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THE DISTRIBUTION OF THE ABSOLUTE VALUE OF THE RATIO OF TWO CORRELATED NORMAL RANDOM VARIABLES

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SUMMARY

The aim of this paper is to study the distribution of the absolute quotient of two Correlated Normal random variables (r.v.s). This study may have many applications, as often the researcher expects a ratio to be positive or considers the sign of the ratio unimportant. Taking into account a form of the distribution of the ratio of two Correlated Normal r.v.s proposed by Oksoy and Aroian in 1986 and further studied in 1994, we find the distribution of the absolute quotient of two correlated Normal r.v.s. The form proposed here is simple to compute, as it is a function of the $T(h,\lambda)$ *studied by Owen in 1956. We also represent the distribution of the absolute ratio as a function of the distribution of the Arctangent r.v.. Some examples regarding the confidence intervals for indicators used to evaluate the relative errors of the invoices reveals the importance of the distribution of the absolute ratio of two Correlated Normal r.v.s, here presented.*

Keywords: *Absolute Ratio of two Correlated Normal Random Variables, Owen's T function, Arctangent r.v., Confidence Intervals for Indicators.*

1. INTRODUCTION

Often we are interested studying the distribution of a ratio the sign of which is irrelevant and assumes only positive values. In industrial practice, measurements are frequently recorded without their algebraic sign in order to add negative values to positive values. The present study may be useful to the researcher interested in the distribution of the ratio of two measures of this kind.

The distribution of the ratio has been studied by many authors. In particular, we mention Geary (1930), Fieller (1932), Hinkley (1969), Oksoy and Aroian (1986), Marsaglia (2006), Galeone (2007) and Galeone and Pollastri (2012). The distribution of the ratio has been used for the construction of confidence intervals (indicated also by CI), for the cost-effectiveness ratio proposed by Galeone and Pollastri (2013). The study of the ratio of two Correlated Normal r.v.s is worthly because in many situations its distribution is skewed and so the approximation to the Normal distribution is not convenient. Moreover the confidence intervals based on the distribution are always bounded contrary to the widely used parametric method proposed by Fieller (1932, 1954).

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Kim (2006) considered the ratio of two independent Folded Normal r.v.s. But often the researcher has to deal situations in which the variable in the numerator is correlated with the variable in the denominator. In this paper we want to study this situation.

Taking into account the result of Oksoy and Aroian (1994) regarding the Cumulative Distribution Function (CDF) of the ratio of two Correlated Normal r.v.s, we propose a way of formulating the CDF for the absolute value of the ratio that is simple and very easy to compute.

We consider some examples to underline the importance of considering the correlation coefficient. We underline also the difference in the limits of the confidence interval of a ratio if we include the knowledge of the positive value of the ratio.

2. DERIVATION OF THE CUMULATIVE DISTRIBUTION FUNCTION

Let us consider a Bivariate Correlated Normal r.v.
$(Y, X) \sim N(\mu_Y, \mu_X, \sigma_Y^2, \sigma_X^2, \rho)$.

The r.v. $W = \dfrac{Y}{X}$ has the CDF (Oksoy and Aroian, 1986) given by

$$F_W(w) = L\left(\frac{a - bt_w}{\sqrt{1+t_w^2}}, -b, \frac{t_w}{\sqrt{1+t_w^2}}\right) + L\left(\frac{bt_w - a}{\sqrt{1+t_w^2}}, b, \frac{t_w}{\sqrt{1+t_w^2}}\right),$$

where$w \in \Re$,

$$a = \sqrt{\frac{1}{1-\rho^2}}\left(\frac{\mu_Y}{\sigma_Y} - \rho\frac{\mu_X}{\sigma_X}\right), b = \left(\frac{\mu_X}{\sigma_X}\right), t_w = \sqrt{\frac{1}{1-\rho^2}}\left(\frac{\sigma_X}{\sigma_Y}w - \rho\right),$$

and $L(h, k, \rho)$ is the bivariate normal integral according to Kotz, Balakrishnan, Johnson (2000).

An alternative formula for $F_W(w)$ involving the $V(h, q)$ function of Nicholson (1943) or the $T(h, \lambda)$ function of Owen (1956) is (see Oksoy and Aroian, 1994)

$$F_W(w) = \frac{1}{2} + \frac{1}{\pi}\text{ arctan }(t_w) + 2V\left\{\frac{bt_w - a}{\sqrt{1+t_w^2}}, \frac{b + at_w}{\sqrt{1+t_w^2}}\right\} - 2V(b, a)$$

where

$$V(h, q) = \int_0^h \int_0^y \Phi(x)\Phi(y)dxdy, \quad y = \frac{q}{h}x,\ T(h, \lambda) = \frac{1}{2\pi}\text{ arctan }\lambda - V(h, \lambda h).$$

It is easy to find that

$$F_W(w) = \frac{1}{2} + \frac{1}{\pi} \arctan(t_w) + \frac{1}{\pi} \arctan\left(\frac{b + at_w}{bt_w - a}\right) +$$

$$- 2T\left(\frac{bt_w - a}{\sqrt{1 + t_w^2}}, \frac{b + at_w}{bt_w - a}\right) - \frac{1}{\pi} \arctan\left(\frac{a}{b}\right) + 2T\left(b, \frac{a}{b}\right)$$

Now we consider the CDF of the r.v. $V = \left|\frac{Y}{X}\right|$:

$$F_{(v)} = P\left\{\frac{|Y|}{|X|} < v\right\} = P\left\{-v < \frac{Y}{X} < v\right\} = P\left\{\frac{Y}{X} < v\right\} - P\left\{\frac{Y}{X} < -v\right\} =$$

$$= F_W(v) - F_W(-v) =$$

$$= \frac{1}{2} + \frac{1}{\pi} \arctan(t_v) + 2V\left\{\frac{bt_v - a}{\sqrt{1 + t_v^2}}, \frac{b + at_v}{\sqrt{1 + t_v^2}}\right\} +$$

$$- 2V(b, a) - \frac{1}{2} - \frac{1}{\pi} \arctan(t_{-v}) +$$

$$- 2V\left\{\frac{bt_{-v} - a}{\sqrt{1 + t_{-v}^2}}, \frac{b + at_{-v}}{\sqrt{1 + t_{-v}^2}}\right\} + 2V(b, a) =$$

$$= \frac{1}{\pi}[\arctan(t_v) - \arctan(t_{-v})] +$$

$$+ 2V\left\{\frac{bt_v - a}{\sqrt{1 + t_v^2}}, \frac{b + at_v}{\sqrt{1 + t_v^2}}\right\} - 2V\left\{\frac{bt_{-v} - a}{\sqrt{1 + t_{-v}^2}}, \frac{b + at_{-v}}{\sqrt{1 + t_{-v}^2}}\right\} =$$

$$= \frac{1}{\pi}[\arctan(t_v) - \arctan(t_{-v})] +$$

$$+ \frac{1}{\pi}\left[\arctan\left(\frac{b + at_v}{bt_v - a}\right) - \arctan\left(\frac{b + at_{-v}}{bt_{-v} - a}\right)\right] + \quad (1)$$

$$- 2T\left(\frac{bt_v - a}{\sqrt{1 + t_v^2}}, \frac{b + at_v}{bt_v - a}\right) + 2T\left(\frac{bt_{-v} - a}{\sqrt{1 + t_{-v}^2}}, \frac{b + at_{-v}}{bt_{-v} - a}\right)$$

In the case that $t_v t_{-v} > -1$ and $\left(\frac{b + at_v}{bt_v - a}\right)\left(\frac{b + at_{-v}}{bt_{-v} - a}\right) > -1$, remembering the formula

$$\arctan(x_1) - \arctan(x_2) = \arctan\left(\frac{x_1 - x_2}{1 + x_1 x_2}\right),$$

we obtain

$$F_V(v) = \frac{1}{\pi}\left[\arctan\left(\frac{t_v - t_{-v}}{1 + t_v t_{-v}}\right)\right] + \frac{1}{\pi}\left[\arctan\left(\frac{t_{-v} - t_v}{1 + t_v t_{-v}}\right)\right] +$$

$$- 2T\left(\frac{bt_v - a}{\sqrt{1 + t_v^2}}, \frac{b + at_v}{bt_v - a}\right) + 2T\left(\frac{bt_{-v} - a}{\sqrt{1 + t_{-v}^2}}, \frac{b + at_{-v}}{bt_{-v} - a}\right)$$

$$= 2T\left(\frac{bt_{-v} - a}{\sqrt{1 + t_{-v}^2}}, \frac{b + at_{-v}}{bt_{-v} - a}\right) - 2T\left(\frac{bt_v - a}{\sqrt{1 + t_v^2}}, \frac{b + at_v}{bt_v - a}\right).$$

The above formula is very efficient from the computational point of view and it can be easily implemented in R.

The R code to compute the CDF and the quantiles of V is reported in De Capitani and Pollastri (2012).

Note that because

$$F_W(u) \geq F_V(u) \qquad \text{for all } u$$

we can assert that the r.v. V first-order stochastically dominates the r.v. W.

As a consequence, the relation between the quantiles of the r.v. W and the r.v. V is

$$F_V^{-1}(p) \leq F_W^{-1}(p).$$

3. RELATIONSHIP BETWEEN THE CDF OF V AND THE CDF OF THE ARCTANGENT RANDOM VARIABLE

The Arctangent distribution was proposed by Zenga in (1979). In 2004 Pollastri and Tornaghi studied the properties of the distribution. This distribution has been utilized to obtain the simultaneous confidence regions and for testing hypotheses regarding the probabilities of a multinomial distribution (Zenga and Fedrizzi, 1981; Brunazzo, 1979; Brunazzo and Fedrizzi, 1980; Pollastri, 1979, 1980, 1982).

Recalling the CDF of the Arctangent r.v.

$$F * (h; \lambda) = 1 - 2\pi \frac{T(h, \lambda)}{\arctan(\lambda)}$$

from 1) we obtain

$$F_V(v) = \frac{1}{\pi}[\arctan(t_v) - \arctan(t_{-v})] +$$

$$+ \frac{1}{\pi}\left[\arctan\left(\frac{b + at_v}{bt_v - a}\right) - \arctan\left(\frac{b + at_{-v}}{bt_{-v} - a}\right)\right] +$$

$$- 2\left\{\frac{1}{2\pi}\arctan\left(\frac{b + at_v}{bt_v - a}\right)\left[1 - F^*\left(\frac{bt_v - a}{\sqrt{1 + t_v^2}}, \frac{b + at_v}{bt_v - a}\right)\right]\right\} +$$

$$+2\left\{\frac{1}{2\pi}\arctan\left(\frac{b+at_{-v}}{bt_{-v}-a}\right)\left[1-F^*\left(\frac{bt_{-v}-a}{\sqrt{1+t_{-v}^2}},\frac{b+at_{-v}}{bt_{-v}-a}\right)\right]\right\}=$$

$$=\frac{1}{\pi}[\arctan(t_v)-\arctan(t_{-v})]+$$

$$+\frac{1}{\pi}\arctan\left(\frac{b+at_v}{bt_v-a}\right)F^*\left(\frac{bt_v-a}{\sqrt{1+t_v^2}};\frac{b+at_v}{bt_v-a}\right)+ \qquad (2)$$

$$-\frac{1}{\pi}\arctan\left(\frac{b+at_{-v}}{bt_{-v}-a}\right)F^*\left(\frac{bt_{-v}-a}{\sqrt{1+t_{-v}^2}};\frac{b+at_{-v}}{bt_{-v}-a}\right)$$

Therefore, we can assert that the CDF of the absolute ratio of two Correlated Normal variables is a function of the CDF of the Arctangent r.v.

The result may help in finding the characteristics of the Distribution of the r.v. V.

4. SOME APPLICATIONS

In order to consider the importance of studying the distribution of the absolute value of the ratio of two Correlated Normal r.v.s., we consider a proposal for building confidence intervals because the interval estimation generally is not considered. Then we consider some applications in order to show the importance of considering the relevance of the correlation coefficient between the variable of the numerator and the one of the denominator. Then we expose other applications to underline the importance of the absolute value of the ratio with respect to the ratio if the researcher is not interested in the sign of the ratio.

4.1 *Proposal of confidence intervals for an absolute ratio*

The CDF of V, indicated as $F_V(v)$, may be estimated by substituting in it the Maximum Likelihood (ML) estimates of $\mu_Y, \mu_X, \sigma_Y^2, \sigma_X^2$ and ρ. The ML estimates of the means μ_y and μ_x are indicated respectively by $\bar{y}$ and $\bar{x}$. The ML estimates of σ_y^2, σ_x^2 and ρ are respectively given by

$$s_y^2=\sum_{i=1}^{n}(y_i-\bar{y})^2/n, \quad s_x^2=\sum_{i=1}^{n}(x_i-\bar{x})^2/n, \quad r=\frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i-\bar{x})^2\sum_{i=1}^{n}(y_i-\bar{y})^2}}.$$

We can estimate the CDF of$V$ as follows:

$$\hat{F}_V(v)=2T\left(\frac{\hat{b}t_{-v}-\hat{a}}{\sqrt{1+t_{-v}^2}},\frac{\hat{b}+\hat{a}t_{-v}}{\hat{b}t_{-v}-\hat{a}}\right)-2T\left(\frac{\hat{b}t_v-\hat{a}}{\sqrt{1+t_v^2}},\frac{\hat{b}+\hat{a}t_v}{\hat{b}t_v-\hat{a}}\right)$$

where

$$\hat{a} = \sqrt{\frac{n}{1-r^2}}\left(\frac{\bar{y}}{s_y} - r\frac{\bar{x}}{s_x}\right), \quad \hat{b} = \sqrt{n}\left(\frac{\bar{x}}{s_x}\right), \quad t_v = \sqrt{\frac{1}{1-r^2}}\left(\frac{s_x}{s_y}w - r\right).$$

We propose to use the following confidence interval for the indicator at level approximately equal to $(1-\alpha)$

$$\left(\hat{F}_V^{-1}(\alpha/2), \hat{F}_V^{-1}(1-\alpha/2)\right).$$

4.2 *First set of examples*

First of all we consider a comparison between the distribution of the ratio and the one of the absolute ratio.

We consider also the comparison between the CI when the sample size increases.

Let us suppose that we wish to build the confidence interval of an indicator (e.g. the ratio of the production of a crop per hectare in two different regions or in two different periods of time). Let us select two random samples in the two different situations and let us estimate the numerator and the denominator of the indicator. The two estimators are asymptotically normally distributed.

Let us consider the comparison of the confidence interval when we take into account that the indicator is positive because we are not interested in the sign and when we do not consider the constraint of positiveness.

Let us suppose that the numerator and the denominator of the estimator of the indicator are distributed as follows:

$$(Y,X) \sim N(\bar{y} = 0.8, \bar{x} = 0.4, s_Y^2 = 0.1, s_X^2 = 0.1, r = 0.7)$$

Table 1 reports the limits $\left(\hat{F}_W^{-1}(0.05), \hat{F}_W^{-1}(0.95)\right)$ for the ratio and $\left(\hat{F}_V^{-1}(0.05), \hat{F}_V^{-1}(0.95)\right)$ for the absolute ratio and the size of the confidence interval at level $(1-\alpha) = 0.90$ in the situation we are examining.

TABLE 1. - *Comparison of the limits of CI for an indicator using the distribution of the ratio and of the absolute ratio*

	Lower limit	Upper limit	width
Ratio	-0.35	0.99	1.34
Absolute ratio	0.09	1.09	1.00

Note that, if we take into account the information that the indicator must be positive because the sign of the ratio is irrelevant, given that the *p*-quantile of *V* is less than or equal to the *p*-quantile of *W*, as underlined in Section 2, the confidence interval shifts to the right and, in the present situation, the size decreases. This fact is evident observing Figure 1 which reports the CDF of the ratio r.v. and of the absolute ratio r.v.

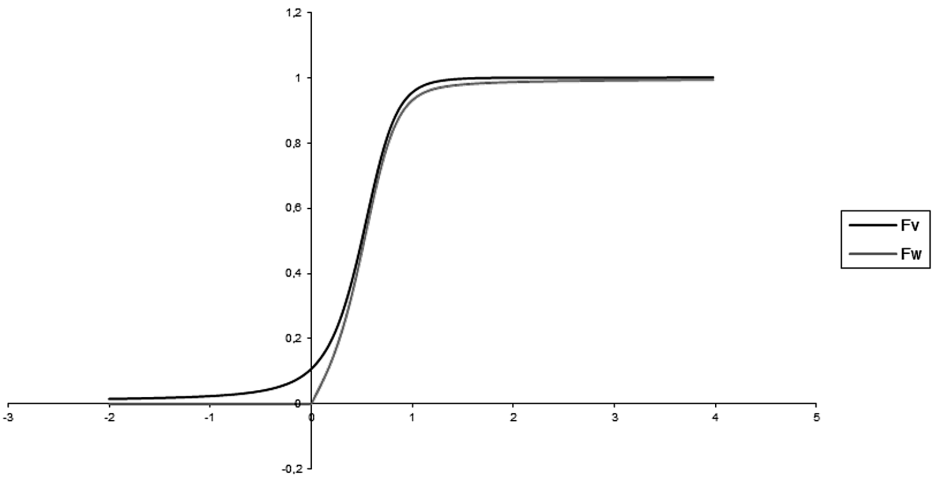

FIGURE 1. - *Comparison of the CDF of the ratio and of the absolute ratio*

4.3 *Second set of examples*

In the present work we consider the distribution of the absolute ratio of two correlated normal r.v. This is a new proposal because, as underlined in the introduction, Kim in 2006 studied the ratio of two independent Folded Normal random variables. So we want to give some examples in order to show that it is important to consider the correlation between the r.v. of the numerator and the one of the denominator of the absolute ratio of two Correlated Normal r.v.s.

In a firm we want to evaluate the skill of the employees in invoicing through a ratio in which the difference between the real amount and the value recorded is at the numerator and the amount of the invoice is at the denominator.

Let us suppose we select a sample of $n = 25$ invoices and we observe the amount of error Y and the value of the invoice X where both the variables are expressed in thousands of €.

In the present situation the estimator of the numerator and the denominator is distributed as

$$(\bar{Y}, \bar{X}) \sim N(\bar{y} = 0.05, \bar{x} = 1, s^2_{\bar{Y}} = 0.0002, s^2_{\bar{X}} = 0.005, r).$$

TABLE 2. - *Limits and length of confidence intervals having fixed* $(1 - \alpha) = 0.90$

r	$\hat{F}_V^{-1}(0.05)$	$\hat{F}_V^{-1}(0.95)$	length
0.0	0.01605	0.08385	0.0678
0.5	0.0165	0.0797	0.0632
0.7	0.01728	0.078	0.0607

Table 2 shows that the length of the CI decreases when the value of the correlation coefficient r increases.

TABLE 3. - *Limits and length of confidence intervals having fixed* $(1-\alpha)=0.95$

R	$\hat{F}_V^{-1}(0.025)$	$\hat{F}_V^{-1}(0.975)$	length
0.0	0.00975	0.09055	0.0808
0.5	0.01035	0.08525	0.0749
0.7	0.01065	0.08305	0.0724

Table 3 shows the limits and the length of the CI for different values of the correlation coefficient r, having set $(1-\alpha)=0.95$.

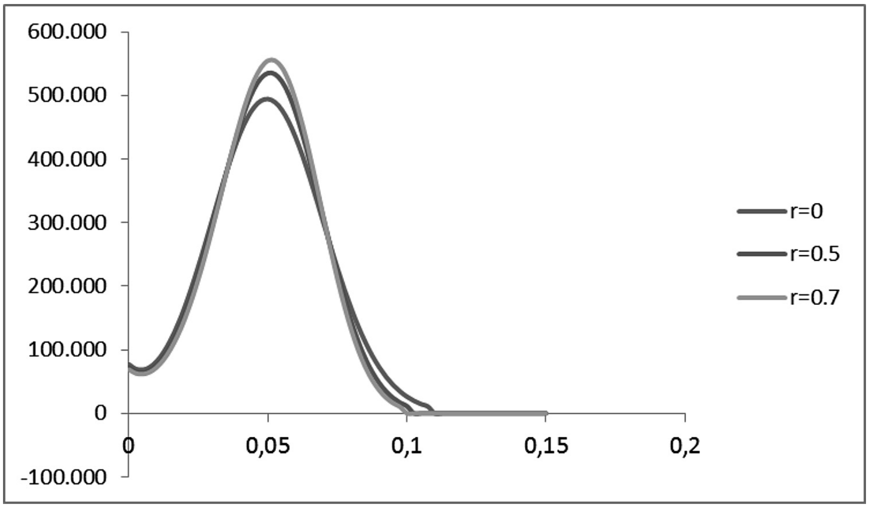

FIGURE 2. - *Comparison of density functions for different values of r when the sample size is* $n=25$

Now we consider the same example when the sample size is $n=75$.

TABLE 4. - *Limits and lengths of the CI for relative error ratio for different values of r when* $(1-\alpha)=0.90$

r	$\hat{F}_V^{-1}(0.05)$	$\hat{F}_V^{-1}(0.95)$	length
0.0	0.0309	0.0694	0.0385
0.5	0.0319	0.0674	0.0355
0.7	0.03235	0.06655	0.0342

TABLE 5. - *Limits and lengths of the CI for relative error ratio for different values of r when* $(1-\alpha)=0.95$

r	$\hat{F}_V^{-1}(0.025)$	$\hat{F}_V^{-1}(0.975)$	length
0.0	0.02725	0.07315	0.0459
0.5	0.02825	0.07065	0.0424
0.7	0.02865	0.06965	0.0410

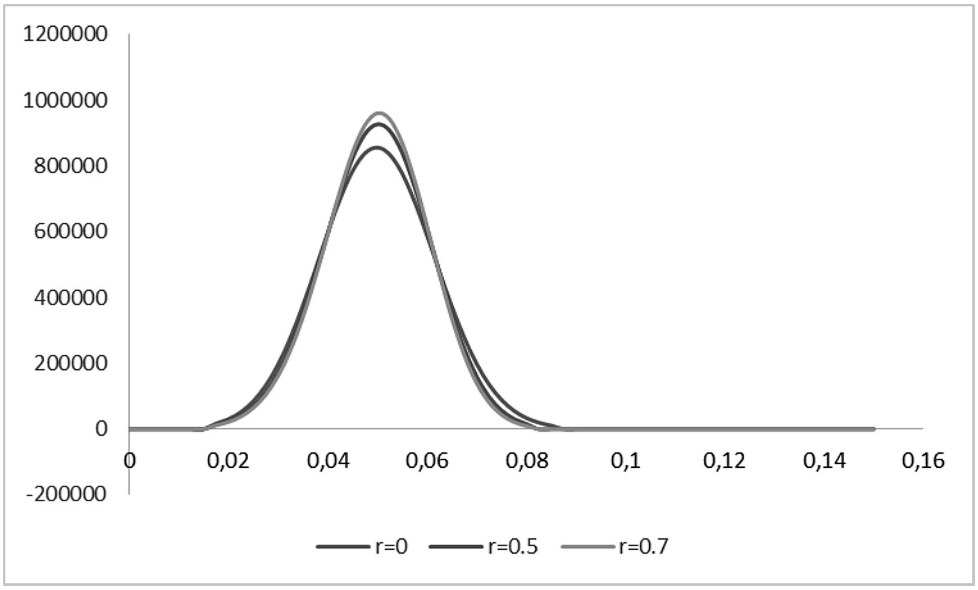

FIGURE 3. - *Comparison of density functions for different values of r when the sample size is* $n=75$

Considering the density functions when $n=25$ and the density functions when $n=75$, it looks evident that the skewness decreases when n increases. Moreover the distribution is more concentrated on the true value of the absolute ratio when n increases.

4.4 *Third set of examples*

Now we will consider the situation of the previous set of examples but where the mean of the difference between the real amount and the value recorded, indicated by μ_Y, is estimated equal to $\bar{y}=0.01$. In Figures 4 and 5 we show the density functions of the ratio and the absolute ratio respectively when $n=25$ and $n=75$.

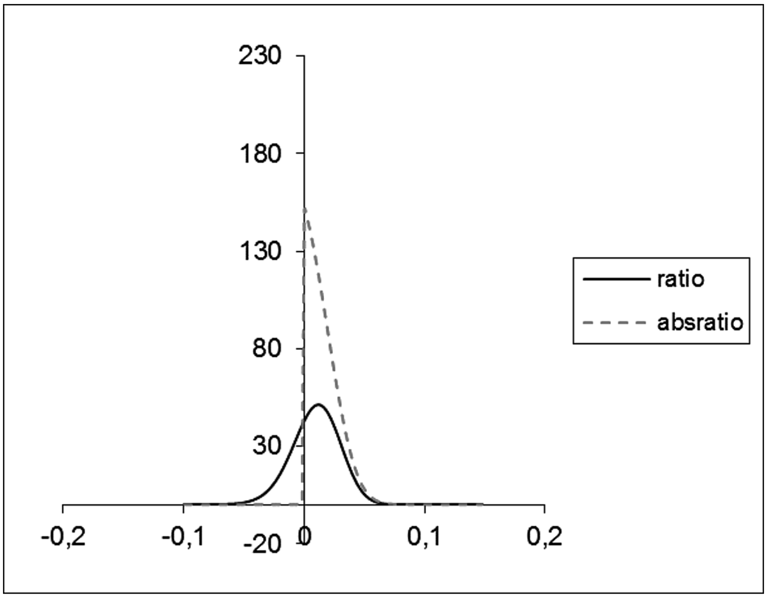

FIGURE 4. - *Density functions of the ratio and of the absolute ratio when* $n = 25$ *and* $r = 0.7$

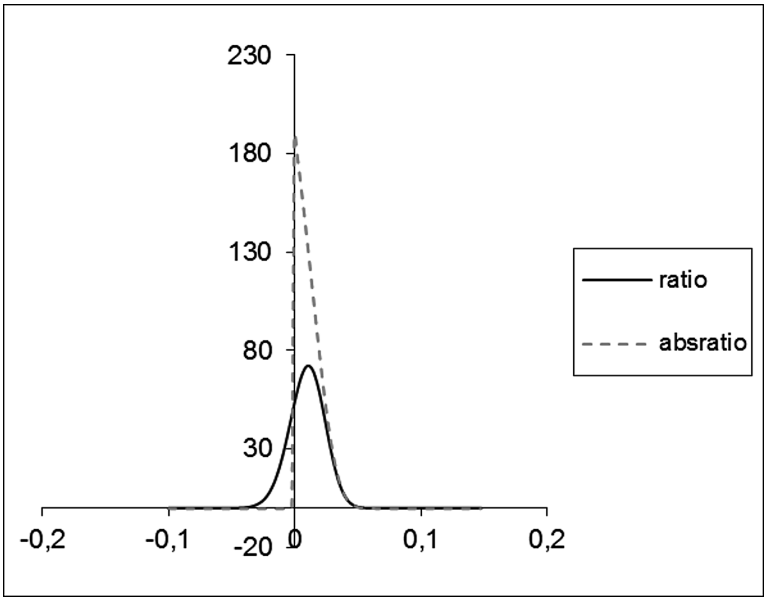

FIGURE 5. - *Density functions of the ratio and of the absolute ratio when* $n = 50$ *and* $r = 0.7$

In Tables 6 and 7 are reported the limits of the CI having fixed $(1-\alpha) = 0.95$ for the two sample sizes considered.

TABLE 6. - *Limits and lengths of the CI for relative error ratio when* $n = 25$ *and* $(1-\alpha) = 0.95$

	$\hat{F}_V^{-1}(0.025)$	$\hat{F}_V^{-1}(0.975)$	length
Ratio	-0.0805	0.0733	0.1538
Absolute Ratio	0.0118	0.0853	0.0735

TABLE 7. - *Limits and lengths of the CI for relative error ratio when* $n = 50$ *and* $(1-\alpha) = 0.95$

	$\hat{F}_V^{-1}(0.025)$	$\hat{F}_V^{-1}(0.975)$	length
Ratio	-0.0721	0.0667	0.1388
Absolute Ratio	0.0012	0.0350	0.0338

This example underlines the importance of using the Absolute Ratio distribution instead of the Ratio distribution especially when there is a consistent part of the density function regarding the negative values of v.

Moreover, increasing the size of the sample, the density function is more concentrated on the real value of the ratio and, as a consequence, the length of the CI decreases significantly.

5. CONCLUSIONS

The absolute ratio is important in the study of indices in many practical situations. Here we propose a simple way of computing the CDF of the absolute value of the ratio of two Correlated Normal variables.

Through the theory and the examples, we show that, if the researcher is not interested in the sign of ratio, it is more convenient to use the distribution of the absolute ratio instead of the distribution of the ratio.

When the variable of the numerator and the one at the denominator of the ratio are correlated, using the distribution proposed in the present paper, we obtain confidence intervals whose length is shorter than in the situation in which the variables are considered independent.

Moreover, when the sample size increases, the skewness decreases and the length of the confidence interval reduces.

The examples reported underline the importance of the distribution proposed and studied here when it is necessary to build confidence intervals for only positive indicators.

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