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A SUBSPACE CORRECTION METHOD FOR DISCONTINUOUS GALERKIN DISCRETIZATIONS OF LINEAR ELASTICITY EQUATIONS

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Abstract. We study preconditioning techniques for discontinuous Galerkin discretizations of isotropic linear elasticity problems in primal (displacement) formulation. We propose subspace correction methods based on a splitting of the vector valued piecewise linear discontinuous finite element space, that are optimal with respect to the mesh size and the Lamé parameters. The pure displacement, the mixed and the traction free problems are discussed in detail. We present a convergence analysis of the proposed preconditioners and include numerical examples that validate the theory and assess the performance of the preconditioners.

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1. Introduction

The finite element approximation of the equations of isotropic linear elasticity may be accomplished in various ways. The most straightforward approach is to use the primal formulation and conforming finite elements. It is well known that the resulting method, in general, does not provide approximation to the displacement field when the material is nearly incompressible (the Poisson ratio is close to 1/2). This phenomenon is called *volume locking*. To alleviate locking, several approaches exist. Among the possible solutions, we mention the use of mixed methods, reduced integration techniques, stabilization techniques, nonconforming methods, and the use of discontinuous Galerkin methods. We refer to [9, 11] for further discussions on such difficulties and their remedies. In this work we focus on design of robust solvers for the Symmetric Interior Penalty discontinuous Galerkin (SIPG) approximation of isotropic linear elasticity [11, 12, 16, 17]. We have chosen to work with these DG discretizations, since we have in mind a method that is simple but still applicable to different types of boundary conditions. In fact, unlike classical low order non–conforming methods (see [9]), the Interior Penalty

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(IP) stabilization methods introduced in [11, 12] can be shown to be stable in the case of essential (Dirichlet or pure displacement) boundary conditions, or natural (Neumann type, or traction free) boundary conditions. As a consequence, these IP methods provide a robust approximation to the displacement field and avoid the volume locking regardless the boundary conditions of the problem.

For the design of the preconditioners we follow the ideas introduced in [4] for second order elliptic problems. However, such extensions are not straightforward, since we aim at constructing preconditioners that work well for three different types of boundary conditions: essential, natural and mixed boundary conditions, used in linear elasticity. This complicates the matters quite a bit. We consider a splitting of the vector valued, piecewise linear, discontinuous finite element space, into two subspaces: the vector valued Crouzeix–Raviart space and a space complementary to it which consists of functions whose averages are L^2-orthogonal to the constants on every edge/face of the partition. This space decomposition is direct and the spaces are orthogonal with respect to a bilinear form obtained *via* using "reduced integration" to calculate the contributions of the penalty terms in SIPG.

In the pure displacement case (essential boundary conditions), the restriction of the bilinear form based on reduced integration is coercive on the Crouzeix–Raviart space and is spectrally equivalent to the SIPG bilinear form. The space decomposition mentioned above is then orthogonal in this reduced integration bilinear form. Thus, in case of essential boundary conditions we have a natural block diagonal preconditioner for the linear elasticity problem: (1) a solution of a problem arising from discretization by nonconforming Crouzeix–Raviart elements; (2) solution of a well-conditioned problem on the complementary space.

For traction free problems or problems with Dirichlet conditions only on part of the boundary, the situation is quite different. In this case, the reduced integration bilinear form when restricted to the Crouzeix–Raviart space has a null space whose dimension depends on the size of the problem (see [9]). Therefore, the full SIPG rather than the reduced integration bilinear form has to be used for both approximation and preconditioning, and consequently the space splitting discussed above is no longer orthogonal in the associated energy norm. Our approach in resolving these issues is based on a delicate estimate given in Section 3.1 which shows a uniform bound on the angle between the Crouzeix–Raviart and its complementary space in the SIPG bilinear form for all types of boundary conditions. Once such a bound is available we show that a uniform block diagonal preconditioner can be constructed.

The rest of the paper is organized as follows. We present the linear elasticity problem, the basic notation and discuss the DG discretizations considered in Section 2. Next, in Section 3 we introduce the splitting of the vector valued piecewise linear DG space and discuss some properties of the related subspaces. In Section 4, we introduce the subspace correction methods, and we prove that they give rise to a uniform preconditioner for the symmetric IP method. The last Section 5 contains several numerical tests that support the theoretical results.

2. Interior Penalty discontinuous Galerkin methods for linear elasticity

In this section, we introduce the linear elasticity problem together with the basic notation and the derivation of the Interior Penalty (IP) methods and we discuss the stability of these methods.

2.1. Problem formulation and notation

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a polygon or polyhedron (not necessarily convex) and let $\boldsymbol{u}$ be a vector field in $\mathbb{R}^d$, defined on Ω such that $\boldsymbol{u} \in [H^1(\Omega)]^d$. The elasticity tensor, which we denote by $\mathcal{C}$, is a linear operator, *i.e.*, $\mathcal{C} : \mathbb{R}^{d\times d}_{\text{sym}} \mapsto \mathbb{R}^{d\times d}_{\text{sym}}$, acting on a symmetric matrix $A \in \mathbb{R}^{d\times d}_{\text{sym}}$, in the following way:

$$\mathcal{C}\, A = 2\mu A + \lambda\,\text{trace}(A) I,$$

where μ and λ are the Lamé parameters and satisfy $0 < \mu_1 < \mu < \mu_2$ and $0 \leq \lambda < \infty$. In terms of the modulus of elasticity (Young's modulus), $\mathfrak{E}$, and Poisson's ratio, ν, the Lamé parameters can be rewritten in the case of plane strain as: $\mu = \mathfrak{E}/(2(1+\nu))$ and $\lambda = \nu\mathfrak{E}/((1+\nu)(1-2\nu)$. The material tends to the incompressible limit (becomes incompressible) when the Lamé parameter $\lambda \to \infty$ or equivalently when the Poisson's ratio $\nu \to 1/2$.

One can show that the linear operator $\mathcal{C}$ is selfadjoint and has two eigenvalues: (1) a simple eigenvalue equal to $(2\mu + d\lambda)$ corresponding to the identity matrix; (2) an eigenvalue equal to 2μ, corresponding to the $\frac{d(d+1)}{2} - 1$ dimensional space of traceless, symmetric, real matrices. Thus for $d = 2, 3$, we always have that

$$2\mu\langle A : A\rangle \leq \langle \mathcal{C}A : A\rangle \leq (2\mu + \mathrm{d}\lambda)\langle A : A\rangle, \tag{2.1}$$

where $\langle \cdot : \cdot \rangle$ denotes the Frobenius inner product of two tensors in $\mathbb{R}^{d\times d}$, *i.e.*,

$$\langle \boldsymbol{v} : \boldsymbol{w}\rangle = \sum_{j=1}^{d}\sum_{k=1}^{d} v_{jk}w_{jk}.$$

The Euclidean inner product in $\mathbb{R}^d$ is denoted by $\langle \cdot, \cdot \rangle$, *i.e.*, $\langle \boldsymbol{v}, \boldsymbol{w}\rangle = \sum_{k=1}^{d} v_k w_k$. The corresponding inner products in $[L^2(\Omega)]^d$ and $[L^2(\Omega)]^{d\times d}$ are denoted by

$$(\boldsymbol{v} : \boldsymbol{w}) = \int_\Omega \langle \boldsymbol{v} : \boldsymbol{w}\rangle, \qquad (\boldsymbol{v}, \boldsymbol{w}) = \int_\Omega \langle \boldsymbol{v}, \boldsymbol{w}\rangle.$$

We write $\partial\Omega = \Gamma_N \cup \Gamma_D$ with Γ_N and Γ_D referring respectively to the subsets of $\partial\Omega$ where Neumann and Dirichlet boundary conditions are imposed.

Let $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ be the symmetric part of the gradient of a vector valued function $\boldsymbol{u}$. The equations of linear elasticity for the unknown displacement field $\boldsymbol{u}$ of a body occupaying a domain Ω are as follows:

$$\begin{aligned} -\mathrm{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) &= \boldsymbol{f} && \text{on } \Omega,\\ (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n} &= \boldsymbol{0}, && \text{on } \Gamma_N,\\ \boldsymbol{u} &= \boldsymbol{0} && \text{on } \Gamma_D. \end{aligned} \tag{2.2}$$

In the above equations, $\boldsymbol{f} \in [L^2(\Omega)]^d$ is a given volume force and $\boldsymbol{n}$ is the outward unit normal vector to $\partial\Omega$. The solution $\boldsymbol{u}$ vanishes on a closed part of the boundary Γ_D (Dirichlet boundary) and the normal stresses are prescribed on Γ_N (Neumann part of the boundary). In the traction free case ($\Gamma_D = \emptyset$ and $\Gamma_N = \partial\Omega$), the existence of a unique solution to (2.2) is guaranteed if the data satisfy the compatibility condition

$$\int_\Omega \boldsymbol{f}\cdot \boldsymbol{v}dx = 0 \quad \forall \boldsymbol{v} \in \mathbf{RM}(\Omega),$$

where $\mathbf{RM}(\Omega)$ is the space of rigid motions, defined by:

$$\mathbf{RM}(\Omega) := \left\{ \boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b}\boldsymbol{x} \quad : \quad \boldsymbol{a} \in \mathbb{R}^d \quad \boldsymbol{b} \in so(d) \ \right\}. \tag{2.3}$$

Here $\boldsymbol{x}$ is the position vector function in Ω and $so(d)$ is the Lie algebra of skew-symmetric $d \times d$ matrices. In this case, the uniqueness of solution is guaranteed up to a rigid motion (and is unique, if we require that the solution is orthogonal to any element from $\mathbf{RM}(\Omega)$). In the case of $\Gamma_D \neq \emptyset$ and closed with respect to $\partial\Omega$ no extra conditions are required to guarantee uniqueness. We also assume that the solution of the linear elasticity system has appropriate regularity, namely, $\boldsymbol{u} \in [H^2(\Omega)]^d \cap [H^1_{0,\Gamma_D}(\Omega)]^d$.

By considering the variational formulation of (2.2), the issue of solvability and uniqueness of the problem reduces to show coercivity of the associated bilinear form. As it is well known, for linear elasticity, this hinges on the classical Korn's inequality [8] which guarantees the existence of a generic positive constant $C_\Omega > 0$ such that:

$$\|\nabla \boldsymbol{v}\|^2_{0,\Omega} \leq C_\Omega \left(\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\Omega} + \|\boldsymbol{v}\|^2_{0,\Omega}\right), \qquad \forall \boldsymbol{v} \in [H^1(\Omega)]^d. \tag{2.4}$$

The second term on the right hand side can be omitted as follows from the Poincaré or Poincaré–Friedrich's inequality, obtaining thus first Korn's inequality for $\boldsymbol{v} \in [H^1_{0,\Gamma_D}(\Omega)]^d$ and second Korn's inequality for $\boldsymbol{v} \in [H^1(\Omega)]^d/\mathbf{RM}(\Omega)$.

2.2. Interior penalty methods: Preliminaries and notation

We now introduce the basic notations and tools needed for the derivation of the DG methods.

Domain partitioning. Let $\mathcal{T}_h$ be a shape-regular partition of Ω into d–dimensional simplices T (triangles if $d = 2$ and tetrahedrons if $d = 3$). We denote by h_T the diameter of T and we set $h = \max_{T\in\mathcal{T}_h} h_T$. We also assume that $\mathcal{T}_h$ is conforming in the sense that it does not contain hanging nodes. A face (shared by two neighboring elements or being part of the boundary) is denoted by E. Clearly, such a face is a $(d-1)$ dimensional simplex, that is, a line segment in two dimensions and a triangle in three dimensions. We denote the set of all faces by $\mathcal{E}_h$, and the collection of all interior faces and boundary faces by $\mathcal{E}_h^o$ and $\mathcal{E}_h^\partial$, respectively. Further, the set of Dirichlet faces is denoted by $\mathcal{E}_h^D$, and the set of Neumann faces by $\mathcal{E}_h^N$. We thus have,

$$\mathcal{E}_h = \mathcal{E}_h^o \cup \mathcal{E}_h^\partial, \quad \mathcal{E}_h^D = \mathcal{E}_h^\partial \cap \Gamma_D, \quad \mathcal{E}_h^N = \mathcal{E}_h^\partial \cap \Gamma_N, \quad \mathcal{E}_h^\partial = \mathcal{E}_h^D \cup \mathcal{E}_h^N.$$

Trace operators (average and jump) on $E \in \mathcal{E}_h$. To define the average and jump trace operators for an interior face $E \in \mathcal{E}_h^o$, and any $T \in \mathcal{T}_h$, such that $E \in \partial T$ we set $\boldsymbol{n}_{E,T}$ to be the unit outward (with respect to T) normal vector to E. With every face $E \in \mathcal{E}_h^o$ we also associate a unit vector $\boldsymbol{n}_E$ which is orthogonal to the $(d-1)$ dimensional affine variety (line in 2D and plane in 3D) containing the face. For the boundary faces, we always set $\boldsymbol{n}_E = \boldsymbol{n}_{E,T}$, where T is the *unique* element for which we have $E \subset \partial T$. In our setting, for the interior faces, the particular direction of $\boldsymbol{n}_E$ is not important, although it is important that this direction is fixed. For every face $E \in \mathcal{E}_h$, we define $T^+(E)$ and $T^-(E)$ as follows:

$$\begin{aligned} T^+(E) &:= \{T \in \mathcal{T}_h \text{ such that } E \subset \partial T, \text{ and } \langle \boldsymbol{n}_E, \boldsymbol{n}_{E,T}\rangle > 0\}, \\ T^-(E) &:= \{T \in \mathcal{T}_h \text{ such that } E \subset \partial T, \text{ and } \langle \boldsymbol{n}_E, \boldsymbol{n}_{E,T}\rangle < 0\}. \end{aligned} \tag{2.5}$$

It is immediate to see that both sets defined above contain *no more than* one element, that is: for every face we have exactly one $T^+(E)$ and for the interior faces we also have exactly one $T^-(E)$. For the boundary faces we only have $T^+(E)$. In the following, we write $T^\pm$ instead of $T^\pm(E)$, when this does not cause confusion and ambiguity.

For a given function $\boldsymbol{w} \in [L^2(\Omega)]^d$ the average and jump trace operators for a fixed $E \in \mathcal{E}_h^o$ are as follows:

$$\{\!\{\boldsymbol{w}\}\!\} := \left(\frac{\boldsymbol{w}^+ + \boldsymbol{w}^-}{2}\right), \qquad [\![\boldsymbol{w}]\!] := (\boldsymbol{w}^+ - \boldsymbol{w}^-), \tag{2.6}$$

where $\boldsymbol{w}^+$ and $\boldsymbol{w}^-$ denote respectively, the traces of $\boldsymbol{w}$ onto E taken from within the interior of T^+ and T^-. On boundary faces $E \in \mathcal{E}_h^\partial$, we set $\{\!\{\boldsymbol{w}\}\!\} = \boldsymbol{w}$ and $[\![\boldsymbol{w}]\!] = \boldsymbol{w}$. We remark that our notation differs from the ones used in [1–3]. We have chosen a notation that is consistent with the one used in [12], where the IP method we consider was first introduced for the pure displacement problem. In addition, it seems that such a choice leads to a shorter and simpler description of the preconditioners we propose here.

Finite Element Spaces. The piecewise linear DG space is defined by

$$V^{\mathrm{DG}} := \{u \in L^2(\Omega) \text{ such that } u\big|_T \in \mathbb{P}^1(T), \quad \forall\, T \in \mathcal{T}_h\},$$

where $\mathbb{P}^1(T)$ is the space of linear polynomials on T. The corresponding space of vector valued functions is defined as

$$\boldsymbol{V}^{\mathrm{DG}} := [V^{\mathrm{DG}}]^d.$$

For a given face E, we denote by $\mathcal{P}_E^0 : L^2(E) \mapsto \mathbb{P}^0(E)$ the L^2-projection onto the constant (vector valued or scalar valued) functions on E defined by

$$\mathcal{P}_E^0 w = \frac{1}{|E|}\int_E w \quad \text{for all} \quad w \in L^2(E), \tag{2.7}$$

$$\mathcal{P}_E^0 \boldsymbol{w} = \frac{1}{|E|}\int_E \boldsymbol{w} \quad \text{for all} \quad \boldsymbol{w} \in [L^2(E)]^d. \tag{2.8}$$

Observe that for $\boldsymbol{w} \in \boldsymbol{V}^{\rm DG}$ the mid–point integration rule implies that $\mathcal{P}_E^0 \boldsymbol{w} = \boldsymbol{w}(m_E)$ for all $E \in \mathcal{E}_h$, with m_E denoting the barycenter of the edge or face E.

The classical Crouzeix–Raviart finite element space can be defined as a subspace of V^{DG}, as follows:

$$V^{\rm CR} = \left\{ v \in V^{\rm DG} \quad : \quad \mathcal{P}_E^0 [\![v]\!] = 0, \ \forall E \in \mathcal{E}_h^o \right\}. \tag{2.9}$$

The corresponding space of vector valued functions is

$$\boldsymbol{V}^{\rm CR} := [V^{\rm CR}]^d. \tag{2.10}$$

2.3. Weighted residual derivation of the IP methods

In [12] the authors introduced a symmetric interior penalty method for the problem of linear elasticity (2.2) in the pure displacement case (*i.e.*, $\Gamma_D = \partial\Omega, \ \Gamma_N = \emptyset$). We define the function space

$$[H^2(\mathcal{T}_h)]^d = \left\{ \boldsymbol{u} \in [L^2(\Omega)]^d \text{ such that } \boldsymbol{u}\big|_T \in [H^2(T)]^d, \quad \forall T \in \mathcal{T}_h \right\}.$$

For any pair of vector fields (or tensors) $\boldsymbol{v}$ and $\boldsymbol{w}$, we denote

$$(\boldsymbol{v}, \boldsymbol{w})_{\mathcal{T}_h} = \sum_{T \in \mathcal{T}_h} \int_T \langle \boldsymbol{v}, \boldsymbol{w} \rangle.$$

For scalar and vector valued functions we also use the notation

$$(v, w)_{\mathcal{E}} = \sum_{E \in \mathcal{E}} \int_E vw, \quad \text{and} \quad (\boldsymbol{v}, \boldsymbol{w})_{\mathcal{E}} = \sum_{E \in \mathcal{E}} \int_E \langle \boldsymbol{v}, \boldsymbol{w} \rangle. \tag{2.11}$$

We now derive, using the weighted residual framework [6], the IP methods for the more general case of mixed boundary conditions. In order to have a transparent derivation of the methods, we assume $\boldsymbol{u} \in [H^2(\Omega)]^d$ and remark here, that this assumption is not needed in the construction, implementation, and analysis of the preconditioners that we propose.

By assuming that the solution of (2.2) is *a priori* discontinuous, $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$, we may rewrite the continuous problem (2.2) as follows: Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that

$$\begin{cases} -\mathrm{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) = \boldsymbol{f} & \text{on } T \in \mathcal{T}_h, \\ [\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}]\!]_E = \boldsymbol{0} & \text{on } E \in \mathcal{E}_h^o \cup \mathcal{E}_h^N, \\ [\![\boldsymbol{u}]\!]_E = \boldsymbol{0} & \text{on } E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D, \end{cases} \tag{2.12}$$

where we recall that $\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda\,\mathrm{trace}(\boldsymbol{\varepsilon}(\boldsymbol{u}))I$. Following [6], we next introduce a variational formulation of (2.12) by considering the following five operators

$$\begin{aligned} &\mathcal{B}_0 : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{T}_h)]^d, && \\ &\mathcal{B}_1 : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^o)]^d, && \mathcal{B}_1^\partial : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^D)]^d \\ &\mathcal{B}_2 : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^o)]^d, && \mathcal{B}_2^\partial : [H^2(\mathcal{T}_h)]^d \longrightarrow [L^2(\mathcal{E}_h^N)]^d, \end{aligned}$$

and weighting each equation in (2.12) appropriately. This then amounts to considering the following problem: Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that for all $\boldsymbol{v} \in [H^2(\mathcal{T}_h)]^d$

$$\begin{aligned} (-\mathrm{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})) - \boldsymbol{f}, \mathcal{B}_0(\boldsymbol{v}))_{\mathcal{T}_h} + ([\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}]\!], \mathcal{B}_2(\boldsymbol{v}))_{\mathcal{E}_h^o} + ([\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v}))_{\mathcal{E}_h^o} \\ + ([\![\boldsymbol{u}]\!], \mathcal{B}_1^\partial(\boldsymbol{v}))_{\mathcal{E}_h^D} + ([\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}]\!], \mathcal{B}_2^\partial(\boldsymbol{v}))_{\mathcal{E}_h^N} = \boldsymbol{0}. \end{aligned} \tag{2.13}$$

Different choices of the operators $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_1^\partial$ and $\mathcal{B}_2^\partial$ above give rise to different variational formulations and, consequently to different DG methods. We refer to [6], Theorem 6 for sufficient conditions on the operators $\mathcal{B}_0$, $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{B}_1^\partial$ and $\mathcal{B}_2^\partial$ to guarantee[5] the uniqueness of the solution of (2.13).

To derive the IP method of interest, we take $\boldsymbol{v}$ piecewise smooth and we set $\mathcal{B}_0(\boldsymbol{v}) = \boldsymbol{v}$, $\mathcal{B}_2(\boldsymbol{v}) = \{\!\{\boldsymbol{v}\}\!\}$ and $\mathcal{B}_2^\partial(\boldsymbol{v}) = \boldsymbol{v}$ in (2.13), to obtain that

$$(-\mathrm{div}(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u})), \boldsymbol{v})_{\mathcal{T}_h} + ([\![(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}]\!], \{\!\{\boldsymbol{v}\}\!\})_{\mathcal{E}_h^o \cup \mathcal{E}_h^D} + ([\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v}))_{\mathcal{E}_h^o \cup \mathcal{E}_h^D} = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}. \tag{2.14}$$

Defining $\mathcal{F}(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_{\mathcal{T}_h}$, and integrating by parts the first term on the left side of (2.14) then leads to

$$(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{T}_h} - (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}\}\!\}, [\![\boldsymbol{v}]\!])_{\mathcal{E}_h^o \cup \mathcal{E}_h^D} + ([\![\boldsymbol{u}]\!], \mathcal{B}_1(\boldsymbol{v}))_{\mathcal{E}_h^o \cup \mathcal{E}_h^D} = \mathcal{F}(\boldsymbol{v}). \tag{2.15}$$

For a fixed edge $E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D$ the operator $\mathcal{B}_1(\boldsymbol{v})$ is defined by

$$\mathcal{B}_1(\boldsymbol{v}) := -\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}))\boldsymbol{n}\}\!\} + \alpha_0\beta_0 h_E^{-1}\mathcal{P}_E^0[\![\boldsymbol{v}]\!] + \alpha_1\beta_1 h_E^{-1}[\![\boldsymbol{v}]\!], \tag{2.16}$$

where, following [12], the parameters β_0 and β_1 are chosen depending on the Lamé constants λ and μ:

$$\beta_0 := \mathrm{d}\lambda + 2\mu, \qquad \beta_1 := 2\mu. \tag{2.17}$$

The remaining two parameters, α_0 and α_1, are still at our disposal to ensure (later on) stability and to avoid locking of the resulting method. In what follows we choose α_0 sufficiently large to ensure the coercivity of the discrete bilinear form, and α_1 to be a positive constant bounded away from 0. We stress that these two parameters are independent of h, λ, and μ (see Sect. 2.4 and also [12], Prop. 2.2).

We define

$$\begin{aligned} a_{j,0}([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]) &:= \alpha_0\beta_0 \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \int_E \langle h_E^{-1}[\![\boldsymbol{u}]\!], \mathcal{P}_E^0[\![\boldsymbol{v}]\!]\rangle, \\ a_{j,1}([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]) &:= \alpha_1\beta_1 \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \int_E \langle h_E^{-1}[\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]\rangle, \end{aligned} \tag{2.18}$$

and set

$$a_j([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]) = a_{j,0}([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]) + a_{j,1}([\![\boldsymbol{u}]\!], [\![\boldsymbol{v}]\!]).$$

Then, the weak formulation of problem (2.12) reads: Find $\boldsymbol{u} \in [H^2(\mathcal{T}_h)]^d$ such that

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{w}) = \mathcal{F}(\boldsymbol{w}), \qquad \forall\ \boldsymbol{w} \in [H^2(\mathcal{T}_h)]^d. \tag{2.19}$$

The bilinear form $\mathcal{A}(\cdot,\cdot)$ is given by

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{w}) = \mathcal{A}_0(\boldsymbol{u}, \boldsymbol{w}) + a_{j,1}([\![\boldsymbol{u}]\!], [\![\boldsymbol{w}]\!]), \tag{2.20}$$

where

$$\begin{aligned} \mathcal{A}_0(\boldsymbol{u}, \boldsymbol{w}) = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{w}))_{\mathcal{T}_h} &- (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}\}\!\}, [\![\boldsymbol{w}]\!])_{\mathcal{E}_h^o \cup \mathcal{E}_h^D} \\ &- ([\![\boldsymbol{u}]\!], \{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{w}))\boldsymbol{n}\}\!\})_{\mathcal{E}_h^o \cup \mathcal{E}_h^D} + a_{j,0}([\![\boldsymbol{u}]\!], [\![\boldsymbol{w}]\!]). \end{aligned} \tag{2.21}$$

To obtain the discrete formulation, we replace in (2.19) the function space $[H^2(\mathcal{T}_h)]^d$ by $\boldsymbol{V}^{\mathrm{DG}}$, and we get the **IP-1 approximation** to the problem: Find $\boldsymbol{u}_h \in \boldsymbol{V}^{\mathrm{DG}}$ such that:

$$\mathcal{A}(\boldsymbol{u}_h, \boldsymbol{w}) = \mathcal{F}(\boldsymbol{w}), \qquad \forall\ \boldsymbol{w} \in \boldsymbol{V}^{\mathrm{DG}}. \tag{2.22}$$

[5] We note that in [6] the focus is on the scalar Laplace equation. The arguments for the elasticity problem, are basically the same.

Remark 2.1. Although we do not consider non-symmetric IP methods in this paper, let us remark that non-symmetric versions can easily be incorporated in the definition of $\mathcal{B}_1(\boldsymbol{v})$. For example, by setting:

$$\mathcal{B}_1(\boldsymbol{v}) := \theta\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}))\boldsymbol{n}\}\!\} + \alpha_0\beta_0 h_E^{-1}\mathcal{P}_E^0[\![\boldsymbol{v}]\!] + \alpha_1\beta_1 h_E^{-1}[\![\boldsymbol{v}]\!],$$

we obtain a non-symmetric bilinear form for the values $\theta = 0$ or $\theta = 1$. Such values of θ correspond to the Incomplete Interior Penalty (IIPG, $\theta = 0$) and Non-symmetric Interior Penalty (NIPG, $\theta = 1$) discretizations, respectively.

2.4. Stability analysis

We close this section presenting the stability and continuity results pertinent to our work. We start by introducing some norm notation. For $\boldsymbol{v} \in [H^2(\mathcal{T}_h)]^d$ we define the seminorms

$$\begin{aligned} \|\nabla\boldsymbol{v}\|_{0,\mathcal{T}_h}^2 &= \sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{v}\|_{0,T}^2, & \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 &= \sum_{T\in\mathcal{T}_h}\int_T\langle\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\varepsilon}(\boldsymbol{v})\rangle, \\ |\mathcal{P}_E^0[\![\boldsymbol{v}]\!]|_*^2 &= \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{v}]\!]\|_{0,E}^2, & |[\![\boldsymbol{v}]\!]|_*^2 &= \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|[\![\boldsymbol{v}]\!]\|_{0,E}^2\,. \end{aligned} \tag{2.23}$$

Moreover, for $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{DG}}$ we define the seminorms (which if $\mathcal{E}_h^D \neq \emptyset$ are norms)

$$\|\boldsymbol{v}\|_{DG0}^2 = \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 + \beta_0|\mathcal{P}_E^0[\![\boldsymbol{v}]\!]|_*^2, \tag{2.24}$$

$$\|\boldsymbol{v}\|_{DG}^2 = \|\boldsymbol{v}\|_{DG0}^2 + \beta_1|[\![\boldsymbol{v}]\!]|_*^2, \tag{2.25}$$

and, finally, the norm

$$|\boldsymbol{v}|_{H^1(\mathcal{T}_h)}^2 = \|\nabla\boldsymbol{v}\|_{0,\mathcal{T}_h}^2 + \beta_0|\mathcal{P}_E^0[\![\boldsymbol{v}]\!]|_*^2 + \beta_1|[\![\boldsymbol{v}]\!]|_*^2. \tag{2.26}$$

Coercivity of the **IP-1** bilinear form (see also [12], Prop. 2.2) with respect to the norm (2.25) can easily be shown by taking $\boldsymbol{u} = \boldsymbol{w} = \boldsymbol{v}$ in (2.20):

$$\begin{aligned} \mathcal{A}(\boldsymbol{v},\boldsymbol{v}) = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v}):\boldsymbol{\varepsilon}(\boldsymbol{v}))_{\mathcal{T}_h} &+ \alpha_0\beta_0\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{v}]\!]\|_{0,\mathcal{E}_h^o\cup\Gamma_D}^2 + \alpha_1\beta_1\|h_E^{-1/2}[\![\boldsymbol{v}]\!]\|_{0,\mathcal{E}_h^o\cup\Gamma_D}^2 \\ &- 2(\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{n})\}\!\},[\![\boldsymbol{v}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D}. \end{aligned}$$

Using Cauchy−Schwarz, trace and inverse inequalities together with the arithmetic-geometric inequality and the bound (2.1) on the maximum eigenvalue of $\mathcal{C}$ it follows that

$$\begin{aligned} (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{n})\}\!\},[\![\boldsymbol{v}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} &= (\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v})\boldsymbol{n})\}\!\},\mathcal{P}_E^0[\![\boldsymbol{v}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} \\ &\leq \frac{C_t(1+C_{\mathrm{inv}})}{\alpha_0\beta_0}\|\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 + \frac{\alpha_0\beta_0}{4}\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{v}]\!]\|_{0,\mathcal{E}_h^o\cup\Gamma_D}^2 \\ &\leq \frac{C_t(1+C_{\mathrm{inv}})}{\alpha_0}\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 + \frac{\alpha_0\beta_0}{4}\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{v}]\!]\|_{0,\mathcal{E}_h^o\cup\Gamma_D}^2 \end{aligned} \tag{2.27}$$

where C_t and C_{inv} denote the constants in the trace and inverse inequalities, respectively. Hence, we finally have

$$\begin{aligned} \mathcal{A}(\boldsymbol{v},\boldsymbol{v}) \geq \left(1 - \frac{2C_t(1+C_{\mathrm{inv}})}{\alpha_0}\right)&\|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\mathcal{T}_h}^2 + \alpha_1\beta_1\|h_E^{-1/2}[\![\boldsymbol{v}]\!]\|_{0,\mathcal{E}_h^o\cup\Gamma_D}^2 \\ &+\frac{\alpha_0}{2}\beta_0\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{v}]\!]\|_{0,\mathcal{E}_h^o\cup\Gamma_D}^2, \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}^{\mathrm{DG}}, \end{aligned}$$

and therefore by taking $\alpha_0 = \max\left(1, 4C_t(1+C_{\mathrm{inv}})\right)$ (sufficiently large) we ensure the coercivity of $\mathcal{A}(\cdot,\cdot)$ with respect to the $\|\cdot\|_{DG}$-norm with constant independent of h, μ, and λ. The continuity of the **IP-1** bilinear form follows from (2.27). Then, standard arguments can be used to derive optimal error estimates and it also follows that the **IP-1** method defined by (2.20) provides a robust approximation to (2.2) and does not lock as $\lambda \to \infty$. We refer to [12] for further details on the error analysis.

3. Space decomposition

We present now a decomposition of the DG space of piecewise linear vector valued functions that plays a key role in the construction of iterative solvers. This decomposition was introduced in [4] for scalar functions and also in [7] in a different context. Its extension to vector valued functions is more or less straightforward. We omit those proofs which are just an easy modification of the corresponding proofs in the scalar case. However, we review the main ingredients and ideas behind such proofs, since they play an important role in the analysis of the preconditioner given later on. In the last part of the section we give some properties of the spaces entering in the decomposition and prove a result that is essential for showing that the proposed preconditioner is uniform.

Following [4] we introduce the space complementary to V^{CR} in V^{DG},

$$\mathcal{Z} = \left\{ z \in V^{\mathrm{DG}} \text{ and } \mathcal{P}^0_E \{\!\!\{ z \}\!\!\} = 0, \text{ for all } E \in \mathcal{E}^o_h \right\}. \tag{3.1}$$

The corresponding space of vector valued functions is

$$\boldsymbol{\mathcal{Z}} = [\mathcal{Z}]^d. \tag{3.2}$$

To describe the basis functions associated with the spaces (2.10) and (3.2), let $\varphi_{E,T}$ denote the scalar basis function on T, dual to the degree of freedom at the mass center of the face E, and extended by zero outside T. For $E \in \partial T$, $E' \in \partial T$, the function $\varphi_{E,T}$ satisfies

$$\varphi_{E,T}(m_{E'}) = \begin{cases} 1 & \text{if} \quad E = E', \\ 0 & \text{otherwise,} \end{cases}$$

and also we have $\varphi_{E,T} \in \mathbb{P}^1(T)$, where $\varphi_{E,T}(x) = 0, \forall\, x \notin T$. For all $\boldsymbol{u} \in \boldsymbol{V}^{\mathrm{DG}}$ we then have

$$\boldsymbol{u}(x) = \sum_{T \in \mathcal{T}_h} \sum_{E \in \partial T} \boldsymbol{u}_T(m_E) \varphi_{E,T}(x) = \sum_{E \in \mathcal{E}_h} \boldsymbol{u}^+(m_E) \varphi^+_E(x) + \sum_{E \in \mathcal{E}^o_h} \boldsymbol{u}^-(m_E) \varphi^-_E(x), \tag{3.3}$$

where in the last identity we have just changed the order of summation and used the short hand notation $\varphi^{\pm}_E(x) := \varphi_{E,T^{\pm}}(x)$ together with

$$\begin{aligned} \boldsymbol{u}^{\pm}(m_E) &:= \boldsymbol{u}_{T^{\pm}}(m_E) = \frac{1}{|E|} \int_E \boldsymbol{u}_{T^{\pm}} \mathrm{d}s, && \forall\, E \in \mathcal{E}^o_h, \text{ i.e., } E = \partial T^+ \cap \partial T^-, \\ \boldsymbol{u}(m_E) &:= \boldsymbol{u}_T(m_E) = \frac{1}{|E|} \int_E \boldsymbol{u}_T \mathrm{d}s, && \forall\, E \in \mathcal{E}^{\partial}_h, \text{ i.e., } E = \partial T \cap \partial \Omega. \end{aligned}$$

Recalling now the definitions of $T^+(E)$ and $T^-(E)$ given in (2.5) we set

$$\begin{cases} \varphi^{CR}_E = \varphi_{E,T^+(E)} + \varphi_{E,T^-(E)}, & \forall\, E \in \mathcal{E}^o_h, \\ \varphi^{CR}_E = \varphi_{E,T^+(E)}, & \forall\, E \in \mathcal{E}^N_h. \end{cases} \tag{3.4}$$

and

$$\begin{cases} \psi^z_E = \dfrac{\varphi_{E,T^+(E)} - \varphi_{E,T^-(E)}}{2}, & \forall\, E \in \mathcal{E}^o_h, \\ \psi^z_E = \varphi_{E,T^+(E)}, & \forall\, E \in \mathcal{E}^D_h. \end{cases} \tag{3.5}$$

Some clarification is needed here. Note that from the definition of $\varphi_{E,T^+(E)}$ and $\varphi_{E,T^-(E)}$ for an interior edge $E \in \mathcal{E}^o_h$, it does not follow that their sum is even defined on the edge E, since it is just a sum of two functions from $L^2(\Omega)$. However, the sum $(\varphi_{E,T^+(E)} + \varphi_{E,T^-(E)})$ has a representative, which is continuous across E and this representative is denoted here with $\boldsymbol{\varphi}^{CR}_E$, see Figure 1.

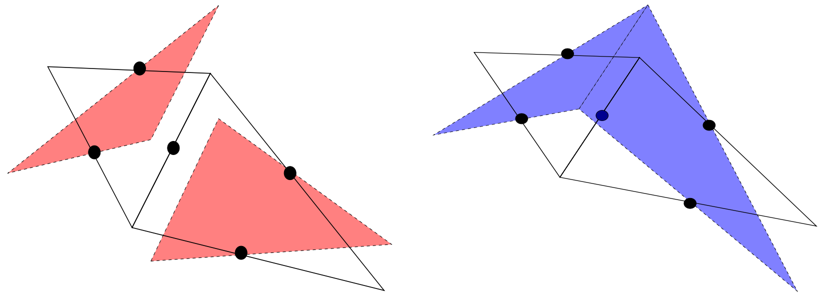

FIGURE 1. Basis functions associated with the face E: ψ_E^z (left) and φ_E^{CR} (right).

Clearly, $\{\varphi_E^{CR}\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}$ are linearly independent, and $\{\boldsymbol{\psi}_E^z\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}$ are linearly independent. A simple argument then shows that

$$\boldsymbol{V}^{\rm CR} = \operatorname{span}\left\{\{\varphi_E^{CR}\boldsymbol{e}_k\}_{k=1}^d\right\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}, \quad \boldsymbol{\mathcal{Z}} = \operatorname{span}\left\{\{\psi_E^z\boldsymbol{e}_k\}_{k=1}^d\right\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}.$$

Here $\boldsymbol{e}_k$, $k=1,\dots,d$ is the k-th canonical basis vector in $\mathbb{R}^d$. Hence by performing a change of basis in (3.3), we have obtained a "natural" splitting of

$$\boldsymbol{V}^{\rm DG} = \boldsymbol{V}^{\rm CR} \oplus \boldsymbol{\mathcal{Z}}$$

and the set

$$\{\boldsymbol{\psi}_E^z\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \cup \{\varphi_E^{CR}\}_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}, \tag{3.6}$$

provides a natural basis for the DG finite element space.

Proposition 3.1. *For any $\boldsymbol{u}\in\boldsymbol{V}^{DG}$ there exist unique $\boldsymbol{v}\in\boldsymbol{V}^{CR}$ and $\boldsymbol{z}\in\boldsymbol{\mathcal{Z}}$ such that*

$$\boldsymbol{u}=\boldsymbol{v}+\boldsymbol{z} \quad \text{and} \quad \begin{array}{l} \boldsymbol{v} = \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^N}\left(\frac{1}{|E|}\int_E \{\!\{\boldsymbol{u}\}\!\}\mathrm{d}s\right)\varphi_E^{CR}(x)\in\boldsymbol{V}^{CR}, \\ \boldsymbol{z} = \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}\left(\frac{1}{|E|}\int_E [\![\boldsymbol{u}]\!]\mathrm{d}s\right)\boldsymbol{\psi}_E^z(x)\in\boldsymbol{\mathcal{Z}}. \end{array} \tag{3.7}$$

The proof of the above result follows by arguing as for the scalar case in [4], Proposition 3.1, but proceeding componentwise. The next Lemma shows that the splitting we have proposed is orthogonal with respect to the inner product defined by $\mathcal{A}_0(\cdot,\cdot)$.

Lemma 3.2. *The splitting $\boldsymbol{V}^{DG}=\boldsymbol{V}^{CR}\oplus\boldsymbol{\mathcal{Z}}$ is $\mathcal{A}_0$–orthogonal. That is*

$$\mathcal{A}_0(\boldsymbol{v},\boldsymbol{z}) = \mathcal{A}_0(\boldsymbol{z},\boldsymbol{v}) = 0 \qquad \forall\,\boldsymbol{v}\in\boldsymbol{V}^{CR}, \quad \forall\,\boldsymbol{z}\in\boldsymbol{\mathcal{Z}}. \tag{3.8}$$

The proof follows straightforwardly by using the weighted residual formulation (2.14)–(2.21) and the definition of the spaces $\boldsymbol{V}^{\rm CR}$ and $\boldsymbol{\mathcal{Z}}$. See [4], Proposition 3.1 for details.

3.1. Some properties of the space $\boldsymbol{\mathcal{Z}}$

We now present some properties of the functions in the space $\boldsymbol{\mathcal{Z}}$. We start with a simple observation. From the definition of the spaces $\boldsymbol{V}^{\rm CR}$ and $\boldsymbol{\mathcal{Z}}$ it is easy to see that

$$\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{z}\|_{0,T}^2 = ([\![\boldsymbol{z}]\!],\{\!\{(\nabla\boldsymbol{z})\boldsymbol{n}\}\!\})_{\mathcal{E}_h^o\cup\mathcal{E}_h^D}.$$

Applying the Schwarz inequality, one then gets the following estimate

$$\sum_{T\in\mathcal{T}_h}\|\nabla\boldsymbol{z}\|_{0,T}^2 \le C\|h^{-1/2}P_E^0[\![\boldsymbol{z}]\!]\|_{0,\mathcal{E}_h}^2,$$

which is a straightforward way to see that the restriction of the **IP-1** bilinear form (even for $\theta = 0, 1$ as in Rem. 2.1) to the space $\boldsymbol{\mathcal{Z}}$ is coercive not only in the DG norm defined in (2.25) but also in the $|\cdot|_{H^1(\mathcal{T}_h)}$-seminorm (2.26) (regardless whether the boundary conditions are Dirichlet, Neumann or mixed type). Therefore the resulting stiffness matrices are positive definite.

The next result provides bounds on the spectrum of the operators associated to $\mathcal{A}_0(\cdot,\cdot)$ and $\mathcal{A}(\cdot,\cdot)$, when restricted to $\boldsymbol{\mathcal{Z}}$.

Lemma 3.3. *Let $\boldsymbol{\mathcal{Z}}$ be the space defined in (3.2). Then for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$, the following estimates hold*

$$\alpha_0\beta_0 h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \mathcal{A}_0(\boldsymbol{z},\boldsymbol{z}) \lesssim \alpha_0\beta_0 h^{-2}\|\boldsymbol{z}\|_0^2, \tag{3.9}$$

and also,

$$[\alpha_0\beta_0 + \alpha_1\beta_1]h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \mathcal{A}(\boldsymbol{z},\boldsymbol{z}) \lesssim [\alpha_0\beta_0 + \alpha_1\beta_1]h^{-2}\|\boldsymbol{z}\|_0^2, \tag{3.10}$$

where β_0 and β_1 are as defined in (2.17).

Proof. We first prove the lower bounds. Arguing as in [4], Lemma 5.3 (but now componentwise for vector valued functions) one can show that (due the special structure of the space $\boldsymbol{\mathcal{Z}}$)

$$h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \lesssim h^{-2}\|\boldsymbol{z}\|_0^2. \tag{3.11}$$

From the coercivity of $\mathcal{A}_0$ it follows then

$$\alpha_0\beta_0 h^{-2}\|\boldsymbol{z}\|_0^2 \lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \le \mathcal{A}_0(\boldsymbol{z},\boldsymbol{z}).$$

Similarly, the $L^2(\mathcal{E}_h)$ stability of the projection $\mathcal{P}_E^0$ together with the coercivity of $\mathcal{A}$ gives

$$\begin{aligned}(\alpha_0\beta_0+\alpha_1\beta_1)h^{-2}\|\boldsymbol{z}\|_0^2 &\lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 + \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 \\ &\lesssim \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 + \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} h_E^{-1}\|[\![\boldsymbol{z}]\!]\|_{0,E}^2 \\ &\le \mathcal{A}(\boldsymbol{z},\boldsymbol{z}),\end{aligned}$$

and so, the lower bounds in (3.9) and (3.10) follow.

We next show the upper bound in (3.9), the one in (3.10) is obtained in an analogous fashion and we will omit the details. Using (2.27) together with (2.1) we get

$$\begin{aligned}\mathcal{A}_0(\boldsymbol{z},\boldsymbol{z}) &\le \alpha_0\beta_0 \sum_{E\in\mathcal{E}_h^o\cup\Gamma_D} h_E^{-1}\|\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 + \|\mathcal{C}^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h}^2 \\ &\le \beta_0\left(\alpha_0\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{\mathcal{E}_h^o\cup\Gamma_D}^2 + \|\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h}^2\right).\end{aligned}$$

Hence, the upper bound in (3.9) follows in a straightforward fashion using the trace and inverse inequalities together with the obvious inequality $\|\boldsymbol{\varepsilon}(\boldsymbol{z})\|_{0,\mathcal{T}_h} \le \|\nabla\boldsymbol{z}\|_{0,\mathcal{T}_h}$. ☐

We close this section with establishing a uniform bound on the angle between $\boldsymbol{V}^{\mathrm{CR}}$ and $\boldsymbol{Z}$ in the inner product given by the bilinear form $\mathcal{A}(\cdot,\cdot)$. The estimate is given in Proposition 3.4. It plays a crucial role in bounding the condition number of the preconditioned system.

We remind that $E \in \mathcal{E}_h$ denotes a $(d-1)$-dimensional simplex (a face), which is either the intersection of two d-dimensional simplices $T \in \mathcal{T}_h$ or an intersection of a d-dimensional simplex $T \in \mathcal{T}_h$ and the complement of Ω, *i.e.*, $E = T \cap (\mathbb{R}^d \setminus \Omega)$. In the former case, the face E is called an interior face and in the latter it is called a boundary face.

The proof of Proposition 3.4 requires arguments involving the incidence relations between simplices $T \in \mathcal{T}_h$ and faces $E \in \mathcal{E}_h$, and estimates on the cardinality of these incidence sets. For the readers' convenience, we provide a list of such estimates below.

- We define $\mathcal{N}_0(E)$ to be the set of d-dimensional $T \in \mathcal{T}_h$ simplices that contain E:

$$\mathcal{N}_0(E) := \{T \in \mathcal{T}_h, \quad \text{such that} \quad E \in T\}$$

By definition, for the cardinality of this set we have $|\mathcal{N}_0(E)| = 2$ for the interior faces and $|\mathcal{N}_0(E)| = 1$ for the boundary faces.

- We define the set of neighbor (or neighboring) faces $\mathcal{N}_1(E)$ to be the set of faces which share an element with E:

$$\mathcal{N}_1(E) := \{E' \in \mathcal{E}_h, \quad \text{such that} \quad \mathcal{N}_0(E) \cap \mathcal{N}_0(E') \neq \emptyset\}$$

From Proposition A.1 (see Appendix 5) we have that $|\mathcal{N}_1(E)| \leq (2d + 1)$.

- Next, we define $\mathcal{N}_2(E)$ to be the set of faces which share at least one neighboring face with E:

$$\mathcal{N}_2(E) := \{E' \in \mathcal{E}_h, \quad \text{such that} \quad \mathcal{N}_1(E) \cap \mathcal{N}_1(E') \neq \emptyset\}$$

From Proposition A.1 we have the estimate $|\mathcal{N}_2(E)| \leq (2d + 1)^2$.

- For the basis functions $\{\psi_E^z\}_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D}$ we have the following relations:

$$\frac{1}{|E|} \int_E [\![\psi_{E'}^z]\!] = \delta_{EE'}, \quad \text{and} \quad [\![\psi_E^z]\!](x) = 1, \quad \text{for all} \quad x \in E, \tag{3.12}$$

$$|[\![\psi_E^z]\!](x)| \leq 1, \quad \text{for all} \quad x \in E', \quad \text{and all} \quad E' \in \mathcal{N}_2(E). \tag{3.13}$$

All the above relations follow from the definition of $\psi_E^z(x)$ and the fact that $[\![\psi_E^z]\!]$ is a linear function on every face in $\mathcal{E}_h$, and therefore $\int_E [\![\psi_{E'}^z]\!] = |E| [\![\psi_{E'}^z]\!](m_E)$.

- Finally, for $E \in \mathcal{E}_h$, $E' \in \mathcal{E}_h$, and $E'' \in \mathcal{E}_h$ it is straightforward to see that we have:

$$\text{If} \quad E \notin \mathcal{N}_1(E') \cap \mathcal{N}_1(E'') \quad \text{then} \quad \int_E [\![\psi_{E'}^z]\!][\![\psi_{E''}^z]\!] = 0. \tag{3.14}$$

An easy consequence from the definitions then is the following:

$$\text{If} \quad E' \notin \mathcal{N}_2(E'') \quad \text{then} \quad \int_E [\![\psi_{E'}^z]\!][\![\psi_{E''}^z]\!] = 0, \quad \text{for all} \quad E \in \mathcal{E}_h. \tag{3.15}$$

We finally give Proposition 3.4. To avoid unnecessary complications with the notation, we state and prove the result for scalar valued functions. The proof for vector valued functions is easy to obtain, and with the same constant, by just applying the scalar valued result component-wise.

Proposition 3.4. *The following inequality holds for all $z \in \mathcal{Z}$:*

$$\sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \|h_E^{-1/2}([\![z]\!] - \mathcal{P}_E^0[\![z]\!])\|_{0,E}^2 \leq \left(1 - \frac{1}{\rho}\right) \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \|h_E^{-1/2}[\![z]\!]\|_{0,E}^2, \tag{3.16}$$

with a constant $\rho \geq 1$ which depends on the shape regularity of the mesh.

Proof. Since $\mathcal{P}_E^0$ is the L^2-orthogonal projection on the constants, we have that

$$\|h_E^{-1/2}([\![z]\!] - \mathcal{P}_E^0[\![z]\!])\|_{0,E}^2 = \|h_E^{-1/2}[\![z]\!]\|_{0,E}^2 - \|h_E^{-1/2}\mathcal{P}_E^0[\![z]\!]\|_{0,E}^2. \tag{3.17}$$

Let $z \in \mathcal{Z}$, *i.e.*, $z = \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'}\psi_{E'}^z$. From (3.12) we have that $\mathcal{P}_E^0[\![\psi_{E'}^z]\!] = \delta_{EE'}$, and hence, we may conclude that

$$\begin{aligned}\|h_E^{-1/2}\mathcal{P}_E^0[\![\boldsymbol{z}]\!]\|_{0,E}^2 &= \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E' \in \mathcal{E}_h} \delta_{EE'} \frac{|E|}{h_E} z_E z_{E'} \\ &= \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \mathbb{D}_{EE} z_E^2 = \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle.\end{aligned}$$

Here we have denoted by $\mathbb{D} : \mathbb{R}^{|\mathcal{E}_h|} \mapsto \mathbb{R}^{|\mathcal{E}_h|}$ a diagonal matrix with non-zero elements $\mathbb{D}_{EE} := \frac{|E|}{h_E}$ and by $\tilde{z} \in \mathbb{R}^{|\mathcal{E}_h|}$ the vector of coefficients $\tilde{z} = \{z_E\}_{E \in \mathcal{E}_h}$ in the expansion of $z \in \mathcal{Z}$ via the basis $\{\psi_E^z\}_{E \in \mathcal{E}_h}$.

Further we consider the right hand side of (3.16) and we have

$$\begin{aligned}\sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|_{0,E}^2 &= \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} h_E^{-1} \left\| \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'}[\![\psi_{E'}^z]\!] \right\|_{0,E}^2 \\ &= \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \int_E h_E^{-1} \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E'' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'} z_{E''} [\![\psi_{E'}^z]\!][\![\psi_{E''}^z]\!] \\ &= \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E'' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'} z_{E''} \left(\sum_{E \in \mathcal{E}_h} \int_E h_E^{-1} [\![\psi_{E'}^z]\!][\![\psi_{E''}^z]\!] \right) \\ &= \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E'' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} z_{E'} z_{E''} \mathbb{S}_{E'E''} = \langle \mathbb{S}\tilde{z}, \tilde{z}\rangle.\end{aligned}$$

Here, $\mathbb{S} : \mathbb{R}^{|\mathcal{E}_h|} \mapsto \mathbb{R}^{|\mathcal{E}_h|}$ denotes the symmetric real matrix with elements

$$\mathbb{S}_{E'E''} = \sum_{E \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \int_E h_E^{-1} [\![\psi_{E'}^z]\!][\![\psi_{E''}^z]\!] = \sum_{E \in \mathcal{N}_1(E') \cap \mathcal{N}_1(E'')} \int_E h_E^{-1} [\![\psi_{E'}^z]\!][\![\psi_{E''}^z]\!]. \tag{3.18}$$

In the last identity above, we have used (3.14). Note that according to (3.15), if $E' \notin \mathcal{N}_2(E'')$ then $\mathbb{S}_{E'E''} = 0$. Thus,

$$\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle = \sum_{E' \in \mathcal{E}_h^o \cup \mathcal{E}_h^D} \sum_{E'' \in \mathcal{N}_2(E)} z_{E'} z_{E''} \mathbb{S}_{E'E''}.$$

From this identity and (3.12) and (3.13), we obtain that

$$|\mathbb{S}_{E'E''}| \le |\mathcal{N}_1(E') \cap \mathcal{N}_1(E'')| \max_{E \in \mathcal{N}_1(E') \cap \mathcal{N}_1(E'')} \frac{|E|}{h_E} \le (2d+1) \max_{E \in \mathcal{N}_1(E') \cap \mathcal{N}_1(E'')} \frac{|E|}{h_E}.$$

Introducing

$$\rho = \sup_{\tilde{w} \in \mathbb{R}^{|\mathcal{E}_h|}} \frac{\langle \mathbb{S}\tilde{w}, \tilde{w}\rangle}{\langle \mathbb{D}\tilde{w}, \tilde{w}\rangle} = \sup_{\tilde{w} \in \mathbb{R}^{|\mathcal{E}_h|}} \frac{\langle \mathbb{D}^{-1/2}\mathbb{S}\mathbb{D}^{-1/2}\tilde{w}, \tilde{w}\rangle}{\langle \tilde{w}, \tilde{w}\rangle},$$

we obtain that

$$\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle = \langle \mathbb{D}^{-1/2}\mathbb{S}\mathbb{D}^{-1/2}\mathbb{D}^{1/2}\tilde{z}, \mathbb{D}^{1/2}\tilde{z}\rangle \le \rho \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle. \tag{3.19}$$

This inequality can be rewritten as $\frac{1}{\rho}\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle \le \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle$ and hence

$$\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle - \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle \le \langle \mathbb{S}\tilde{z}, \tilde{z}\rangle - \frac{1}{\rho}\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle = \left(1 - \frac{1}{\rho}\right)\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle.$$

Note that (3.17) implies that

$$\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle = \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle + \sum_{E\in\mathcal{E}_h} \|h_E^{-1/2}([\![z]\!] - \mathcal{P}_E^0[\![z]\!])\|_{0,E}^2, \tag{3.20}$$

and thus $\langle \mathbb{S}\tilde{z}, \tilde{z}\rangle \ge \langle \mathbb{D}\tilde{z}, \tilde{z}\rangle$. This shows that $\rho \ge 1$ in (3.19).

It remains to show that ρ can be bounded by quantities depending only on the shape regularity of the mesh. Again, by (3.15) we have that: if $E' \notin \mathcal{N}_2(E'')$ then $\mathbb{S}_{E'E''} = 0$. Hence,

$$\begin{aligned}\rho &\le \|\mathbb{D}^{-1/2}\mathbb{S}\mathbb{D}^{-1/2}\|_{\ell^\infty} \le \max_{E''\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \sum_{E'\in\mathcal{N}_2(E'')} \frac{|\mathbb{S}_{E''E'}|}{\sqrt{\mathbb{D}_{E'E'}\mathbb{D}_{E''E''}}}\\ &\le \max_{E''\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}\left[|\mathcal{N}_2(E'')| \max_{E'\in\mathcal{N}_2(E'')} \frac{|\mathbb{S}_{E'E''}|}{\sqrt{\mathbb{D}_{E'E'}\mathbb{D}_{E''E''}}}\right]\\ &\le (2d+1)^3 \max_{E''\in\mathcal{E}_h^o\cup\mathcal{E}_h^D}\ \max_{E'\in\mathcal{N}_2(E'')}\ \max_{E\in\mathcal{N}_1(E')\cap\mathcal{N}_1(E'')} \frac{|E|}{h_E}\sqrt{\frac{h_{E'}h_{E''}}{|E'||E''|}}.\end{aligned}$$

The quantity on the right side of this estimate only depends on the shape regularity of the mesh and the proof is complete. □

Remark 3.5. We remark that the constants in Proposition 3.4 can be sharpened, at the price of further complicating the proof. The result given above is sufficient for our purposes, and we do not further comment on the possible "optimal" value of the constant ρ above. Another relevant observation is that the inequality in Proposition 3.4 holds true, with the same or even smaller ρ, if we replace $\mathcal{E}_h^o \cup \mathcal{E}_h^D$ with a subset of edges $\mathcal{E} \subset (\mathcal{E}_h^o \cup \mathcal{E}_h^D)$ in (3.16). The proof is completely analogous (just $\mathcal{E}_h^o \cup \mathcal{E}_h^D$ is replaced by $\mathcal{E}$).

4. Preconditioning

In this section, we present the construction and convergence analysis of the preconditioners we propose for the considered IP-methods.

To construct the preconditioners, we use the subspace splitting given in Proposition 3.1, which suggests a simple change of basis. We have that for any $\boldsymbol{u}, \boldsymbol{w} \in \boldsymbol{V}^{\rm DG}$, we can write $\boldsymbol{u} = \boldsymbol{z}+\boldsymbol{v}$, and $\boldsymbol{w} = \boldsymbol{\zeta}+\boldsymbol{\varphi}$, where $\boldsymbol{z}, \boldsymbol{\zeta} \in \boldsymbol{\mathcal{Z}}$ and $\boldsymbol{v}, \boldsymbol{\varphi} \in \boldsymbol{V}^{\rm CR}$. Therefore, by performing this change of basis we can write $\mathcal{A}(\boldsymbol{u}, \boldsymbol{w}) = \mathcal{A}((\boldsymbol{z}, \boldsymbol{v}), (\boldsymbol{\zeta}, \boldsymbol{\phi}))$. The $\mathcal{A}_0$-orthogonality (3.8) of the subspaces in the splitting gives

$$\mathcal{A}_0((\boldsymbol{z}, \boldsymbol{v}), (\boldsymbol{\zeta}, \boldsymbol{\phi})) = \mathcal{A}_0(\boldsymbol{z}, \boldsymbol{\zeta}) + \mathcal{A}_0(\boldsymbol{v}, \boldsymbol{\phi}),$$

which implies that the resulting stiffness matrix of $\mathcal{A}_0$ in this new basis is block diagonal. Therefore it is enough to study how to efficiently solve each of the blocks in the above block diagonal structure of $\mathcal{A}_0$: the subproblem resulting from the restriction of $\mathcal{A}_0$ to $\boldsymbol{\mathcal{Z}}$ and the subproblem on the space $\boldsymbol{V}^{\rm CR}$.

For traction free or mixed type of boundary conditions, a preconditioner based only on $\mathcal{A}_0$ does not result in an optimal and robust solution method. However, the block structure of $\mathcal{A}_0$ in the new basis still suggests that a reasonable choice for an approximation of $\mathcal{A}(\cdot, \cdot)$ is

$$\mathcal{B}((\boldsymbol{z}, \boldsymbol{v}), (\boldsymbol{\zeta}, \boldsymbol{\phi})) = \mathcal{A}(\boldsymbol{z}, \boldsymbol{\zeta}) + \mathcal{A}(\boldsymbol{v}, \boldsymbol{\phi}). \tag{4.1}$$

The following algorithm describes the application of a preconditioner, which is based on the bilinear form in equation (4.1).

Algorithm 4.1. *Let $\boldsymbol{r} \in [L^2(\Omega)]^d$ be given. Then the action of the preconditioner on $\boldsymbol{r}$ is the function $\boldsymbol{u} \in \boldsymbol{V}^{DG}$ which is obtained from the following three steps.*

1. *Find $z \in \boldsymbol{\mathcal{Z}}$ such that*
$$\mathcal{A}(\boldsymbol{z}, \boldsymbol{\zeta}) = (\boldsymbol{r}, \boldsymbol{\zeta})_{\mathcal{T}_h} \quad \text{for all} \quad \boldsymbol{\zeta} \in \boldsymbol{\mathcal{Z}}.$$
2. *Find $\boldsymbol{v} \in \boldsymbol{V}^{CR}$ such that*
$$\mathcal{A}(\boldsymbol{v}, \boldsymbol{\varphi}) = (\boldsymbol{r}, \boldsymbol{\varphi})_{\mathcal{T}_h} \quad \text{for all} \quad \boldsymbol{\varphi} \in \boldsymbol{V}^{CR}.$$
3. *Set $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{v}$.*

As before, the application of this preconditioner corresponds to solving the subproblem of the restriction of $\mathcal{A}(\cdot, \cdot)$ to $\boldsymbol{\mathcal{Z}}$ and the subproblem of the restriction of $\mathcal{A}(\cdot, \cdot)$ to $\boldsymbol{V}^{\mathrm{CR}}$.

We now briefly discuss how the two smaller sub-problems can be efficiently solved in both cases: (1) the case of Dirichlet boundary conditions on all of $\partial\Omega$; and (2) the case of Neumann or mixed boundary conditions.

Solution in the subspace $\mathcal{Z}$: Lemma 3.3 guarantees that the restriction of $\mathcal{A}(\cdot, \cdot)$ and $\mathcal{A}_0(\cdot, \cdot)$ to $\boldsymbol{\mathcal{Z}}$ is well-conditioned with respect to both, the mesh size and the Lamé constants λ, μ. Therefore, the linear system corresponding to the subproblem of the restriction to $\boldsymbol{\mathcal{Z}}$ can be efficiently solved by the method of Conjugate Gradients (CG). A simple consequence of the well known estimate on the convergence of CG (see, *e.g.*, [13,15]) shows that the number of CG iterations required to achieve a fixed error tolerance is uniformly bounded, independently of the size of the problem and the parameters.

Solution in $\boldsymbol{V}^{\mathrm{CR}}$: We now briefly discuss how to construct a uniform preconditioner for the corresponding subproblem on the space $\boldsymbol{V}^{\mathrm{CR}}$. Rather than developing a completely new method, the idea is to use the optimal preconditioners that have already been studied in literature, and modify them if needed so that they fit in the present framework. For our discussion, we distinguish two cases: the pure displacement problem ($\Gamma_N = \emptyset$) and the case with mixed or traction free boundary conditions ($\Gamma_N \neq \emptyset$).

- For the case of Dirichlet boundary conditions on the entire boundary-the so-called pure displacement problem-it is known how to construct optimal order multilevel preconditioners that are robust with respect to the parameter λ, see *e.g.* [5,10,14] and the references therein.
- The traction free problem or the case of mixed boundary conditions is more difficult to handle because the (discrete) Korn inequality is not satisfied for the standard discretization by Crouzeix–Raviart elements without additional stabilization, as was shown in [9]. The design of optimal and robust solution methods for stabilized discretizations is still an open problem, however, auxiliary space techniques might bridge this gap soon.

4.1. Convergence analysis

We now prove that the proposed block preconditioners are indeed optimal so that their convergence is uniform with respect to mesh size and the Lamé parameters. This result is given in Theorem 4.3. The following Lemma is crucial for this proof, since it gives estimates on the norm of the off-diagonal blocks in the 2×2 block form of the stiffness matrix associated to $\mathcal{A}(\cdot, \cdot)$, corresponding to the space splitting $\boldsymbol{V}^{\mathrm{DG}} = \boldsymbol{V}^{\mathrm{CR}} \oplus \boldsymbol{\mathcal{Z}}$. The result provides a measure of the angle between the subspaces $\boldsymbol{V}^{\mathrm{CR}}$ and $\boldsymbol{\mathcal{Z}}$, with respect to the $\mathcal{A}$-norm. The proof of this result uses Proposition 3.4.

Lemma 4.2. Strengthened Cauchy−Schwarz inequality: *The following inequality holds for any $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and any $\boldsymbol{v} \in \boldsymbol{V}^{CR}$*
$$[\mathcal{A}(\boldsymbol{z}, \boldsymbol{v})]^2 \leq \gamma^2 \mathcal{A}(\boldsymbol{z}, \boldsymbol{z}) \mathcal{A}(\boldsymbol{v}, \boldsymbol{v})$$
where $\gamma \leq q < 1$ and q is a constant that depends only on α_0, α_1, and the constant from Proposition 3.4 and is independent of the Lamé parameters λ and μ and the meshsize h.

Proof. We know that we can choose α_0 large enough, such that for all $\boldsymbol{u} \in \boldsymbol{V}^{\rm DG}$ we have

$$\mathcal{A}_0(\boldsymbol{u},\boldsymbol{u}) = (\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{u}))_{\mathcal{T}_h} - 2(\{\!\{(\mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}))\boldsymbol{n}\}\!\}, [\![\boldsymbol{u}]\!])_{\mathcal{E}_h^o\cup\mathcal{E}_h^D} + \alpha_0 a_{j,0}([\![\boldsymbol{u}]\!],[\![\boldsymbol{u}]\!]) \geq 0.$$

Then it is sufficient to prove that there exists $\gamma = \gamma(\alpha_1) < 1$ such that for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and for all $\boldsymbol{v} \in \boldsymbol{V}^{\rm CR}$ the inequality

$$[a_{j,1}([\![\boldsymbol{z}]\!],[\![\boldsymbol{v}]\!])]^2 \leq \gamma^2 a_{j,1}([\![\boldsymbol{z}]\!],[\![\boldsymbol{z}]\!]) a_{j,1}([\![\boldsymbol{v}]\!],[\![\boldsymbol{v}]\!]),$$

holds. By the definition of the spaces $\boldsymbol{\mathcal{Z}}$ and $\boldsymbol{V}^{\rm CR}$, on the boundary edges $E \in \mathcal{E}_h^\partial$ we have either $\mathcal{P}_E^0[\![\boldsymbol{z}]\!] = 0$ (if $E \in \mathcal{E}_h^N$) or $\mathcal{P}_E^0[\![\boldsymbol{v}]\!] = 0$ (if $E \in \mathcal{E}_h^D$). Hence, from the symmetry of $\mathcal{P}_E^0$ we conclude that

$$\int_E \langle [\![\boldsymbol{z}]\!], \mathcal{P}_E^0[\![\boldsymbol{v}]\!]\rangle = \int_E \langle \mathcal{P}_E^0[\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!]\rangle = 0, \quad \text{for all} \quad E \in \mathcal{E}_h^\partial, \quad \text{and all} \quad \boldsymbol{z} \in \boldsymbol{\mathcal{Z}}, \quad \boldsymbol{v} \in \boldsymbol{V}^{\rm CR}.$$

Since for the interior edges $E \in \mathcal{E}_h^o$ we also have $\mathcal{P}_E^0[\![\boldsymbol{v}]\!] = 0$, the above relation and the definition of $\mathcal{P}_E^0$ altogether imply that for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$, and $\boldsymbol{v} \in \boldsymbol{V}^{\rm CR}$

$$\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1}\mathcal{P}_E^0[\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!]\rangle = \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1}[\![\boldsymbol{z}]\!], \mathcal{P}_E^0[\![\boldsymbol{v}]\!]\rangle = 0. \tag{4.2}$$

Equation (4.2) and the Schwarz inequality then lead to

$$\begin{aligned}
[a_{j,1}([\![\boldsymbol{z}]\!],[\![\boldsymbol{v}]\!])]^2 &= \left[\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1}[\![\boldsymbol{z}]\!], [\![\boldsymbol{v}]\!]\rangle\right]^2 \\
&= \left[\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \int_E \langle h_E^{-1}([\![\boldsymbol{z}]\!] - \mathcal{P}_E^0[\![\boldsymbol{z}]\!]), [\![\boldsymbol{v}]\!]\rangle\right]^2 \\
&\leq a_{j,1}([\![\boldsymbol{v}]\!],[\![\boldsymbol{v}]\!]) \left[\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \|h_E^{-1/2}([\![\boldsymbol{z}]\!] - \mathcal{P}_E^0[\![\boldsymbol{z}]\!])\|_{0,E}^2\right].
\end{aligned}$$

Next, the result in Proposition 3.4 (more precisely its vector valued form) implies that

$$\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \|h_E^{-1/2}([\![\boldsymbol{z}]\!] - \mathcal{P}_E^0[\![\boldsymbol{z}]\!])\|_{0,E}^2 \leq \left(1 - \frac{1}{\rho}\right) \alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|_{0,E}^2.$$

Therefore, we have

$$\begin{aligned}
[a_{j,1}([\![\boldsymbol{z}]\!],[\![\boldsymbol{v}]\!])]^2 &\leq \left(1 - \frac{1}{\rho}\right) a_{j,1}([\![\boldsymbol{v}]\!],[\![\boldsymbol{v}]\!]) \left[\alpha_1\beta_1 \sum_{E\in\mathcal{E}_h^o\cup\mathcal{E}_h^D} \|h_E^{-1/2}[\![\boldsymbol{z}]\!]\|_{0,E}^2\right] \\
&\leq \left(1 - \frac{1}{\rho}\right) a_{j,1}([\![\boldsymbol{z}]\!],[\![\boldsymbol{z}]\!]) a_{j,1}([\![\boldsymbol{v}]\!],[\![\boldsymbol{v}]\!]),
\end{aligned}$$

which shows the desired inequality. □

We are now in a position to prove that the preconditioner given by Algorithm 4.1 is uniform with respect to the mesh size and the problem parameters.

TABLE 1. Observed CBS constant γ^2 for $\Omega = (0,1)^2$.

γ^2	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.49999$
$\ell = 1$	0.0664	0.025	0.0024	2.4024×10^{-4}	2.4015×10^{-6}
$\ell = 2$	0.0678	0.0255	0.0025	2.4567×10^{-4}	2.4559×10^{-6}
$\ell = 3$	0.0684	0.0258	0.0025	2.4866×10^{-4}	2.4857×10^{-6}
$\ell = 4$	0.0686	0.0259	0.0025	2.4974×10^{-4}	2.4966×10^{-6}

Theorem 4.3. *Let $\mathcal{A}(\cdot,\cdot)$ be the symmetric bilinear form defined by (2.20) where $\theta = -1$ and $\mathcal{B}(\cdot,\cdot)$ be the bilinear form defined by (4.1). Then the following estimates hold for all $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}$ and for all $\boldsymbol{v} \in \boldsymbol{V}^{CR}$ with the constant $\gamma < 1$ from Lemma 4.2:*

$$\frac{1}{1+\gamma}\mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \frac{1}{1-\gamma}\mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})). \tag{4.3}$$

Proof. Using Lemma 4.2 we have

$$-2\gamma\sqrt{\mathcal{A}(\boldsymbol{z},\boldsymbol{z})\,\mathcal{A}(\boldsymbol{v},\boldsymbol{v})} \le 2\mathcal{A}(\boldsymbol{z},\boldsymbol{v}) \le 2\gamma\sqrt{\mathcal{A}(\boldsymbol{z},\boldsymbol{z})\,\mathcal{A}(\boldsymbol{v},\boldsymbol{v})}$$

and since $-a^2 - b^2 \le 2ab \le a^2 + b^2$ for any real numbers a and b we obtain

$$(1-\gamma)\left(\mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v})\right) \le \mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v}) + 2\mathcal{A}(\boldsymbol{z},\boldsymbol{v}) \le (1+\gamma)\left(\mathcal{A}(\boldsymbol{z},\boldsymbol{z}) + \mathcal{A}(\boldsymbol{v},\boldsymbol{v})\right)$$

which is the same as

$$(1-\gamma)\mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le \mathcal{A}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})) \le (1+\gamma)\mathcal{B}((\boldsymbol{z},\boldsymbol{v}),(\boldsymbol{z},\boldsymbol{v})).$$

Thus (4.3) holds with the same constant $\gamma < 1$ as used in the estimate of Lemma 4.2. □

5. Numerical experiments

In this section we present a set of numerical tests that illustrate our theoretical results. We consider the SIPG discretization of the model problem (2.2) on the unit square in $\mathbb{R}^2$ with mixed boundary conditions. For the penalty parameters in (2.18) we choose the values $\alpha_0 = 4$ and $\alpha_1 = 1$. The coarsest mesh (at level 0) consists of eight triangles and is refined four times. Each refined mesh at level ℓ, $\ell = 1, 2, 3, 4$ is obtained by subdividing every triangle at level $(\ell - 1)$ into four congruent triangles. The CBS constants and the spectral condition numbers summarized in the tables below have been computed using MATLAB.

In Table 1 we list the values of the constant γ^2 in the inequality stated in Lemma 4.2 for different levels of refinement. Evidently, γ is uniformly bounded with respect to the mesh size and with respect to the material parameters, Young's modulus $\mathfrak{E}$ and Poisson ratio ν.

It can also be seen from Table 2 that even in the case of a jump in the Poisson ratio (on the coarsest mesh) the two subspaces $\boldsymbol{V}^{\mathrm{CR}}$ and $\boldsymbol{\mathcal{Z}}$ still remain nearly $\mathcal{A}$-orthogonal; In our experiment we set $\nu = \nu_1 = 0.3$ (and $\mathfrak{E} = \mathfrak{E}_1 = 1$) in the subdomain $\Omega_1 = [0, 0.5] \times [0, 0.5] \cup [0.5, 1] \times [0.5, 1]$, and varied $\nu = \nu_2$ ($\mathfrak{E}_2 = 1$) in the subdomain $\Omega_2 = \Omega \setminus \Omega_2$, respectively.

Next we consider an L-shaped domain $\Omega = [0, 1] \times [0, 1] \setminus (0.5, 1] \times (0.5, 1]$ with Neumann boundary conditions on the sides $y = 0$ and $y = 1$ and Dirichlet boundary conditions on the remaining part of the boundary. The initial triangulation (level 0) consists of 4 similar triangles. The angle is almost the same as for the square domain, see Table 3.

Furthermore, we computed the relative condition number of the preconditioner B corresponding to the bilinear form (4.1) for the model problem on the L-shaped domain. The results of this experiment confirm the uniform bound provided by Theorem 4.3, see Table 4

TABLE 2. Observed CBS constant γ^2 for $\Omega = (0,1)^2$ and jumps in ν.

γ^2	$\nu_2 = 0.3$	$\nu_2 = 0.4$	$\nu_2 = 0.49$	$\nu_2 = 0.499$	$\nu_2 = 0.49999$
$\ell = 1$	0.0451	0.0177	0.0442	0.0509	0.0517
$\ell = 2$	0.0460	0.0180	0.0689	0.0803	0.0816
$\ell = 3$	0.0464	0.0182	0.0689	0.0802	0.0816
$\ell = 4$	0.0466	0.0182	0.0689	0.0802	0.0816

TABLE 3. Observed CBS constant γ^2 for L-shaped domain.

γ^2	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.49999$
$\ell = 1$	0.0561	0.0202	0.0019	1.8918×10^{-4}	1.8906×10^{-6}
$\ell = 2$	0.0631	0.0233	0.0022	2.2118×10^{-4}	2.2106×10^{-6}
$\ell = 3$	0.0672	0.0252	0.0024	2.4216×10^{-4}	2.4207×10^{-6}
$\ell = 4$	0.0682	0.0257	0.0025	2.4810×10^{-4}	2.4801×10^{-6}

TABLE 4. Tabulated values of $\kappa(B^{-1}A)$ for L-shaped domain.

$\kappa(B^{-1}A)$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.49999$
$\ell = 1$	1.6204	1.3314	1.0912	1.0279	1.0028
$\ell = 2$	1.6713	1.3606	1.0990	1.0302	1.0030
$\ell = 3$	1.6997	1.3774	1.1037	1.0316	1.0031
$\ell = 4$	1.7073	1.3820	1.1050	1.0320	1.0032

TABLE 5. Values of $\kappa(A_{zz})$ for L-shaped domain.

$\kappa(A_{zz})$	$\nu = 0.25$	$\nu = 0.4$	$\nu = 0.49$	$\nu = 0.499$	$\nu = 0.49999$
$\ell = 1$	8.9067	7.1484	6.4788	6.4220	6.4158
$\ell = 2$	9.0875	7.1932	6.4829	6.4229	6.4164
$\ell = 3$	9.1577	7.2080	6.4841	6.4230	6.4164
$\ell = 4$	9.1794	7.2118	6.4844	6.4230	6.4164

Finally, we computed the condition number $\kappa(A_{zz})$ of the matrix A_{zz} related to the restriction of $\mathcal{A}(\cdot,\cdot)$ to the space $\boldsymbol{\mathcal{Z}}$, again for the model problem on the L-shaped domain. In view of Lemma 3.3 we already know that A_{zz} is well-conditioned, and this is clearly seen in Table 5 where the values of $\kappa(A_{zz})$ are listed.

APPENDIX A. AUXILIARY RESULTS

A.1. Bounds on the cardinality of $\mathcal{N}_1(E)$ and $\mathcal{N}_2(E)$

In the proof of the strengthened Cauchy−Schwarz inequality in Section 3.1 we needed several estimates on the cardinality of the sets $\mathcal{N}_0(E)$, $\mathcal{N}_1(E)$, and $\mathcal{N}_2(E)$, which have been defined in Section 3.1. These estimates are given in the proposition below. We remind the reader that we have $|\mathcal{N}_0(E)| \le 2$.

Proposition A.1. *The following inequalities hold:*

$$|\mathcal{N}_1(E)| \le (2d+1) \quad \textit{and} \quad |\mathcal{N}_2(E)| \le (2d+1)^2. \tag{A.1}$$

Proof. Let $E \in \mathcal{E}_h$ be fixed. To prove the bound on $|\mathcal{N}_1(E)|$ we consider the elements $T \in \mathcal{T}_h$, such that $E \in T$. In each such element T, there are exactly d faces $E' \in T$, $E' \neq E$. Since there are at most two elements $T \in \mathcal{T}_h$ containing E we have at most $2d$ faces $E' \in \mathcal{E}_h$ such that $E' \in \mathcal{N}_1(E)$, and $E' \neq E$. Adding E itself to the total count gives $|\mathcal{N}_1(E)| \leq (2d+1)$.

The second bound given in (A.1) follows from the first and the following inclusion:

$$\mathcal{N}_2(E) \subset \bigcup_{E' \in \mathcal{N}_1(E)} \mathcal{N}_1(E').$$

To show the above inclusion, we consider an arbitrary $E'' \in \mathcal{N}_2(E)$. By the definition of $\mathcal{N}_2(E)$, the intersection of $\mathcal{N}_1(E'')$ and $\mathcal{N}_1(E)$ is not empty. Equivalently, there exists $E' \in \mathcal{E}_h$ such that $E' \in \mathcal{N}_1(E'')$ and $E' \in \mathcal{N}_1(E)$. On the other hand, from the definition of $\mathcal{N}_1(E'')$, we have that $E' \in \mathcal{N}_1(E'')$ implies that $E'' \in \mathcal{N}_1(E')$, *i.e.*, if E' is a neighbor of E'', then E'' is a neighbor of E'.

Putting this together, we conclude that: if $E'' \in \mathcal{N}_2(E)$, then there exists $E' \in \mathcal{N}_1(E)$, such that $E'' \in \mathcal{N}_1(E')$, and this is exactly the inclusion we wanted to show.

To prove the desired bound is then straightforward:

$$\left| \bigcup_{E' \in \mathcal{N}_1(E)} \mathcal{N}_1(E') \right| \leq \sum_{E' \in \mathcal{N}_1(E)} |\mathcal{N}_1(E')| \leq \sum_{E' \in \mathcal{N}_1(E)} (2d+1) = (2d+1)|\mathcal{N}_1(E)| \leq (2d+1)^2. \qquad \square$$

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