Part Two

THE $X_1^{(L)}$ – LEAST SQUARES BOUNDARY COEFFICIENTS

Thm J1.1

I) $\forall\ \epsilon > 0$, $\exists$ *at least an approximation order* $L[\epsilon]$ *and a vector of coefficients* $\mathbf{c}^{(L[\epsilon])}$ *such that* $B_{DV}^{(L[\epsilon])} < \epsilon^2$.

II) *Select one such* $L[\epsilon]$. *Let* $\varrho >> r_{\max}$ *and* $C[\Gamma, \varrho, k]$ *be a quantity, which depends on* Γ, ϱ *and* k. *Then the corresponding* $\mathbf{c}^{(L[\epsilon])}$ *is related to the far zone coefficients,* $\{f_\lambda \mid \lambda \in \Lambda[L[\epsilon]]\}$ *by the error bound*

$$|f_\lambda - c_\lambda^{(L[\epsilon])}|^2 < \frac{k^2\epsilon^2}{\pi} C[\Gamma, \varrho, k],\ \forall \lambda \in \Lambda(L[\epsilon]).$$

Remarks.

♠ In general, the determination of $\mathbf{c}^{(L)}$ is asymptotically ill posed, because $\mathbf{P}^{(L)}$ is asymptotically ill conditioned [Ramm, 1986].

♠ The dependence of $L[\epsilon]$ on ϵ has not yet been investigated.

Def An obstacle is of Rayleigh class if $\sum_\lambda f_\lambda v_\lambda[\mathbf{x}]$ converges, $\forall \mathbf{x} \in \Gamma$.

Thm J1.2 [Barantsev *et al.*, 1971].

Ellipses, $r[\phi] = \dfrac{1}{\sqrt{1-\varepsilon^2\cos^2\phi}}$, *of eccentricity* $\varepsilon < \dfrac{1}{\sqrt{2}}$ *are Rayleigh obstacles and*

$$f_\lambda \approx \frac{C l^{-\gamma}}{\sqrt{2\pi l}} \left[\frac{ek}{2l}\right]^l \left[\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right]^l \text{ as } l \to \infty, \text{ where } \gamma > 0.$$

Proof : by asymptotics of $H_l^{(1)}$ and J_l, singular integral equations, and saddle point method.

... MORE ABOUT RAYLEIGH OBSTACLES ...

THM J1.3 [MILLAR, 1973]. *On the boundary of a RAYLEIGH obstacle*

$$\lim_{L \to \infty} |f_\lambda - c_\lambda^{(L)}| = 0, \forall \lambda \in \Lambda(L).$$

DEF (ROUND obstacle). $\Omega \in$ RAYLEIGHObstacleUniformNormalDerivative class if $\sum_\lambda f_\lambda \partial_N v_\lambda[\mathbf{x}]$ converges <u>uniformly</u> to $\partial_N E_z^{(sc)}|_\Gamma$, $\forall \mathbf{x} \in \Gamma$.

THM J1.4. *Ellipses, such that* $\varepsilon < \frac{1}{\sqrt{2}}$ *are a class of ROUND obstacles.*

Proof: Denote $\zeta := kr[\phi]$. Let $r[\phi] \geq 1$.

Since $\frac{\mathrm{d}H_l^{(1)}(\zeta)}{\mathrm{d}\zeta} f_\lambda \approx \mathrm{i}\frac{Cl^{-\gamma}}{kr\pi}\left[\frac{\varepsilon}{r\sqrt{1-\varepsilon^2}}\right]^l$ as $l \to \infty$, then $\sum_\lambda f_\lambda \partial_N v_\lambda[\mathbf{x}]$ converges uniformly on Γ.

Moreover [KLEINMAN, ROACH, STRØM, 1984] the infinite dimensional matrix $\mathbf{L} := -\frac{\mathrm{i}}{4}\left[\langle v_\lambda|_\Gamma \, \partial_N v_\mu \rangle\right]$ is invertible and therefore

$$\sum_\lambda f_\lambda \partial_N v_\lambda|_\Gamma = \partial_N E_z^{(sc)}|_\Gamma . \qquad \blacksquare$$

THE $X_1^{(L)}$ ALS (FWD PROPAGATED) COEFFICIENTS: CONSISTENCY AND OTHER PROPERTIES

THM J2.1 (consistency)

If $\Gamma = S_R^{n-1}$ i.e., Ω is a disk ($n = 2$) or a sphere ($n = 3$) of radius R,

then the $X_1^{(L)}$ ALS scheme is consistent i.e.,

I) $c_\lambda^{(L)}$ *and* $p_\lambda^{(L)}$ *are independent of* L

II) $\forall L \geq 0 : c_\lambda^{(L)} = f_\lambda = p_\lambda^{(L)}, \ \forall \lambda \in \Lambda(L)$.

Proof: I) orthogonality of $\{v_\lambda\} \Rightarrow$ independence of L.
II) uniform convergence of $\sum_\lambda f_\lambda \, \partial_N v_\lambda$ on $S_R^{n-1} \Rightarrow$ direct verification. ■

PROP J2.2 (properties of $RL^{(L)}$ and RL).

I) *Let $n = 2$ (or $= 3$) and kR = constant, then*

$$\lim_{m \to \infty} \left| -\frac{i}{4} \langle u_{p,m} |_\Gamma \, \partial_N v_{p,m} \rangle \right| = \frac{1}{2}, p = 0, 1.$$

II) [KLEINMAN, ROACH, STRØM, 1984]
If Ω is RAYLEIGH, $k^2 \notin \sigma[-\Delta_D]$ and $L = \infty$, then $\exists [RL]^{-1}$.

Proof: I) use the asymptotics of of $H_l^{(1)}$ and J_l.
II) By the properties of complete families and dual bases in BANACH spaces. ■

$\Gamma \equiv \partial\Omega$ $\qquad$ $\underline{b}^{(L)} \equiv \vec{\mathcal{B}}^{(L)}$

THE $X_1^{(L)}$ ALS SCHEME AND SUCCESSIVE APPROXIMATIONS

DEF: $\Omega \in$ RAYLEIGHObstacleUniformNormalDerivative class if $\sum_\lambda f_\lambda\, \partial_N\, v_\lambda[\mathbf{x}]$ converges <u>uniformly</u> to $\partial_N E_z^{(sc)}|_\Gamma$, $\forall \mathbf{x} \in \Gamma$.

THM J2.3 (successive approximations in the infinite dimensional case). *Let* $\Omega \in$ ROUND. *Assume* $\mathbf{f} \in \ell_2$ *and* RL: $\ell_2 \to \ell_2$ *is bounded. If* $r_\sigma[RL] < 1$, *then,* $\forall\ \mathbf{b} \in \ell_2$, *there exists a unique solution,* $\mathbf{f}$, *to*

$$\mathbf{f} = \mathbf{b} + RL\,.\,\mathbf{f},$$

obtained by successive approximations, where t *is the iteration index i.e.,*

$$\mathbf{p}^{[t+1]} = \mathbf{b} + RL\,.\,\mathbf{p}^{[t]}\,, t = 0, 1, 2, \ldots$$

started with an arbitrary $\mathbf{p}^{[0]} \in \ell_2$.

Proof: $\Omega \in$ ROUND $\Rightarrow f_\lambda = -\frac{i}{4}\langle u_\lambda|_\Gamma\, \partial_N E_z^{(inc)} + \sum_\mu f_\mu\, \partial_N v_\mu\rangle$; then

recall the contraction mapping THEOREM.

THM J2.4 (infinite *vs.* finite dimensional propagators)
<u>*Let*</u> $\Omega \in$ ROUND, $\mathbf{f}$, $\mathbf{b}$ *and* RL *as in* THM *J2.3.* <u>*Assume*</u>

$$r_\sigma[RL^{(L)}] < 1\,, \forall\, L \text{ and } r_\sigma[RL] < 1$$

<u>*then*</u> *the following hold*

I) $\forall L\ \exists!$ *fixed point* $\bar{\mathbf{c}}^{(L)}$ *such that*

$$\bar{\mathbf{c}}^{(L)} = \mathbf{b}^{(L)} + RL^{(L)}.\bar{\mathbf{c}}^{(L)} \text{ i.e., } \bar{\mathbf{c}}^{(L)} = [\,\mathbf{1}^{(L)} - RL^{(L)}\,]^{-1}.\mathbf{b}^{(L)}$$

II) $\lim_{L\to\infty} \bar{\mathbf{c}}^{(L)} = \mathbf{f}$.

III) $\lim_{L\to\infty} \mathbf{p}^{(L)} = \mathbf{f}$.

Proof : Since $\mathbf{b}^{(L)} \to \mathbf{b}$, $\mathbf{1}^{(L)} - RL^{(L)} \to \mathbf{1} - RL$ and $[\mathbf{1}^{(L)} - RL^{(L)}]^{-1} \to [\mathbf{1} - RL]^{-1}$ then $\bar{\mathbf{c}}^{(L)} \to \mathbf{f}$ (convergence of the projection method).