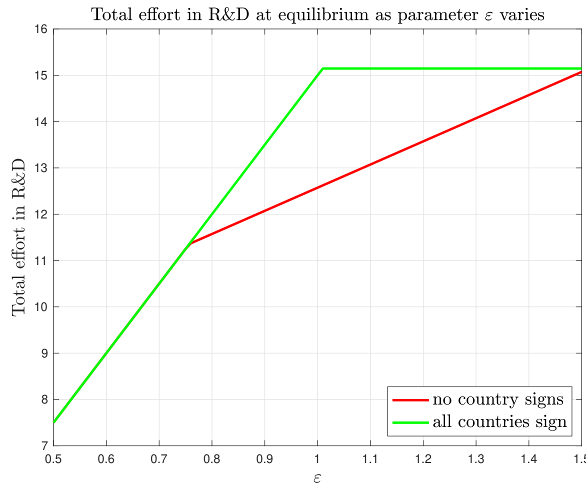
REFERENCES

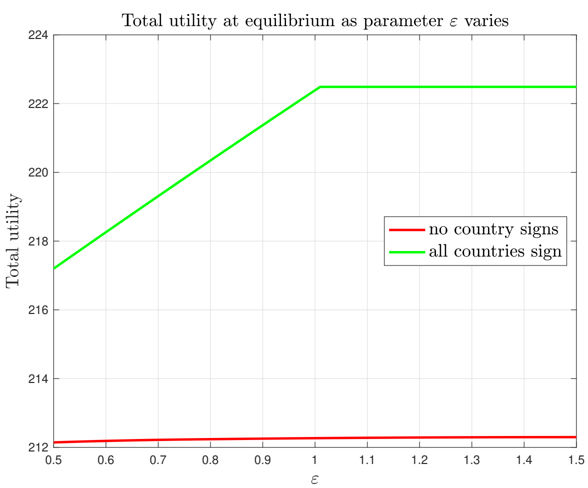
[1] M. Aghassi, D. Bertsimas, G. Perakis, Solving asymmetric variational inequalities via convex optimization, Oper. Res. Lett. 34 (2006), 481-490.
[2] P. Bonacich, Power and centrality: a family of measures. Am. J., Sociol. 92 (1987), 1170-1182.



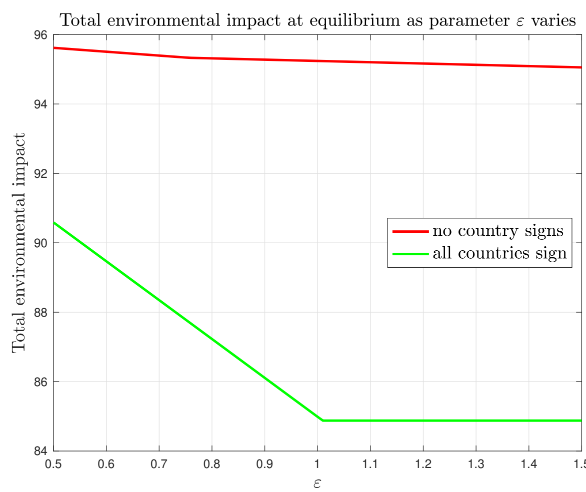
(A) Total pollutant emission at equilibrium.



(B) Total effort in R&D at equilibrium.



(C) Total utility at equilibrium.



(D) Total environmental impact at equilibrium.

FIGURE 3. Total pollutant emission, total effort in R&D, total utility, total environmental impact at equilibrium, for each of the two Nash equilibria at first-stage, as a function of parameter ε.

[3] S. Buob, G. Stephan, To mitigate or to adapt: How to confront global climate change. Eur. j. Political Econ. 27 (2011), 1-16.
[4] M. Breton, L. Sbragia, The impact of adaptation on the stability of international environmental agreements, Environmental and Resource Economics 74 (2019), 697-725.
[5] F. Facchinei, A. Fischer, V. Piccialli, On generalized Nash games and variational inequalities, Oper. Res. Lett. 35 (2007), 159-164.
[6] F. Facchinei, J-S. Pang, Finite-Dimensional Variational Inequalities and Complementarity Problems, Springer, New York, (2003).
[7] M. Farnham, P. Kennedy, Adapting to climate change: equilibrium welfare implications for large and small economies, Environ. Resour. Econ. 61 (2015), 345-363.
[8] F. Faraci, F. Raciti, On generalized Nash equilibrium problems in infinite dimension: the Lagrange multipliers approach, Opitimization 64 (2015), 321-338.
[9] S. Hallegatte, Strategies to adapt to an uncertain climate change, Glob. Environ. Change 19 (2009), 240-247.

[10] B. Jadamba, F. Raciti, On the Modeling of Some Environmental Games with Uncertain Data, J. Optim. Theory Appl. 167 (2015), 959-968.

[11] M.O. Jackson, Y. Zenou, Games on Networks, In: Handbook of Game Theory with Economic Applications, vol. 4, pp. 95-163, Elsevier, Amsterdam, 2015.

[12] B. Jadamba, F. Raciti, On the modelling of some environmental games with uncertain data, J. Optim. Theory Appl. 167 (2015), 959-968.

[13] I. Konnov, Equilibrium models and variational inequalities, Elsevier, Amsterdam, 2007.

[14] W. Marrouch, A.R. Chaudhuri, International environmental agreements in the presence of adaptation, FEEM working paper No. 35, 2011.

[15] W. Marrouch, A.R. Chaudhuri, International Environmental Agreements: Doomed to fail or Destined to Succeed? A review of the Literature, Int. Rev. Environ. Resour. Econ. 9 (2016), 245-319.

[16] N. Masoudi, G. Zaccour, Adapting to climate change: Is cooperation good for the environment? Econ. Lett. 153 (2017), 1-5.

[17] N. Masoudi, G. Zaccour, Adaptation and International Environmental Agreements Environ. Resource Econ. 71 (2018), 1–21.

[18] G. Mastroeni, M. Pappalardo, F. Raciti, Generalized Nash equilibrium problems and variational inequalities in Lebesgue spaces, Minimax Theory Appl. 5 (2020), 47-64.

[19] K. Nabetani, P. Tseng, M. Fukushima, Parametrized variational inequality approaches to generalized Nash equilibrium problems with shared constraints, Comput. Optim. Appl. 48 (2011), 423-452.

[20] A. Nagurney, Network Economics: A Variational Inequality Approach, Springer, Dordrecht, 1999.

[21] M. Passacantando, Game Theory models of international agreements on adaptation to climate change, In: Optimization in Green Sustainability and Ecological Transition. AIRO Springer Series, vol. 12, pp. 49-58, Springer Nature, Cham (Switzerland), 2024.

[22] M. Passacantando, F. Raciti, A note on generalized Nash games played on networks, In: Nonlinear Analysis, Differential Equations, and Applications, Springer Optimization and Its Applications, (ed. by T. M. Rassias), vol. 173, pp. 365-380, Springer, New York, 2021.

[23] M. Passacantando, F. Raciti, A performance measure analysis for traffic networks with random data and general monotone cost functions, Optimization 71 (2022), 2375-2401.

[24] M. Passacantando, F. Raciti, A finite convergence algorithm for solving network games with strategic complements and bounded strategies, Optimization Methods and Software 38 (2023), 1105–1128.

[25] J.B. Rosen, Existence and uniqueness of equilibrium points for concave n-person games, Econometrica 33 (1965), 520-534.