HIGHER INTEGRABILITY FOR SOLUTIONS TO VARIATIONAL PROBLEMS WITH FAST GROWTH

ARRIGO CELLINA AND MARCO MAZZOLA

Abstract. We prove higher integrability properties of solutions to variational problems of minimizing

$$\int_\Omega [e^{f(\|\nabla u(x)\|)} + g(x,u(x))]\, dx \tag{1}$$

where f is a convex function satisfying some additional conditions.

1. Introduction

In this paper we consider the properties of a solution $\tilde{u}$ to the problem of minimizing

$$\int_\Omega [e^{f(\|\nabla u(x)\|)} + g(x,u(x))]\, dx. \tag{2}$$

In general, in order to establish the validity of the Euler Lagrange equation for the solution to this problem , i.e., in order to prove that, for every admissible variation η, the equation

$$\int_\Omega \{e^{f(\|\nabla \tilde{u}(x)\|)} f'(\|\nabla \tilde{u}(x)\|)\langle \frac{\nabla \tilde{u}(x)}{\|\nabla \tilde{u}(x)\|}, \nabla \eta(x)\rangle + g_u(x,\tilde{u}(x))\eta(x)\}dx = 0 \tag{3}$$

holds, one has preliminarly to prove that the integrand is in L^1, in particular, that $e^{f(\|\nabla \tilde{u}(\cdot)\|)} f'(\|\nabla \tilde{u}(\cdot)\|) \in L^1_{loc}$. However, for Lagrangeans L growing faster than exponential, the integrability of a term like

$$\int_\Omega L(\|\nabla u(x)\|)\, dx$$

does not imply the integrability of

$$\int_\Omega \nabla L(\|\nabla u(x)\|)\, dx.$$

In fact, consider $L(s) = e^{s^2}$, so that $L' = 2se^{s^2}$. For $n = 1$, the function $\xi(\cdot)$ whose derivative is

$$\xi'(t) = \sqrt{-\ln(|t|(|\ln(t)|)^{\frac{3}{2}})}$$

is such that $e^{\xi'(t)^2} = \frac{1}{|t||\ln(t)|^{\frac{3}{2}}}$ is integrable on $(-\frac{1}{2}, \frac{1}{2})$; however, for $|t|$ small,

$$\xi'(t)e^{\xi'(t)^2} = \frac{1}{|t|(|\ln(t)|)^{\frac{3}{2}}}\sqrt{-\ln(|t|(|\ln(t)|)^{\frac{3}{2}})} >$$

$$\frac{1}{|t|(|\ln(t)|)^{\frac{3}{2}}}\sqrt{\frac{-1}{2}|\ln(t)|} = \frac{1}{\sqrt{2}|t||\ln(t)|},$$

hence $L'(\xi'(\cdot))$ is not locally integrable.

This problem does not occur when we are able to prove some additional regularity properties of the solution $\tilde{u}$. When $g = 0$, by using a barrier as in [7], one can prove that the gradient of the solution is in $L^\infty(\Omega)$; alternatively, taking advantage of the regularity properties of solutions to elliptic equations, as in [2] for the case $L(t) = e^{t^2}$, and in [4],[5] for the case $L(t) = e^{f(t)}$, under general assumptions on f, one proves that the gradient of the solution is in L^∞_{loc}. Both these methods demand additional smoothness assumptions: smoothness of the boundary and of the second derivative of f, in the case of a barrier; smoothness of the second derivative of f in the other case.

In the present paper we prove a higher integrability result for $\tilde{u}$: our result is weaker than the local boundedness of $\nabla\tilde{u}$, the result proved in [2], [4], [5]; however, it holds for a larger class of functionals, where, possibly, the stronger boundedness result might not hold. In fact, we do not assume further regularity on f besides its being convex and differentiable: in particular, we do not assume the existence of a second derivative of f, nor we assume its strict convexity. Moreover, we allow also a dependence on x and on u, assuming that g is a standard Carathéodory function. Our method of proof is based on a simple variation and on the properties of polarity.

2. Higher integrability

In what follows, Ω is a bounded open subset of $\mathbb{R}^N$. The function f^* is the *polar* or *conjugate* [6] of f, a possibly extended valued function. Moreover, since there is no assumption of strict convexity of f, the map f^* is convex but not necessarily differentiable: its subgradient will be denoted by ∂f^*: it is a maximal monotone map. In the Theorem that follows, we use the notation $\frac{1}{p\partial f^*(p)}$: we mean 0 when $p \notin Dom(\partial f^*)$ and, when $p \in Dom(\partial f^*)$, we mean any selection from the set-valued map $p \to \frac{1}{p\partial f^*(p)}$: since $\frac{1}{p\partial f^*(p)}$ is strictly decreasing, it is multi-valued at most on a countable set, and any two selections will differ only on a set of measure zero.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be convex, differentiable, symmetric, $f(0) = 0$ and assume that*

$$\int^\infty \frac{1}{p\partial f^*(p)} dp < \infty.$$

Let g be differentiable with respect to u, and let g and g_u be Carathéodory functions, and assume that for every U there exists $\alpha_U \in L^1_{loc}$ such that $|v| \leq U$ implies $|g_u(x, v)| \leq \alpha_U(x)$. Let $\tilde{u} \in u^0 + W^{1,1}_0(\Omega)$ be a locally bounded solution to the problem of minimizing

$$\int_\Omega [e^{f(\|\nabla u(x)\|)} + g(x, u(x))]\, dx.$$

Then, for every function ξ such that $\int_\Omega e^{f(\xi(x))}\, dx < \infty$, we have that

$$e^{f(\|\nabla\tilde{u}(\cdot)\|)} f'(\|\nabla\tilde{u}(\cdot)\|)\xi(\cdot) \in L^1_{loc}(\Omega).$$

The result applies, in particular, to the function $\xi(x) = \|\nabla\tilde{u}(x)\|$, so that we have $e^{f(\|\nabla\tilde{u}(\cdot)\|)} f'(\|\nabla\tilde{u}(\cdot)\|)\|\nabla\tilde{u}(\cdot)\| \in L^1_{loc}(\Omega)$.

Examples. The map $f(s) = s - 2\sqrt{s+1} + 2$ is convex, differentiable and of linear growth. Its conjugate is the extended-valued function $f^*(p) = \frac{p^2}{1-|p|}$ for $|p| < 1$, $= \infty$ elsewhere. The conditions of the theorem are satisfied.

A map satisfying the assumption of the Theorem is $f(s) = 2e^{s^{\frac{1}{2}}}(s^{\frac{1}{2}} - 1) - s$; then $f^{*\prime}(p) = (\ln(p+1))^2$ and $\int^{\infty} \frac{1}{p(\ln(p+1))^2} dp < \infty$.

A map f that does not satisfy the assumption of the Theorem is $f(s) = \frac{1}{e}(e^s - s - 1)$; in this case, we have $f^{*\prime}(p) = 1 + \ln(p + \frac{1}{e})$.

Remark 1. *In Theorem 1 we assume the solution $\tilde{u}$ to be locally bounded. The validity of this assumption can be guaranteed:*

i) when $g = 0$, assuming that the boundary datum u^0 is in L^{∞}, through a standard comparison result, noticing that, with the exception of the case $f \equiv 0$, $e^{f(\|v\|)}$ is a strictly convex function of z.

ii) in general, assuming that there exist $p \in \mathbb{R}^+$, $\alpha \in L^1(\Omega)$ and $\beta \in \mathbb{R}$ such that $u_0 \in W^{1,p}(\Omega)$ and

$$|g(x,u)| \le \alpha(x) + \beta |u|^p.$$

In fact, with the exception of the case $f \equiv 0$, there are A and $B > 0$ such that that $f(t) \ge A + Bt$; hence, fix N^* larger than $\sup\{N, p\}$. For suitable constants, we have

$$\infty > \int_{\Omega} [e^{f(\|\nabla \tilde{u}(x)\|)} + g(x, \tilde{u}(x))]\, dx \ge \int_{\Omega} [e^{A + B\|\nabla \tilde{u}(x)\|} - |\alpha(x)| - |\beta||\tilde{u}(x)|^p]\, dx$$

$$\ge A_1 + B_1 \|\nabla \tilde{u}(x)\|^{N^*}_{L^{N^*}(\Omega)} - |\beta| \|\tilde{u}\|^p_{L^p(\Omega)}$$

$$\ge A_1 + B_1 \|\nabla \tilde{u}(x)\|^{N^*}_{L^{N^*}(\Omega)} - C_1 \|u_o\|^p_{L^p(\Omega)} - C_1 \|\tilde{u} - u_o\|^p_{L^p(\Omega)}.$$

By Poincaré's inequality,

$$\infty > A_2 + B_1 \|\nabla \tilde{u}(x)\|^{N^*}_{L^{N^*}(\Omega)} - C_2 \|\nabla \tilde{u} - \nabla u_0\|^p_{L^p(\Omega)}.$$

By Holder's inequality,

$$\infty > A_2 + B_1 \|\nabla \tilde{u}(x)\|^{N^*}_{L^{N^*}(\Omega)} - C_3 \|\nabla u_0\|^p_{L^p(\Omega)} - D \|\nabla \tilde{u}\|^p_{L^{N^*}(\Omega)},$$

so that there are positive constants h and k such that

$$\infty > -h + k \|\nabla \tilde{u}(x)\|^{N^*}_{L^{N^*}(\Omega)}.$$

Hence, $\tilde{u}$ belongs to $C_B(\Omega)$ ([1]).

The proof of Theorem 1 relies on directly comparing the value of the functional on the solution $\tilde{u}$ and on a variation $\tilde{u} + \varepsilon v$. For it, we shall need the following Lemmas.

Lemma 1. *Let $G : \mathbb{R} \to 2^{\mathbb{R}}$ be upper semicontinuous, strictly increasing and such that $G(0) = \{0\}$. Assume that, for a selection g from G,*

$$\int^{\infty} g(\frac{1}{s}) ds < \infty. \tag{4}$$

Then, the implicit Cauchy problem

$$x(t) \in G(x'(t)), \qquad x(0) = 0$$

admits a solution $\tilde{x}$, positive on some interval $(0, \tau)$.

Notice that the condition expressed by (4) is independent on the selection g; in fact, G is multi-valued at most on countably many points, and the map $s \to \frac{1}{s}$ is strictly monotonic.

Proof. Set $\gamma = G^{-1}$: γ is single-valued, continuous and $\gamma(0) = 0$. We claim that for every $z > 0$

$$\int_{(0,z)} \frac{1}{\gamma(y)} dy = z\frac{1}{\gamma(z)} + meas(R),$$

where $R = \{(y,x) : 0 \le y \le z; \frac{1}{\gamma(z)} \le x \le \frac{1}{\gamma(y)}\}$. In fact, we have also that $R = \{(x,y) : 0 \le y \le \gamma^{-1}(\frac{1}{x}); \frac{1}{\gamma(z)} \le x < \infty\}$, so that $meas(R) = \int_{\frac{1}{\gamma(z)}}^{\infty} \gamma^{-1}(\frac{1}{y})dy = \int_{\frac{1}{\gamma(z)}}^{\infty} g(\frac{1}{s})ds$, that is finite by assumption.

Hence, the map $\Phi(x) = \int_0^x \frac{1}{\gamma(y)}dy$ is well defined, differentiable, positive for $x > 0$ and $\Phi(0) = 0$. Define $\tilde{x}(t)$ implicitely by

$$\Phi(\tilde{x}(t)) - t = 0;$$

then, $\tilde{x}$ is a differentiable function, $\tilde{x}(0) = 0$ and $x'(t) = \gamma(x(t))$. □

Let $O \subset\subset \Omega$, set $O_\delta = O + B(0,\delta)$ and let $\delta > 0$ be such that $\overline{O_\delta}$ is in Ω.

Lemma 2. *Let f be as in Theorem 1. Then, for every non-negative ξ in $L^1(O_\delta)$ and $U \in \mathbb{R}$, there exist $\eta \in C_c^1(O_\delta)$ and K such that*

$$f(\xi(1-\varepsilon\eta) + \varepsilon\|\nabla\eta\|U) - f(\xi) \le \varepsilon K.$$

Proof. Consider the function

$$G(z) = z\frac{2U}{\partial f^*(\frac{1}{z})}. \tag{5}$$

We claim that G satisfies the assumptions of Lemma 1. In fact, $G(0) = \{0\}$ and G is a strictly increasing multi-valued map (single-valued except on a countable set); we have

$$G(\frac{1}{x'}) = \frac{2U}{x'\partial f^*(x')}$$

so that, by the assumptions of Lemma 2, the condition of Lemma 1 is satisfied.

Consider $\tilde{x}$, the solution to $\tilde{x} \in G(\tilde{x}')$, provided by Lemma 1. Define η as follows. Let $d(x)$ be the distance from a point $x \in O_\delta$ to ∂O_δ and set

$$\eta(x) = \inf\left\{\frac{1}{\tilde{x}(\delta)}\tilde{x}(d(x)), 1\right\}$$

so that, in particular, $\eta = 1$ on O. Almost everywhere, d is differentiable with $\|\nabla d\| = 1$ and, at a point of differentiability, we have

$$\nabla\eta(x) = \begin{cases} 0 & \text{if } d(x) > \delta \\ \frac{1}{\tilde{x}(\delta)}\tilde{x}'(d(x))\nabla d(x) & \text{if } d(x) < \delta \end{cases}.$$

Hence, a.e., we have that $\|\nabla\eta\| \le \frac{1}{\tilde{x}(\delta)}\tilde{x}'(\delta)$ and that, either $\nabla\eta = 0$, or that

$$\eta = \frac{1}{\tilde{x}(\delta)}\tilde{x} = \frac{1}{\tilde{x}(\delta)}\tilde{x}'\frac{2U}{\partial f^*(\frac{1}{\tilde{x}'})} = \|\nabla\eta\|h(\tilde{x}(\delta)\|\nabla\eta\|)$$

with $h(z) = \frac{2U}{\partial f^*(\frac{1}{z})}$, an increasing function.

Set $F(\varepsilon,\xi) = f((1-\varepsilon\eta(x))\xi(x) + \varepsilon\|\nabla\eta(x)\|U)$. From the convexity of f, we obtain

(6)

$$F(\varepsilon,\xi)-f(\xi) \le \begin{cases} \varepsilon f'(\xi(1-\varepsilon\eta)+\varepsilon\|\nabla\eta\|U)[-\eta\xi+\|\nabla\eta\|U] & \text{if } -\eta\xi+\|\nabla\eta\|U>0 \\ \varepsilon f'(\xi)[-\eta\xi+\|\nabla\eta\|U] & \text{if } -\eta\xi+\|\nabla\eta\|U\le 0 \end{cases}.$$

In the second case, take K to be 0. In the first case, we cannot have $\nabla\eta = 0$, hence we have, a.e., $\eta = \|\nabla\eta\|h(\tilde{x}(\delta)\|\nabla\eta\|)$ and

$$f(\xi(1-\varepsilon\eta)+\varepsilon\|\nabla\eta\|U) - f(\xi) \le \varepsilon\|\nabla\eta\|f'(\xi(1-\varepsilon\eta)+\varepsilon\eta U)[-h(\tilde{x}(\delta)\|\nabla\eta\|)\xi + U].$$

In addition, from $-\eta\xi + \|\nabla\eta\|U > 0$, we infer $\xi \le \frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)}$, so that

$$\xi(1-\varepsilon\eta)+\varepsilon\|\nabla\eta\| \le \frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)} + \varepsilon\|\nabla\eta\|U$$

and

$$\|\nabla\eta\|f'(\xi(1-\varepsilon\eta)+\varepsilon\eta U) \le \|\nabla\eta\|f'(\frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)} + \varepsilon\|\nabla\eta\|U).$$

There exists σ such that, for $\|\nabla\eta\| < \sigma$, we have $\frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)} + \varepsilon\|\nabla\eta\|U \le \frac{2U}{h(\tilde{x}(\delta)\|\nabla\eta\|)}$. For those x such that $\|\nabla\eta(x)\| < \sigma$, recalling (5),

$$\|\nabla\eta\|f'(\frac{U}{h(\tilde{x}(\delta)\|\nabla\eta\|)} + \varepsilon\|\nabla\eta\|U) \le \|\nabla\eta\|f'(\frac{2U}{h(\tilde{x}(\delta)\|\nabla\eta\|)})$$

$$= \|\nabla\eta\|f'(\partial f^*(\frac{1}{\tilde{x}(\delta)\|\nabla\eta\|})) = \frac{1}{\tilde{x}(\delta)} = K.$$

It is left to consider the case $\|\nabla\eta\| \ge \sigma$: in this case, $\xi \le \frac{U}{h(\tilde{x}(\delta)\sigma)}$ and the result follows from the boundedness of $\|\nabla\eta\|$. □

Lemma 3. *Let ψ non negative and such that*

$$\int_O \psi e^{f(\psi)} f'(\psi) \le M.$$

Then, for any ξ such that $\int_O e^{f(\xi)}$ is bounded, we have that

$$\int_O \xi e^{f(\psi)} f'(\psi)$$

is bounded.

Proof. a) Consider the strictly increasing function $z(t) = f'(t)e^{f(t)}$ and call $t = i(z)$ its inverse, so that we have

$$z = e^{f(i(z))} f'(i(z)). \qquad (7)$$

We have that $i(v) \to \infty$ as $v \to \infty$. Define the function ϕ as $\phi(z) = i(z)z$, hence, in terms of t,

$$\phi(f'e^{f(t)}) = tf'e^{f(t)}. \qquad (8)$$

b) We wish to compute the polar g^* of the function $g(b) = e^{f(b)}$. Define b_z implicitely, setting

$$z = g'(b_z) = e^{f(b_z)} f'(b_z),$$

and notice that the previous equality defines b_z uniquely and we have $b_z = i(z)$, where i is defined in a). Then

$$g^*(z) = \sup_b bz - g(b) = b_z e^{f(b_z)} f'(b_z) - e^{f(b_z)} = b_z z - e^{f(b_z)} = i(z)z - e^{f(i(z))}$$

so that, by (8) and (7), $g^*(z) \le \phi(f'(b_z)e^{f(b_z)}) = \phi(f'(i(z))e^{f(i(z))}) = \phi(z)$.

For any t and b, we have

$$bf'(t)e^{f(t)} = bv(t) \le g^*(v(t)) + g(b) \le \phi(v(t)) + g(b)$$

Set, in the previous inequality, $t = \psi$ and $b = \xi$. From the definition of ϕ, we obtain

$$\begin{aligned} \xi f'(\psi)e^{f(\psi)} &\le \phi(f'(\psi)e^{f(\psi)}) + e^{f(\xi)} \\ &= \psi f'(\psi)e^{f(\psi)} + e^{f(\xi)}. \end{aligned}$$

From the assumptions of the Lemma, the proof is completed. □

Proof of Theorem 1. In the proof, we shall first prove the higher integrability result for the special case where $\xi(\cdot) = \|\nabla\tilde{u}(\cdot)\|$ and then extend this result to the general case.

a) Let O and O_δ as before. Set $U = \sup\{|\tilde{u}(x)| : x \in O_\delta\}$. Since $\tilde{u}$ is a minimum, for every variation v we have

$$\int_\Omega [e^{f(\|\nabla\tilde{u}(x)+\varepsilon\nabla v\|)} + g(x, \tilde{u}(x) + \varepsilon v(x))]\, dx \ge \int_\Omega [e^{f(\|\nabla\tilde{u}(x)\|)} + g(x, \tilde{u}(x))]\, dx.$$

set $v = -\eta\tilde{u}$, so that $\nabla v = -\tilde{u}\nabla\eta - \eta\nabla\tilde{u}$ and $|v| \le U$. For $\varepsilon > 0$ (and $\varepsilon < 1$), we obtain

(9)
$$\int_\Omega \left(\frac{e^{f(\|\nabla\tilde{u}(x)(1-\varepsilon\eta)-\varepsilon\tilde{u}\nabla\eta\|)} - e^{f(\|\nabla\tilde{u}(x)\|)}}{\varepsilon}\right) dx \ge -\int_\Omega \frac{g(x, \tilde{u}(x) + \varepsilon v(x)) - g(x, \tilde{u}(x))}{\varepsilon}.$$

b) By Lemma 2, η can be defined so that, for some $K \ge 0$, we have:

(10)
$$f(\|\nabla\tilde{u}(1-\varepsilon\eta) - \varepsilon\tilde{u}\nabla\eta\|) - f(\|\nabla\tilde{u}\|) \le f(\|\nabla\tilde{u}\|(1-\varepsilon\eta) + \varepsilon U\|\nabla\eta\|) - f(\|\nabla\tilde{u}\|) \le \varepsilon K.$$

Set $F(\varepsilon, \nabla\tilde{u}) = f((1 - \varepsilon\eta(x))\|\nabla\tilde{u}(x)\| + \varepsilon\|\nabla\eta(x)\|U)$. From (9), we have

$$\begin{aligned} -\int_\Omega \frac{g(x, \tilde{u}(x) + \varepsilon v(x)) - g(x, \tilde{u}(x))}{\varepsilon} &\le \int_\Omega \left(\frac{e^{F(\varepsilon,\nabla\tilde{u})} - e^{f(\|\nabla\tilde{u}(x)\|)}}{\varepsilon}\right) dx \\ &= \int_\Omega \left(\frac{e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K+\varepsilon K} - e^{f(\|\nabla\tilde{u}(x)\|)}}{\varepsilon}\right) dx \\ &= \int_\Omega e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K}\left[\frac{e^{\varepsilon K} - e^{f(\|\nabla\tilde{u}(x)\|)-F(\varepsilon,\nabla\tilde{u})+\varepsilon K)}}{\varepsilon}\right] dx = \\ \int_\Omega e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K} &\left[\frac{e^{\varepsilon K} - 1 + 1 - e^{f(\|\nabla\tilde{u}(x)\|)-F(\varepsilon,\nabla\tilde{u})+\varepsilon K)}}{\varepsilon}\right] dx. \end{aligned}$$

The previous inequality can be written as

$$\begin{aligned} (11) \qquad \int_\Omega e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K}\left[\frac{e^{\varepsilon K} - 1}{\varepsilon}\right] dx + \int_\Omega \frac{g(x, \tilde{u}(x) + \varepsilon v(x)) - g(x, \tilde{u}(x))}{\varepsilon} \ge \\ \int_\Omega e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K}\left[\frac{e^{f(\|\nabla\tilde{u}\|)-(F(\varepsilon,\nabla\tilde{u})-\varepsilon K)} - 1}{\varepsilon}\right] dx. \end{aligned}$$

c) From (10) we infer that $F(\varepsilon,\nabla\tilde{u})-\varepsilon K\le f(\|\nabla\tilde{u}(x)\|)$; moreover, $\frac{e^{\varepsilon K}-1}{\varepsilon}\le Ke^K$. In addition,

$$|\frac{g(x,\tilde{u}(x)+\varepsilon v(x))-g(x,\tilde{u}(x))}{\varepsilon}|=|g_u(x,u_{\varepsilon,x})\eta\tilde{u}(x)|$$

for some value $u_{\varepsilon,x}$ in the interval of extremes $\tilde{u}(x)$ and $\tilde{u}(x)-\varepsilon\eta(x)\tilde{u}(x)$, so that

$$|g_u(x,u_{\varepsilon,x})\eta(x)\tilde{u}(x)|\le[\alpha_U(x)]U.$$

Hence, the left hand side of (11) is bounded by some M, independent of ε.

d) Consider the right hand side. For some $t_{\varepsilon,x}$ in the interval of extremes $\|\nabla\tilde{u}\|$ and $(1-\varepsilon\eta)\|\nabla\tilde{u}\|+\varepsilon\|\nabla\eta\|U$, we have

$$f((1-\varepsilon\eta)\|\nabla\tilde{u}\|+\varepsilon\|\nabla\eta\|U)-f(\|\nabla\tilde{u}\|)=\varepsilon f'(t_{\varepsilon,x})(-\eta\|\nabla\tilde{u}\|+\|\nabla\eta\|U).$$

As $\varepsilon\to0$, $t_{\varepsilon,x}\to\|\nabla\tilde{u}(x)\|$ pointwise, so that $f'(t_{\varepsilon,x})$ converges to $f'(\|\nabla\tilde{u}(x)\|)$; moreover, $f(\|\nabla\tilde{u}\|)-(F(\varepsilon,\nabla\tilde{u})-\varepsilon K)=-\varepsilon f'(t_{\varepsilon,x})(-\eta\|\nabla\tilde{u}\|+\|\nabla\eta\|U)+\varepsilon K=-\varepsilon f'(\|\nabla\tilde{u}\|)(-\eta\|\nabla\tilde{u}\|+\|\nabla\eta\|U)+\varepsilon K+\varepsilon o(1)$, so that

$$\frac{e^{f(\|\nabla\tilde{u}\|)-(F(\varepsilon,\nabla\tilde{u})-\varepsilon K)}-1}{\varepsilon}$$

converges pointwise to $K+f'(\|\nabla\tilde{u}\|)(\eta\|\nabla\tilde{u}\|-\|\nabla\eta\|U)$. In addition, by (10), $\varepsilon K-f((1-\varepsilon\eta)\|\nabla\tilde{u}\|+\varepsilon\|\nabla\eta\|U)+f(\|\nabla\tilde{u}\|)\ge0$, so that the integrand at the right hand side is non negative. Finally, pointwise, $e^{F(\varepsilon,\nabla\tilde{u})-\varepsilon K}\to e^{f(\|\nabla\tilde{u}(x)\|)}$. Hence, applying Fatou's lemma, we obtain

$$\int_\Omega e^{f(\|\nabla\tilde{u}\|)}\left[K+f'(\|\nabla\tilde{u}\|)(\eta\|\nabla\tilde{u}\|-\|\nabla\eta\|U)\right]\le M.$$

Since $K+f'(\|\nabla\tilde{u}\|)(\eta\|\nabla\tilde{u}\|-\|\nabla\eta\|U)\ge0$, and $\nabla\eta=0$ and $\eta=1$ on O, we have obtained that

$$\int_O\|\nabla\tilde{u}\|e^{f(\|\nabla\tilde{u}\|)}f'(\|\nabla\tilde{u}\|)\le M_1. \tag{12}$$

This proves the result for the case $\xi=\|\nabla\tilde{u}\|$. An application of Lemma 3 completes the proof.

□

Notice that $\tilde{u}$ does not have to be a minimizer: a local minimizer would do.

References

[1] R. A. Adams, Sobolev spaces, Academic Press, New York, 1975

[2] G. Lieberman, *On the regularity of the minimizer of a functional with exponential growth.* Comment. Math. Univ. Carolin. 33 (1992), 45–49

[3] O. A. Ladyenskaja, N. N. Uralceva, Linear and quasi-linear equations of elliptic type. Izdat. "Nauka", Moscow 1964

[4] P. Marcellini , *Everywhere regularity for a class of elliptic systems without growth conditions.* Annali Sc. Norm. Sup. di Pisa 23 (1996) 1–25.

[5] P. Marcellini, G. Papi, *Nonlinear elliptic systems with general growth.* J. Differential Equations 221 (2006), no. 2, 412–443.

[6] R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1972.

[7] J. Serrin, *The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables.* Philos. Trans. Roy. Soc. London Ser. A 264 (1969) 413–496.

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano
E-mail address: arrigo.cellina@unimib.it

Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi 53, 20125 Milano
E-mail address: m.mazzola7@campus.unimib.it