## THE NUMERICAL DETERMINATION OF SPECTRAL RADII

Scope: find a non trivial, non empty set such that, Thm J2.3 holds.

**Prop J2.5** [Démidovitch & Maron, 1979, Ch. XII, § 11]

$$r_\sigma[RL^{(L)}] \approx r_\sigma{}^{(M)}[RL^{(L)}] := \left|\frac{\mathrm{Tr}[RL^{(L)}]^{M+1}}{\mathrm{Tr}[RL^{(L)}]^{M}}\right|, \ M \text{ large.}$$

(Stop @ $M$ such that, $|r_\sigma{}^{(M+1)} - r_\sigma{}^{(M)}| < 10^{-7}$).

**Remarks**

♦ If $\Gamma$ is a disk, then $r_\sigma[RL^{(L)}] = \max\limits_{0 \le l \le L} |RL^{(L)}{}_{plpl}|$, where

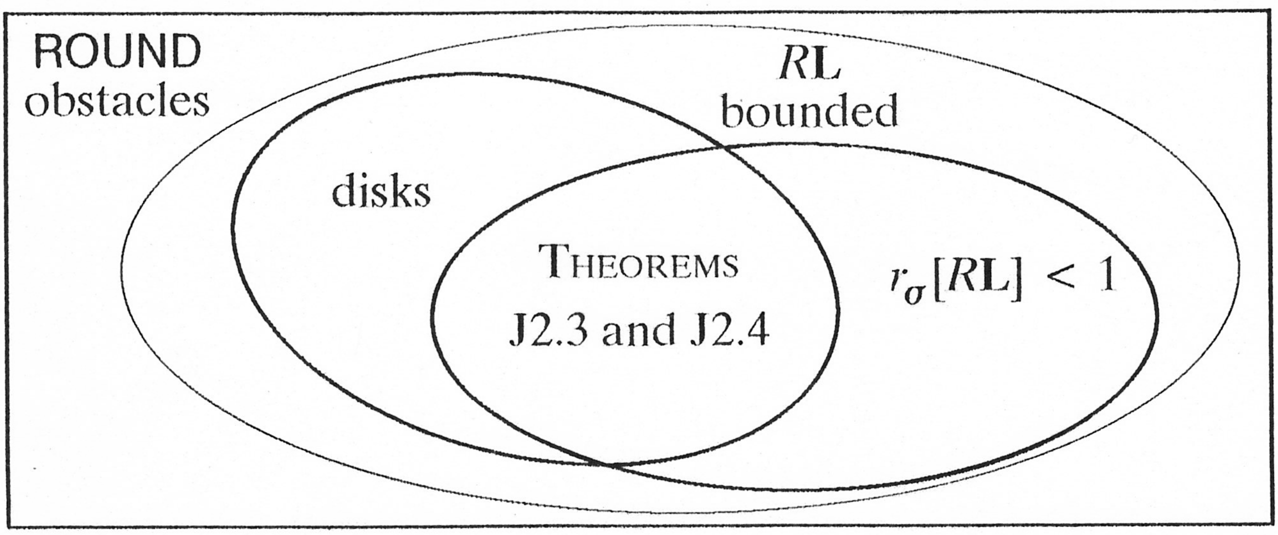
$$RL^{(L)}{}_{plpl} = -\frac{i}{2}\pi k R\, J_l[kR] \frac{d}{dz} H_l^{(1)}[kR],\ p = 0, 1.$$

♦ $r_\sigma[RL^{(L)}]$ depends on $\{ L, k, \vec{\psi} \}$, **not** on $\hat{\mathbf{k}}$ .

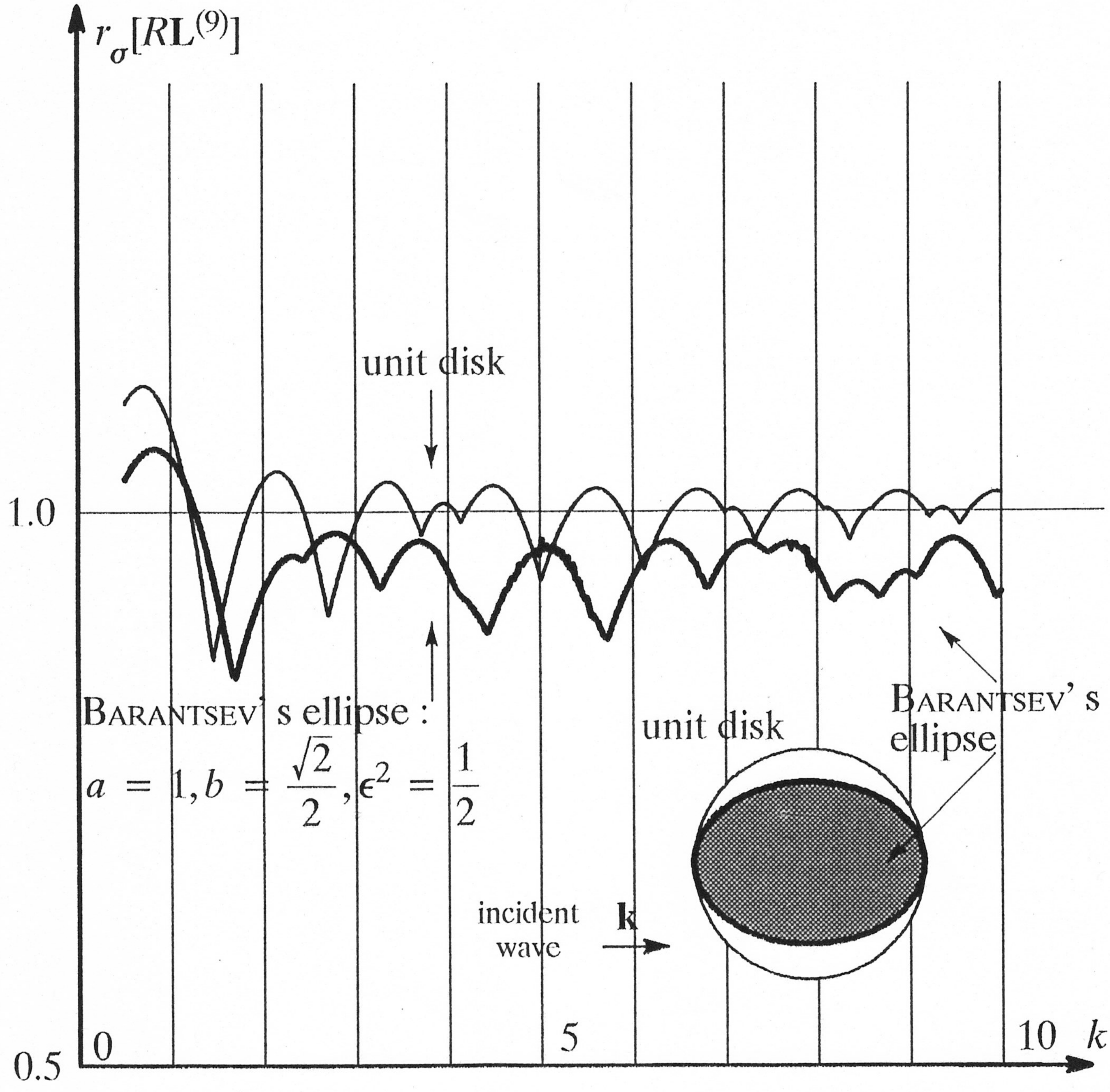
♦ When $\Gamma = S_R^{n-1}$, there is no contradiction between the consistency property (Thm J2.1) and the result $r_\sigma[RL^{(L)}] > 1$ for some $\{L, kR\}$. Namely, consistency is verified directly and does not involve any iterative scheme.

♠ Thms J2.3 and J2.4 only provide a sufficient condition for the convergence of forward propagation.

The relationships known to date among the conditions on $RL$.



# COMPUTED SPECTRAL RADII




Unit disk: direct calculation.
BARANTSEV' s ellipse: by PROP J2.5.
$10 < M[k] < 2000$, strongly affected by $k$.

## THE TRUSTED METHOD INEQUALITY

Recall: $\mathbf{c}^{(L)}$ = initial value ; $\mathbf{p}^{(L)}$ = first iterate.

**COR J2.6** (to THM J2.4)

*Fix L , then* $\left\|\mathbf{c}^{(L)} - \bar{\mathbf{c}}^{(L)}\right\| < \left\|\mathbf{p}^{(L)} - \bar{\mathbf{c}}^{(L)}\right\|$ . (Norms in $\mathbb{C}^{\,\mathrm{card}[\Lambda(L)]}$).

**DEF** A method is <u>trusted</u> with respect to **f** if it returns $\bar{\mathbf{c}}^{(L)}$ such that, given $\epsilon > 0$,

$$|f_\lambda - \bar{c}_\lambda^{(L)}| < \epsilon\, |\bar{c}_\lambda^{(L)}|,\ \forall \lambda \in \Lambda(L).$$

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**PROP J2.7** (a sufficient condition for the effectiveness of forward propagation)

*<u>If</u> $\exists\, \epsilon > 0$, small such that, $\forall \lambda \in \Lambda(L)$*

*a)* $\quad |f_\lambda - \bar{c}_\lambda^{(L)}| < \epsilon\, |\bar{c}_\lambda^{(L)}|$ <u>*and*</u>

*b)* $\quad |p_\lambda^{(L)} - \bar{c}_\lambda^{(L)}| + 2\epsilon\, |\bar{c}_\lambda^{(L)}| < |c_\lambda^{(L)} - \bar{c}_\lambda^{(L)}|$

*<u>then</u> ALS forward propagation is effective i.e.,*

$$|p_\lambda^{(L)} - f_\lambda| < |c_\lambda^{(L)} - f_\lambda|.$$

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**REMARKS**

- ♠ COR J2.6 $\not\Rightarrow |c_\lambda^{(L)} - \bar{c}_\lambda^{(L)}| < |p_\lambda^{(L)} - \bar{c}_\lambda^{(L)}|$ , $\lambda \in \Lambda(L)$.
- ♠ The effectiveness of **ALS** remains an open problem.

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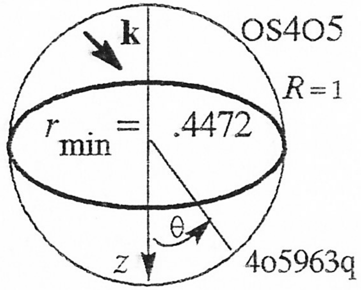
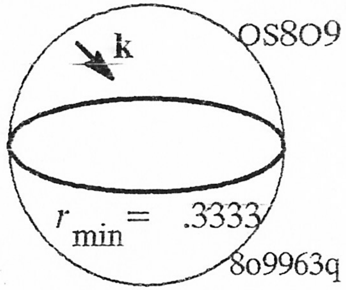
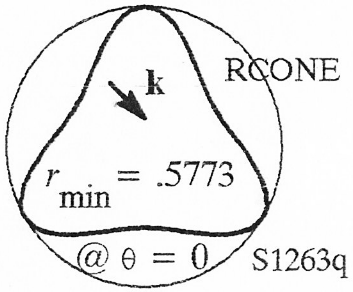
**PROP J2.8** (How small shall $\epsilon$ be for a method to be trusted ?)

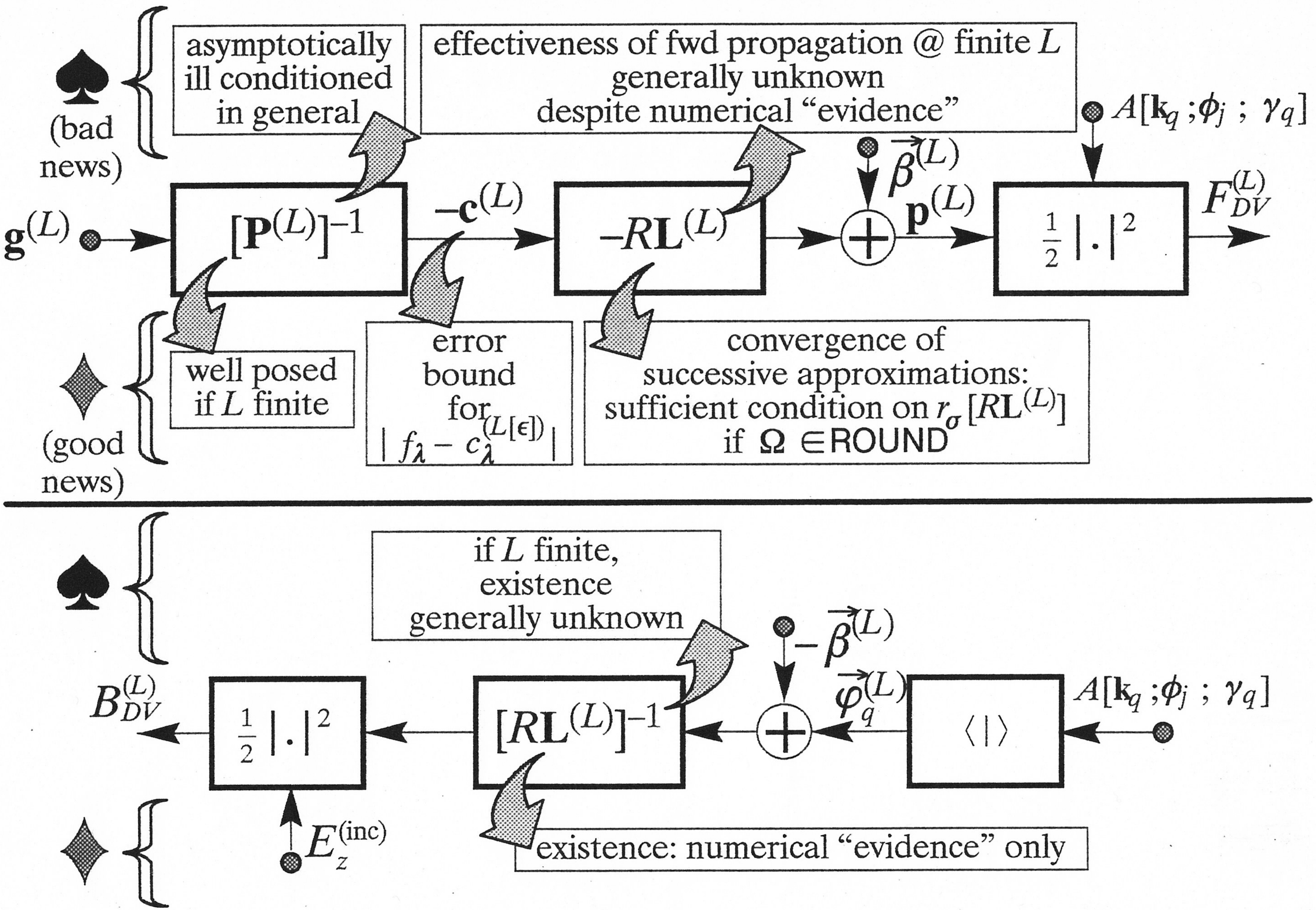
*<u>If</u> **f** unknown, <u>then</u> a method, which returns $\bar{\mathbf{c}}^{(L)}$, is a candidate for trust provided it is $\epsilon$–accurate, where*

$$\epsilon < \max_{\lambda \in \Lambda(L)} \frac{|\bar{c}_\lambda^{(L)} - c_\lambda^{(L)}| - |\bar{c}_\lambda^{(L)} - p_\lambda^{(L)}|}{2|\bar{c}_\lambda^{(L)}|}\,.$$

# Error Analysis of Forward Propagators : May $\vec{\varphi}^{(L)}$ Come from a *Trusted Method* ?

| Obstacle | $\mathbf{k}, L, n_p$ | $B^{(L)}_{\min}$ | $B^{(L)}[\vec{\varphi}^{(L)}]$ | $B^{(L)}[\mathbf{p}^{(L)}]$ | $\epsilon_1$ | $F^2_1$ | $F^2_p$ |
| --- | --- | --- | --- | --- | --- | --- | --- |
|   | $\hat{\mathbf{q}}$, 9, 63 | .681–**1** | .26+**3** | .537+**2** | <u>.40–**11**</u> | .21–**2** | .81–**3** |
|   | $\hat{\mathbf{q}}$, 9, 63 | .27+**0** | .94+**5** | .15+**5** | <u>.45–**12**</u> | .10–**1** | .61–**2** |
|   | $\hat{\mathbf{q}}$, 12, 63 | .23–**1** | .15+**2** | .44+**1** | <u>.50–**4**</u> | .68–**3** | .26–**3** |

# FORWARD AND BACKWARD PROPAGATION: WHY DO THEY WORK ? OR, WHY SHOULDN' T THEY ?



# MORE QUESTIONS AND OPEN PROBLEMS

Q1 How many IPSWICH data sets are needed to imply the uniqueness (if any) of the unknown shape ?

Q2 (approximation) In view of THM J1.2, are cylindrical wave functions the appropriate basis ?

Q3 (approximation) What is the role of approximation and parameterization orders, $\{ I, L \}$, in uniqueness and stability ?

Q4 Is $\vec{\gamma}$ the proper unknown to represent phase corrections of the IPSWICH data ?

Q5 Are conjugate directions the best descent rule ?

Q6 How will other backpropagation methods e.g., the $\mathbf{W}^{(L)}$ method [1994, 1995] perform ?

Q7 How will penalty methods (ANGELL–KLEINMAN–ROACH [1987], RAMM [1994]) compare ?