Simple Undirected Graphs as Formal Contexts*

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Abstract. The adjacency matrix of a graph is interpreted as a formal context. Then, the counterpart of Formal Concept Analysis (FCA) tools are introduced in graph theory. Moreover, a formal context is seen as a Boolean information table, the structure at the basis of Rough Set Theory (RST). Hence, we also apply RST tools to graphs. The peculiarity of the graph case, put in evidence and studied in the paper, is that both FCA and RST are based on a (different) binary relation between objects.

1 Introduction

The aim of this work is to define a framework that enables us to apply Formal Concept Analysis (FCA) tools, and to some extent also Rough Set Theory (RST) tools, to graphs. In order to do so, we will view the adjacency matrix of a graph as a formal context (Boolean Information Table in case of RST). It is well known that RST and FCA are similar but complementary disciplines that can be integrated in several ways. A key difference between the two theories is the binary relation on which they are based, in the RST case it is a relation between objects and in the FCA case between objects and properties. However, in this particular framework the two theories are even closer, since objects coincide with attributes. The two relations remain different but they can be interpreted in the same setting, understanding their complementarity. We will consider not only the standard operators: formal concepts in FCA and lower/upper approximations in RST but a more general framework arising from the theory of oppositions [5].

The relationship between graphs and FCA is not new, however it is has not yet been clearly outlined and developed. The paper [10] defines a bipartite graph from a formal context and proves that (X, Y) is a concept iff $X \cup Y$ is a maximal bi-clique of the corresponding graph. The same result is mentioned briefly in [6]. Here, we work in the other direction: starting from a general graph, we use the adjacency matrix to define a formal context. Then, we show that concepts coincide with bipartitions of the maximal bi-cliques (see Theorem 3.1). This

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result is also mentioned in [13], but with no formalization nor proof. We focus then on complete and complete bipartite graphs studying their concept lattice. In Section 3.3 some considerations on other Galois connections than the standard one are given. Section 4 is devoted to rough sets: we study the partition and the approximations that can be introduced on a given graph.

2 Preliminary Notions

The basic notions of Graph Theory, Formal Concept Analysis and Rough Set Theory are recalled.

2.1 Graphs

We denote by $G = (V(G), E(G))$ a finite simple (i.e. no loops and no multiple edges are allowed) undirected graph, with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. If $v, v' \in V(G)$, we will write $v \sim v'$ if $\{v, v'\} \in E(G)$ and $v \nsim v'$ otherwise. We denote by $Adj(G)$ the adjacency matrix of G. We recall that $Adj(G)$ is a $n \times n$ matrix (a_{ij}) such that $a_{ij} := 1$ if $v_i \sim v_j$ and $a_{ij} := 0$ otherwise. If $v \in V(G)$, we set

$$N_G(v) := \{w \in V(G) : \{v, w\} \in E(G)\}.$$

$N_G(v)$ is usually called *neighborhood* of v in G. Graph of particular interest for our discussion will be complete and bipartite ones.

Definition 2.1. *The* complete graph *on n vertices, denoted by K_n, is the graph with vertex set $\{v_1, \ldots, v_n\}$ and such that $\{v_i, v_j\}$ is an edge, for each pair of indexes $i \neq j$.*

Definition 2.2. *A graph $B = (V(B), E(B))$ is said* bipartite *if there exist two non-empty subsets B_1 and B_2 of $V(B)$ such that $B_1 \cap B_2 = \emptyset$, $B_1 \cup B_2 = V(B)$ and $E(B) \subseteq \{\{x, y\} : x \in B_1,\, y \in B_2\}$. In this case the pair (B_1, B_2) is called a* bipartition *of B and we write $B = (B_1|B_2)$. It is said that $B = (B_1|B_2)$ is a* complete bipartite graph *if $E(B) = \{\{x, y\} : x \in B_1,\, y \in B_2\}$. If p and q are two positive integers and $B_1 = \{x_1, \ldots, x_p\}$, $B_2 = \{y_1, \ldots, y_q\}$, we denote by $K_{p,q}$ the complete bipartite graph having bipartition (B_1, B_2).*

Definition 2.3. *A* biclique *B of G is a complete bipartite subgraph of G. We say that a biclique $B = (B_1|B_2)$ of G is* maximal *if for any biclique $B' = (B_1'|B_2')$ of G such that $B_1 \subseteq B_1'$ and $B_2 \subseteq B_2'$ it results that $B_1 = B_1'$ and $B_2 = B_2'$.*

2.2 Formal Concept Analysis

We start by recalling the general definition of formal contexts and their basic properties (see [9]).

Definition 2.4. *A* Formal Context *is a triple* $\mathbb{K} = (Z, M, \mathcal{R})$*, where* Z *and* M *are sets and* $\mathcal{R} \subseteq Z \times M$ *is the binary relation involving them. The elements of* Z *and* M *are called* objects *and* attributes *(or* properties*) respectively. We write* $g\mathcal{R}m$ *instead of* $(g, m) \in \mathcal{R}$*. If* $O \subseteq Z$ *and* $Q \subseteq M$*, we set*

$$O^{\uparrow} := \{m \in M : (\forall g \in O)\, g\mathcal{R}m\} \subseteq M$$

and

$$Q^{\downarrow} := \{g \in Z : (\forall m \in Q)\, g\mathcal{R}m\} \subseteq Z.$$

In this way the following two mappings are defined: $^{\uparrow} : \mathcal{P}(Z) \to \mathcal{P}(M)$, $O \mapsto O^{\uparrow}$ and $^{\downarrow} : \mathcal{P}(M) \to \mathcal{P}(Z)$, $Q \mapsto Q^{\downarrow}$. By suitable compositions of these two mappings we are able to construct the two new mappings $^{*} : \mathcal{P}(Z) \to \mathcal{P}(Z)$, $O \mapsto O^{\uparrow\downarrow}$ and $^{\diamond} : \mathcal{P}(M) \to \mathcal{P}(M)$, $Q \mapsto Q^{\downarrow\uparrow}$, which are closure operators on, respectively, $\mathcal{P}(Z)$ and $\mathcal{P}(M)$ [9].

Definition 2.5. *A* concept *of the Formal Context* $\mathbb{K} = (Z, M, \mathcal{R})$ *is a pair* (O, Q)*, where* $O \subseteq Z$*,* $Q \subseteq M$*,* $O^{\uparrow} = Q$ *and* $Q^{\downarrow} = O$*. If* (O, Q) *is a concept,* O *is called* extent *of* (O, Q) *and* Q *is called* intent *of* (O, Q)*. We denote by* $\mathfrak{B}(\mathbb{K})$ *the set of all the concepts of the Formal Context* $\mathbb{K}$*.*

If (O_1, Q_1) and (O_2, Q_2) are two concepts in $\mathfrak{B}(\mathbb{K})$, it is usual to consider the relation $(O_1, Q_1) \sqsubseteq (O_2, Q_2)$ if and only if $O_1 \subseteq O_2$ (that is equivalent to $Q_1 \supseteq Q_2$). Then $\sqsubseteq$ is a partial order on $\mathfrak{B}(\mathbb{K})$ and $(\mathfrak{B}(\mathbb{K}), \sqsubseteq)$ is a complete lattice, called *concept lattice* (or also *Galois lattice*) of the Formal Context $\mathbb{K}$, whose meet and join operations on an arbitrary family of formal concepts $\{(O_\alpha, Q_\alpha) : \alpha \in A\}$ are the following:

$$\bigwedge_{\alpha \in A} (O_\alpha, Q_\alpha) = \Big(\bigcap_{\alpha \in A} O_\alpha, (\bigcup_{\alpha \in A} Q_\alpha)^{\diamond}\Big)$$
$$\bigvee_{\alpha \in A} (O_\alpha, Q_\alpha) = \Big((\bigcup_{\alpha \in A} O_\alpha)^{*}, \bigcap_{\alpha \in A} Q_\alpha\Big)$$

2.3 Rough Set Theory

In the context of RST a table representing a formal context is named Boolean information table (or Boolean information system). More formally, a *Boolean information table* is a structure $\mathcal{I} = \langle U, Att, Val, F\rangle$, where U (called *universe set*) is a non empty set of *objects*, Att (called *attribute set*) is a non empty set of *attributes*, $Val = \{0, 1\}$ is called the *value set* (in the general case it is not assumed to be Boolean) and $F : U \times Att \to Val$ (called *information map*) is an application from the direct product $U \times Att$ into the value set Val.

If $A \subseteq Att$, it is usual to consider the binary relation I_A on the universe set U defined as follows: if $u, u' \in U$ then

$$uI_Au' \iff F(a, u) = F(a, u'), \forall a \in A. \tag{1}$$

The binary relation I_A is an equivalence relation on U and it is called *A-indiscernibility relation.* If $u \in U$, we denote by $[u]_A$ the equivalence class of u with respect to I_A. We also set $\pi_A(\mathcal{I}) := \{[u]_A : u \in U\}$ and we call $\pi_A(\mathcal{I})$ the A-indiscernibility partition of the information system $\mathcal{I}$.

Definition 2.6. *Let $\mathcal{I} = \langle U, Att, Val, F\rangle$ be an information table, $A \subseteq Att$ and $Y \subseteq U$. The A-*lower approximation *of Y is the following subset of U:*

$$\mathbf{l}_A(Y) := \{x \in U : [x]_A \subseteq Y\} = \bigcup \{C \in \pi_A(\mathcal{I}) : C \subseteq Y\}.$$

The A-upper approximation of Y is defined as:

$$\mathbf{u}_A(Y) := \{x \in U : [x]_A \cap Y \neq \emptyset\} = \bigcup \{C \in \pi_A(\mathcal{I}) : C \cap Y \neq \emptyset\}.$$

*The subset Y is called A-*exact *if and only if $\mathbf{l}_A(Y) = \mathbf{u}_A(Y)$ and A-*rough *otherwise.*

The lower approximation represents the elements that *certainly*, with respect to our knowledge expressed by A, belongs to Y. On the other hand, the upper approximation is the set of objects *possibly* belonging to A.

We will denote by $\mathbb{CO}_A(\mathcal{I})$ the set of all the A-exact subsets. The following result is well known (where $\hat{s} = \{1, 2, \ldots s\}$).

Proposition 2.1. *(i) If $\pi_A(\mathcal{I})$ contains exactly s elements (i.e. equivalence classes), then $\mathbb{CO}_A(\mathcal{I})$ is a Boolean algebra isomorphic to $\langle \mathcal{P}(\hat{s}), \subseteq, \cap, \cup, ^c, \emptyset, \hat{s}\rangle$. (ii) More specifically, a non-empty subset Y of the universe U is A-exact if and only if Y is a set theoretical union of blocks of the set-partition $\pi_A(\mathcal{I})$.*

2.4 The Cube of Oppositions

Starting from a binary relation $R \subseteq X \times Y$ and generalizing the Aristotelian square of oppositions, it is possible to define a cube of oppositions [8]. Given a subset $S \subseteq Y$, the eight vertices of the cube are defined by $R(S) = \{x \in X | \exists s \in S, xRs\}$ and all the interaction of three kinds of negation: the complement on X, on Y and the negation of the relation R. More in detail, let us assume that R and its negation $\overline{R}$ ($x\overline{R}y$ if and only if $\neg(xRy)$) are both not empty and serial, and define $xR = \{y \in Y | xRy\}$. Then, we can obtain from R four vertices, that form a classical square of oppositions (in what follows $\overline{S} := Y \setminus S$):

(I) $R(S) = \{x \in X | \exists s \in S, xRs\} = \{x \in X | S \cap xR \neq \emptyset\}$
(O) $R(\overline{S}) = \{x \in X | \exists s \in \overline{S}, xRs\}$
(E) $\overline{R(S)} = \{x \in X | \forall s \in S, \neg(xRs)\}$
(A) $\overline{R(\overline{S})} = \{x \in X | \forall s \in \overline{S}, \neg(xRs)\} = \{x \in X | xR \subseteq S\}$

We remark that E and A are the complement of I and O, respectively, and that A is a subset of I and E a subset of O. The other four corners are obtained using the complementary relation $\overline{R}$:

(o) $\overline{R}(S) = \{x \in X | \exists s \in S, \neg(xRs)\}$
(i) $\overline{R}(\overline{S}) = \{x \in X | \exists s \in \overline{S}, \neg(xRs)\} = \{x \in X | S \cup xR \neq Y\}$
(a) $\overline{\overline{R}(S)} = \{x \in X | \forall s \in S, xRs\} = \{x \in X | S \subseteq xR\}$
(e) $\overline{\overline{R}(\overline{S})} = \{x \in X | \forall s \in \overline{S}, xRs\}$

All these sets can have a nice interpretation both in FCA and RST [5]. In the case of FCA, R is the standard relation $\mathcal{R}$ defining a formal context, and in the case of RST, R is the indiscernibility relation I_A, hence it is defined on the same domain $X \times X$. As we will discuss, in our particular case, also for FCA we have $X = Y$, so both relations are defined on $X \times X$ even if they are not the same relation.

3 Simple Undirected Graphs viewed as Formal Contexts

We begin now the study of the finite simple undirected graphs as particular types of formal contexts.

Definition 3.1. *Let $G = (V(G), E(G))$ be a finite simple undirected graph, with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. We call* Formal Context of the graph G *the Formal Context $\mathbb{K}[G] := (V(G), V(G), \mathcal{R}_G)$, where $v\mathcal{R}_G v'$ if and only if $\{v, v'\} \in E(G)$ for all $v, v' \in V(G)$.*

Hence the object subset and the attribute subset of the Formal Context $\mathbb{K}[G]$ are both equal to the vertex set $V(G)$, whereas the binary relation which defines this formal context is exactly the incidence relation between vertices of the graph G. Let us also note that, since the graph G is undirected, the relation $\mathcal{R}_G$ is symmetric.

3.1 Concepts of a Graph

Given the above considerations, in the Formal Context $\mathbb{K}[G]$ induced by a simple undirected graph G, the maps $^{\uparrow} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ and $^{\downarrow} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ are coincident. Therefore in the sequel we denote with the same symbol $'$ the map $' : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ such that $O \mapsto O' := O^{\uparrow} = O^{\downarrow}$, when O is any vertex subset of G. This implies obviously that also the two operators $^{*} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ and $^{\diamond} : \mathcal{P}(V(G)) \to \mathcal{P}(V(G))$ coincide. Therefore in the sequel we set $O \mapsto O'' := O^{*} = O^{\diamond}$, for all $O \subseteq V(G)$.

Let us now see how O' and O'' are defined in terms of neighborhood of vertices.

Proposition 3.1. *If $O \subseteq V(G)$ then*

$$O' = \bigcap_{v \in O} N_G(v) = \{w \in V(G) : O \subseteq N_G(w)\} \tag{2}$$

Proof. We have that

$$O' := \{w \in V(G) : (\forall v \in O) v \mathcal{R}_G w\} = \{w \in V(G) : (\forall v \in O) w \in N_G(v)\},$$

that is $O' = \bigcap_{v \in O} N_G(v)$. For the other set equality, if $w \in V(G)$ is such that $O \subseteq N_G(w)$ and $v \in O$, then $w \in N_G(w)$, therefore $w \in \bigcap_{v \in O} N_G(v)$. On the other hand, if $w \in \bigcap_{v \in O} N_G(v)$ and v_0 is an arbitrary vertex in O then $w \in N_G(v_0)$. Hence $v_0 \in N_G(w)$, and this shows that $O \subseteq N_G(w)$. $\square$

Remark 3.1. The identity in (2) is valid also when the subset $O = \emptyset$. In fact, in this case, we always have $\emptyset^{\uparrow} = M$ and $\emptyset^{\downarrow} = Z$, that is $O' = O^{\uparrow} = O^{\downarrow} = V(G)$ in the formal context $\mathbb{K}[G]$. On the other hand, it is usual (in elementary set theory) to interpret the intersection $\bigcap_{v \in O} N_G(v)$ as coincident with the whole set $V(G)$ when O is the empty set.

Corollary 3.1. *If $O \subseteq V(G)$ then*

$$O' \subseteq V(G) \setminus O \tag{3}$$

Proof. If $w \in O'$ and $w \in O$, by (2) it follows that $w \in N_G(w)$, i.e. $\{w, w\} \in E(G)$, but this contradicts the hypothesis that G is a simple graph. This proves (3). $\square$

Remark 3.2. If G is a finite simple undirected graph, a vertex subset $O \subseteq V(G)$ is the extent [intent] of some concept of the Formal Context $\mathbb{K}[G]$ if and only if $O'' = O$. In this case, both the pairs (O, O') and (O', O) are concepts of $\mathbb{K}[G]$. Moreover, since G has no loops, the cross table of the Formal Context $\mathbb{K}[G]$ (that is, the adjacency matrix of G) has zeroes in all its diagonal places, and this obviously implies that $V(G)' = \emptyset$. Hence both the pairs $(\emptyset, V(G))$ and $(V(G), \emptyset)$ are always concepts of the Formal Context $\mathbb{K}[G]$.

We re-interpret now the notion of concept in the case of the formal context $\mathbb{K}[G]$. Recalling the definition of biclique of a graph (see definition 2.3), we have then the following characterization.

Theorem 3.1. *Let O and Q be two subsets of $V(G)$. Then, the pair (O, Q) is a concept in $\mathbb{K}[G]$ if and only if (O, Q) is a bipartition of some maximal biclique of G. On the other hand, if $B = (B_1|B_2)$ is a maximal biclique of G, then the pair (B_1, B_2) is a concept in $\mathbb{K}[G]$. Hence the concepts in $\mathbb{K}[G]$ are exactly the bipartitions of the maximal bicliques of G.*

Proof. Let (O, Q) be a concept in $\mathbb{K}[G]$. By definition of concept we have in this case that:

$$O' := \{v \in V(G) : (\forall u \in O)\, u \sim v\} = Q$$

and

$$Q' := \{u \in V(G) : (\forall v \in Q)\, u \sim v\} = O.$$

Since G has no loops, the subsets O and Q are disjoint. Moreover, if $u \in O$ and $v \in Q$, then $u \sim v$. Thus $(O|Q)$ is a biclique of G.

Let $B = (B_1|B_2)$ be a biclique of G such that $O \subseteq B_1$ and $Q \subseteq B_2$. By definition of bipartite graph, if $u \in O \subseteq B_1$ and $v \in B_2$, $u \sim v$, then $B_2 \subseteq O' = Q$ and thus $B_2 = Q$. Similarly if $v \in Q \subseteq B_2$ and $u \in B_1$, $u \sim v$, then $B_1 \subseteq Q' = O$ and thus $B_1 = O$. It follows that $(O|Q)$ is a maximal biclique of G.

Let now $(O|Q)$ be a maximal biclique of G. Then, by definition of biclique, $Q \subseteq O'$ and $O' \subseteq Q$. Moreover we have:

$$O'' := \{u \in V(G) : (\forall v \in O')\, u \sim v\}.$$

It follows that $O \subseteq O''$ and that $(O''|O')$ is a biclique of G. Then, by maximality of $(O|Q)$, we obtain that $Q = O'$ and $O = O'' = Q'$, so (O, Q) is a concept in $\mathbb{K}[G]$. □

One of the consequences of this result is the possibility to apply algorithms developed for formal concept generation [11] to improve results to compute maximal bicliques on graphs [1]. We leave this comparison to a future study.

When $G = K_n$ is a complete graph, the context coincide with the contranominal scale $(V(G), V(G), \neq)$ [9], hence we obtain that the map $'$ behaves as the set complement.

Proposition 3.2. *If $G = K_n$ and $O \subseteq V(G)$ we have that $O' = V(G) \setminus O$ and $O'' = O$.*

In the case of a complete bipartite graph $G = K_{p,q}$ we obtain:

Proposition 3.3. *If $G = K_{p,q} = (B_1|B_2)$ and O is a non-empty subset of $V(G)$ then*

$$O' = \begin{cases} B_1 & \text{if } \; O \subseteq B_2 \\ B_2 & \text{if } \; O \subseteq B_1 \\ \emptyset & \text{otherwise} \end{cases} \tag{4}$$

and

$$O'' = \begin{cases} B_1 & \text{if } \; O \subseteq B_1 \\ B_2 & \text{if } \; O \subseteq B_2 \\ V(G) & \text{otherwise} \end{cases} \tag{5}$$

Proof. Let $B_1 = \{x_1, \ldots, x_p\}$ and $B_2 = \{y_1, \ldots, y_q\}$. By definition of $K_{p,q}$ we have that $N_G(x_i) = B_2$ for $i = 1, \ldots, p$ and $N_G(y_j) = B_1$ for $j = 1, \ldots, q$. Therefore, if $O \subseteq B_2$, then $\bigcap_{v \in O} N_G(v) = B_1$, hence $O' = B_1$ by (2). Analogously if $O \subseteq B_1$. Finally, we assume that $x_i \in O$, for some $i = 1, \ldots, p$, and also $y_j \in O$, for some $j = 1, \ldots, q$. Then, by (2) it follows that $O' \subseteq N_G(x_i) \cap N_G(y_j) = B_2 \cap B_1 = \emptyset$ since $B_1|B_2$ is a set-partition of the vertex set of G. This proves (4). On the other hand, if $O \subseteq B_1$, by (4) we deduce that $O' = B_2$, therefore $O'' = (O')' = B_2' = B_1$ again by (4). Analogously, we obtain $O'' = B_2$ if $O \subseteq B_2$. Finally, if $O \cap B_1 \neq \emptyset$ and $O \cap B_2 \neq \emptyset$, by (4) we have that $O' = \emptyset$, hence $O'' = (\emptyset)' = V(G)$. This proves (5). □

3.2 The Concept Lattice of a Graph

We explicitly introduce now the notion of concept lattice for a finite simple undirected graph.

Definition 3.2. *We call* concept lattice *(or also* Galois lattice*) of the graph G the concept lattice of the Formal Context $\mathbb{K}[G]$ and we denote it simply by $(\mathfrak{B}(G), \sqsubseteq)$ instead of $(\mathfrak{B}(\mathbb{K}[G]), \sqsubseteq)$.*

At first let us recall some basic notions about posets. If $P = (X, \leq)$ is a partially ordered set (briefly *poset*), we can consider the usual *dual poset* of P, that is the poset $P^* = (X, \leq^*)$, where $\leq^*$ is the partial order on X defined by $x \leq^* y :\iff y \leq x$, for all $x, y \in X$. A poset $P = (X_1, \leq_1)$ is said *isomorphic* to another poset $P_2 = (X_2, \leq_2)$ if there exists a bijective map $\phi : X_1 \to X_2$ such that $x \leq_1 y \iff \phi(x) \leq_2 \phi(y)$, for all $x, y \in X_1$. A poset P is called *self-dual* if P is isomorphic to its dual poset P^*.

Then, the following basic result about concept lattices of a graph holds.

Proposition 3.4. *Let G be a finite simple undirected graph. Then the concept lattice $(\mathfrak{B}(G), \sqsubseteq)$ is self-dual.*

Proof. By Remark 3.2 we know that a pair $(O, Q) \in \mathcal{P}(V(G)) \times \mathcal{P}(V(G))$ is a concept if and only if also (Q, O) is a concept, that is, $(O, Q) \in \mathfrak{B}(G)$ if and only if $(Q, O) \in \mathfrak{B}(G)$. We define then the map $\phi : \mathfrak{B}(G) \to \mathfrak{B}(G)$ such that $\phi((O, Q)) := (Q, O)$. Obviously the map ϕ is surjective, therefore, since the set $\mathfrak{B}(G)$ is finite, it is also bijective. Finally, if (O_1, Q_1) and (O_2, Q_2) are any two concepts in $\mathfrak{B}(G)$, by definition of the partial order $\sqsubseteq$ and definition of dual order $\sqsubseteq^*$ we have that

$$(O_1, Q_1) \sqsubseteq (O_2, Q_2) \iff (Q_2, O_2) \sqsubseteq (Q_1, O_1) \iff \phi((O_1, Q_1)) \sqsubseteq^* \phi((O_2, Q_2))$$

Hence the map ϕ is an order-isomorphism between the concept lattice $(\mathfrak{B}(G), \sqsubseteq)$ and its dual lattice $(\mathfrak{B}(G), \sqsubseteq^*)$. ☐

In the next result we determine the concept lattice when G is the complete graph K_n.

Proposition 3.5. *If $n \geq 1$ then $\mathfrak{B}(K_n) = \{(O, O^c) : O \subseteq V(K_n)\}$ and $(\mathfrak{B}(K_n), \sqsubseteq) \cong (\mathcal{P}(V(K_n)), \subseteq)$.*

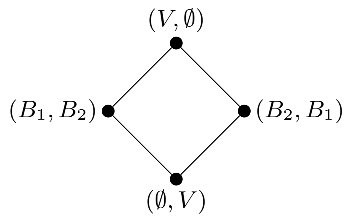
Proof. It is a consequence of the equivalence of $\mathbb{K}(K_n)$ with the contranominal scale [9]. ☐

For the complete bipartite graph we have the following result.

Proposition 3.6. *Let $K_{p,q} = (B_1|B_2)$ and $V = V(K_{p,q})$. Then*

$$\mathfrak{B}(K_{p,q}) = \{(\emptyset, V), (B_1, B_2), (B_2, B_1), (V, \emptyset)\} \tag{6}$$

and the Hasse diagram of the concept lattice $(\mathfrak{B}(K_{p,q}), \sqsubseteq)$ is the following:



Hence $(\mathfrak{B}(K_n), \sqsubseteq) \cong (\mathcal{P}(\hat{2}), \subseteq)$.

Proof. By Remark 3.2, a concept of $\mathfrak{B}(K_{p,q})$ is a pair (O, O'), where $O'' = O$. Therefore, by (5) we deduce that the unique concepts of $\mathfrak{B}(K_{p,q})$ are $(\emptyset, V)$, $(B_1, B_2), (B_2, B_1), (V, \emptyset)$. This proves (6). Finally, by definition of the partial order $\sqsubseteq$ we immediately deduce that the Hasse diagram of $(\mathfrak{B}(K_{p,q}), \sqsubseteq)$ is that given above. $\square$

3.3 Other Operations in FCA

The operation $'$ is one of the four operations that can be introduced in FCA in analogy with possibility theory [7]. These four operations generate the sets A,I,a,i defined in section 2.4 (the other four are just their complement). In the particular case of formal contexts induced by graphs, they read as:

- $R^{\Delta}(O) := \overline{\overline{R}(O)} = O'$;
- $R^{\nabla}(O) := \overline{R}(\overline{O}) = \{v \in V | N_G(v) \cup O \neq V\}$ the set of vertices that are missing at least a link outside O;
- $R^{\Pi}(O) := R(O) = \{v \in V | N_G(v) \cap O \neq \emptyset\}$ the set of vertices connected with at least one vertex in O;
- $R^{N}(O) := \overline{R(\overline{O})} = \{v \in V | N_G(v) \subseteq O\}$ the set of vertices connected with no vertex outside O.

As discussed above, the Galois connection induced by R^{Δ} is of particular interest in the case of graphs. The interpretation of the Galois connections induced by the other operations in terms of graphs is not so easy. In [10], the Galois connection induced by R^{Π} is nicely interpreted in terms of maximal connected components. However, this result can be hardly translated to our framework (let us remark that the graph in [10] is obtained from a given formal context, we operate in the other direction). The problem lies in the fact that X and $Y = R^{\Pi}(X)$ are generally not disjoint hence they do not form a bipartition of $X \cup Y$ as it happens in [10]. More constraints needs to be considered on the starting graph in order to have some geometrical interpretation of this kind of operator. We deserve this issue to a further investigation.

Finally, let us notice that as an easy consequence of the definitions of R^{Π} and $N_G(v)$, R^{Π} can be expressed in terms of neighborhoods as

$$O' \subseteq R^{\Pi}(O) = \bigcup_{v \in O} N_G(v) \tag{7}$$

4 Simple Graphs as Boolean Information Tables

Analogously to the formal context case, the adjacency matrix of a graph G can be interpreted as a Boolean information table $\mathcal{I}[G]$, where the universe set and the attribute set are both V and the information map is defined as $F(v_i, v_j) := 1$ if $v_i \sim v_j$ and $F(v_i, v_j) := 0$ otherwise.

The equivalence relation I_A (where A is a set of verteces) is in relation with the notion of neighborhood as can be seen in the following theorem.

Theorem 4.1. *Let $A \subseteq V(G)$ and $v, v' \in V(G)$. The following conditions are equivalent:*
(i) vI_Av'.
(ii) For all $z \in A$ it results that $v \sim z$ if and only if $v' \sim z$.
(iii) $N_G(v) \cap A = N_G(v') \cap A$.

Proof. $(i) \implies (ii)$: Let $z \in A$ and $v \sim v'$, we show that $v' \sim z$. By (i) we have that $F(v, a) = F(v', a)$ for all $a \in A$, therefore $F(v, z) = F(v', z)$. Since $v \sim z$ it follows that $F(v, z) = 1$, and hence also $F(v', z) = 1$, that is $v' \sim z$. By symmetry of the relation I_A, if we assume that $v' \sim z$, we obtain $v \sim z$. This proves (ii)
$(ii) \implies (iii)$: By symmetry of the condition (ii), it is sufficient to prove that $N_G(v) \cap A \subseteq N_G(v') \cap A$. Let therefore $z \in N_G(v) \cap A$, then $v \sim z$ and $z \in A$. By (ii) we have then that $v' \sim z$, that is $z \in N_G(v')$. Hence $z \in N_G(v') \cap A$.
$(iii) \implies (i)$: Let $a \in A$. We show that $F(v, a) = F(v', a)$. Let us note that

$$F(v, a) = F(v', a) \iff (v \sim a \iff v' \sim a). \tag{8}$$

Then, if $v \sim a$, we have that $a \in N_G(v) \cap A$ =(by (iii))= $N_G(v') \cap A$, hence $a \in N_G(v')$, that is $v' \sim a$. Analogously, by symmetry of (iii), if $v' \sim a$ then $v \sim a$. By (8) we deduce therefore that $F(v, a) = F(v', a)$. Since $a \in A$ is arbitrary, this proves (i). □

Corollary 4.1. *If $v \in V(G)$ and $A \subseteq V(G)$, then $[v]_A = \{v' : N_G(v) \cap A = N_G(v') \cap A\}$.*

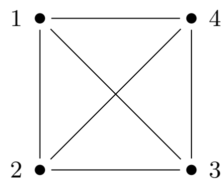
That is two vertices are equivalent if they have the same neighborhood (relatively to A). The Theorem 4.1 also provides a sufficient condition for two vertices of the graph to have no common edges.

Corollary 4.2. *If vI_Av' and $\{v, v'\} \cap A \neq \emptyset$, then $v \nsim v'$.*

Proof. It follows directly by Theorem 4.1 because there are no loops into G. □

4.1 The Partitions of a Graph

Now, we turn our attention to the partition generated by the relation I_A on complete and bipartite graphs. Let us start with an example.



	1	2	3	4
1	0	1	1	1
2	1	0	1	1
3	1	1	0	1
4	1	1	1	0

Fig. 1. The complete graph K_4.

Example 4.1. Let us consider now the complete graph K_4 and the corresponding information table in Figure 4.1.

In this case we can easily compute all the set partitions $\pi_A(K_4)$, where $A \subseteq \{1,2,3,4\}$. Once denoted a partition $\pi_A = X_1|\cdots|X_n$ with X_i the equivalence classes induced by I_A, we have :
$\pi_\emptyset = 1234$, $\pi_{\{1\}} = 1|234$, $\pi_{\{2\}} = 2|134$, $\pi_{\{3\}} = 3|124$, $\pi_{\{4\}} = 4|123$, $\pi_{\{1,2\}} = 1|2|34$, $\pi_{\{1,3\}} = 1|3|24$, $\pi_{\{1,4\}} = 1|4|23$, $\pi_{\{2,3\}} = 14|2|3$, $\pi_{\{2,4\}} = 13|2|4$, $\pi_{\{3,4\}} = 12|3|4$, $\pi_{\{1,2,3\}} = \pi_{\{1,2,4\}} = \pi_{\{1,3,4\}} = \pi_{\{2,3,4\}} = \pi_{\{1,2,3,4\}} = 1|2|3|4$.

As the previous example suggests, we can determine the general form of any partition $\pi_A(K_n)$, for all $n \geq 1$ and all $A \subseteq V(K_n)$.

Proposition 4.1. *Let $n \geq 1$ and let $A = \{w_1, \dots, w_k\}$ be a subset of $V(K_n) = \{v_1, \dots, v_n\}$. Then*

$$\pi_A(K_n) = w_1|w_2|\dots|w_k|A^c, \tag{9}$$

where A^c is the complementary subset of A in $V(K_n)$.

Proof. Let v, $v' \in V(K_n)$, with $v \neq v'$. By Corollary 4.2, since $v \sim v'$, it holds that if vI_Av', then v, $v' \in A^c$. On the other hand, if v, $v' \in A^c$, then $\forall z \in A$, $F(z,v) = F(z,v') = 1$, namely vI_Av'. The proposition is proved. □

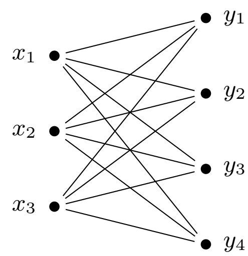
Example 4.2. Let us consider now the complete graph $K_{3,4}$ in Figure 4.2.

It is easy to verify then that in this case we have only two possibilities: $\pi_\emptyset = x_1x_2x_3y_1y_2y_3y_4$ and $\pi_A = x_1x_2x_3|y_1y_2y_3y_4$ if $A \neq \emptyset$.

Also in this case we can generalize the previous example to any complete bipartite graph.

Proposition 4.2. *Let p and q be two positive integers. Let $K_{p,q} = (B_1|B_2)$, where $B_1 = \{x_1, \dots, x_p\}$ and $B_2 = \{y_1, \dots, y_q\}$. Then $\pi_A(K_{p,q}) = x_1 \dots x_p|y_1 \dots y_q$ for each subset $A \subseteq V(K_{p,q})$ such that $A \neq \emptyset$.*

Proof. Let $A \subseteq V(G)$ be a non-empty subset of $V(G)$ and let v, $v' \in V(G)$. If v, $v' \in B_1$ or v, $v' \in B_2$, then for each $z \in A$ we have $F(z,v) = F(z,v')$, so vI_Av'. If $v \in B_1$ and $v' \in B_2$, then for each $z \in A$ we have $F(z,v) \neq F(z,v')$, so $\neg(vI_Av')$. Thus $\pi_A(G) = B_1|B_2$. □



	x_1	x_2	x_3	y_1	y_2	y_3	y_4
x_1	0	0	0	1	1	1	1
x_2	0	0	0	1	1	1	1
x_3	0	0	0	1	1	1	1
y_1	1	1	1	0	0	0	0
y_2	1	1	1	0	0	0	0
y_3	1	1	1	0	0	0	0
y_4	1	1	1	0	0	0	0

Fig. 2. The graph $K_{3,4}$ and the corresponding information table.

4.2 Upper and Lower Approximations

In this section we provide some results and discussion on rough set approximations, at first in the general graph case and, then, in the case of complete and bipartite graphs.

Proposition 4.3. *Let $G = (V(G), E(G))$ be a simple undirected graph and let $\mathcal{I}[G]$ be the Boolean information system associated to G. Let A and Y be two subsets of $V(G)$. Then:*
(i) $\mathbf{l}_A(Y) = \{v \in V(G) : (u \in V(G) \wedge N_G(u) \cap A = N_G(v) \cap A) \Longrightarrow u \in Y\}$.
(ii) $\mathbf{u}_A(Y) = \{v \in V(G) : \exists u \in Y : N_G(u) \cap A = N_G(v) \cap A\}$.

Proof. It follows directly by (iii) of Theorem 4.1 and the definitions of the approximations. □

The lower approximation of a set of vertices Y represents a subset of Y such that there are no elements outside Y with the same connections of any vertex in $\mathbf{l}_A(Y)$ (relatively to A). The upper approximation of Y is the set of vertices with the same connections (w.r.t. A) of at least one element in Y.

We study now the cases of complete $G = K_n$ and bipartite $G = K_{p,q}$ graphs.

Proposition 4.4. *Let $G = K_n$ be the complete graph on n vertices and let A and Y be two subsets of $V(G) = \{v_1, \dots, v_n\}$. Then:*
(i) the A-lower approximation of Y is

$$\mathbf{l}_A(Y) = \begin{cases} Y \cup A^c & \text{if } A^c \subseteq Y \\ A \cap Y & \text{otherwise .} \end{cases}$$

(ii) The A-upper approximation of Y is

$$\mathbf{u}_A(Y) = \begin{cases} Y & \text{if } Y \subseteq A \\ Y \cup A^c & \text{otherwise} \end{cases}$$

(iii) Y is A-exact if and only if $Y \subseteq A$ or $A^c \subseteq Y$.

Proof. In this proof we denote $V(G)$ simply by V. If $v \in V$, by definition of K_n we have $N_G(v) = V \setminus \{v\}$, therefore $N_G(v) \cap A = A \setminus \{v\}$. By Corollary 4.1 we obtain then $[v]_A = \{v' \in V : A \setminus \{v\} = A \setminus \{v'\}\}$, hence

$$[v]_A = \begin{cases} \{v\} & \text{if } v \in A \\ A^c & \text{otherwise .} \end{cases} \tag{10}$$

By definition of A-lower approximation of Y and by (10) we have then

$$\mathbf{l}_A(Y) = \{v \in V : (v \in A \Longrightarrow v \in Y) \vee (v \in A^c \Longrightarrow A^c \subseteq Y)\}. \tag{11}$$

It is immediate to note then that (11) is equivalent to (i). This proves (i). By definition of A-upper approximation of Y and by (10) we have then

$$\mathbf{u}_A(Y) = \{v \in V : (v \in A \Longrightarrow v \in Y) \vee (v \in A^c \Longrightarrow A^c \cap Y \neq \emptyset)\}. \tag{12}$$

It is immediate to note then that (12) is equivalent to (ii). This proves (ii). In order to prove (iii), if $Y \subseteq A$ then $A^c \nsubseteq Y$, therefore $\mathbf{u}_A(Y) = Y$ by (ii) and $\mathbf{l}_A(Y) = A \cap Y = Y$ by (i), hence Y is A-exact. If $A^c \subseteq Y$ and $A^c \neq \emptyset$ then $Y \nsubseteq A$ therefore $\mathbf{l}_A(Y) = (A \cap Y) \cup A^c$ by (i) and $\mathbf{u}_A(Y) = (A \cap Y) \cup A^c$ by (ii), hence Y is A-exact. If $A^c = \emptyset$ then $A = V(G)$, therefore $\mathbf{l}_A(Y) = Y$ by (i) and $\mathbf{u}_A(Y) = Y$ by (ii), hence Y is A-exact. On the other hand, if $Y \nsubseteq A$ and $A^c \nsubseteq Y$, then $A^c \neq \emptyset$ and $\mathbf{u}_A(Y) = (A \cap Y) \cup A^c$ by (ii), $\mathbf{l}_A(Y) = A \cap Y$ by (ii). Since $A^c \neq \emptyset$, we obtain then $\mathbf{l}_A(Y) \neq \mathbf{u}_A(Y)$, hence Y is not A-exact. This proves (iii). ◻

We now examine for $K_{p,q}$ the results similar to those described previously for K_n.

Proposition 4.5. *Let $K_{p,q} = (B_1|B_2)$, where $B_1 = \{x_1, \ldots, x_p\}$ and $B_2 = \{y_1, \ldots, y_q\}$. Let A and Y be two non-empty subsets of $V = V(K_{p,q})$ such that $Y \neq V$. Then:*
(i) the A-lower approximation of Y is

$$\mathbf{l}_A(Y) = \begin{cases} B_1 & \text{if } B_1 \subseteq Y \\ B_2 & \text{if } B_2 \subseteq Y \\ \emptyset & \text{otherwise .} \end{cases}$$

(ii) The A-upper approximation of Y is

$$\mathbf{u}_A(Y) = \begin{cases} B_1 & \text{if } Y \subseteq B_1 \\ B_2 & \text{if } Y \subseteq B_2 \\ V & \text{otherwise .} \end{cases}$$

(iii) Y is A-exact if and only if $Y = B_1$ or $Y = B_2$.

Proof. (i) Let $B_1 \subseteq Y$. If $x \in B_1$, by Proposition 4.2 follows that $[x]_A = B_1 \subseteq Y$, therefore, by definition of $\mathbf{l}_A(Y)$, we obtain $B_1 \subseteq \mathbf{l}_A(Y)$. On the other hand, if it were $x \in B_2 \cap \mathbf{l}_A(Y)$, for some vertex $x \in V$, then, again by Proposition 4.2

and by definition of $\mathbf{l}_A(Y)$, we would have that $B_2 = [x]_A \subseteq Y$. Since $B_1|B_2$ is a set-partition of V, the last inclusion implies that $Y = V$, which is contrary to our hypothesis. Hence $B_1 \subseteq \mathbf{l}_A(Y)$ and $B_2 \cap \mathbf{l}_A(Y) = \emptyset$, and since $B_1|B_2$ is a set-partition of V we deduce that $\mathbf{l}_A(Y) = B_1$ if $B_1 \subseteq Y$. A similar reasoning also shows that if $B_2 \subseteq Y$ then $\mathbf{l}_A(Y) = B_2$. Finally, let $B_1 \nsubseteq Y$ and $B_2 \nsubseteq Y$. Since each vertex $x \in V$ is such that $x \in B_1$ or $x \in B_2$, by Proposition 4.2 we have respectively $[x]_A = B_1 \nsubseteq Y$ and $[x]_A = B_2 \nsubseteq Y$, that is $x \notin \mathbf{l}_A(Y)$ for each vertex $x \in V$, hence $\mathbf{l}_A(Y) = \emptyset$.
(ii) Let $Y \subseteq B_1$. If $x \in B_1$, by Proposition 4.2 follows that $[x]_A = B_1 \cap Y \neq \emptyset$ because Y is non-empty subset of B_1. Hence $x \in \mathbf{u}_A(Y)$. On the other hand, if $x \in \mathbf{u}_A(Y)$ by definition of $\mathbf{u}_A(Y)$ we have $[x]_a \cap Y \neq \emptyset$. Let $y \in [x]_A \cap Y$. Since $y \in Y \subseteq B_1$, by Proposition 4.2 we obtain $B_1 = [y]_A = [x]_A$, therefore, again by Proposition 4.2 we deduce that $x \in B_1$. Hence $\mathbf{u}_A(Y) = B_1$. The case $Y \subseteq B_2$ is exactly similar. Finally, let $Y \nsubseteq B_1$ and $Y \nsubseteq B_2$. Since $B_1|B_2$ is a set-partition of V, we deduce that $B_1 \cap Y \neq \emptyset$ and $B_2 \cap Y \neq \emptyset$. Now, if we take an arbitrary vertex $x \in V$, then $x \in B_1$ or $x \in B_2$. If $x \in B_1$, then, by Proposition 4.2 it follows that $[x]_A \cap Y = B_1 \cap Y \neq \emptyset$, therefore $x \in \mathbf{u}_A(Y)$. Analogously if $x \in B_2$. This shows that $V \subseteq \mathbf{u}_A(Y)$, that is $V = \mathbf{u}_A(Y)$.
(iii) It follows at once by Proposition 2.1 (ii) and by Proposition 4.2. □

4.3 Other Operations in RST

Let us consider the sets introduced in subsection 2.4. The vertex (A) corresponds to the lower approximation and the corner (I) to the upper one [5]. Then, (E) is the negation of the upper approximation, called the *exterior region* $\mathbf{e}$ and it represents the objects (vertices in our case) surely not belonging to the set under approximation. In the graph case, a vertex x belongs to $\mathbf{e}(O)$ if there is no vertex in O sharing all the connections with x. As a simple corollary of the results on the upper approximation we get the following.

Corollary 4.3. *Let $G = (V(G), E(G))$ be a simple undirected graph and let $\mathcal{I}[G]$ be the Boolean information system associated to G. Let A and Y be two subsets of $V(G)$. Then:*
(i) $\mathbf{e}_A(Y) = \{v \in V(G) : \nexists u \in V(G) : N_G(u) \cap A = N_G(v) \cap A)\}$.
(ii) If G is complete, then

$$\mathbf{e}_A(Y) = \begin{cases} Y^c & \text{if } Y \subseteq A \\ A \cap Y^c & \text{otherwise} \end{cases}$$

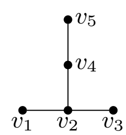
(iii) If the graph is bipartite, i.e., $G = K_{p,q} = (B_1|B_2)$, then

$$\mathbf{e}_A(Y) = \begin{cases} B_2 & \text{if } Y \subseteq B_1 \\ B_1 & \text{if } Y \subseteq B_2 \\ \emptyset & \text{otherwise .} \end{cases}$$

The corner (a) is named in RST a *sufficiency* operator and (i) is the dual sufficiency. In case of R being an equivalence relation, the sufficiency operator is trivial since it gives either the emptyset or the set O under approximation. Similarly, the dual sufficiency either results in the complement of O or in the universe. Both the operators become more interesting in a generalized setting, for instance when R is a similarity instead of an equivalence relation. However, this generalized situation is out scope of the present work.

Let us stress once more that, in the particular case of formal context induced by graphs, objects coincide with attributes and the relation $\mathcal{R}$ is defined on the same set, as in case of rough-set indiscernibility relation. Hence, FCA tools can be compared and/or combined with RST ones. For instance, from the fact that both $''$ and $\mathbf{u}$ are closure operators on objects, we have that $O \subseteq O''$ and $O \subseteq \mathbf{u}(O)$. So, we can wonder which is the relationship among the two mappings O'' and $\mathbf{u}(O)$. In case of complete bipartite graphs we have that $O'' = \mathbf{u}(O)$ (and also $O' = \mathbf{e}(O)$), as can be seen by propositions 3.3 and 4.5. Also in case of complete graphs and $A = V(G)$ we have $O'' = \mathbf{u}(O) = O$ (by propositions 3.2 and 4.4). However, in the general case, nothing can be said as it is shown by the following example.

Example 4.3. Let us consider the following (bipartite and not complete) graph:



If we set $O = \{v_1, v_3\}$, then we get $O'' = \{v_1, v_3, v_4\}$ and $\mathbf{u}(O) = O$. So, $\mathbf{u}(O) \subset O''$. On the other hand, considering the complete graph of Figure 4.1 with $O = \{2, 3\}$ and $A = \{1, 2\}$, we have $\mathbf{u}(O) = \{2, 3, 4\}$ and $O'' = O$ leading to $O'' \subset \mathbf{u}(O)$.

5 Conclusion

We laid bare the possibility to investigate graphs using techniques from Formal Concept Analysis and Rough Set Theory. Several results exploring the corresponding on graphs of operators in the two theories have been given. The picture, however, is far from being complete. Indeed, a complete description of the structure of oppositions arising from FCA and RST in the case of graphs, as well as the interaction between FCA and RST operators is still missing. Moreover, as far as RST is concerned, we only explored the approximations defined by the standard indiscernibility relation. A natural extension would be to consider more general rough set models and to explore other concepts such as rough membership, attribute dependencies and reducts.

References

1. G. Alexe, S. Alexe, Y. Crama, S. Foldes, P.L. Hammer, B. Simeone, Consensus algorithms for the generation of all maximal bicliques, *Discrete Applied Mathematics*, **145**, 2004, 11–21

2. C. Berge, Hypergraphs: Combinatorics of Finite Sets, Elsevier, Amsterdam, 1984.
3. G. Birkhoff, Lattice Theory, American Mathematical Society, Providence, Rhode Island, Third Edition, 1967
4. G. Cattaneo, Generalized Rough Sets (Preclusivity Fuzzy-Intuitionistic (BZ) Lattices), *Studia Logica*, **58**, 1997, 47–77
5. D. Ciucci, D. Dubois, H. Prade.: The structure of oppositions in rough set theory and formal concept analysis - Toward a new bridge between the two settings. In: C. Beierle, C. Meghini (eds.) Proc. FoIKS'14, *LNCS*, vol. 8367, pp. 154–173. Springer (2014)
6. M. Dawande, P. Keskinocak, J. M. Swaminathan, S. Tayur: On Bipartite and Multipartite Clique Problems. *J. Algorithms*, **41**, 2001, 388–403
7. D. Dubois, F. Dupin de Saint Cyr, H. Prade, A possibility-theoretic view of formal concept analysis, *Fundamenta Informaticae*, **75**, 2007, 195–213
8. D. Dubois and H. Prade, From Blanché's hexagonal organization of concepts to formal concept analysis and possibility theory, *Logica Universalis*, **6**, 2012, 149–169.
9. B. Ganter, R. Wille, Formal Concept Analysis. Mathematical Foundations, Springer-Verlag, 1999.
10. Gaume, B., Navarro, E., Prade, H., A Parallel between Extended Formal Concept Analysis and Bipartite Graphs Analysis In: E. Hüllermeier, R. Kruse, F. Hoffmann(eds.) Proc. IPMU'10, *LNCS*, vol. 6178, pp. 270–280. Springer (2010)
11. S.O. Kuznetsov, S.A. Obiedkov, Comparing performance of algorithms for generating concept lattices, *J. Exp. Theor. Artif. Intell.* ,**14**, 2002,189–216
12. T.T. Lee, An Information-Theoretic Analysis of Relational Databases - part I: Data Dependencies and Metric. *IEEE Transactions on Software Engineering*, **SE-13** (1987), pp. 1049–1061.
13. J. Li, G. Liu, H. Li, L. Wong: Maximal Biclique Subgraphs and Closed Pattern Pairs of the Adjacency Matrix: A One-to-One Correspondence and Mining Algorithms, *IEEE Trans. Knowl. Data Eng.*, **19**, 2007, 1625–1637
14. Pawlak Z., Rough sets. Theoretical Aspects of Reasoning about Data. Kluwer Academic Publisher, 1991.
15. T. Qiu, X. Chen, Q. Liu, H. Huang, Granular Computing Approach to Finding Association Rules in Relational Database, *International Journal of Intelligent Systems*, **25**, 2010, 165–179.
16. A. Skowron, C. Rauszer, The Discernibility Matrices and Functions in Information Systems, Intelligent Decision Support, Theory and Decision Library series , vol. 11, Springer Netherlands, 1992, pp. 331–362.
17. Y.Y. Yao, On modeling data mining with granular computing, COMPSAC 2001. IEEE, 2001, pp. 638–643.
18. Y.Y. Yao, Information granulation and rough set approximation, *International Journal of Intelligent Systems*, 2001 Vol. 16, No. 1, 87–104.
19. Y.Y. Yao, A Partition Model of Granular Computing, in Transactions on Rough sets I *LNCS*, vol. 3100, Springer-Verlag, 2004, pp. 232–253.
20. Y.Y. Yao, B. Yao, Covering based rough set approximations, *Information Sciences*, **200**, 2012, 91–107
21. Y.Y. Yao, N. Zhong, Granular Computing using Information Tables, in Data Mining, Rough Sets and Granular Computing, Physica-Verlag, 2002, pp. 102–124.
22. L.A. Zadeh, Towards a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets and Systems*, **19**, 1997, 111–127.

23. L.A. Zadeh, Fuzzy sets and information granularity, in: *Advances in Fuzzy Set Theory and Applications*, Gupta, N., Ragade, R. and Yager, R. (Eds.), North-Holland, Amsterdam, 3–18, 1979.
24. M.J. Zaki, M. Ogihara, Theoretical foundation of association rules, Proc. 3rd SIGMOD Workshop on Research Issues in Data Mining and Knowledge Discovery, 1998