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Schwartz correspondence for real motion groups in low dimensions

Francesca Astengo[1] · Bianca Di Blasio[2] · Fulvio Ricci[3]

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Abstract
For a Gelfand pair (G, K) with G a Lie group of polynomial growth and K a compact subgroup, the *Schwartz correspondence* states that the spherical transform maps the bi-K-invariant Schwartz space $\mathcal{S}(K\backslash G/K)$ isomorphically onto the space $\mathcal{S}(\Sigma_{\mathcal{D}})$, where $\Sigma_{\mathcal{D}}$ is an embedded copy of the Gelfand spectrum in $\mathbb{R}^{\ell}$, canonically associated to a generating system $\mathcal{D}$ of G-invariant differential operators on G/K, and $\mathcal{S}(\Sigma_{\mathcal{D}})$ consists of restrictions to $\Sigma_{\mathcal{D}}$ of Schwartz functions on $\mathbb{R}^{\ell}$. Schwartz correspondence is known to hold for a large variety of Gelfand pairs of polynomial growth. In this paper we prove that it holds for the strong Gelfand pair (M_n, SO_n) with $n = 3, 4$. The rather trivial case $n = 2$ is included in previous work by the same authors.

Keywords Lie groups · Groups of polynomial growth · Gelfand pairs · Spherical transform · Schwartz space

Mathematics Subject Classification 43A85 · 43A90 · 22E30

1 Introduction

Let (G, K) be a Gelfand pair, with G a connected Lie group and K a compact subgroup of it. By definition, this means that the convolution algebra $L^1(K\backslash G/K)$ of bi-K-invariant integrable functions on G, i.e. satisfying

$$f(k_1 g k_2) = f(g) \qquad \forall\, g \in G \quad k_1, k_2 \in K \tag{1.1}$$

✉ Fulvio Ricci
fulvio.ricci@sns.it

Francesca Astengo
astengo@dima.unige.it

Bianca Di Blasio
bianca.diblasio@unimib.it

[1] Dipartimento di Matematica, Dipartimento di Eccellenza 2023–2027, Università di Genova, Via Dodecaneso 35, 16146 Genoa, Italy

[2] Dipartimento di Matematica e Applicazioni, Università di Milano Bicocca, Via Cozzi 53, 20125 Milan, Italy

[3] Scuola Normale Superiore, Piazza dei Cavalieri 7, 56126 Pisa, Italy

is commutative, or, equivalently, that the composition algebra $\mathbb{D}(G/K)$ of G-invariant differential operators on G/K is commutative.

The Gelfand spectrum Σ of $L^1(K\backslash G/K)$ is the space of bounded spherical functions on G with the topology induced by the weak* topology on $L^\infty(K\backslash G/K)$. For each choice of a finite generating subset $\mathcal{D} = \{D_1, \dots, D_\ell\}$ of $\mathbb{D}(G/K)$, Σ can be homeomorphically embedded into $\mathbb{C}^\ell$, by assigning to each spherical function $\varphi \in \Sigma$ the ℓ-tuple $\xi = (\xi_1, \dots, \xi_\ell)$ if $D_j\varphi = \xi_j\varphi$ for $j = 1, \dots, \ell$. The image $\Sigma_\mathcal{D}$ of this embedding is a closed subset of $\mathbb{C}^\ell$ [7].

Denoting by φ_ξ the spherical function associated with $\xi \in \Sigma_\mathcal{D}$, the spherical transform $\mathcal{G}f$ of $f \in L^1(K\backslash G/K)$ can then be regarded as a map defined on $\Sigma_\mathcal{D}$ by

$$\mathcal{G}f(\xi) = \int_G f(x)\,\varphi_\xi(x^{-1})\,dx. \tag{1.2}$$

If G has polynomial volume growth (in which case we also say that the pair (G, K) has polynomial growth) and the chosen generators $D_j \in \mathcal{D}$ are symmetric, the eigenvalues are real, so that $\Sigma_\mathcal{D} \subset \mathbb{R}^\ell$. We refer to [3] for a presentation of Gelfand pairs of polynomial growth and the proofs of various preliminary results that will be used in this paper.

Gelfand pairs were originally introduced in the context of spherical analysis on symmetric spaces [12]. The interest in pairs of polynomial growth, and in particular of the nilpotent Gelfand pairs mentioned below, is more recent [5, 6, 13–15, 19, 22–24].

We say that *Schwartz correspondence* holds for a Gelfand pair (G, K) of polynomial growth if the following property is satisfied:

(S) *The spherical transform maps the bi-K-invariant Schwartz space $\mathcal{S}(K\backslash G/K)$ isomorphically onto the space $\mathcal{S}(\Sigma_\mathcal{D})$ of restrictions to $\Sigma_\mathcal{D}$ of Schwartz functions on $\mathbb{R}^\ell$.*

This is an intrinsic property of the pair because it does not depend on the choice of the generating system $\mathcal{D}$ [2, 3, 9].

The range of Gelfand pairs for which property (S) has been proved includes those with G compact (compact pairs) [3], various families with $G = K \ltimes H$ and H nilpotent (nilpotent pairs) [1, 2, 8–10], and those with $G = K \ltimes H$ and K abelian [3, Thm. 8.1]. No example in which condition (S) fails has been found so far.

An interesting class of Gelfand pairs for which Schwartz correspondence has not been established yet consists of the *strong Gelfand pairs*.

Denoting by $\mathrm{Int}(K)$ the group of inner automorphisms of G induced by elements of K and calling K-central a function on G that is $\mathrm{Int}(K)$-invariant, the pair (G, K) is called a strong Gelfand pair if the algebra $L^1(G)^{\mathrm{Int}(K)}$ of K-central integrable functions is commutative. The term "strong" is then justified by the obvious fact that bi-K-invariant functions are K-central.

It is easy to verify that $L^1(G)^{\mathrm{Int}(K)} \approx L^1\big(K\backslash(K \ltimes_{\mathrm{Int}} G)/K\big)$, so that (G, K) is strong Gelfand if and only if $(K \ltimes_{\mathrm{Int}} G, K)$ is Gelfand.

According to a general structure of Gelfand pairs due to Vinberg [23, Thm. 13.3.20], Gelfand pairs (G, K) must have $G = L \ltimes H$, with H nilpotent of step ≤ 2 and $L \supseteq K$ orbit-equivalent to K. The case of a strong Gelfand pairs corresponds to $L = K \ltimes_{\mathrm{Int}} K$.

A careful analysis of the classification in [24] shows that the non-compact strong Gelfand pairs (G, K) with polynomial growth have $G = K \ltimes H$ and one of the following forms:

- $K = SO_n$ and $H = \mathbb{R}^n$,
- $K = U_n$ and $H = \mathbb{C}^n$ or the Heisenberg group H_n,
- direct products of the above.

Objects and notions given above for general Gelfand pairs can be reformulated in the following terms for a strong Gelfand pair (G, K), cf. [3, Sect. 2.1]:

- $\mathbb{D}(G)^{\mathrm{Int}(K)}$ is the algebra of K-central differential operators on G,
- the spherical functions are the K-central eigenfunction of all operators in $\mathbb{D}(G)^{\mathrm{Int}(K)}$, normalized so that they take value 1 at the identity.

Then the Gelfand spectrum Σ consists of the bounded spherical functions and, given a finite generating system $\mathcal{D}$ of $\mathbb{D}(G)^{\mathrm{Int}(K)}$, the embedded copy $\Sigma_{\mathcal{D}}$ of Σ is defined as the set of $(\dim\mathcal{D})$-tuples of eigenvalues like in the general case, and the spherical transform is defined as in (1.2). The formulation of Schwartz correspondence for strong Gelfand pairs with polynomial growth is the same as in (S), only with $\mathcal{S}(K\backslash G/K)$ replaced by $\mathcal{S}(G)^{\mathrm{Int}(K)}$.

The cases in which K is abelian are already covered by the positive results in [3]. The authors have recently proved property (S) for the strong pair $\big(M_2(\mathbb{C}), U_2\big)$, where $M_2(\mathbb{C}) = U_2 \ltimes \mathbb{C}^2$ is the complex motion group in two dimensions [4].

In this paper we consider the strong pairs $\big(M_n(\mathbb{R}), SO_n\big)$ with $n = 3, 4$ and prove the following.

Theorem 1.1 *The strong Gelfand pairs $\big(M_n(\mathbb{R}), SO_n\big)$, with $n = 3, 4$, satisfy property* (S)*, i.e., the spherical transform maps $\mathcal{S}(M_n(\mathbb{R}))^{\mathrm{Int}(SO_n)}$ isomorphically onto $\mathcal{S}(\Sigma_{\mathcal{D}})$.*

One important tool in the analysis of K-central functions for general strong Gelfand pairs is decomposition into K-types.

Given $\tau \in \widehat{K}$, let f_τ be the K-type component of $f \in L^1(G)^{\mathrm{Int}(K)}$ defined by

$$f_\tau(g) = f *_K (d_\tau \overline{\chi_\tau})(g) = (d_\tau \overline{\chi_\tau}) *_K f(g) = \int_K f(gk^{-1})(d_\tau \overline{\chi_\tau})(k)\, dk \tag{1.3}$$

where d_τ and χ_τ are character and dimension of τ, respectively. We say that f has K-type τ if $f_\tau = f$. The algebra $L^1(G)^{\mathrm{Int}(K)}$ decomposes as the direct sum of the subalgebras $L^1(G)_\tau^{\mathrm{Int}(K)}$ of K-central functions of K-type τ. Morever, for $\tau \neq \tau'$,

$$L^1(G)_\tau^{\mathrm{Int}(K)} * L^1(G)_{\tau'}^{\mathrm{Int}(K)} = \{0\} \tag{1.4}$$

and, denoting by τ_0 the trivial representation, $L^1(G)_{\tau_0}^{\mathrm{Int}(K)} = L^1(K\backslash G/K)$.

It follows that (G, K) is a strong Gelfand pair if and only (G, K, τ) is a *commutative triple* (i.e., $L^1(G)_\tau^{\mathrm{Int}(K)}$ is commutative) for every $\tau \in \widehat{K}$ [20]. If this is the case, then Σ is the disjoint union of the spectra Σ^τ of $L^1(G)_\tau^{\mathrm{Int}(K)}$, consisting of the bounded spherical function that are of type τ.

We can then extend the notion of Schwartz correspondence to the commutative triple (G, K, τ) as follows.

(S_τ) *The τ-spherical transform maps $\mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ isomorphically onto the space $\mathcal{S}(\Sigma_{\mathcal{D}}^\tau)$, where $\Sigma_{\mathcal{D}}^\tau$ is the Gelfand spectrum of $L^1(G)_\tau^{\mathrm{Int}(K)}$.*

It is quite clear that, if the strong Gelfand pair (G, K) satisfies property (S), then property (S_τ) is satisfied for every $\tau \in \widehat{K}$. Conversely, if we assume that property (S_τ) has been proved for every τ, it is still necessary to take into account the dependence on τ of the Schwartz norm estimates involved. The following reduction principle of property (S) to single K-types has been used in [4] for the strong Gelfand pair $\big(M_2(\mathbb{C}), U_2\big)$ and will be used here to prove Theorem 1.1.

Theorem 1.2 ([3, Prop. 5.2 and Thm. 7.1]) *Property* (S) *holds for a strong Gelfand pair (G, K) of polynomial growth if and only if the following condition is satisfied:*

(S') *given $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$ and $N \in \mathbb{N}$, for each K-type component f_τ of f, $\tau \in \widehat{K}$, $\mathcal{G} f_\tau$ admits a Schwartz extension $u_{\tau,N}$ such that $\|u_{\tau,N}\|_{(N)}$ is rapidly decaying in τ.*

In the two cases considered here, $n = 3, 4$, the copy $\Sigma_{\mathcal{D}}$ of the spectrum is embedded in $\mathbb{R}^n$ and its subsets $\Sigma^\tau_{\mathcal{D}}$ lie in distinct parallel 2-dimensional affine subspaces, so that, as long as we treat each τ separately, $\Sigma^\tau_{\mathcal{D}}$ can be regarded as being embedded in $\mathbb{R}^2$ and $\mathcal{G}f_\tau$ as a function of two variables only. In this picture $\Sigma^\tau_{\mathcal{D}}$ is the union of finitely many half-parabolas if $n = 3$, and of finitely many half-lines if $n = 4$, all exiting from the origin. The number of half-parabolas, or of half-lines, increases with the dimension of τ.

In order to obtain property (S') we prove the following two facts:

(i) for general $f \in \mathcal{S}\big(M_n(\mathbb{R})\big)^{\mathrm{Int}(SO_n)}$ and any $\tau \in \widehat{SO_n}$, it is possible to determine a formal two-variable power series $s_\tau(\xi_1, \xi_2) = \sum_{p,q} \frac{a^\tau_{p,q}}{p!q!}\xi_1^p \xi_2^q$ with the property that, for any $N \in \mathbb{N}$, there are functions $h_{\tau,N} \in \mathcal{S}(\mathbb{R}^2)$ with rapidly decaying N-th Schwartz norm, Taylor expansion at 0 equal to s_τ, and $\mathcal{G}f_\tau - h_{\tau,N}$ vanishing of infinite order at 0 (Propositions 6.2 and 6.4);

(ii) an explicit extension formula for $\mathcal{G}f_\tau$ exists, with the decay in τ prescribed in (S'), under the assumption that $\mathcal{G}f_\tau$ vanishes of infinite order at $0 \in \Sigma^\tau_{\mathcal{D}}$ (Proposition 6.5).

Once (i) and (ii) are verified, an application of [16, Prop. 4.2.1] gives the desired extensions $u_{\tau,N}$ of $\mathcal{G}f_\tau$, satisfying condition (S') (Proposition 6.6). Following this approach, the major obstacle consists in proving the existence of the formal power series s_τ with the properties in (i), because the analysis of $\mathcal{G}f_\tau$ near 0 only gives, at any degree, a certain number of linear relations among the unknown coefficients $a^\tau_{p,q}$. These relations lead to infinitely many, overdetermined in general, linear systems in the $a^\tau_{p,q}$ and the crucial problem is their solvability, a property apparently impossible to obtain by plain linear algebra.

Once solvability is proved the decay estimates in (i) can be proved in a relatively easy way. In order to bypass this problem, we introduce an analytic argument to directly obtain (S_τ) for every τ, though not necessarily with the decay estimates required in (S').

This argument exploits the identification, via an isomorphism A_τ, of $L^1\big(M_n(\mathbb{R})\big)^{\mathrm{Int}(SO_n)}_\tau$ with the algebra $L^1\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^{SO_n}$ of $\mathrm{End}(\mathcal{V}_\tau)$-valued, SO_n-equivariant, integrable functions on $\mathbb{R}^n$, $\mathcal{V}_\tau$ being a representation space of τ. The map A_τ also gives a correspondence between the two sets of spherical functions and induces a homomorphisms from $\mathbb{D}\big(M_n(\mathbb{R})\big)^{\mathrm{Int}(SO_n)}$ onto $\big(\mathbb{D}(\mathbb{R}^n) \otimes \mathrm{End}(\mathcal{V}_\tau)\big)^{SO_n}$, the algebra of $\mathrm{End}(\mathcal{V}_\tau)$-valued, SO_n-equivariant differential operators on $\mathbb{R}^n$.

For both $n = 3$ and $n = 4$, this last algebra admits a single generator $\mathbf{D}_\tau$ as a $\mathbb{D}(\mathbb{R}^n)^{SO_n}$-module. For $F \in \mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^{SO_n}$ it is then possible to obtain the representation formula (Proposition 5.1)

$$F = \sum_{j=0}^{m} \mathbf{D}^j_\tau g_j \tag{1.5}$$

with m depending on τ and the g_j radial, scalar-valued, Schwartz functions on $\mathbb{R}^n$. This reduces the problem to the existence of a Schwartz extension for the Hankel transform of each g_j, and this is a trivial consequence of the Whitney representation theorem for even functions on $\mathbb{R}$.

Though the general plan described above corresponds to that already used in [4], the detailed treatment presents here some new complications. In principle, the same strategy of proof of Schwartz correspondence can be extended to real and complex motion pairs in higher dimensions, but the algebraic problems arising in Sects. 4 and 5 become much more delicate.

Our paper is organized as follows. In Sect. 2 we recall some preliminary facts about strong Gelfand pairs of polynomial growth and describe the decomposition into K-types. Moreover, specializing to the case where $G = K \ltimes_\delta H$, we describe the isomorphism A_τ between $\mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ and $\mathcal{S}\big(H, \mathrm{End}(\mathcal{V}_\tau)\big)^K$. In Sect. 3 we fix $\tau \in \widehat{SO_n}$ and, for any n, we recall from [20] some general facts about the algebra $\big(\mathbb{D}(\mathbb{R}^n) \otimes \mathrm{End}(\mathcal{V}_\tau)\big)^{SO_n}$ and the Gelfand spectrum of $L^1\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^{SO_n}$. Starting from Sect. 4, we limit ourselves to the cases $n = 3, 4$. First we choose a priviliged generating system of $\big(\mathbb{D}(\mathbb{R}^n) \otimes \mathrm{End}(\mathcal{V}_\tau)\big)^{SO_n}$ and spot the main differences between the cases $n = 3$ and $n = 4$. Then we fix notation in order to treat the two cases in a unified way as far as possible and we determine the Gelfand spectrum $\Sigma^\tau_{\mathcal{D}}$. In Sect. 5 we apply the results of the previous section to obtain the decomposition formula (1.5) and we show that property (S_τ) holds for each $\tau \in \widehat{SO_n}$. In Sect. 6 we derive, for fixed τ, the linear systems whose solutions allow us to obtain the infinite order local approximations $\{h_{\tau,N}\}_N$ of $\mathcal{G}_\tau f_\tau$ with rapid decay, according to (i) above. At this stage, dealing with the linear systems, we have to introduce new technical arguments with respect to [4]. Finally we sum up all our results proving that the Schwartz correspondence holds for $\big(M_n(\mathbb{R}), SO_n\big)$ with $n = 3, 4$.

2 Spherical transforms for strong Gelfand pairs

In this section we fix notation, recall basic facts and prove some statements in the general context of a strong Gelfand pair (G, K) of polynomial growth. For details and unproven statements, we refer to [3, 20].

The main result in the first subsection is the polynomial growth of the norms of $\mathcal{G}_\tau^{-1}$, the inverse of the τ-spherical transform on $\mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$.

In the second subsection we specialize to the case where $G = K \ltimes_\delta H$. In the analysis of functions of a given K-type, a matrix realization is often used, see [20] and the references therein. Specifically, in Proposition 2.4 we prove that $\mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ is algebraically and topologically isomorphic to $\mathcal{S}\big(H, \mathrm{End}(\mathcal{V}_\tau)\big)^K$, the space of Schwartz $\mathrm{End}(\mathcal{V}_\tau)$-valued equivariant functions on H, i.e. satisfying (2.1).

2.1 The τ-spherical transform and its inverse.

Let (G, K) be a strong Gelfand pair. We need notational conventions in order to distinguish among the different Gelfand structures that intervene.

The plain symbols Σ and $\mathcal{G}$ denote, respectively, Gelfand spectrum and spherical transform of (G, K) as strong pair, i.e., the set of multiplicative linear functionals of the full algebra $L^1(G)^{\mathrm{Int}(K)}$ endowed with the weak* topology. The symbols Σ^τ and $\mathcal{G}_\tau$ denote, respectively, Gelfand spectrum and spherical transform (or *τ-spherical transform*) of the commutative triple (G, K, τ), i.e., of the algebra $L^1(G)_\tau^{\mathrm{Int}(K)}$. As usual, each multiplicative linear functional is realized by integration with a bounded spherical function. It is a fact [20] that every bounded spherical function is of K-type τ for some, hence unique, $\tau \in \widehat{K}$. Hence Σ is the disjoint union of the spectra Σ^τ, consisting of the bounded spherical functions of type τ (or *τ-spherical functions*). In particular, Σ^{τ_0} and $\mathcal{G}_{\tau_0}$ are Gelfand spectrum and spherical transform of the (ordinary) Gelfand pair (G, K). By (1.4), the spherical transform $\mathcal{G} f$ of $f \in L^1(G)^{\mathrm{Int}(K)}$ restricts to Σ^τ as the τ-spherical transform $\mathcal{G}_\tau f_\tau$ of its τ-component.

If G is connected and denoting by $\mathbb{D}(G)^{\mathrm{Int}(K)}$ the algebra of left-invariant differential operators on G that are also $\mathrm{Int}(K)$-invariant, the bounded spherical functions are the elements of $C^\infty(G)^{\mathrm{Int}(K)}$ that are eigenfunctions of all elements of $\mathbb{D}(G)^{\mathrm{Int}(K)}$, normalized to attain value 1 at the identity.

By the general facts about Gelfand pairs mentioned in the introduction, it follows that, if G has also polynomial growth, the choice of a finite system $\mathcal{D} = \{D_1, \dots, D_\ell\}$ of symmetric generators of $\mathbb{D}(G)^{\mathrm{Int}(K)}$ determines a homeomorphism $\iota_{\mathcal{D}}$ of Σ with a closed subset $\Sigma_{\mathcal{D}}$ of $\mathbb{R}^\ell$.

We shall always assume that G is connected with polynomial growth.

It is convenient to choose $\mathcal{D}$ of the form $\mathcal{D} = \mathcal{D}' \cup \mathcal{D}''$, where $\mathcal{D}' = \{D_1, \dots, D_{\ell'}\}$ generates the center of $\mathbb{D}(K)$. Then the homeomorphism $\iota_{\mathcal{D}} = (\iota'_{\mathcal{D}}, \iota''_{\mathcal{D}})$ maps $\varphi \in \Sigma$ to a point

$$\xi = (\xi', \xi'') = \big(\iota'_{\mathcal{D}}(\varphi), \iota''_{\mathcal{D}}(\varphi)\big) \in \mathbb{R}^{\ell'} \times \mathbb{R}^{\ell''} \qquad (\ell'' = \ell - \ell').$$

When φ is τ-spherical, the coordinates of $\xi'_\tau = \iota'_{\mathcal{D}}(\varphi)$ are the eigenvalues of $\overline{\chi_\tau}$ under $\{D_1, \dots, D_{\ell'}\}$ and

$$\Sigma_{\mathcal{D}} = \bigcup_{\tau \in \widehat{K}} \{\xi'_\tau\} \times \Sigma^\tau_{\mathcal{D}}$$

where

$$\Sigma^\tau_{\mathcal{D}} = \big\{\iota''_{\mathcal{D}}(\varphi) : \varphi \in \Sigma^\tau\big\} \subset \mathbb{R}^{\ell''}$$

is homeomorphic to Σ^τ. Note also that rapid decay in τ can be measured in terms of powers of $|\xi'_\tau|$ (see [3, formula (7.2)]).

With a language abuse the spherical transform of a function in $\mathcal{S}(G)^{\mathrm{Int}(K)}$ will be considered as a function on $\Sigma_{\mathcal{D}} \subset \mathbb{R}^\ell$, and the τ-spherical transform of a function in $\mathcal{S}(G)^{\mathrm{Int}(K)}_\tau$ will be considered as a function on $\Sigma^\tau_{\mathcal{D}} \subset \mathbb{R}^{\ell''}$. In formulae, for every $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$,

$$\mathcal{G}f(\xi) = \int_G f(x)\,\varphi(x^{-1})\,dx \qquad \forall \xi = \iota_{\mathcal{D}}(\varphi) \in \Sigma_{\mathcal{D}}$$
$$\mathcal{G}_\tau f_\tau(\xi'') = \mathcal{G}f(\xi'_\tau, \xi'') \qquad \forall \xi'' = \iota''_{\mathcal{D}}(\varphi) \in \Sigma^\tau_{\mathcal{D}}.$$

In particular, when f is of type τ,

$$\mathcal{G}f(\xi'_\sigma, \xi'') = \begin{cases} \mathcal{G}_\tau f(\xi'') & \text{if } \sigma = \tau \\ 0 & \text{if } \sigma \neq \tau. \end{cases}$$

For a fixed compact, symmetric neighborhood U of the identity e of G, define

$$|x| = \begin{cases} 0 & \text{if } x = e \\ \min\{n : x \in U^n\} & \text{if } x \neq e. \end{cases}$$

For our convenience we will use the following Schwartz norms. For $N \in \mathbb{N}$,

$$\|u\|_{(N)} = \max_{|\alpha| \le N} \|\big(1 + |\cdot|\big)^N \partial^\alpha u\|_{L^2} \qquad \forall u \in \mathcal{S}(\mathbb{R}^\ell)$$
$$\|f\|_{(N)} = \max_{|\alpha| \le N} \|\big(1 + |\cdot|\big)^N D^\alpha f\|_{L^2} \qquad \forall f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$$

where for every multiindex $\alpha = (\alpha_1, \dots, \alpha_\ell)$

$$\partial^\alpha = \partial^{\alpha_1}_{\xi_1} \partial^{\alpha_1}_{\xi_2} \cdots \partial^{\alpha_\ell}_{\xi_\ell} \qquad \text{and} \qquad D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_\ell^{\alpha_\ell}.$$

When C is a closed set of $\mathbb{R}^\ell$, $\mathcal{S}(C)$ denotes the space of restrictions to C of Schwartz functions on $\mathbb{R}^\ell$, that is

$$\mathcal{S}(C) = \left\{u_{|C} : u \in \mathcal{S}(\mathbb{R}^\ell)\right\} \cong \mathcal{S}(\mathbb{R}^\ell)/\{u : u_{|C} = 0\}$$

endowed with the quotient topology, induced by the norms i.e.

$$\|\psi\|_{(N)} = \inf\{\|u\|_{(N)} : u_{|C} = \psi\} \qquad \psi \in \mathcal{S}(C) \quad N \in \mathbb{N}.$$

By the Plancherel-Godement Theorem [23, p. 193] the spherical transform $\mathcal{G}$ is a unitary operator from $L^2(G)^{\mathrm{Int}(K)}$ onto $L^2(\Sigma_\mathcal{D}, \beta)$, where β is the Plancherel measure on $\Sigma_\mathcal{D}$. Since the symmetric operators D_j are essentially self-adjoint [18], the set $\Sigma_\mathcal{D} \subset \mathbb{R}^\ell$ is their joint $L^2(G)^{\mathrm{Int}(K)}$-spectrum (see [3, Proposition 4.4]). In spectral analytic terms, $\mathcal{G}^{-1}\psi$ is the convolution kernel of $\psi(D_1, \dots, D_\ell)$.

We can then rephrase a multiplier theorem by A. Martini [16, Prop. 4.2.1] in the following way.

Proposition 2.1 *The map $\mathcal{G}^{-1}$ is continuous from $\mathcal{S}(\Sigma_\mathcal{D})$ in $\mathcal{S}(G)^{\mathrm{Int}(K)}$, i.e., for every $N \in \mathbb{N}$ there exist $M = M(N)$ and C_N such that*

$$\|\mathcal{G}^{-1}u\|_{(N)} \le C_N \, \|u\|_{(M)} \qquad \forall u \in \mathcal{S}(\Sigma_\mathcal{D}).$$

The use of this proposition is twofold. First, Proposition 2.1, combined with the open mapping theorem for Fréchet spaces, allows us to disregard continuity of $\mathcal{G}$. In other words, in order to prove that property (S) holds for the strong Gelfand pair (G, K), we only need to prove that $\mathcal{G}^{-1}$ is surjective, i.e., for every $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$, the spherical transform $\mathcal{G}f$ extends to a Schwartz function on $\mathbb{R}^\ell$. This will be proved by reduction to K–types according to the criterion in Theorem 1.2.

The second application of Proposition 2.1 consists in establishing an analogous result for a fixed K–type τ. This will be used in the proof of our main result (Proposition 6.6) as a tool in the construction of the extension with norms of rapid decay in τ as required in Theorem 1.2.

Corollary 2.2 *Let τ be in $\widehat{K}$. Every $v \in \mathcal{S}(\Sigma_\mathcal{D}^\tau)$ is the spherical transform of a unique function $\mathcal{G}_\tau^{-1}v \in \mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$. Moreover for every $N \in \mathbb{N}$ there exist $M = M(N)$, $M' = M'(N)$ and C_N such that*

$$\|\mathcal{G}_\tau^{-1}v\|_{(N)} \le C_N \, (1 + |\xi'_\tau|)^{M'} \, \|v\|_{(M)} \qquad \forall v \in \mathcal{S}(\Sigma_\mathcal{D}^\tau).$$

Proof The starting point is the existence, for each $\tau \in \widehat{K}$, of a Schwartz function $u_\tau \in \mathcal{S}(\mathbb{R}^{\ell'})$ such that $u_\tau(\xi'_\tau) = 1$, $u_\tau(\xi'_\sigma) = 0$ if $\sigma \in \widehat{K}$, $\sigma \neq \tau$ and for every N there exist $N' = N'(N)$ and C_N such that

$$\|u_\tau\|_{(N)} \le C_N \, (1 + |\xi'_\tau|)^{N'}.$$

This fact has been proved in [3, Lemma 7.2].

Suppose now we are given $v \in \mathcal{S}(\Sigma_\mathcal{D}^\tau)$ and let $v_* \in \mathcal{S}(\mathbb{R}^{\ell''})$ be an extension of it. Define a Schwartz function u_* on $\mathcal{S}(\mathbb{R}^\ell)$ by

$$u_*(\xi', \xi'') = u_\tau(\xi') \, v_*(\xi'') \qquad \forall (\xi', \xi'') \in \mathbb{R}^\ell$$

and denote by u its restriction to $\Sigma_\mathcal{D}$. By Proposition 2.1 the function $f = \mathcal{G}^{-1}u$ belongs to $\mathcal{S}(G)^{\mathrm{Int}(K)}$ and for every $N \in \mathbb{N}$ there exist $M = M(N)$, $M' = M'(N)$ and C_N such that

$$\|f\|_{(N)} \le C_N \, \|u\|_{(M)} \le C_N \, (1 + |\xi'_\tau|)^{N'} \|v\|_{(M)}.$$

Finally,

$$\mathcal{G}f(\xi'_\sigma, \xi'') = u(\xi'_\sigma, \xi'') = u_\tau(\xi'_\sigma)\, v(\xi'') = 0 \qquad \sigma \neq \tau$$

which implies that f is of type τ and

$$v(\xi'') = u(\xi'_\tau, \xi'') = \mathcal{G}f(\xi'_\tau, \xi'') = \mathcal{G}_\tau f(\xi'') \qquad \forall \xi'' \in \Sigma^\tau_{\mathcal{D}}.$$

Uniqueness of v is obvious, because $L^1(G)^{\mathrm{Int}(K)}_\tau$ is semisimple. □

2.2 End($\mathcal{V}_\tau$)-valued Schwartz functions

For the purposes of this paper, we will introduce $\mathrm{End}(\mathcal{V}_\tau)$-valued functions only in the case of a strong Gelfand pair (G, K) with $G = K \ltimes_\delta H$ and K is compact and acting on the nilpotent group H by δ.

We denote by $\mathcal{V}_\tau$ the representation space of τ, by d_τ its dimension and by $\mathrm{End}(\mathcal{V}_\tau)$ the algebra of linear endomorphisms of $\mathcal{V}_\tau$. The space of integrable $\mathrm{End}(\mathcal{V}_\tau)$-valued functions on H is an algebra with convolution defined by

$$F_1 * F_2(h) = \int_H F_1(z^{-1}h) F_2(z)\, dz$$

and norm $\|F\|_1 = \int_H \|F(h)\|_{\mathrm{op}}\, dh$.

We say that an $\mathrm{End}(\mathcal{V}_\tau)$-valued function F on H is K-equivariant if it satisfies

$$F(\delta(k)h) = \tau(k) F(h) \tau(k^{-1}) \qquad \forall\, k \in K \tag{2.1}$$

and denote by $L^1\big(H, \mathrm{End}(\mathcal{V}_\tau)\big)^K$, and similarly for other function spaces, the subspace of integrable K-equivariant functions.

The two maps

$$\begin{array}{rccl} A_\tau : & f(k,h) & \longmapsto & F(h) = \int_K f(k,h)\tau(k)\, dk \\ A_\tau^{-1} : & F(h) & \longmapsto & f(k,h) = d_\tau\ \mathrm{tr}\big(\tau(k^{-1})F(h)\big) \end{array}$$

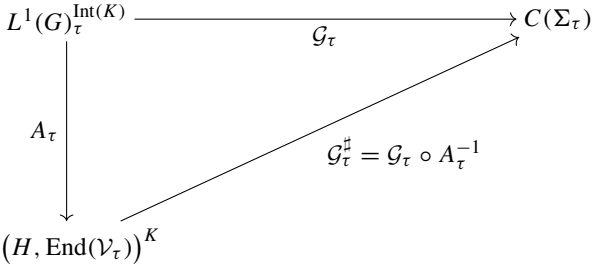
establish a one-to-one correspondence between K-central functions f on G of K-type τ and K-equivariant $\mathrm{End}(\mathcal{V}_\tau)$-valued functions on H.

In particular, A_τ is an isomorphism of algebras from $L^1(G)^{\mathrm{Int}(K)}_\tau$ onto $L^1\big(H, \mathrm{End}(\mathcal{V}_\tau)\big)^K$ and $\sqrt{d_\tau} A_\tau$ is unitary from $L^2(G)^{\mathrm{Int}(K)}_\tau$ onto $L^2\big(H, \mathrm{End}(\mathcal{V}_\tau)\big)^K$, where

$$\|F\|_2^2 = \int_H \big\|F(h)\big\|^2_{\mathrm{HS}}\, dh \qquad \forall F \in L^2\big(H, \mathrm{End}(\mathcal{V}_\tau)\big)^K$$

and $\|\cdot\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm.

Via the map A_τ, we can then redefine the ingredients of the spherical analysis of K-central functions of type τ on G in terms of the above $\mathrm{End}(\mathcal{V}_\tau)$–valued model.



This will be explained in detail in the next section in the case of the euclidean motion group, where we explicitly determine the $\mathrm{End}(\mathcal{V}_\tau)$–valued spherical functions, their eigenvalues and we deduce, via A_τ, the τ-spherical ones. We denote by $\big(\mathbb{D}(H)\otimes \mathrm{End}(\mathcal{V}_\tau)\big)^K$ the algebra of "$\mathrm{End}(\mathcal{V}_\tau)$–valued" differential operators on H which commute with translations and with the action of K on smooth $\mathrm{End}(\mathcal{V}_\tau)$–valued functions F given by

$$k : F \longmapsto \tau(k)F(\delta(k^{-1})h)\tau(k^{-1}).$$

Conjugation by A_τ is a homomorphism of $\mathbb{D}(G)^{\mathrm{Int}(K)}$ onto $\big(\mathbb{D}(H)\otimes \mathrm{End}(\mathcal{V}_\tau)\big)^K$ and its kernel consists of the operators which vanish on functions of K-type τ.

The following lemma allows us to disregard the operators in $\mathcal{D}'$, the center of $\mathbb{D}(K)$, in the evaluation of the Schwartz norms of functions in $\mathcal{S}\big(H,\mathrm{End}(\mathcal{V}_\tau)\big)^K$.

Lemma 2.3 *Let $\tau\in\widehat{K}$ and D_j be in $\mathcal{D}'$. Then $A_\tau D_j A_\tau^{-1}$ is the scalar multiplication operator by ξ_j so that for every F in $C^\infty\cap L^2\big(H,\mathrm{End}(\mathcal{V}_\tau)\big)^K$*

$$\|A_\tau D A_\tau^{-1}F(h)\|_{\mathrm{HS}}\le|\xi'_\tau|\,\|F(h)\|_{\mathrm{HS}}\qquad\forall h\in H,\ \forall D\in\mathcal{D}'.$$

Proof Let $\xi'_\tau=(\xi_1,\dots,\xi_{\ell'})$, where ξ_j is the eigenvalue of $\overline{\chi}_\tau$ under $D_j\in\mathcal{D}'$. Given F in $C^\infty\cap L^2\big(H,\mathrm{End}(\mathcal{V}_\tau)\big)^K$, the function $f=A_\tau^{-1}F$ is in $C^\infty\cap L^2(G)^{\mathrm{Int}(K)}_\tau$, so that by (1.3)

$$D_j f=D_j(f*_K(d_\tau\overline{\chi_\tau}))=f*_K(d_\tau D_j\overline{\chi_\tau})=\xi_j\,f*_K(d_\tau\overline{\chi_\tau})=\xi_j f.$$

Therefore applying the isomorphism A_τ we obtain

$$A_\tau D_j A_\tau^{-1}F=A_\tau D_j f=\xi_j\,A_\tau f=\xi_j\,F$$

and the thesis follows. □

The following proposition establishes that A_τ is a topological isomorphisms from $\mathcal{S}(G)^{\mathrm{Int}(K)}_\tau$ onto $\mathcal{S}\big(H,\mathrm{End}(\mathcal{V}_\tau)\big)^K$.

As N-order Schwartz norm of a function F in $\mathcal{S}\big(H,\mathrm{End}(\mathcal{V}_\tau)\big)^K$ we take

$$\|F\|_{(N)}=\max_{|\beta|\le N}\|(1+|\cdot|)^N\,(D^\tau)^\beta F\|_{L^2}$$

where for every multiindex $\beta=(\beta_1,\dots,\beta_{\ell''})$

$$(D^\tau)^\beta=A_\tau\,D_{\ell'+1}^{\beta_1}\cdots D_\ell^{\beta_{\ell''}}\,A_\tau^{-1}.$$

Proposition 2.4 *Let $\tau\in\widehat{K}$ and N in $\mathbb{N}$. The following estimates hold*

$$\sqrt{d_\tau}\,\|A_\tau f\|_{(N)}\le\|f\|_{(N)}\le(1+|\xi'_\tau|)^N\sqrt{d_\tau}\,\|A_\tau f\|_{(N)}\qquad\forall f\in\mathcal{S}(G)^{\mathrm{Int}(K)}_\tau.$$

Proof Let $f\in\mathcal{S}(G)^{\mathrm{Int}(K)}_\tau$. Since $\sqrt{d_\tau}A_\tau$ is an isometry on L^2-spaces, we have

$$\begin{aligned}\|A_\tau f\|_{(N)}&=\max_{|\beta|\le N}\|(1+|\cdot|)^N\,(D^\tau)^\beta A_\tau f\|_{L^2}\\&=\max_{|\beta|\le N}\|A_\tau(1+|\cdot|)^N\,(D'')^\beta f\|_{L^2}\\&=\frac{1}{\sqrt{d_\tau}}\max_{|\beta|\le M}\|(1+|\cdot|)^N\,(D'')^\beta f\|_{L^2}\\&\le\frac{1}{\sqrt{d_\tau}}\|f\|_{(N)}\end{aligned}$$

where $(D'')^{\beta} = (D_{\ell'+1})^{\beta_1} \cdots (D_{\ell})^{\beta_{\ell''}}$. Conversely, using Lemma 2.3 with $F = A_\tau f$

$$\begin{aligned}\|f\|_{(N)} &= \max_{|\beta|\le N} \|(1+|\cdot|)^N\, D^\beta A_\tau^{-1} F\|_{L^2}\\ &= \max_{|\beta|\le N} \|A_\tau^{-1}(1+|\cdot|)^N\, A_\tau D^\beta A_\tau^{-1} F\|_{L^2}\\ &\le \sqrt{d_\tau}\,(1+|\xi'_\tau|)^N \max_{|\beta|\le N} \|(1+|\cdot|)^N\,(D^\tau)^\beta F\|_{L^2}\\ &= \sqrt{d_\tau}\,(1+|\xi'_\tau|)^N\, \|F\|_{(N)}\,.\end{aligned}$$

□

Corollary 2.5 *The map A_τ is a topological isomorphism from $\mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ onto $\mathcal{S}\big(H, \mathrm{End}(\mathcal{V}_\tau)\big)^K$.*

3 τ-spherical analysis on the euclidean motion group $SO_n \ltimes \mathbb{R}^n$

In this section we introduce the euclidean motion group $G = SO_n \ltimes_\delta \mathbb{R}^n$ for general n. We recall some basic facts about the algebra $\big(\mathbb{D}(\mathbb{R}^n) \otimes \mathrm{End}(\mathcal{V}_\tau)\big)^K$ and the Gelfand spectrum Σ^τ of $L^1\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$.

3.1 The euclidean motion group

The euclidean motion group is the semidirect product $G = SO_n \ltimes_\delta \mathbb{R}^n$ where the action δ of $K = SO_n$ on $\mathbb{R}^n$ is the natural one, i.e., $\delta(k)y = k\,y$. We write its elements as pairs $(k, y) \in SO_n \times \mathbb{R}^n$ with product law

$$(k, y)(k', y') = (kk', y + k\,y')$$

so that the action of K on G by inner automorphisms is given by

$$k\ :\ (k', y') \longrightarrow (k\,k'\,k^{-1}, k\,y').$$

It is well known that the algebra $L^1(G)^{\mathrm{Int}(K)}$ is commutative, i.e., the pair (G, K) is a strong Gelfand pair [20, 22, 24].

3.2 Invariant differential operators

Differential operators in $\mathbb{D}(G)^{\mathrm{Int}(K)}$ are obtained from $\mathrm{Ad}(K)$–invariant polynomials on the Lie algebra $\mathfrak{g}$ of G via the symmetrization procedure described in [20, formula (2.4)].

Indeed, let p be in $\mathcal{P}(\mathfrak{g})^{\mathrm{Ad}(K)}$ and note that $\mathfrak{g}$ can be identified with $\mathfrak{k} \oplus \mathbb{R}^n$, where $\mathfrak{k} = \mathfrak{so}_n$, so that p satisfies

$$p(X, y) = p(Ad(k)(X, y)) = p(kXk^{-1}, ky) \qquad \forall k \in K, \quad (X, y) \in \mathfrak{k} \oplus \mathbb{R}^n.$$

Then, using the symmetrization operator λ on the universal enveloping algebra $\mathfrak{U}(\mathfrak{k})$ [21, p. 180] and Fourier transform in y, we construct the corresponding operator $D_p \in \mathbb{D}(G)^{\mathrm{Int}(K)}$. Explicitly, for the choice of an orthonormal basis $\{X_1, \ldots, X_{n(n-1)/2}\}$ of $\mathfrak{so}_n$ we write $X = \sum_\ell x_\ell\, X_\ell$ and if

$$p(X, y) = \sum_{\alpha,\beta} c_{\alpha,\beta}\, x^\alpha\, y^\beta \tag{3.1}$$

we define

$$D_p = \sum_{\alpha,\beta} c_{\alpha,\beta} \lambda((ix)^\alpha)\, (-i\partial_y)^\beta. \tag{3.2}$$

The coefficient i in both factors has been introduced in order to map real polynomials into symmetric operators.

For the reader's convenience we recall that

$$(\lambda((ix)^\alpha) f)(g) = i^{|\alpha|}\, \partial_t^\alpha f(g \exp(t_1 X_1 + \cdots + t_m X_m))|_{t=(t_1,\ldots,t_m)=0}$$

with $m = n(n-1)/2$, and that we choose the following normalization of the Fourier transform of an integrable function f on $\mathbb{R}^n$

$$\mathcal{F} f(\eta) = \widehat{f}(\eta) = \int_{\mathbb{R}^n} f(y)\, e^{-iy\cdot\eta}\, dy.$$

Explicit families of generating polynomials of $\mathcal{P}(\mathfrak{g})^{\mathrm{Ad}(K)}$ can be found in [FRY1, Thm. 7.5(1)]. It is convenient for us to choose a set of generators $\{p_1, \ldots p_{\ell'}, q_1, \ldots, q_{\ell''}\}$, where the polynomials p_j depend only on the variables in $\mathfrak{k}$, the polynomial $q_1 = |y|^2$ depends only on the $\mathbb{R}^n$ variables and the polynomials q_j with $j > 1$ may depend on both groups of variables.

Calling D_{p_j}, D_{q_j} the symmetrizations of the above polynomials, the operator D_{q_1} is (minus) the laplacian on $\mathbb{R}^n$, i.e.,

$$D_{q_1} = \Delta = -\sum_{j=1}^n \partial_j^2$$

and the operators D_{p_j} form the set $\mathcal{D}'$ in Sect. 2 so that their eigenvalues on each spherical function depend only on its type.

3.3 *K*-type subalgebras and End($\mathcal{V}_\tau$)-valued spherical analysis

We now focus on the commutative triple (G, K, τ) with $\tau \in \widehat{K}$ and understand the structure of End($\mathcal{V}_\tau$)–valued functions F on $\mathbb{R}^n$ satisfying the K-equivariance condition (2.1), where $\delta(k)y = k\, y$.

This condition implies that such a function F is uniquely determined by its values for $y = ro$ where $o = (1, 0, \ldots, 0)$ and $r \geq 0$, and that $F(ro)$ commutes with $\tau(k)$ for all k in the stabilizer K_o of o, i.e., $k = \begin{bmatrix} 1 & 0 \\ 0 & k' \end{bmatrix}$ with $k' \in SO_{n-1}$.

Since (SO_n, SO_{n-1}) is also a strong Gelfand pair, the restriction of τ to K_o decomposes without multiplicities,

$$\tau_{|SO_{n-1}} = \sigma_0 + \cdots + \sigma_{a_\tau} \qquad \mathcal{V}_\tau = \mathcal{U}_{\sigma_0} \oplus \cdots \oplus \mathcal{U}_{\sigma_{a_\tau}} \tag{3.3}$$

where $\mathcal{U}_{\sigma_j}$ is the representation space of σ_j and the σ_j are mutually inequivalent. Then, by Schur's lemma,

$$F(ro) = \sum_{s=1}^{a_\tau} b_{s,\tau}(F, r) P_{s,\tau} \qquad b_{s,\tau}(F, r) \in \mathbb{C} \tag{3.4}$$

where $P_{s,\tau}$ is the orthogonal projection of $\mathcal{V}_\tau$ onto $\mathcal{U}_{\sigma_s}$, so that

$$b_{s,\tau}(F,r) = \frac{1}{d_{\sigma_s}} \operatorname{tr}\big(P_{s,\tau} F(ro)\big),$$

where $d_{\sigma_s} = \dim \mathcal{U}_{\sigma_s}$.

Spherical functions have been determined in [20, Sect. 7, 11] and we recall them in the following proposition. The Fourier transform of $\mathrm{End}(\mathcal{V}_\tau)$-valued functions is defined componentwise.

Proposition 3.1 *The Gelfand spectrum Σ^τ of $L^1\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$ consists of the multiplicative functionals*

$$F \longmapsto b_{s,\tau}(\widehat{F},\rho) = \int_{\mathbb{R}^n} \frac{1}{d_\tau}\, tr\big(F(y)\Phi^\tau_{\rho,s}(-y)\big)\, dy$$

where $\Phi^\tau_{\rho,s}$ is the $\mathrm{End}(\mathcal{V}_\tau)$-valued spherical function

$$\Phi^\tau_{\rho,s}(y) = \frac{d_\tau}{d_{\sigma_s}} \int_K e^{i\rho o\cdot(ky)} \tau(k^{-1}) P_{s,\tau}\, \tau(k)\, dk \tag{3.5}$$

with $d_\tau = \dim \mathcal{V}_\tau$ and $d_{\sigma_s} = \dim \mathcal{U}_{\sigma_s}$, $s = 0, \dots, a_\tau$.

Proof For the reader's convenience we explicit here the link between the two formulas in the statement and we refer to [20] for the other details. By (2.1), we have

$$\begin{aligned}
b_{s,\tau}(\widehat{F},\rho) &= \frac{1}{d_{\sigma_s}} \operatorname{tr}\big(P_{s,\tau}\widehat{F}(\rho o)\big) \\
&= \frac{1}{d_{\sigma_s}} \int_{\mathbb{R}^n} e^{-i\rho o\cdot y} \operatorname{tr}\big(P_{s,\tau} F(y)\big)\, dy \\
&= \frac{1}{d_{\sigma_s}} \int_K \int_{\mathbb{R}^n} e^{-i\rho o\cdot(ky)} \operatorname{tr}\big(P_{s,\tau} F(ky)\big)\, dy\, dk \\
&= \frac{1}{d_{\sigma_s}} \int_K \int_{\mathbb{R}^n} e^{-i\rho o\cdot(ky)} \operatorname{tr}\big(\tau(k^{-1}) P_{s,\tau}\, \tau(k) F(y)\big)\, dy\, dk \\
&= \int_{\mathbb{R}^n} \frac{1}{d_\tau} \operatorname{tr}\big(F(y)\Phi^\tau_{\rho,s}(-y)\big)\, dy\,.
\end{aligned}$$

□

Via the map A_τ we infer that the multiplicative functionals of $L^1(G)^{\mathrm{Int}(K)}_\tau$ are given by

$$f \longmapsto \int_{K \ltimes \mathbb{R}^n} f(k,y)\varphi^\tau_{\rho,s}\big((k,y)^{-1}\big)\, dk\, dy$$

where the functions

$$\varphi^\tau_{\rho,s}(k,y) = \frac{1}{d_\tau^2} A_\tau^{-1} \Phi^\tau_{\rho,s}(k,y) = \frac{1}{d_\tau} \operatorname{tr}\big(\tau(k)^{-1}\Phi^\tau_{\rho,s}(y)\big)$$

are the τ-spherical functions for the Gelfand pair $(K \ltimes_{\mathrm{Int}} G, K)$. Constants are arranged in such a way that $\Phi^\tau_{\rho,s}(0)$ is the identity operator and $\varphi^\tau_{\rho,s}(e,0) = 1$.

3.4 Differential operators in $\left(\mathbb{D}(\mathbb{R}^n)\otimes \mathrm{End}(\mathcal{V}_\tau)\right)^K$

Suppose that p is in $\mathcal{P}(\mathfrak{g})^{\mathrm{Ad}(K)}$. In Sect. 3.2 we developed a rule that generates the differential operator D_p in $\mathbb{D}(G)^{\mathrm{Int}(K)}$ (see formulas (3.1) and (3.2)).

We can then obtain the differential operator D_p^τ in $\left(\mathbb{D}(\mathbb{R}^n)\otimes \mathrm{End}(\mathcal{V}_\tau)\right)^K$ simply conjugating by A_τ.

The purpose of the following lemma is to relate conjugation by A_τ to the symmetrization procedure, specializing [20, Corollary 2.3] to our case.

Lemma 3.2 *Suppose that p is as in (3.1). Then for every F in $\mathcal{S}\left(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\right)^K$*

$$D_p^\tau F(y) = \sum_{\alpha\beta} c_{\alpha\beta}\,(-i\partial_y)^\beta F(y)\, d\tau(\lambda((-ix)^\alpha))$$

so that

$$\widehat{D_p^\tau F}(\eta) = \widehat{F}(\eta)\sum_{\alpha\beta} c_{\alpha\beta}\,\eta^\beta d\tau(\lambda((-ix)^\alpha))$$

Proof Let $f_\tau = A_\tau^{-1}F$. Then with $m = n(n-1)/2$

$$\begin{aligned}
D_p^\tau F(y) &= A_\tau D_p f_\tau(y)\\
&= \int_K (D_p f_\tau)(k,y)\,\tau(k)\,dk\\
&= \sum_{\alpha\beta} c_{\alpha\beta}\, i^{|\alpha|}\int_K (\lambda(x^\alpha)(-i\partial_y)^\beta f_\tau)(k,y)\,\tau(k)\,dk\\
&= \sum_{\alpha\beta} c_{\alpha\beta}\, i^{|\alpha|}\int_K (-i\partial_y)^\beta\,\partial_t^\alpha f_\tau(k\exp(t_1X_1+\cdots+t_mX_m),y)|_{t=(t_1,\dots,t_m)=0}\,\tau(k)\,dk\\
&= \sum_{\alpha\beta} c_{\alpha\beta}\, i^{|\alpha|}\,(-i\partial_y)^\beta\,\partial_t^\alpha \int_K f_\tau(k\exp(t_1X_1+\cdots+t_mX_m),y)\,\tau(k)\,dk\Big|_{t=(t_1,\dots,t_m)=0}\\
&= \sum_{\alpha\beta} c_{\alpha\beta}\, i^{|\alpha|}(-i\partial_y)^\beta\,\partial_t^\alpha \int_K f_\tau(k',y)\,\tau(k')\tau(\exp(-(t_1X_1+\cdots+t_mX_m))\,dk'\Big|_{t=(t_1,\dots,t_m)=0}\\
&= \sum_{\alpha\beta} c_{\alpha\beta}\, i^{|\alpha|}(-i\partial_y)^\beta \int_K f_\tau(k',y)\,\tau(k')\,dk'\,\partial_t^\alpha\tau(\exp(-(t_1X_1+\cdots+t_mX_m))|_{t=(t_1,\dots,t_m)=0}\\
&= \sum_{\alpha\beta} c_{\alpha\beta}\,(-i\partial_y)^\beta F(y)\,d\tau(\lambda((-ix)^\alpha))\,.
\end{aligned}$$

□

4 Equivariant polynomials and the Gelfand spectrum $\Sigma_{\mathcal{D}}^\tau$ when $n = 3, 4$

From now on we will consider the cases where $n = 3, 4$. In this section we will compute, for any fixed τ in $\hat{K}$, the eigenvalues ξ'_τ. Next we will consider the representation $\tilde{\tau}$ on $\mathrm{End}(\mathcal{V}_\tau)$ given by

$$\tilde{\tau}(k)A = \tau(k)A\tau(k^{-1}) \qquad \forall k\in K \tag{4.1}$$

and decompose $\mathrm{End}(\mathcal{V}_\tau)$ into irreducible invariant subspaces. The lower dimensional non-trivial subspace contains an element which turns out to be important in our analysis, because it coincides with the Fourier transform of one of the generators D_q^τ at the basepoint. We shall denote this element by B_τ. The main result is Lemma 4.2 where we determine which polynomials in B_τ lie in the various irreducible invariant subspaces, and Proposition 4.4, where we determine the Gelfand spectrum $\Sigma_{\mathcal{D}}^\tau$.

4.1 The case where $n = 3$

We denote by τ_μ, with $\mu \in (1/2)\mathbb{N}$ the irreducible representation of SU_2 of dimension $2\mu+1$. For $\mu \in \mathbb{N}$, τ_μ projects to a representation of $SO_3 \equiv SU_2/\mathbb{Z}_2$ and $\widehat{SO_3} = \{\tau_\mu : \mu \in \mathbb{N}\}$.

The restriction of τ_μ to $K_o \equiv SO_2$ in (3.3) decomposes as

$$\tau_{\mu\,|_{SO_2}} = \sum_{s=0}^{2\mu} \sigma_s$$

where $\sigma_s \left(\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \right)$ is the character $e^{i(s-\mu)t}$. So $\mathcal{V}_{\tau_\mu} = \mathcal{U}_{\sigma_0} \oplus \cdots \oplus \mathcal{U}_{\sigma_{2\mu}}$ and the spaces $\mathcal{U}_{\sigma_s} \sim \mathbb{C}$ are all one dimensional.

We choose the basis of $\mathfrak{so}_3$ formed by the vectors

$$X_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \qquad X_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \qquad X_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and we identify

$$X = \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} = \sum_{j=1}^{3} x_j\, X_j \in \mathfrak{so}_3$$

with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The following family of invariant polynomials of $\mathcal{P}(\mathfrak{g})^{\mathrm{Ad}(K)}$ is well known

$$p_1 = |x|^2 \qquad q_1 = |y|^2 \qquad q_2 = x \cdot y.$$

Then the symmetrization map of Sect. 3.2 produces $D_{p_1} = -\sum_{j=1}^3 X_j^2$, a constant multiple of the Casimir operator on SO_3, and

$$\xi'_{\tau_\mu} = \mu(\mu+1).$$

According to Lemma 3.2, we obtain the following generating system of $\big(\mathbb{D}(\mathbb{R}^3) \otimes \mathrm{End}(\mathcal{V}_\tau)\big)^K$

$$D_{q_1}^{\tau_\mu} = \Delta I \qquad \mathbf{D}_{\tau_\mu} = D_{q_2}^{\tau_\mu} = -\sum_{j=1}^{3} \partial_{y_j}\, d\tau_\mu(X_j).$$

Identifying $\mathrm{End}(\mathcal{V}_{\tau_\mu})$ with $\mathcal{V}'_{\tau_\mu} \otimes \mathcal{V}_{\tau_\mu}$, the representation $\tilde{\tau}_\mu$ defined in (4.1) is equivalent to the tensor product $\tau'_\mu \otimes \tau_\mu$, where $\tau'_\mu \sim \tau_\mu$ is the contragredient representation of τ_μ.

Therefore $\tilde{\tau}_\mu$ decomposes as

$$\tilde{\tau}_\mu \sim \tau'_\mu \otimes \tau_\mu \sim \tau_{2\mu} \oplus \tau_{2\mu-1} \oplus \cdots \oplus \tau_1 \oplus \tau_0$$

and accordingly

$$\mathrm{End}(\mathcal{V}_{\tau_\mu}) = \sum_{\ell=0}^{2\mu} \mathcal{W}^\ell_{\tau_\mu} \qquad \mathcal{W}^\ell_{\tau_\mu} \sim \tau_\ell, \tag{4.2}$$

where $\dim \mathcal{W}^\ell_{\tau_\mu} = 2\ell + 1$.

Let

$$B_{\tau_\mu} = \widehat{\mathbf{D}_{\tau_\mu}}(o) = -id\tau_\mu(X_1) = \sum_{s=0}^{2\mu} (-\mu + s) P_{s,\tau} \tag{4.3}$$

and consider the $\tilde{\tau}_\mu$-invariant subspace $\mathcal{W}$ generated by $d\tau_\mu(X_1)$. Then

$$\mathcal{W} = \mathrm{span}_{\mathbb{C}} \left\{ \tau_\mu(k) d\tau_\mu(X_1) \tau_\mu(k)^{-1} \,:\, k \in SO_3 \right\} = d\tau_\mu(\mathfrak{so}_3)$$

which is a 3-dimensional invariant subspace. So it must coincide with the component $\mathcal{W}^1_{\tau_\mu}$ in (4.2).

For every $\ell = 0, \ldots, 2\mu$, let $\mathcal{B}^\ell_{\tau_\mu}$ be the subspace of K_o-invariant elements in $\mathcal{W}^\ell_{\tau_\mu}$. Since the representation $\tilde{\tau}_\mu$ restricted to $\mathcal{W}^\ell_{\tau_\mu}$ is equivalent to τ_ℓ, it contains the null weight with multiplicity one, and so $\mathcal{B}^\ell_{\tau_\mu}$ is one-dimensional. In particular, B_{τ_μ} spans $\mathcal{B}^1_{\tau_\mu}$.

4.2 The case where $n = 4$

When $n = 4$, it is convenient to use the identifications

$$SO_4 \cong (SU_2 \times SU_2)/\mathbb{Z}_2 \cong (\mathrm{Sp}_1 \times \mathrm{Sp}_1)/\mathbb{Z}_2 \qquad \mathbb{R}^4 \cong \mathbb{H}$$

with one copy of $\mathrm{Sp}_1 = \{u \in \mathbb{H} : |u| = 1\}$ acting by left multiplication and the other by right multiplication on $\mathbb{H}$,

$$(u, v) \cdot y = uyv^{-1}. \tag{4.4}$$

According to the previous identifications, the unitary dual $\widehat{SO_4}$ consists of the exterior tensor products $\tau_{\nu,\mu} = \tau_\nu \boxtimes \tau_\mu$, with $\nu, \mu \in (1/2)\mathbb{N}$ and $\nu + \mu \in \mathbb{N}$.

The Lie algebra $\mathfrak{so}_4$ can be identified with $\mathfrak{su}_2 \times \mathfrak{su}_2 \cong \mathrm{Im}\,\mathbb{H} \times \mathrm{Im}\,\mathbb{H}$. For $\alpha, \beta \in \mathrm{Im}\,\mathbb{H}$ we shall write $X_{\alpha,\beta}$ for the corresponding element in $\mathfrak{so}_4$. Differentiating (4.4) we obtain

$$X_{\alpha,\beta} y = \alpha y - y\beta.$$

Let

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

be the standard basis of $\mathfrak{su}_2$. Denote these elements by U_1, U_2, U_3 if they belong to the first copy of $\mathfrak{su}_2$, and by V_1, V_2, V_3 if they belong to the second. The adjoint action of SO_4 on $\mathfrak{so}_4 \times \mathbb{R}^4$ is given by

$$\big(\mathrm{Ad}(u, v)\big)(\alpha, \beta, y) = (u\alpha u^{-1}, v\beta v^{-1}, uyv^{-1})$$

and we choose the following family of generating polynomials of $\mathcal{P}(\mathfrak{g})^{\mathrm{Ad}(K)}$

$$p_1 = |\alpha|^2 \qquad p_2 = |\beta|^2 \qquad q_1 = |y|^2 \qquad q_2 = \mathrm{Re}\,(\alpha y\, \overline{y\beta}) = \langle \alpha y, y\beta \rangle.$$

Then $\lambda(p_i) = C_i$, a constant multiple of Casimir operator on the i-th copy of SU_2, so

$$\xi'_{\tau_{\nu,\mu}} = \big(2\nu(2\nu+2), 2\mu(2\mu+2)\big)$$

and the $\mathrm{End}(\mathcal{V}_{\tau_{\nu,\mu}})$-valued differential operators are

$$D^{\tau_{\nu,\mu}}_{q_1} = \Delta I \qquad \mathbf{D}_{\tau_{\nu,\mu}} = D^{\tau_{\nu,\mu}}_{q_2}.$$

One could work out an explicit formula for $\mathbf{D}_{\tau_{\nu,\mu}}$, but for our purposes it will suffice to know that for $o = 1 \in \mathbb{H}$

$$\widehat{\mathbf{D}_{\tau_{\nu,\mu}}}(o) = -\sum_{j=1}^{3} d\tau_\nu(U_j) \otimes d\tau_\mu(V_j)$$

which follows immediately from Lemma 3.2.

The stabilizer K_o of o is $\mathrm{diag}(SU_2 \times SU_2)/\mathbb{Z}_2 \cong SO_3$. Since $(\tau_\nu \boxtimes \tau_\mu)_{|_{\mathrm{diag}(SU_2\times SU_2)}} = \tau_\nu \otimes \tau_\mu$, the analogue of formula (3.3) becomes

$$\tau_{\nu,\mu}{}_{|_{K_o}} = \tau_{\mu+\nu} \oplus \tau_{\mu+\nu-1} \oplus \cdots \oplus \tau_{|\mu-\nu|}.$$

Therefore for $j = 0, \ldots, 2(\mu \wedge \nu)(= 2\min\{\mu, \nu\})$, we let $\sigma_j \sim \tau_{\mu+\nu-j}$ and $\mathcal{U}_{\sigma_j}$ for its representation space.

Since every SU_2-factor acts on the corresponding factor space, for the action on $\mathrm{End}(\mathcal{V}_{\tau_{\nu,\mu}})$ we have the decomposition

$$\tilde{\tau}_{\nu,\mu} \sim \tau'_{\nu,\mu} \otimes \tau_{\nu,\mu} \sim \sum_{j=0}^{2\nu}\sum_{k=0}^{2\mu} \tau_j \boxtimes \tau_k$$

corresponding to

$$\mathrm{End}(\mathcal{V}_{\tau_{\nu,\mu}}) = \sum_{j=0}^{2\nu}\sum_{k=0}^{2\mu} \mathcal{W}^{j,k}_{\tau_{\nu,\mu}} \qquad \mathcal{W}^{j,k}_{\tau_{\nu,\mu}} = \mathcal{W}^{j}_{\tau_\nu} \otimes \mathcal{W}^{k}_{\tau_\mu} \sim \tau_j \boxtimes \tau_k.$$

We remark that, restricting the representation to K_o, $\mathcal{W}^{j,k}_{\tau_{\nu,\mu}}$ contains a fixed element if and only if $j = k$.

For every $\ell = 0, 1, \ldots, 2(\mu \wedge \nu)$, we denote by $\mathcal{B}^{\ell}_{\tau_{\nu,\mu}}$ the subspace of K_o-fixed matrices in $\mathcal{W}^{\ell,\ell}_{\tau_{\nu,\mu}}$. The subspace $\mathcal{B}^{\ell}_{\tau_{\nu,\mu}}$ is one-dimensional and consists of operators which are constant multiples of the identity operator on each $\mathcal{U}_{\sigma_s}$.

Let

$$B_{\tau_{\nu,\mu}} = \widehat{\mathbf{D}_{\tau_{\nu,\mu}}}(o) = -\sum_{j=1}^{3} d\tau_\nu(U_j) \otimes d\tau_\mu(V_j).$$

As noted in the previous subsection, $d\tau_\nu(U_j)$ is in $\mathcal{W}^1_{\tau_\nu}$. Therefore $B_{\tau_{\nu,\mu}}$ is the only nontrivial element in $\mathcal{B}^1_{\tau_{\nu,\mu}}$.

Since K_o–invariant elements act as constants on $\mathcal{U}_{\sigma_s}$, we may define coefficients $\beta_{s,\tau_{\nu,\mu}}$ so that

$$B_{\tau_{\nu,\mu}} = \sum_{s=0}^{2(\mu\wedge\nu)} \beta_{s,\tau_{\nu,\mu}} P_{s,\tau_{\nu,\mu}}.$$

Lemma 4.1 $\beta_{s,\tau_{\nu,\mu}} = 4(\mu - s)(\nu - s) - 2s(s+1)$.

Proof In the polynomial model $\mathcal{V}_{\tau_\nu} \otimes \mathcal{V}_{\tau_\mu} = \mathcal{P}^{2\nu,2\mu}(\mathbb{C}^2)$, we obtain

$$\mathcal{U}_{\sigma_s} = |z|^{2s}\ \mathcal{H}^{2\nu-s,2\mu-s}$$

where $\mathcal{H}^{\alpha,\beta}$ denotes the space of harmonic polynomials of bidegree (α, β).

Then $d\tau_\nu(U_j)$ and $d\tau_\mu(V_j)$ act respectively on the holomorphic and the antiholomorphic variables. Therefore we shall more briefly write $Z_j = d\tau_\nu(U_j)$ and $\bar{Z}_j = d\tau_\mu(V_j)$ so that

$$\begin{aligned} Z_1 &= -iz_1\partial_{z_1} + iz_2\partial_{z_2} & Z_2 &= -z_2\partial_{z_1} + z_1\partial_{z_2} & Z_3 &= -iz_2\partial_{z_1} - iz_1\partial_{z_2} \\ \bar{Z}_1 &= i\bar{z}_1\partial_{\bar{z}_1} - i\bar{z}_2\partial_{\bar{z}_2} & \bar{Z}_2 &= -\bar{z}_2\partial_{\bar{z}_1} + \bar{z}_1\partial_{\bar{z}_2} & \bar{Z}_3 &= i\bar{z}_2\partial_{\bar{z}_1} + i\bar{z}_1\partial_{\bar{z}_2} \end{aligned}$$

and

$$\begin{aligned} Z_2 + iZ_3 &= 2z_1\partial_{z_2} & Z_2 - iZ_3 &= -2z_2\partial_{z_1} \\ \bar{Z}_2 + i\bar{Z}_3 &= -2\bar{z}_2\partial_{\bar{z}_1} & \bar{Z}_2 - i\bar{Z}_3 &= 2\bar{z}_1\partial_{\bar{z}_2}. \end{aligned}$$

According to the above notation

$$\begin{aligned} B_{\tau_{\nu,\mu}} &= -\sum_{j=1}^{3} Z_j\bar{Z}_j = -Z_1\bar{Z}_1 - \frac{1}{2}(Z_2 + iZ_3)(\bar{Z}_2 - i\bar{Z}_3) - \frac{1}{2}(Z_2 - iZ_3)(\bar{Z}_2 + i\bar{Z}_3) \\ &= -(z_1\partial_{z_1} - z_2\partial_{z_2})(\bar{z}_1\partial_{\bar{z}_1} - \bar{z}_2\partial_{\bar{z}_2}) - 2z_1\bar{z}_1\partial_{z_2}\partial_{\bar{z}_2} - 2z_2\bar{z}_2\partial_{z_1}\partial_{\bar{z}_1}. \end{aligned}$$

Consider the polynomial in $\mathcal{U}_{\sigma_s}$ given by

$$P(z,\bar{z}) = |z|^{2s} z_1^{2\nu-s}\bar{z}_2^{2\mu-s} = \sum_{k=0}^{s}\binom{s}{k} z_1^{2\nu-s+k}\bar{z}_1^{k} z_2^{s-k}\bar{z}_2^{2\mu-k}.$$

Explicit computations show that

$$B_{\tau_{\nu,\mu}} P = \big(4(\mu - s)(\nu - s) - 2s(s+1)\big)P.$$

□

4.3 Polynomials in B_τ

In order to treat the cases $n = 3, 4$ together as far as possible, it is convenient to use the following notation. We recall that

$$a_\tau = \begin{cases} 2\mu & n = 3 \quad \tau = \tau_\mu \\ 2\,(\mu \wedge \nu) & n = 4 \quad \tau = \tau_{\nu,\mu} \end{cases}$$

and for $j = 0, \ldots, a_\tau$ we put

$$\mathcal{W}_\tau^j = \begin{cases} \mathcal{W}_{\tau_\mu}^j & n = 3 \quad \tau = \tau_\mu \\ \mathcal{W}_{\tau_{\nu,\mu}}^{j,j} & n = 4 \quad \tau = \tau_{\nu,\mu}. \end{cases}$$

Lemma 4.2 *For every $\ell = 0, \ldots, a_\tau$, there is a monic polynomial q_τ^ℓ of degree ℓ such that $\mathcal{B}_\tau^\ell$ consists of the scalar multiples of $q_\tau^\ell(B_\tau)$.*

Moreover, for a polynomial p of degree at most a_τ, we have $p(B_\tau) \in \sum_{j\le\ell}\mathcal{W}_\tau^j$ if and only if $\deg(p) \le \ell$.
Finally, when $n = 3$, q_τ^ℓ is of the same parity as ℓ.

Proof We start giving the proof in the case where $n = 3$, so $\tau = \tau_\mu$ and $B_\tau = B_{\tau_\mu}$, given in (4.3).

We proceed by induction proving all statements together. They are trivial for $\ell = 0$ and have been proved in Sect. 4.1 for $\ell = 1$.

Let $\ell \geq 2$ and consider the polynomial $t \mapsto p(t) = t\, q_\tau^{\ell-1}(t)$. Then p is of degree ℓ and $p(B_\tau)$ is a K_o–invariant element. By the inductive hypothesis, $q_\tau^{\ell-1}(B_\tau)$ is in $\mathcal{W}_{\tau_\mu}^{\ell-1}$ and B_τ is in $\mathcal{W}_{\tau_\mu}^1$; moreover we have $\tau'_\mu \otimes \tau_\mu|_{\mathcal{W}_{\tau_\mu}^{\ell-1}} \sim \tau_{\ell-1}$ and the decomposition

$$\tau_{\ell-1} \otimes \tau_1 \sim \tau_\ell \oplus \tau_{\ell-1} \oplus \tau_{\ell-2}$$

in irreducible summands with multiplicity one for τ_ℓ. Therefore $p(B_\tau)$ is in $\mathcal{W}_{\tau_\mu}^\ell \oplus \cdots \oplus \mathcal{W}_{\tau_\mu}^0$ with a nontrivial component in $\mathcal{W}_{\tau_\mu}^\ell$. We call $q_\tau^\ell(B_\tau)$ this component. Hence $(p - q_\tau^\ell)(B_\tau)$ is in $\sum_{j\in L} \mathcal{W}_{\tau_\mu}^j$, which, by the inductive hypothesis, is a polynomial in B_τ of degree at most $\ell - 1$. Then q_τ^ℓ has degree ℓ and its leading term is the same as p, so it is monic.

As for the parity statement, note that the inner product is proportional to the Hilbert–Schmidt one so that

$$\langle (B_\tau)^\ell, (B_\tau)^{\ell'} \rangle = c \sum_{j=-\mu}^{\mu} j^{\ell+\ell'} = 0$$

when $\ell + \ell'$ is odd.

In the case where $n = 4$, the proof is almost identical to the previous one; the only difference lies in the fact that in the decomposition

$$\begin{aligned}(\tau_{\ell-1} \boxtimes \tau_{\ell-1}) \otimes (\tau_1 \boxtimes \tau_1) &\sim (\tau_\ell \oplus \tau_{\ell-1} \oplus \tau_{\ell-2}) \boxtimes (\tau_\ell \oplus \tau_{\ell-1} \oplus \tau_{\ell-2}) \\ &\sim \sum_{j=\ell-2}^{\ell} \sum_{k=\ell-2}^{\ell} \tau_j \boxtimes \tau_k\end{aligned}$$

in irreducible summands with multiplicity one for $\tau_\ell \boxtimes \tau_\ell$, K_o–invariant elements arise only when $j = k$. □

4.4 Equivariant homogeneous polynomials

Denote by Q_τ the K–equivariant polynomial arising from the symmetrization of the polynomial q_2 and Lemma 3.2. Explicity,

$$\widehat{\mathbf{D}_\tau F} = \widehat{F}\, Q_\tau \qquad \forall F \in \mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K. \tag{4.5}$$

This polynomial is also completely determined by the following properties:

(i) $Q_\tau(o) = B_\tau$;
(ii) Q_τ is K-equivariant;
(iii) Q_τ is of least degree.

Note that Q_τ is homogeneous of degree κ where

$$\kappa = \begin{cases} 1 & \text{if } n = 3 \\ 2 & \text{if } n = 4 \end{cases}.$$

Suppose now that $P : \mathbb{R}^n \to \mathrm{End}(\mathcal{V}_\tau)$ is a K-equivariant polynomial. Then each homogeneous component of P is also equivariant and P can be uniquely decomposed as the sum of $\mathcal{W}_\tau^\ell$-valued homogeneous polynomials.

Proposition 4.3 *Suppose that there exists a nontrivial K–equivariant polynomial, homogeneous of degree d and $\mathcal{W}_\tau^\ell$–valued. Then $d \geq \ell\kappa$ and $d - \ell\kappa$ is even.*

Proof Suppose that $P : \mathbb{R}^n \to \mathrm{End}(\mathcal{V}_\tau)$ is a nontrivial K–equivariant polynomial homogeneous of degree d and $\mathcal{W}_\tau^\ell$-valued. Since $P(o)$ is in $\mathcal{B}_\tau^\ell$, by Lemma 4.2

$$P(o) = c\, q_\tau^\ell(B_\tau)$$

for some constant c. Setting $y' = y/|y|$ for $y \neq 0$,

$$P(y) = c\,|y|^d \tau(k_{y'}) q_\tau^\ell(B_\tau) \tau(k_{y'}^{-1}) = c\,|y|^d q_\tau^\ell(Q_\tau(y/|y|)).$$

Since in the last equation there are no negative powers, we deduce that $d \geq \ell\kappa$. Finally, since P is a polynomial, $d - \ell\kappa$ is even. □

4.5 End($\mathcal{V}_\tau$)-valued spherical functions as derivatives of scalar spherical functions

For $\rho \geq 0$ we denote by φ_ρ the spherical function of (G, K) as an ordinary Gelfand pair with eigenvalue ρ^2 relative to Δ. In formulae,

$$\varphi_\rho(y) = \varphi_1(\rho y) \quad \text{where} \quad \varphi_1(y) = \int_{S^{n-1}} e^{i\, y\cdot\eta}\, d\sigma(\eta) \quad \Big(= |y|^{-\frac{n}{2}+1}\, J_{\frac{n}{2}-1}\big(|y|\big)\Big)$$

and σ is the normalized surface measure of the unit sphere S^{n-1} in $\mathbb{R}^n$.

Coherently with Lemma 4.1, we call $\beta_{s,\tau}$ the eigenvalue of B_τ on the subspace $\mathcal{U}_{\sigma_s}$. In formulae,

$$\beta_{s,\tau} = \begin{cases} -\mu + s & n = 3, \quad \tau = \tau_\mu \\ 4(\mu - s)(\nu - s) - 2s(s+1) & n = 4, \quad \tau = \tau_{\nu,\mu} \end{cases} \qquad s = 0, \dots, a_\tau,$$

so that $B_\tau = \sum_{s=0}^{a_\tau} \beta_{s,\tau} P_{s,\tau}$. Note that, with respect to the notation in (3.4) $\beta_{s,\tau}$ is a shorthand notation

$$\beta_{s,\tau} = b_{s,\tau}(Q_\tau, 1). \tag{4.6}$$

Since $\beta_{s,\tau} \neq \beta_{s',\tau}$ when $s \neq s'$, the orthogonal projection $P_{s,\tau}$ of $\mathcal{V}_\tau$ onto $\mathcal{U}_{\sigma_s}$ can be written as a polynomial in B_τ of degree a_τ. Indeed,

$$P_{s,\tau} = p_{s,\tau}(B_\tau) \qquad \text{where} \qquad p_{s,\tau}(t) = \prod_{j\neq s,\, j=0}^{a_\tau} \frac{t - \beta_{j,\tau}}{\beta_{s,\tau} - \beta_{j,\tau}}.$$

In the following proposition we write the $\mathrm{End}(\mathcal{V}_\tau)$-valued spherical functions as derivatives of scalar spherical functions and we observe that the distributional Fourier transform of $\Phi_{\rho,s}^\tau$ is given in terms of the orthogonal projection $P_{s,\tau}$. As a consequence we easily compute the embedded spectrum $\Sigma_{\mathcal{D}}^\tau$ showing that it consists of half parabolas when $n = 3$ and rays through the origin when $n = 4$.

Given an $\mathrm{End}(\mathcal{V}_\tau)$-valued polynomial P, we denote by $P(\partial)$ the differential operator defined by the rule

$$\widehat{P(\partial)F}(\eta) = \widehat{F}(\eta)\, P(\eta) \qquad \eta \in \mathbb{R}^n, \quad \forall F \in \mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K.$$

In particular note that

$$\mathbf{D}_\tau = Q_\tau(\partial).$$

We recall that $\kappa = 1$ when $n = 3$, and $\kappa = 2$ when $n = 4$.

Proposition 4.4 *The spectrum $\Sigma^\tau_{\mathcal{D}}$ is given by*

$$\Sigma^\tau_{\mathcal{D}} = \left\{\left(\rho^2, \beta_{s,\tau}\,\rho^\kappa\right) : \rho \geq 0,\ s = 0, \ldots, a_\tau\right\}.$$

Moreover, the $\mathrm{End}(\mathcal{V}_\tau)$*-valued spherical function $\Phi^\tau_{\rho,s}$ in (3.5) corresponds to the pair of eigenvalues $\left(\rho^2, \beta_{s,\tau}\,\rho^\kappa\right)$ and satisfies*

$$\Phi^\tau_{\rho,s} = (2\pi)^n \frac{d_\tau}{d_{\sigma_s}} (p_{s,\tau} \circ Q_\tau)(\partial)\,\varphi_\rho \qquad \forall \rho > 0, \tag{4.7}$$

Proof The case $\rho = 0$ is trivial. When $\rho > 0$, by formula (3.5) we have

$$\begin{aligned}
\Phi^\tau_{\rho,s}(y) &= \frac{d_\tau}{d_{\sigma_s}} \int_{SO_n} e^{i\rho\, o\cdot(ky)} \tau(k^{-1}) P_{s,\tau}\, \tau(k)\, dk \\
&= \frac{d_\tau}{d_{\sigma_s}} \int_{SO_n} e^{i\rho\, k^{-1}o\cdot y}\, (p_{s,\tau} \circ Q_\tau)(k^{-1}o)\, dk \\
&= \frac{d_\tau}{d_{\sigma_s}} \int_{S^{n-1}} e^{i\rho\,\eta\cdot y}\, (p_{s,\tau} \circ Q_\tau)(\eta)\, d\sigma(\eta) \\
&= (2\pi)^n \frac{d_\tau}{d_{\sigma_s}}\, (p_{s,\tau} \circ Q_\tau)(\partial)\,\varphi_\rho\,(y),
\end{aligned}$$

We now determine the embedded spectrum. From formula (3.5), it is easy to check that $\Delta\Phi^\tau_{\rho,s} = \rho^2\,\Phi^\tau_{\rho,s}$. As for the second eigenvalue by (4.7)

$$\widehat{\Phi}^\tau_{1,s}(o) = C_{s,\tau}\ P_{s,\tau} = C_{s,\tau}\ \ p_{s,\tau}(B_\tau).$$

where $C_{s,\tau} = \frac{d_\tau}{d_{\sigma_s}}\ \frac{(2\pi)^n}{|S^{n-1}|}$. Hence by (3.4) and (4.6) we have

$$\widehat{\mathbf{D}_\tau \Phi^\tau_{1,s}}(o) = \widehat{\Phi}^\tau_{1,s}(o)\, Q_\tau(o) = C_{s,\tau}\ P_{s,\tau}\ B_\tau = \beta_{s,\tau}\ \widehat{\Phi}^\tau_{1,s}(o).$$

By homogeneity and equivariance we have

$$\mathbf{D}_\tau \Phi^\tau_{\rho,s} = \rho^\kappa \beta_{s,\tau}\ \Phi^\tau_{\rho,s} \qquad \forall \rho > 0, \quad s = 0, \ldots a_\tau.$$

□

Since we regard the τ-spherical transform as defined on the embedded spectrum, and according to Lemma 3.5, we shall write

$$\mathcal{G}^\sharp_\tau F(\rho^2, \beta_{s,\tau}\rho^\kappa) = \frac{1}{d_\tau} \int_{\mathbb{R}^n} \mathrm{tr}\left(F(y)\Phi^\tau_{\rho,s}(-y)\right) dy = b_{s,\tau}\left(\widehat{F}, \rho\right). \tag{4.8}$$

The next corollary shows that we can disregard part of the spectrum when dealing with even or odd functions in the case where $n = 3$.

Corollary 4.5 *In the case where $n = 3$,*

$$\Phi^\tau_{\rho,a_\tau - s}(y) = \Phi^\tau_{\rho,s}(-y),$$

so that if F is decomposed into its even part F_+ and its odd part F_-, then

$$\mathcal{G}^\sharp_\tau F_\pm(\rho^2, \beta_{s,\tau}\rho) = \pm\mathcal{G}^\sharp_\tau F_\pm(\rho^2, -\beta_{s,\tau}\rho)$$

Proof Note that when $n = 3$

$$p_{a_\tau - s}(t) = p_s(-t)$$

and since Q_τ is homogeneous of degree $\kappa = 1$,

$$\big(p_{a_\tau - s} \circ Q_\tau\big)(y) = \big(p_s \circ Q_\tau\big)(-y).$$

Moreover $d_{\sigma_s} = 1$, so that

$$\begin{aligned}
\Phi^\tau_{\rho, a_\tau - s}(y) &= d_\tau \int_{S^2} e^{i\rho\, y\cdot\eta}\, p_{a_\tau - s} \circ Q_\tau(\eta)\, d\sigma(\eta) \\
&= d_\tau \int_{S^2} e^{i\rho\, y\cdot\eta}\, p_s \circ Q_\tau(-\eta)\, d\sigma(\eta) \\
&= d_\tau \int_{S^2} e^{-i\rho\, y\cdot\eta}\, p_s \circ Q_\tau(\eta)\, d\sigma(\eta) \\
&= \Phi^\tau_{\rho, s}(-y).
\end{aligned}$$

Finally, by a simple change of variables

$$\mathcal{G}^\sharp_\tau F_\pm(\rho^2, \beta_{s,\tau}\rho) = \pm \frac{1}{d_\tau} \int_{\mathbb{R}^n} \operatorname{tr}\big(F_\pm(y)\Phi^\tau_{\rho, a_\tau - s}(-y)\big)\, dy,$$

and the desired formula follows remembering that, when $\tau = \tau_\mu, a_\tau = 2\mu$ and $\beta_{s,\tau} = -\mu + s$, so $\beta_{a_\tau - s, \tau} = -\beta_{s,\tau}$. □

5 Schwartz correspondence for (G, K, τ)

In this section we give our first argument to prove that $\mathcal{G}^\sharp_\tau$ maps $\mathcal{S}\big(\mathbb{R}^n, \operatorname{End}(\mathcal{V}_\tau)\big)^K$ into $\mathcal{S}(\Sigma^\tau_{\mathcal{D}})$. The implied norm inequalities will be left implicit because their dependence on τ is not sufficiently sharp for our purposes. Sharper estimates will be obtained in Sect. 6.

The analysis of the Banach algebra $L^1(\mathbb{R}^n, \operatorname{End}(\mathcal{V}_\tau))^K$ developed in the previous section leads to the decomposition of $\operatorname{End}(\mathcal{V}_\tau)$–valued equivariant functions into a sum of derivatives of scalar valued functions.

Proposition 5.1 *Let F be in $\mathcal{S}(\mathbb{R}^n, \operatorname{End}(\mathcal{V}_\tau))^K$. Then F can be expressed in a unique way as*

$$F = \sum_{i=0}^{a_\tau} \mathbf{D}^i_\tau g_i$$

with scalar valued functions $g_i \in \mathcal{S}(\mathbb{R}^n)^K$.

Proof Every F in $\mathcal{S}\big(\mathbb{R}^n, \operatorname{End}(\mathcal{V}_\tau)\big)^K$ can be decomposed in a unique way as $F = \sum_{\ell=0}^{a_\tau} F_\ell$, where the component F_ℓ is in $\mathcal{S}(\mathbb{R}^n, \mathcal{W}^\ell_\tau)^K$. Therefore we start with a single F_ℓ.

Since the Fourier transform commutes with rotations, $\widehat{F_\ell}$ is in $\mathcal{S}(\mathbb{R}^n, \mathcal{W}^\ell_\tau)^K$ too, so that, by Lemma 4.2, for every $\rho \neq 0$, $\widehat{F_\ell}(\rho o)$ is a constant multiple, $f_\ell(\rho)$ say, of $q^\ell_\tau(B_\tau)$.

Suppose that $P_{\ell,d}$ is the Taylor polynomial of $\widehat{F_\ell}$ centred at the origin of order d. i.e.,

$$\widehat{F_\ell}(\eta) = P_{\ell,d}(\eta) + o(|\eta|^d) \qquad \eta \to 0.$$

Then $P_{\ell,d}$ and all of its homogeneous components are K–equivariant and $\mathcal{W}^\ell_\tau$–valued. Therefore $P_{\ell,d}(o)$ is a constant multiple of $q^\ell_\tau(B_\tau)$. Using Proposition 4.3 we conclude that $P_{\ell,d} = 0$ if $d < \kappa\ell$.

We can therefore write

$$\widehat{F}_\ell(\rho o) = f_\ell(\rho)\, q_\tau^\ell(B_\tau) = c_\ell\, \rho^{\kappa\ell}\, q_\tau^\ell(B_\tau) + o(\rho^{\kappa\ell}) \qquad \rho \to 0.$$

Clearly the so-obtained function f_ℓ is in $\mathcal{S}(\mathbb{R})$ and of the same parity as $\kappa\ell$.

Hence

$$f_\ell(\rho) = c_\ell\, \rho^{\kappa\ell} + o(\rho^{\kappa\ell}) \qquad \rho \to 0$$

and by Hadamard's division Lemma [17] there exists an even smooth function $\tilde{f}_\ell$ on $\mathbb{R}$ such that $f_\ell(\rho) = \rho^{\kappa\ell}\, \tilde{f}_\ell(\rho)$ and $\tilde{f}_\ell(0) = c_\ell$. With $h_\ell = \tilde{f}_\ell(|\cdot|)$, we conclude that when $\eta = |\eta|\eta'$

$$\widehat{F}_\ell(\eta) = \tilde{f}_\ell(|\eta|)\, |\eta|^{\kappa\ell}\, \tau(k_{\eta'})\, q_\tau^\ell(B_\tau)\tau(k_{\eta'})^* = h_\ell(\eta)\, q_\tau^\ell(Q_\tau(\eta)).$$

Therefore h_ℓ is a uniquely determined scalar invariant Schwartz function on $\mathbb{R}^n$ and taking the inverse Fourier transform, for a scalar invariant Schwartz function γ_ℓ on $\mathbb{R}^n$,

$$F_\ell(y) = q_\tau^\ell(\mathbf{D}_\tau)\, \gamma_\ell(y).$$

Since q_τ^ℓ is a polynomial of degree ℓ, rearranging the terms in the sum

$$F = \sum_{\ell=0}^{a_\tau} F_\ell = \sum_{\ell=0}^{a_\tau} q_\tau^\ell(\mathbf{D}_\tau)\, \gamma_\ell(y)$$

we obtain the desired decomposition. □

Lemma 5.2 *Let F be in $\mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$. Then there exist $\gamma_0, \dots, \gamma_{a_\tau}$ in $\mathcal{S}(\mathbb{R})$ such that*

$$\mathcal{G}_\tau^\sharp F(\rho^2, \beta_{s,\tau}\rho^\kappa) = \sum_{i=0}^{a_\tau} \gamma_i(\rho^2)\, (\rho^\kappa\, \beta_{s,\tau})^i \qquad \forall \rho \geq 0, \quad s = 0, 1, \dots, a_\tau.$$

Proof Let F be in $\mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$ and g_i be the K–invariant scalar functions we have associated to F in Proposition 5.1. Then $\widehat{g_i}$ is K–invariant, so that by Schwartz–Mather there exists a function $\gamma_i \in \mathcal{S}(\mathbb{R})$ such that

$$\widehat{g_i}(\eta) = \gamma_i(|\eta|^2)$$

and by (4.5)

$$\widehat{F}(\rho o) = \sum_{i=0}^{a_\tau} \widehat{g_i}(\rho o)\, \widehat{\mathbf{D}_\tau^i}(\rho o) = \sum_{i=0}^{a_\tau} \gamma_i(\rho^2)\, \rho^{\kappa i}\, B_\tau^i = \sum_{i=0}^{a_\tau} \gamma_i(\rho^2) \sum_{s=0}^{a_\tau} (\rho^\kappa\, \beta_{s,\tau})^i\, P_{s,\tau}$$

Finally, by (4.8),

$$\mathcal{G}_\tau^\sharp F(\rho^2, \beta_{s,\tau}\rho^\kappa) = b_{s,\tau}(\widehat{F}, \rho) = \sum_{i=0}^{a_\tau} \gamma_i(\rho^2)\, (\beta_{s,\tau}\, \rho^\kappa)^i \qquad \forall \rho \geq 0$$

and the thesis follows. □

Given $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$, via the map A_τ, the following result gives the existence of an extension of $\mathcal{G}_\tau f_\tau$ for each $\tau \in \widehat{K}$.

Corollary 5.3 *Let F be in $\mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$. Then*

(i) *There exists $u \in \mathcal{S}(\mathbb{R}^2)$ such that $u_{|_{\Sigma_\mathcal{D}^\tau}} = \mathcal{G}_\tau^\sharp F$.*

(ii) *The map* $\mathcal{G}_\tau^\sharp$ *is an isomorphism from* $\mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$ *onto* $\mathcal{S}(\Sigma_{\mathcal{D}}^\tau)$.
(iii) *The map* $\mathcal{G}_\tau$ *is an isomorphism from* $\mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ *onto* $\mathcal{S}(\Sigma_{\mathcal{D}}^\tau)$.

Proof Let $\mathcal{C}_\tau$ be the closed convex hull of $\Sigma_{\mathcal{D}}^\tau$ and

$$\mathcal{C}_\tau^1 = \{(\xi_1, \xi_2) \,:\, \mathrm{dist}((\xi_1, \xi_2), \mathcal{C}_\tau) < 1\}.$$

Note that if (ξ_1, ξ_2) is in $\mathcal{C}_\tau^1$ and $\xi_1 > 1$, then $|\xi_2| \le R\,\xi_1$ for some $R > 0$.

Let η be a smooth function on $\mathbb{R}^2$ which takes value 1 on $\mathcal{C}_\tau$, vanishes outside $\mathcal{C}_\tau^1$ and with bounded derivatives of any order. Let F be in $\mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$ and let $\gamma_0, \dots, \gamma_n$ in $\mathcal{S}(\mathbb{R})$ be as in Lemma 5.2 such that

$$\mathcal{G}_\tau^\sharp F\big(\rho^2, \beta_{s,\tau}\,\rho^\kappa\big) = \sum_{i=0}^{a_\tau} (\beta_{s,\tau}\,\rho^\kappa)^i\,\gamma_i(\rho^2) \qquad \forall \rho \ge 0, \quad s = 0, 1, \dots, a_\tau.$$

Define the Schwartz function g on $\mathbb{R}^2$ by

$$u(\xi_1, \xi_2) = \eta(\xi_1, \xi_2) \sum_{i=0}^{a_\tau} \xi_2^i\,\gamma_i\,(\xi_1) \qquad \forall (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Then $u_{|_{\Sigma_{\mathcal{D}}^\tau}} = \mathcal{G}_\tau^\sharp F$.

By Corollary 2.2 the map $\mathcal{G}_\tau^{-1}$ is continuous and (ii) and (iii) follow by Corollary 2.5. □

6 Growth control for large τ and conclusion

In this section we conclude the proof of Theorem 1.1. As required in Theorem 1.2, for each τ and each order N, given f in $\mathcal{S}(G)^{\mathrm{Int}(K)}$, we produce an extension of $\mathcal{G}_\tau f_\tau$ with rapidly τ-decaying norms up to order N. For this purpose we will follow steps (i) and (ii) in the Introduction.

We begin by observing that, as in the case of the euclidean Fourier transform, the derivatives of the spherical transform $\mathcal{G}_\tau f_\tau$ are controlled by Schwartz norms of f_τ. This is shown in the next lemma, whose use is twofold. On the one hand it gives estimates of the coefficients of the formal power series s_τ in (i) of the introduction (Proposition 6.2); on the other hand it allows us to control uniformly in τ the norms of the extension of $\mathcal{G}_\tau f_\tau$ under the assumption that $\mathcal{G}_\tau f_\tau$ vanishes of infinite order at the origin and (ii) will follow (Proposition 6.5).

Lemma 6.1 *Let* τ *be in* $\widehat{K}$. *There exist sequences* $\{C_m\}$ *and* $\{N_m\}$, *independent of* τ *and increasing, such that for every* $m \in \mathbb{N}$

$$\big|(d/dt)^m \mathcal{G}_\tau f_\tau\big(t^{2/\kappa}, \beta_{s,\tau} t\big)\big| \le C_m\,\|f_\tau\|_{(N_m)} \qquad \forall t \ge 0 \quad \forall f_\tau \in \mathcal{S}(G)_\tau^{\mathrm{Int}(K)}.$$

Proof Let $f_\tau \in \mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ and $F = A_\tau f_\tau$. By (4.8)

$$\mathcal{G}_\tau f_\tau(t^{2/\kappa}, \beta_{s,\tau} t) = \mathcal{G}_\tau^\sharp F(t^{2/\kappa}, \beta_{s,\tau} t) = \frac{1}{d_{\sigma_s}}\ \mathrm{tr}\,\big(\widehat{F}(t^{1/\kappa}\,o) P_{s,\tau}\big).$$

We recall that when $n = 3$, then $\kappa = 1$ and the result is immediate. When $n = 4$ we have $\kappa = 2$, but also $\widehat{F}(-y) = \widehat{F}(y)$, for every $y \in \mathbb{R}^4$. Therefore the thesis follows easily from Propostition 2.4. □

We introduce the following notation for the derivatives at the origin of a function $\psi \in \mathcal{S}(\Sigma^\tau_{\mathcal{D}})$

$$c_{s,\tau,m}(\psi) = \left(\frac{d}{dt}\right)^m_{|_{t=0^+}} \psi\left(t^{2/\kappa}, \beta_{s,\tau} t\right) \qquad m \in \mathbb{N}, \quad s = 0, 1, \dots a_\tau.$$

Note that the previous lemma gives estimates of $c_{s,\tau,m}(\mathcal{G}_\tau f_\tau)$ for each $f_\tau \in \mathcal{S}(G)^{\mathrm{Int}(K)}_\tau$.

6.1 Control in τ of derivatives at the origin

In this section we prove the following statement.

Proposition 6.2 *Let f_τ be in $\mathcal{S}(G)^{\mathrm{Int}(K)}_\tau$ and u a smooth extension of $\mathcal{G}_\tau f_\tau$ to $\mathbb{R}^2$. Then for every integer $m \geq 0$*

$$\left|\binom{m}{q} \partial_1^{m-q} \partial_2^q u(0,0)\right| \leq C_m \, d_\tau^m \, \|f_\tau\|_{(N_m)} \qquad q = 0, 1, \dots m$$

where the sequences $\{C_m\}$ and $\{N_m\}$ are independent of τ and increasing.

The proof of Proposition 6.2 takes some effort. Indeed, its proof is rather technical and is based on relating (unknown) Maclaurin coefficients of any extension u up to order m with the (known) derivatives of $\mathcal{G}_\tau f_\tau$ along the curves defining the spectrum $\Sigma^\tau_{\mathcal{D}} \subset \mathbb{R}^2$.

In doing so, for each order of derivation m, one obtains a (possibly overdetermined) linear system of the form

$$\Lambda_{\tau,m} x = c_{\tau,m},$$

where the matrix $\Lambda_{\tau,m}$ is expressed in terms of the integers $\beta_{s,\tau}$ and the vectors x and $c_{\tau,m}$ are related to $\partial_1^{m-q}\partial_2^q u(0,0)$ and $c_{s,\tau,m}(\mathcal{G}_\tau f_\tau)$, respectively. By Corollary 5.3, this system is solvable; our task here is to describe the norm of x in terms of the norms of $c_{\tau,m}$.

Estimates are split in Sects. 6.1.1 and 6.1.2, dealing with the cases where $n = 4$ and $n = 3$, respectively. In each case a special approach to the problem is used.

6.1.1 Proof of Proposition 6.2 when $n = 4$

In this case, the expression of the integers $\beta_{s,\tau}$ requires some additional work, starting from the following technical lemma.

Lemma 6.3 *Let $n = 4$. For any positive integer $d \leq a_\tau$ there exists an integer $\omega_{\tau,d}$ such that if*

$$\lambda_{i,\tau,d} = \beta_{i,\tau} - \omega_{\tau,d} \qquad i = 0, 1, \dots a_\tau, \tag{6.1}$$

then

$$\prod_{\substack{i=0 \\ i \neq s}}^{d} \left|\frac{\lambda_{i,\tau,d}}{\lambda_{i,\tau,d} - \lambda_{s,\tau,d}}\right| \leq 1 \qquad \forall s = 0, \dots d.$$

Moreover $|\omega_{\tau,d}| \leq d_\tau$.

Proof Let $d \le a_\tau$ and s_d^* an index in $\{0, 1, \dots d\}$ such that

$$\prod_{\substack{i=0\\ i\neq s_d^*}}^{d} |\beta_{i,\tau} - \beta_{s_d^*,\tau}| \le \prod_{\substack{i=0\\ i\neq s}}^{d} |\beta_{i,\tau} - \beta_{s,\tau}| \qquad \forall s = 0, 1, \dots d.$$

Define $\omega_{\tau,d} := \beta_{s_d^*,\tau}$, then $|\omega_{\tau,d}|$ is an integer, no bigger than d_τ.

Notice that for $i = s_d^*$ we have $\lambda_{i,\tau,d} = \beta_{i,\tau} - \omega_{\tau,d} = \beta_{i,\tau} - \beta_{s_d^*,\tau} = 0$, and $\lambda_{i,\tau,d}$ are nonzero integers when $i \neq s_d^*$. Then, for every s in $\{0, 1, \dots d\}$,

$$\prod_{\substack{i=0\\ i\neq s}}^{d} |\lambda_{i,\tau,d}| \le \prod_{\substack{i=0\\ i\neq s_d^*}}^{d} |\lambda_{i,\tau,d}| = \prod_{\substack{i=0\\ i\neq s_d^*}}^{d} |\beta_{i,\tau} - \beta_{s_d^*,\tau}| \le \prod_{\substack{i=0\\ i\neq s}}^{d} |\beta_{i,\tau} - \beta_{s,\tau}| = \prod_{\substack{i=0\\ i\neq s}}^{d} |\lambda_{i,\tau,d} - \lambda_{s,\tau,d}|.$$

□

We now continue the proof of Proposition 6.2.

Fix $\tau \in \widehat{K}$ and let $f_\tau \in \mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ and u a smooth extension of $\mathcal{G}_\tau f_\tau$ on $\mathbb{R}^2$.

When $m = 0$, the inequality follows from Lemma 6.1. Given an order of derivation $m > 0$, let $\omega_{\tau,m}$ be as in Lemma 6.3 when $m \in (0, a_\tau]$ and $\omega_{\tau,m} = \omega_{\tau,a_\tau}$ when $m > a_\tau$. For each $m > 0$, define a smooth function v_m on $\mathbb{R}^2$ by

$$v_m(\xi_1, \xi_2) = u(\xi_1, \xi_2 + \xi_1\omega_{\tau,m}) \qquad \forall \xi_1, \xi_2 \in \mathbb{R}^2,$$

so that, when $t \ge 0$ and $s = 0, 1, \dots, a_\tau$,

$$\mathcal{G}_\tau f_\tau(t, \beta_{s,\tau} t) = u\big(t, \beta_{s,\tau} t)\big) = v_m(t, (\beta_{s,\tau} - \omega_{\tau,m})t).$$

Let

$$v_m \sim \sum_{p,q} \frac{a_{p,q}}{p!q!}\, t^p \xi_2^q.$$

be the Taylor expansion at the origin of the smooth function v_m. Our next goal is to write the Taylor coefficients $a_{p,q}$ of v_m in terms of derivatives of $\mathcal{G}_\tau f_\tau$ at the origin. For the sake of brevity, we write (6.1) as

$$\lambda_s = \beta_{s,\tau} - \omega_{\tau,m} \qquad s = 0, 1, \dots, a_\tau$$

and we have

$$\begin{aligned} c_{s,\tau,m}(\mathcal{G}_\tau f_\tau) &= \left(\frac{d}{dt}\right)^m\Big|_{t=0^+} v_m(t, \lambda_s t) \\ &= \left(\frac{d}{dt}\right)^m\Big|_{t=0^+} \sum_{p+q=m} \frac{a_{p,q}}{p!q!} t^m\, \lambda_s^q \\ &= \sum_{p+q=m} \binom{m}{q} a_{p,q}\, \lambda_s^q. \end{aligned}$$

Therefore we are led to solve the $(a_\tau + 1) \times (m + 1)$ linear system

$$\sum_{q=0}^{m} \lambda_s^q\, x_q = c_{s,\tau,m}(\mathcal{G}_\tau f_\tau) \qquad s = 0, 1, \dots, a_\tau \tag{6.2}$$

where

$$x_q = \binom{m}{q} a_{m-q,q} = \binom{m}{q} \partial_1^{m-q} \partial_2^q v(0,0). \tag{6.3}$$

Let $m \le a_\tau$. By Corollary 5.3 the system (6.2) is solvable, therefore we can consider the first $m+1$ equations.

Denote by Λ the $(m+1) \times (m+1)$ matrix $\Lambda = \left(\Lambda_{s,q}\right) = \left(\lambda_s^q\right)$. Observe that $\lambda_s \neq \lambda_j$ if $s \neq j$ and that all maximal minors of Λ are essentially nonvanishing Vandermonde determinants. Thus, by Cramer's rule,

$$|x_q| \le \sum_{s=0}^{m} |c_{\tau,s,m}(\mathcal{G}_\tau f_\tau)| \left| \frac{V_{s,q}}{V} \right| \qquad q = 0, 1, \dots, m, \tag{6.4}$$

where V is the full Vandermonde determinant with nodes λ_s and $V_{s,q}$ are its cofactors. Expressing $V_{s,q}$ in terms of Schur polynomials, cf. [11], by Lemma 6.3 we have

$$\left| \frac{V_{s,q}}{V} \right| = \left| \sum_{i_1 < i_2 < \dots < i_{m-q},\, i_j \neq s} \frac{\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_{m-q}}}{\prod_{i \neq s} (\lambda_i - \lambda_s)} \right| \le \binom{m}{q} \prod_{i \neq s, \lambda_i \neq 0} \frac{|\lambda_i|}{|\lambda_i - \lambda_s|} \le \binom{m}{q}.$$

Hence by (6.3), (6.4) and Lemma 6.1

$$|\partial_1^{m-q} \partial_2^q v(0,0)| \le \sum_{s=0}^{m} |c_{\tau,s,m}(\mathcal{G}_\tau f_\tau)| \le C_m\, (m+1)\, \| f_\tau \|_{(N_m)}.$$

Note that

$$\partial_1^{m-q} \partial_2^q u(0,0) = \sum_{i=0}^{m-q} \binom{m-q}{i} \omega_{\tau,m}^i\ \partial_1^{m-q-i} \partial_2^{q+i} v_m(0,0).$$

Thus,

$$\begin{aligned} \binom{m}{q} |\partial_1^{m-q} \partial_2^q u(0,0)| &\le d_\tau^m \binom{m}{q} \sum_{i=0}^{m-q} \binom{m-q}{i} |\partial_1^{m-q-i} \partial_2^{q+i} v_m(0,0)| \\ &\le d_\tau^m \binom{m}{q} \sum_{i=0}^{m-q} \binom{m-q}{i} (m+1)\, |c_{\tau,s,m}(\mathcal{G}_\tau f_\tau)|. \\ &\le d_\tau^m\, m^{m/2}\, 2^m\, (m+1)\, C_m\, \| f_\tau \|_{(N_m)}. \end{aligned}$$

The case $m > a_\tau$ can be reduced to the previous one by looking for a solution $x = (x_0, x_1, \dots, x_q, \dots x_m)$ with $x_q = 0$ for $q > a_\tau$.

6.1.2 Proof of Proposition 6.2 when $n = 3$

When $n = 3$ we can slightly modify the arguments used for the complex motion group on $\mathbb{C}^2$ [4]. Indeed, in this case there is no need to use the technical Lemma 6.3, so the proof is less involved. However, in order to have a solvable system of equations, one has to deal separately with the odd and the even part of $F \in \mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$.

Given $f_\tau \in \mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$, let $F = A_\tau f_\tau$, and denote by F_- and F_+ its odd and its even part, respectively. From Corollary 4.5 we have

$$\mathcal{G}_\tau^\sharp F_\pm(t^2, \beta_{s,\tau} t) = \pm \mathcal{G}_\tau^\sharp F_\pm(t^2, -\beta_{s,\tau} t) \qquad \forall (t^2, \beta_{s,\tau} t) \in \Sigma_{\mathcal{D}}^\tau.$$

Therefore we can restrict ourselves to consider $s = \mu, \mu+1, \dots 2\mu$ corresponding to $\beta_{s,\tau} = -\mu + s = 0, 1, \dots, \mu$.

If u is a smooth extension of $\mathcal{G}_\tau f_\tau = \mathcal{G}_\tau^\sharp F$ to $\mathbb{R}^2$, straightforward computations show that the smooth functions u_- and u_+, defined by

$$u_\pm(\xi_1, \xi_2) = \frac{1}{2}(u(\xi_1, \xi_2) \pm u(\xi_1, -\xi_2)) \qquad \forall (\xi_1, \xi_2) \in \mathbb{R}^2,$$

extend $\mathcal{G}_\tau^\sharp F_-$ and $\mathcal{G}_\tau^\sharp F_+$, respectively.

Let us consider the odd part first. If

$$u_- \sim \sum_{p,q} \frac{a_{p,q}}{p!q!}\, \xi_1^p\, \xi_2^q$$

is the Taylor expansion at the origin of u_- then $a_{p,2q'} = 0$, for every $p, q' \in \mathbb{N}$, and the coefficients $a_{p,q}$ must satisfy the equalities

$$\begin{aligned} c_{\mu+j,\tau,m}(\mathcal{G}_\tau^\sharp F_-) &= \left(\frac{d}{dt}\right)^m_{\big|_{t=0^+}} \mathcal{G}_\tau^\sharp F_-\big(t^2, jt\big) \\ &= \left(\frac{d}{dt}\right)^m_{\big|_{t=0}} \left(\sum_{p,q} \frac{a_{p,q}}{p!q!}\, t^{2p+q}\, j^q\right) \qquad\qquad j = 0, 1, 2, \dots, \mu \quad (6.5) \\ &= \sum_{2p+q=m} \frac{(2p+q)!}{p!q!}\, a_{p,q}\, j^q. \end{aligned}$$

Note that

$$\mathcal{G}_\tau^\sharp F_-\big(\xi_1, 0\big) = u_-(\xi_1, 0) = 0 \qquad \forall \xi_1 \geq 0,$$

so the equality in (6.5) for $j = 0$ is trivial. Moreover,

$$c_{\mu+j,\tau,2m'}(\mathcal{G}_\tau F_-) = \left(\frac{d}{dt}\right)^{2m'}_{\big|_{t=0^+}} \mathcal{G}_\tau^\sharp F_-\big(t^2, jt\big) = 0 \qquad \forall t \geq 0, \quad \forall m' \geq 0,$$

so the equalities (6.5) are trivial when m is even. Hence we write $m = 2m' + 1$, $q = 2q' + 1$ and $p = m' - q'$ and we obtain an $\mu \times (m'+1)$ linear system of the form

$$c_{j,2m'+1} = \sum_{q'=0}^{m'} (j^2)^{q'}\, x_{q'} \qquad j = 1, 2, \dots \mu$$

where

$$c_{j,2m'+1} = j^{-1}\, c_{\mu+j,\tau,2m'+1}(\mathcal{G}_\tau F_-)$$

and

$$x_{q'} = \frac{(2m'+1)!}{(m'-q')!(2q'+1)!} a_{m'-q',2q'+1}.$$

We now apply similar arguments to the even part $F_+ \in \mathcal{S}\big(\mathbb{R}^n, \mathrm{End}(\mathcal{V}_\tau)\big)^K$. If

$$u_+ \sim \sum_{p,q} \frac{a_{p,q}}{p!q!}\, \xi_1^p\, \xi_2^q$$

is the Maclaurin expansion of u_+ then $a_{p,2q'+1} = 0$ for every $p, q' \in \mathbb{N}$ and the coefficients $a_{p,q}$ must satisfy the equalities

$$c_{\mu+j,\tau,m}(\mathcal{G}_\tau F_+) = \sum_{2p+q=m} \frac{(2p+q)!}{p!q!}\, a_{p,q}\, j^q \qquad j = 0, 1, 2, \dots, \mu. \tag{6.6}$$

Note that if m is odd and $m = 2p + q$, then q is odd and $a_{p,q} = 0$. Moreover

$$c_{\mu+j,\tau,2m'+1}(\mathcal{G}_\tau F_-) = \left(\frac{d}{dt}\right)^{2m'+1}_{\big|_{t=0^+}} \mathcal{G}_\tau^\sharp F_-\big(t^2, jt\big) = 0 \qquad \forall t \geq 0, \quad \forall m' \geq 0.$$

Hence the previous equalities (6.6) are trivial when m is odd. Thus we consider $m = 2\,m'$, with $m' \in \mathbb{N}$ and we observe that the equality in (6.6) for $j = 0$ reduces to

$$a_{m',0} = \frac{(m')!}{(2m')!}\, c_{\mu,\tau,m}(\mathcal{G}_\tau F_+).$$

Therefore we are led to solve the $\mu \times (m' + 1)$ linear system

$$c_{j,2m'} = \sum_{q'=0}^{m'} (j^2)^{q'}\, x_{q'} \qquad j = 1, 2, \dots \mu$$

where

$$c_{j,2m'} = c_{\mu+j,\tau,2m'}(\mathcal{G}_\tau F_-)$$

and

$$x_{q'} = \frac{(2m')!}{(m'-q')!(2q')!} a_{m'-q',2q'}.$$

It is now possible to follow the lines of the proof given for $n = 4$ in order to get the desired estimate. Note that the coefficients of the matrix Λ are $\Lambda_{j,q} = (j^2)^q$ and an easy computation gives

$$\prod_{\substack{j=1 \\ j\neq s}}^{m'+1} \frac{j^2}{|j^2 - s^2|} = 2\, \frac{\binom{m'+1}{s}}{\binom{m'+1+s}{s}} \leq 1 \qquad s = 1, 2, \dots m' + 1$$

which allows us to obtain the required estimates in the Vandermonde determinants. Thus the proof of Proposition 6.2 is complete.

6.2 Conclusion of the Proof of Theorem 1.1

In this section we prove condition (S') of Theorem 1.2 adapting the arguments in [4]. We include here the proofs for reader's convenience and we also correct some inaccuracies in the proof of [4, Proposition 7.7].

We use a Borel argument to produce a smooth function with given Taylor's coefficients at the origin and with controlled Schwartz norms up to some fixed order. Indeed, we observed in the proof of Proposition 6.2 that $c_{s,\tau,m}(\mathcal{G}_\tau f_\tau)$ are related to Maclaurin's coefficients of any extension u of $\mathcal{G}_\tau f_\tau$ and in the same proposition we gave estimates for these coefficients. The following lemma is analogous to Lemma 7.6 in [4] and completes the proof of item (i) in the Introduction.

Lemma 6.4 *Let f be in $\mathcal{S}(G)^{\mathrm{Int}(K)}$ and M in $\mathbb{N}$. Then, for every $\tau \in \widehat{K}$ there exists $h_{\tau,M}$ in $C_c^\infty(\mathbb{R}^2)$ such that*

$$c_{s,\tau,m}(\mathcal{G}_\tau f_\tau) = c_{s,\tau,m}\left(\left(h_{\tau,M}\right)_{|\Sigma^\tau_{\mathcal{D}}}\right) \qquad s = 0, 1, \dots a_\tau \quad \forall m \geq 0$$

and $\|h_{\tau,M}\|_{(M)}$ is rapidly decaying in τ.

Proof Let $f \in \mathcal{S}(G)^{\mathrm{Int}(K)}$, $M \in \mathbb{N}$. Let $\varphi \in C_c^\infty(\mathbb{R}^2)$ supported in $\{\xi \in \mathbb{R}^2 \ : \ |\xi| \leq 1\}$ and equal to 1 for $|\xi| \leq \frac{1}{2}$. For any integers m, q with $0 \leq q \leq m$ let

$$\psi_{m,q}(\xi_1, \xi_2) = \varphi(\xi_1, \xi_2)\, \xi_1^{m-q} \xi_2^q .$$

Fix $\tau \in \widehat{K}$ and let u_τ be a smooth extension of $\mathcal{G}_\tau f_\tau$ to $\mathbb{R}^2$. Following the standard Borel's Lemma procedure, define

$$h_m(\xi) = \frac{1}{m!} \sum_{q=0}^{m} \binom{m}{q} \partial_1^{m-q} \partial_2^q u_\tau(0)\, \varepsilon_m^m\, \psi_{m,q}(\xi/\varepsilon_m)$$

and

$$h_{\tau,M}(\xi) = \sum_{m \in \mathbb{N}} h_m(\xi) \tag{6.7}$$

where the coefficients $\varepsilon_m \in (0, 1]$ will be chosen afterwards so that the series in (6.7) converges normally in every C^N-norm and $\|h_{\tau,M}\|_{(M)}$ decays rapidly in τ.

By Proposition 6.2, for every $N \in \mathbb{N}$,

$$\|h_m\|_{C^N} \leq \frac{d_\tau^m}{m!}\, \varepsilon_m^{m-N}\, C_m\, \|f_\tau\|_{(N_m)} \sum_{q=0}^{m} \|\psi_{m,q}\|_{C^N}$$

where the sequences $\{C_m\}$ and $\{N_m\}$ are independent of τ and increasing.

With $\alpha_m = C_m \sum_{q=0}^m \|\psi_{m,q}\|_{C^{m-1}}$, we choose

$$\varepsilon_m = \varepsilon_{m,\tau,M} = \begin{cases} 1 & \text{if } m \leq M \\ \frac{1}{d_\tau!} \frac{1}{(1+\alpha_m \|f_\tau\|_{(N_m)})} & \text{if } m > M. \end{cases}$$

Then

$$\sum_{m>M} \|h_m\|_{C^{m-1}} \leq \frac{1}{d_\tau!} \sum_{m \geq M} \frac{d_\tau^m}{m!} \leq \frac{e^{d_\tau}}{d_\tau!}.$$

This implies that the series $\sum_{m \in \mathbb{N}} \|h_m\|_{C^k}$ converges for every k, so that $h_{\tau,M}$ is smooth. Moreover, since the sequence $\{N_m\}$ is increasing, then, for $N \leq M$,

$$\sum_{m \leq M} \|h_m\|_{C^N} \leq \sum_{m \leq M} \|h_m\|_{C^M} \leq \sum_{m \leq M} \frac{d_\tau^m}{m!}\, C_m\, \|f_\tau\|_{(N_m)} \sum_{q=0}^{m} \|\psi_{m,q}\|_{C^M} \leq s_M\, d_\tau^M\, \|f_\tau\|_{(N_M)},$$

where $s_M = \sum_{d\le M} \frac{C_m}{m!} \sum_{q=0}^{m} \|\psi_{m,q}\|_{C^M}$. Hence, since $\|f_\tau\|_{(N_M)}$ is rapidly decreasing in τ, the function $h_{\tau,M}$ defined in (6.7) with $\varepsilon_m = \varepsilon_{m,\tau,M}$ satisfies our requirements. □

The following proposition is analogous to Proposition 7.7 in [4] and gives (ii) in the introduction.

Proposition 6.5 *Let $f_\tau \in \mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ such that*

$$c_{s,\tau,m}(\mathcal{G}_\tau f_\tau) = 0 \qquad \forall m \in \mathbb{N} \quad \forall s = 0, 1, \dots a_\tau.$$

Then $\mathcal{G}_\tau f_\tau$ can be extended to a Schwartz function v_τ satisfying the following norm bounds: for every $N \in \mathbb{N}$ there exist constants $M_N \in \mathbb{N}$ and $C_N > 0$, independent of τ, such that

$$\|v_\tau\|_{(N)} \le C_N \, \|f_\tau\|_{(M_N)}.$$

Proof Let η be a bump function in $C_c^\infty(\mathbb{R})$ supported in $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and equal to 1 in a neighbourhood of the origin Let $f_\tau \in \mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ and define v_τ on $\mathbb{R}^2$ by the rule

$$v_\tau(\xi_1, \xi_2) = \begin{cases} \displaystyle\sum_{s=0}^{a_\tau} \mathcal{G}_\tau f_\tau(\xi_1^{2/\kappa}, \beta_{s,\tau}\xi_1)\, \eta_s\left(\frac{\xi_2}{\xi_1^{\kappa/2}}\right) & \xi_1 > 0 \\ 0 & \xi_1 \le 0. \end{cases}$$

where

$$\eta_s(x) = \eta\left(x - \beta_{s,\tau}\right), \qquad \forall x \in \mathbb{R} \quad s = 0, 1, \cdots, a_\tau.$$

It is straightforward to show that v_τ extends $\mathcal{G}_\tau f_\tau$ to $\mathbb{R}^2$. We now show that v_τ is smooth and that for every $N \in \mathbb{N}$ there exist constants $M_N \in \mathbb{N}$ and $C_N > 0$, independent of τ, such that

$$\|v_\tau\|_{(N)} \le C_N \, \|f_\tau\|_{(M_N)}.$$

One can check by induction that, for appropriate coefficients $a_{r,h,q}$

$$\partial_{\xi_1}^h \left(\xi_1^{-q\kappa/2}\, \eta_s^{(q)}\left(\frac{\xi_2}{\xi_1^{\kappa/2}}\right)\right) = \xi_1^{-q\kappa/2-h} \sum_{r=0}^{h} a_{r,h,q} \left(\frac{\xi_2}{\xi_1^{\kappa/2}}\right)^r \eta_s^{(q+r)}\left(\frac{\xi_2}{\xi_1^{\kappa/2}}\right).$$

Now fix (ξ_1, ξ_2) with $\xi_1 > 0$ close enough to the spectrum so that there exists s such that $\left|\frac{\xi_2}{\xi_1^{\kappa/2}} - \beta_{s,\tau}\right| < \frac{1}{2}$. Then the sum defining v_τ reduces to a single term and

$$\begin{aligned} \partial_{\xi_1}^p \partial_{\xi_2}^q v_\tau(\xi_1, \xi_2) &= \sum_{h=0}^{p} \binom{p}{h} \partial_{\xi_1}^{p-h}\left(\mathcal{G}_\tau f_\tau(\xi_1^{2/\kappa}, \beta_{s,\tau}\xi_1)\right) \partial_{\xi_1}^h \left(\xi_1^{-q\kappa/2}\, \eta_s^{(q)}\left(\frac{\xi_2}{\xi_1^{\kappa/2}}\right)\right). \\ &= \sum_{h=0}^{p}\sum_{r=0}^{h} \binom{p}{h} a_{r,h,q}\, \xi_1^{-q\kappa/2-h} \left(\frac{\xi_2}{\xi_1^{\kappa/2}}\right)^r \partial_{\xi_1}^{p-h}\left(\mathcal{G}_\tau f_\tau(\xi_1^{2/\kappa}, \beta_{s,\tau}\xi_1)\right) \eta_s^{(q+r)}\left(\frac{\xi_2}{\xi_1^{\kappa/2}}\right). \end{aligned}$$

Therefore we need to estimate

$$(1 + |\xi_1| + |\xi_2|)^N \; \xi_1^{-q\kappa/2-h} \left|\frac{\xi_2}{\xi_1^{\kappa/2}}\right|^r \; \partial_{\xi_1}^{p-h}\left(\mathcal{G}_\tau f_\tau(\xi_1^{2/\kappa}, \beta_{s,\tau}\xi_1)\right) \tag{6.8}$$

for every integer $N \ge 0$ when $\left|\frac{\xi_2}{\xi_1^{\kappa/2}} - \beta_{s,\tau}\right| < \frac{1}{2}$.

Note that

$$|\xi_2| = |\xi_2 - \xi_1^{\kappa/2}\beta_{s,\tau} + \xi_1^{\kappa/2}\beta_{s,\tau}| \leq |\xi_1^{\kappa/2}|\left(\tfrac{1}{2} + |\beta_{s,\tau}|\right)$$

Thus the expression in (6.8) can be controlled by a linear combination, independent of τ, of terms of the type

$$\xi_1^{-\alpha}\,\partial_{\xi_1}^{\ell}\Big(\xi_1{}^{2\beta/\kappa}\,(\xi_1\,\beta_{s,\tau})^{\gamma}\,\mathcal{G}_\tau f_\tau(\xi_1^{2/\kappa},\beta_{s,\tau}\xi_1)\Big) = \xi_1^{-\alpha}\,\partial_{\xi_1}^{\ell}\Big(\mathcal{G}_\tau(\Delta^{\beta}\,\mathbf{D}_\tau^{\gamma} f_\tau)(\xi_1^{2/\kappa},\beta_{s,\tau}\xi_1)\Big)$$

where α, β, γ, ℓ are non negative integers.

Since $g(\xi_1) = \mathcal{G}_\tau(\Delta^{\beta}\,\mathbf{D}_\tau^{\gamma} f)(\xi_1^{2/\kappa},\beta_{s,\tau}\xi_1)$ vanishes of infinite order at the origin, for any integer $q \geq 0$ there exists $\theta \in (0,1)$ such that for any $\xi_1 > 0$

$$\xi_1^{-\alpha}\,g^{(\ell)}(\xi_1) = \frac{1}{\ell!}\,g^{(\ell+\alpha)}(\theta\xi_1).$$

Therefore, in view of Lemma 6.1, $v_\tau \in C^\infty(\mathbb{R}^2)$ and the required norm estimates follow. □

Finally we collect all our results in order to produce an extension with norms rapidly decaying in τ, so proving the Schwartz correspondence for $\left(M_n(\mathbb{R}), SO_n\right)$ according to Theorem 1.2. Given f in $\mathcal{S}(G)^{\mathrm{Int}(K)}$ and τ in $\widehat{K}$, we have $\mathcal{G} f_\tau(\xi'_\tau,\xi'') = \mathcal{G}_\tau f_\tau(\xi'')$, hence condition (S') is equivalent to the following.

Proposition 6.6 *Let f be in $\mathcal{S}(G)^{\mathrm{Int}(K)}$ and N in $\mathbb{N}$. Then, for every $\tau \in \widehat{K}$, $\mathcal{G}_\tau f_\tau$ admits a Schwartz extension $u_{\tau,N}$ from $\Sigma_{\mathcal{D}}^{\tau}$ to $\mathbb{R}^2$ such that $\|u_{\tau,N}\|_{(N)}$ is rapidly decaying in τ.*

Proof Let f be in $\mathcal{S}(G)^{\mathrm{Int}(K)}$, $\tau \in \widehat{K}$ and $N \in \mathbb{N}$. For M to be chosen afterwards, let $h_{\tau,M}$ be the function associated to f_τ, according to Lemma 6.4. By Corollary 2.2 there exists $g_{\tau,M} \in \mathcal{S}(G)_\tau^{\mathrm{Int}(K)}$ such that $\mathcal{G}_\tau g_{\tau,M} = \left(h_{\tau,M}\right)_{|_{\Sigma_{\mathcal{D}}^{\tau}}}$. Since

$$c_{s,\tau,m}\left(\mathcal{G}_\tau(f_\tau - g_{\tau,M})\right) = 0 \qquad \forall m \in \mathbb{N} \quad \forall s = 0,1,\ldots a_\tau,$$

by Proposition 6.5,here exists an extension $v_{\tau,M}$ of $\mathcal{G}_\tau(f_\tau - g_{\tau,M})$ and a positive integer N', depending only on N, such that

$$\|v_{\tau,M}\|_{(N)} \leq C_N\,\|f_\tau - g_{\tau,M}\|_{(N')}.$$

By Corollary 2.2, there exists N'', depending on N', such that

$$\|g_{\tau,M}\|_{(N')} \leq C_{\tau,N}\|h_{\tau,M}\|_{(N'')}$$

where the constant $C_{\tau,N}$ has polynomial growth in τ. Choose M bigger than N and N'', then

$$u_{\tau,N} = v_{\tau,M} + h_{\tau,M}$$

is a Schwartz extension to $\mathbb{R}^2$ of $\mathcal{G}_\tau f_\tau$ whose N-Schwartz norm decays rapidly in τ. Indeed,

$$\begin{aligned}\|u_{\tau,N}\|_{(N)} &\leq C_N\,\|f_\tau\|_{(N')} + C_N\,C_{\tau,N}\|h_{\tau,M}\|_{(N'')} + \|h_{\tau,M}\|_{(N)}\\ &\leq C'_{\tau,N}\left(\|f_\tau\|_{(N')} + \|h_{\tau,M}\|_{(M)}\right)\end{aligned}$$

where the constant $C'_{\tau,N}$ has polynomial growth in τ and both $\|f_\tau\|_{(N')}$ and $\|h_{\tau,M}\|_{(M)}$ decay rapidly in τ. □

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Declarations

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