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Marco Mazzola*

# Properties of Solutions to Variational Problems

Supervisor: Prof. Arrigo Cellina

*: Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, via Cozzi 53, 20125 Milano. e-mail *m.mazzola7@campus.unimib.it*

# Contents

# Chapter 1

# Introduction

This thesis deals with the properties satisfied by the solutions to minimization problems in the Calculus of Variations.

The typical structure of a variational problem is the following: minimize a functional of the type

$$\int_\Omega L\left(x, u\left(x\right), \nabla u\left(x\right)\right)\ \mathrm{d}x,$$

over the function space $u_0 + W_0^{1,1}\left(\Omega\right)$, where $L$ is a Carathéodory map and $u_0$ is an appropriate boundary datum.

Suppose, for the moment, that at least one minimizer $\overline{u}$ does exist. It would be interesting to know whether this solution of the variational problem satisfies any necessary conditions: the Euler-Lagrange equation is the most classical answer to this question, even if its validity has been proved, so far, only for particular cases and we are still far from having a general theorem about its validity.

Another question could be whether $\overline{u}$ belongs to any better function space than $W^{1,1}\left(\Omega\right)$. In other words: does $\overline{u}$ possess any further regularity properties? These issues are not independent, in fact most of the regularity properties follow from the necessary conditions satisfied by the solutions, mainly from the Euler-Lagrange equation. On the other hand, regularity properties could be used, in turn, to prove the validity of the Euler-Lagrange equation.

In the first part of this thesis we obtain the validity of necessary conditions in the form of the Euler-Lagrange equation, or of the Pontryagin maximum principle, for solutions to some particular variational problems; in the second part we prove regularity results concerning higher differentiability properties.

The thesis is organized as follows: in the present chapter we introduce the problems that we will treat in detail later, we recall a (far from being complete) list of known results, with their references, and we state our main results. In the following chapters we will provide detailed arguments and will present the full proofs of the results.

Most of the results exposed in this thesis are contained in the papers [10], [16], [17] and [18].

## 1.1 Necessary conditions for solutions to variational problems

Let us consider a variational problem (P) of the kind

$$\text{minimize} \int_{\Omega} L\left(x, u\left(x\right), \nabla u\left(x\right)\right) \, \mathrm{d}x,$$

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $u \in u_0 + W_0^{1,1}\left(\Omega\right)$ and $L\left(x, u, \xi\right) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory map, that is $L\left(\cdot, u, \xi\right)$ is measurable for fixed $\left(u, \xi\right)$ and $L\left(x, \cdot, \cdot\right)$ is continuous for almost every $x \in \Omega$. Moreover, assume that $L$ is convex with respect to $\xi$, differentiable with respect to $u$ and $\xi$ and such that also $L_u$ and $L_{\xi_i}$, $i = 1, \ldots, N$, are Carathéodory. Classical results show that, under reasonable regularity and growth assumptions from below on the Lagrangian $L$, problem (P) admits a solution $\overline{u}$: we say that $\overline{u}$ satisfies the classical Euler-Lagrange equation if for every $\eta \in C_c^{\infty}\left(\Omega\right)$, we have

$$\int_{\Omega} \left[\left\langle \nabla_{\xi} L\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right), \nabla \eta\left(x\right)\right\rangle + L_u\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right) \eta\left(x\right)\right] \, \mathrm{d}x = 0, \quad (1.1)$$

or, in a more compact form,

$$\mathrm{div}_x \nabla_{\xi} L\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right) = L_u\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right) \quad (1.2)$$

in the sense of distributions.

So far, without any further assumptions on the Lagrangian L, the validity of the Euler-Lagrange equation has to be considered a conjecture. Actually, even in the one-dimensional case, the result cannot be true in this full generality. In fact, a famous example by Ball and Mizel, [4], shows that there exist functionals whose minimizers do not satisfy (1.1).

The main problems occurring in the classical argument for the validity of the Euler-Lagrange equation are the following. Let $\overline{u}$ be a minimizer of the problem (P) and consider a variation $\eta$, that is a smooth function such that $\eta = 0$ at the boundary of $\Omega$. Using $\eta$ to explore the neighborhood of $\overline{u}$, for $\varepsilon$ positive one has

$$\int_{\Omega} \frac{L\left(x, \overline{u}\left(x\right) + \varepsilon\eta\left(x\right), \nabla\overline{u}\left(x\right) + \varepsilon\nabla\eta\left(x\right)\right) - L\left(x, \overline{u}\left(x\right), \nabla\overline{u}\left(x\right)\right)}{\varepsilon} \, \mathrm{d}x \geq 0. \quad (1.3)$$

The integrand converges pointwise to

$$\left[\left\langle \nabla_{\xi} L\left(x, \overline{u}\left(x\right), \nabla\overline{u}\left(x\right)\right), \nabla\eta\left(x\right)\right\rangle + L_u\left(x, \overline{u}\left(x\right), \nabla\overline{u}\left(x\right)\right)\eta\left(x\right)\right],$$

but a theorem that allows to pass to the limit under the integral sign preserving the inequality in (1.3) does not exist and it is easy to build counterexamples. Hence, this passing to the limit under the integral sign remains a difficult point.

Another question is: does the integral in (1.1) make sense? In other words, does the fact that

$$\int_{\Omega} L\left(x, \overline{u}\left(x\right), \nabla\overline{u}\left(x\right)\right) \, \mathrm{d}x < \infty$$

imply $\nabla_{\xi} L\left(\cdot, \overline{u}, \nabla\overline{u}\right)$ and $L_u\left(\cdot, \overline{u}, \nabla\overline{u}\right)$ are, at least locally, integrable? (We shall focus mainly on the term concerning the derivatives with respect to $\xi$, indeed in what follows we will assume

$$\left|L_u\left(x, u, \xi\right)\right| \leq KL\left(x, u, \xi\right),$$

a.e. $x \in \Omega, \forall\left(u, \xi\right) \in \mathbb{R} \times \mathbb{R}^N$).

These obstacles are classically overcome by requiring assumptions of growth type on the Lagrangian $L$, with respect to the variable $\xi$: for instance, for problems of the kind

$$\text{minimize} \int_{\Omega} \left[L\left(\nabla u\left(x\right)\right) + g\left(x, u\left(x\right)\right)\right] \, \mathrm{d}x, \quad (1.4)$$

it is possible to prove that if $L$ has polynomial growth, i.e. there exist $M > 0$ and $p > 1$ such that $\left|L\left(\xi\right)\right| \leq M\left(1 + \|\xi\|^p\right)$ for all $\xi \in \mathbb{R}^N$, from the convexity of $L$ it follows that there is $\overline{M}$ such that $\left\|\nabla_{\xi} L\left(\xi\right)\right\| \leq \overline{M}\left(1 + \|\xi\|^{p-1}\right)$. Under these growth conditions it is classical to prove that, if $\overline{u} \in W^{1,p}\left(\Omega\right)$, then the Euler-Lagrange equation holds for $\overline{u}$. In [13] Cellina shows that if $L$ in (1.4) is of at most exponential growth, then a slightly more accurate argument can be used to prove that the Euler-Lagrange equation holds.

Beyond exponential growth some special results have been obtained: Lieberman, [30], proves that minimizers of the functional

$$\int_{\Omega} \exp\left(\left|\nabla u\left(x\right)\right|^2\right) dx$$

are classical solutions to the corresponding Euler-Lagrange equation. Otherwise, one could try to obtain some regularity properties for the solution $\overline{u}$: for instance, if $\overline{u} \in W^{1,\infty}_{loc}(\Omega)$ one can pass to the limit in (1.3) under the integral sign, then it is easy to prove the validity of the Euler-Lagrange equation. Marcellini, in [33] and [34], gains local Lipschitzianity for solutions to variational problems requiring general growth conditions on the Lagrangian $L$. Cellina, [12], instead, proves global Lipschitzianity through conditions on the set $\Omega$ and the boundary datum $u_0$. More recently, weaker assumptions on $\Omega$ and on $u_0$ were introduced by Bousquet and Clarke, [11], to obtain local Lipschitzianity.

### 1.1.1 Necessary conditions without differentiability assumptions on the Lagrangian

Chapter 2 deals with the necessary conditions satisfied by a solution $\overline{u}$ to the problem of minimizing

$$\int_\Omega \left[f\left(\|\nabla u\left(x\right)\|\right) + g\left(x, u\left(x\right)\right)\right] dx \qquad \text{on } u_0 + W^{1,1}_0\left(\Omega\right) \tag{1.5}$$

where $f$ is a convex function defined on $\mathbb{R}^+$ and $g$ is a Carathéodory function, differentiable with respect to $u$, and whose derivative $g_u$ is also a Carathéodory function. The main point here is that we ask $f$ to be convex and to satisfy a growth assumption, but we do not require any differentiability properties on $f$. Indeed, in variational problems, the assumption of convexity of the Lagrangian is more acceptable than the assumption of differentiability. Now we no longer have the notion of differential but only the one of subdifferential: since in this case we cannot write the Euler-Lagrange equation in the classical form (1.1), what is the right statement of the necessary condition we should look for? A suggestion comes from the Pontryagin maximum principle for optimal control problems [36].

To show what we mean, consider a 1-dimensional domain $\Omega = [a, b] \subset \mathbb{R}$ and express our variational problem in terms of an optimal control problem

$$\min \int_a^b f\left(t, x\left(t\right), u\left(t\right)\right) dt, \quad x\left(a\right) = x_0, x\left(b\right) = x_1 \tag{1.6}$$

where the state $x$ and the control $u$ are linked by the differential condition $x'\left(t\right) = u\left(t\right)$ and the set of the admissible controls $U$ equals to the effective domain of $f$.

The Pontryagin maximum principle states that if $(\overline{x}, \overline{u})$ is a solution to (1.6), then there exist a non negative $p_0$ and a map $p \in W^{1,1}\left([a, b]\right)$ such that $(p_0, p) \neq (0, 0)$ and a.e. in $[a, b]$:

(i) $\dfrac{d}{dt} p\left(t\right) = p_0 \cdot \dfrac{\partial f}{\partial x}\left(t, \overline{x}\left(t\right), \overline{u}\left(t\right)\right);$

(ii) $-p_0 \cdot f\left(t, \overline{x}\left(t\right), \overline{u}\left(t\right)\right) + p\left(t\right) \overline{u}\left(t\right) = \max_{u \in U} \left\{-p_0 \cdot f\left(t, \overline{x}\left(t\right), u\right) + p\left(t\right) u\right\}$.

In the normal case $p_0 = 1$, when we pass to the problem of minimizing functionals of the form

$$\int_{\Omega} \left[F\left(\nabla u\left(x\right)\right) + g\left(x, u\left(x\right)\right)\right] dx$$

with $F$ a convex function defined on $\mathbb{R}^N$, one can conjecture that the suitable form of the Euler-Lagrange equations satisfied by a solution $\overline{u}$ should be

$$\exists p\left(\cdot\right) \in L^1\left(\Omega\right) : \operatorname{div} p\left(\cdot\right) = g_v\left(\cdot, u\left(\cdot\right)\right)$$

in the sense of distributions and, for a.e. $x$ and every $\xi \in \mathbb{R}^N$, we have

$$\left\langle p, \nabla u\left(x\right)\right\rangle - \left[F\left(\nabla u\left(x\right)\right) + g\left(x, u\left(x\right)\right)\right] \geq \left\langle p, \xi\right\rangle - \left[F\left(\xi\right) + g\left(x, u\left(x\right)\right)\right].$$

Equivalently, the condition can be expressed as

$$\exists p\left(\cdot\right) \in L^1\left(\Omega\right), \text{ a selection from } \partial F\left(\nabla u\left(\cdot\right)\right), \text{ such that } \operatorname{div} p\left(\cdot\right) = g_v\left(\cdot, u\left(\cdot\right)\right).$$

In this form, this condition is the equivalent of the Pontryagin Maximum Principle. The purpose of Chapter 2 is to prove this condition for the class of mappings under consideration. Clearly, this cannot be the right condition for those problems such that $Dom\left(F\right) \neq Dom\left(\partial F\right)$. Actually, the functionals we consider are such that the domain of $\partial F$ is the whole space. This is not the most general problem about the validity of necessary conditions for minimization problems, as (1.5), with $f$ convex: our $f$ is defined on $\mathbb{R}$ and in our result are not included problems with restrictions on $\nabla u$, as $\left\|\nabla u\right\| \leq 1$, that would require extended valued convex functions. This is a very active and difficult area of research, with very few results available [19], [6], [7].

Necessary conditions for problems similar to our problem (1.5) have been obtained by F. H. Clarke in [20]: his methods and results are different from ours, although there is some overlapping.

Very recently, the method exposed in Chapter 2 has been generalized in [8] to obtain analogous necessary conditions for general problems of type (P) involving nonradial lagrangians. The result is based on a variant of the Riesz representation Theorem.

### 1.1.2 Higher integrability for solutions to variational problems

In Chapter 3 we consider the variational problem (P) and we look for regularity properties for its solutions. More precisely, our investigation is about higher integrability properties for the gradient of a solution $\overline{u}$ of (P). As we have already noticed, in order to establish the validity of the Euler Lagrange equation for the solution to this problem, i.e., in order to prove that, for every admissible variation $\eta$, the equation

$$\int_\Omega \left[\langle \nabla_\xi L\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right), \nabla \eta\left(x\right)\rangle + L_u\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right)\eta\left(x\right)\right] \, \mathrm{d}x = 0, \tag{1.7}$$

holds, one has preliminarly to prove that the integrand is in $L^1$; hence, in particular, that $\langle \nabla_\xi L\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right), \nabla \eta\left(x\right)\rangle \in L^1_{loc}$. However, for Lagrangeans $L$ growing faster than exponential, the integrability of a term like

$$\int_\Omega L\left(x, u\left(x\right), \nabla u\left(x\right)\right) dx$$

does not imply the integrability of

$$\int_\Omega \nabla_\xi L\left(x, u\left(x\right), \nabla u\left(x\right)\right) dx.$$

In fact, consider $L\left(s\right) = e^{s^2}$, so that $L' = 2se^{s^2}$. For $N = 1$, the function $\xi\left(\cdot\right)$ whose derivative is

$$\xi'\left(t\right) = \sqrt{-\ln\left(|t|\left|\ln\left(t\right)\right|^{\frac{3}{2}}\right)}$$

is such that $e^{\xi'(t)^2} = \frac{1}{|t||\ln(t)|^{\frac{3}{2}}}$ is integrable on $\left(-\frac{1}{2}, \frac{1}{2}\right)$; however, for $|t|$ small,

$$\xi'\left(t\right)e^{\xi'(t)^2} = \frac{1}{|t|\left|\ln\left(t\right)\right|^{\frac{3}{2}}}\sqrt{-\ln\left(|t|\left|\ln\left(t\right)\right|^{\frac{3}{2}}\right)} >$$

$$\frac{1}{|t|\left|\ln\left(t\right)\right|^{\frac{3}{2}}}\sqrt{-\frac{1}{2}\left|\ln\left(t\right)\right|} = \frac{1}{\sqrt{2}\,|t|\left|\ln\left(t\right)\right|},$$

hence $L'\left(\xi'\left(\cdot\right)\right)$ is not locally integrable.

Obviously this problem does not occur when we are able to prove some additional regularity properties of the solution $\overline{u}$. For problems of the type

$$\text{minimize} \int_\Omega L\left(\|\nabla u\left(x\right)\|\right) \, \mathrm{d}x,$$

by using a barrier as in [39], under smoothness conditions on the boundary and on the second derivative of $L$ one can prove that the gradient of the solution is in $L^\infty\left(\Omega\right)$; alternatively, taking advantage of the regularity properties of solutions to elliptic equations, as in [30] for the case $L\left(t\right) = e^{t^2}$, and in [34], [35], under general growth conditions on $L$, one proves that the gradient of the solution is in $L^\infty_{loc}$.

When we do not have any further regularity results, the exponential growth of $L$ in (P), with respect to the variable $\xi$, has to be considered the limit case in which the integrability of $L\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right)$ garantees the local integrability of $\nabla_{\xi} L\left(x, \overline{u}\left(x\right), \nabla \overline{u}\left(x\right)\right)$. Indeed, this fact is valid for any function $u$, not just the solution $\overline{u}$. We wonder whether we can obtain any higher integrability properties for a minimizer; whether these properties still hold for minimizers of functionals having faster growth; finally, whether we can use these properties, that we obtain only using the fact that $\overline{u}$ is a minimum and not a solution of the Euler-Lagrange equation, to prove the validity of the Euler-Lagrange equation itself for functionals having growth faster than exponential.

More precisely, we prove that if $\left|L_u\left(x, u, \xi\right)\right| \leq K L\left(x, u, \xi\right)$ and either

i) $\left|\nabla_{\xi} L\left(x, u, \xi\right)\right| \leq K L\left(x, u, \xi\right)$ or

ii) $\exists \Lambda \in C^1\left(\mathbb{R}\right)$ such that $\left|\frac{\nabla L}{L}\left(x, u, \xi\right)\right| \leq K \frac{\Lambda'}{\Lambda}\left(\left|\xi\right|\right)$, the map $\Gamma\left(t\right) := \log \Lambda\left(t\right)$ is convex and
$$\int^{\infty} \frac{1}{p \partial \Gamma^*\left(p\right)} dp < \infty,$$

where $\partial \Gamma^*$ is the subdifferential of the polar of $\Gamma$, then a locally bounded solution $\overline{u}$ to the problem (P) satisfies
$$\left|\nabla \overline{u}\right| \left|\nabla_{\xi} L\left(\cdot, \overline{u}, \nabla \overline{u}\right)\right| \in L^1_{loc}\left(\Omega\right).$$

The assumptions i) and ii) allow growth slower than exponential as well as faster than exponential. In particular, in these cases the result garantees the existence of the integral in (1.1).

Our result is weaker than the local boundedness of $\nabla \overline{u}$, the result proved in [30], [34], [35]; however, it holds for a larger class of functionals, where, possibly, the stronger boundedness result might not hold. In fact, we do not assume further regularity on $L$ besides its being convex and differentiable: in particular, we do not assume the existence of a second derivative of $L$, nor we assume its strict convexity. Moreover, we allow also a dependence on $x$ and on $u$.

Our method of proof is based on a simple variation and on the properties of polarity. In particular, the validity of the Euler Lagrange equation related to problem (P) is not needed.

### 1.1.3 On the validity of the Euler-Lagrange equation

In Chapter 4 we look for necessary conditions satisfied by solutions to variational problems of type (P) with fast growth. We already saw that the exponential growth of $L$ with respect to the variable $\xi$ has to be considered a limit beyond which it is difficult to prove the validity of the Euler-Lagrange equation; in fact, in this case, the gradient $\nabla_\xi L$ grows faster, with respect to $\xi$, than the function $L$ itself. In Chapter 4 we succeed in overcoming this barrier in two cases. In section 4.1 we consider lagrangians satisfying a growth condition that allows faster than exponential growth; in this case we use the higher integrability properties obtained in Chapter 3 in order to prove the validity of the Euler-Lagrange equation. In section 4.2 we treat the same topic in the case of problems of the type

$$\text{minimize} \int_\Omega \left[L\left(\nabla u\left(x\right)\right) + g\left(x, u\left(x\right)\right)\right] dx \tag{1.8}$$

without growth assumptions on $L$.

Let us consider the problem (P). As remarked above, we obtain higher integrability for a minimizer $\overline{u}$ independently of the validity of the Euler-Lagrange equation. Actually, one would like to use it to establish the validity of this equation. Indeed, as we have noticed, for any admissible variation $\eta$ we have

$$\frac{L\left(x, \overline{u}\left(x\right) + \varepsilon\eta\left(x\right), \nabla\overline{u}\left(x\right) + \varepsilon\nabla\eta\left(x\right)\right) - L\left(x, \overline{u}\left(x\right), \nabla\overline{u}\left(x\right)\right)}{\varepsilon}$$

$$\longrightarrow \left[\left\langle\nabla_\xi L\left(x, \overline{u}\left(x\right), \nabla\overline{u}\left(x\right)\right), \nabla\eta\left(x\right)\right\rangle + L_u\left(x, \overline{u}\left(x\right), \nabla\overline{u}\left(x\right)\right)\eta\left(x\right)\right]$$

pointwise with respect to $x$ and

$$\left|\frac{L\left(x, \overline{u} + \varepsilon\eta, \nabla\overline{u} + \varepsilon\nabla\eta\right) - L\left(x, \overline{u}, \nabla\overline{u}\right)}{\varepsilon}\right|$$

$$\leq \left|\frac{\partial L}{\partial u}\left(x, \overline{u} + \overline{s}\varepsilon\eta, \nabla\overline{u}\right) \cdot \eta\right| + \left|\left\langle\nabla_\xi L\left(x, \overline{u} + \varepsilon\eta, \nabla\overline{u} + \overline{t}\varepsilon\nabla\eta\right), \nabla\eta\right\rangle\right| \tag{1.9}$$

where $0 < \overline{s}, \overline{t} < 1$. Suppose that, for every $U > 0$, there exists a comparison map $\Lambda \in C^1\left(\mathbb{R}\right)$, $\Lambda$ convex, such that

$$K_1\Lambda\left(\left|\xi\right|\right) \leq L\left(x, u, \xi\right) \leq K_2\Lambda\left(\left|\xi\right|\right)$$

$$K_1\Lambda'\left(\left|\xi\right|\right) \leq \left|\nabla_\xi L\left(x, u, \xi\right)\right| \leq K_2\Lambda'\left(\left|\xi\right|\right)$$

a.e. $x \in \Omega$, $\forall u \leq U$. We prove that if there exists $c > 0$ such that either

i) $\Lambda'\left(t\right) \leq c\Lambda\left(t\right)$ or

ii) $\frac{\Lambda'(t)}{\Lambda(t)}$ is non decreasing and, for every $t \geq t_0$, $\Lambda'\left(t\right) \leq c\left(1 + \log t\right)\Lambda\left(t\right)$, then the last term in (1.9) is bounded by $|\nabla \overline{u}| \, |\nabla_\xi L(x, \overline{u}, \nabla \overline{u})| \, |\nabla \eta|$. Hence, one can pass to the limit in (1.3) by higher integrability and dominated convergence.

We observe that assumption ii) allows faster than exponential growth: for instance, we obtain the validity of the Euler-Lagrange equation for the problem of minimizing

$$\int_\Omega L(x, u(x), \nabla u(x))\, dx,$$

when $L(x, u, \xi) \sim |\xi|^{|\xi|}$ as $|\xi| \to \infty$.

In Section 4.2 we prove the validity of the Euler-Lagrange equation related to the minimum problem (1.8) without requiring any growth assumptions on the Lagrangian $L$, but only on the mapping $u \mapsto g(x, u)$. This result is strongly inspired by a recent paper by Degiovanni and Marzocchi [22]. In that work the authors open a new path in order to prove that any minimizer of the functional

$$\int_\Omega L(\nabla u(x)) \, \mathrm{d}x + \varphi(u - u_0), \tag{1.10}$$

where $u \in u_0 + W_0^{1,p}(\Omega)$, $1 \leq p < \infty$ and $\varphi \in W^{-1,p'}(\Omega)$, satisfies the associated Euler-Lagrange equation

$$\int_\Omega \langle \nabla L(\nabla \overline{u}(x)), \nabla \eta \rangle \, \mathrm{d}x = -\varphi(\eta), \qquad \forall \eta \in \mathcal{C}_c^\infty(\Omega).$$

Their main assumptions are convexity and regularity of $L$, without any upper growth condition. At present, this work represents the border of knowledge about the validity of the Euler-Lagrange equation. We use some results of theirs as a main tool applied to the problem (1.8): here we require $L$ to be a convex map and $g$ to be a Carathéodory map such that $u \mapsto g(x, u)$ is concave for almost every $x \in \Omega$ and satisfies some growth assumptions. We prove that, for any minimizer $\overline{u}$ of (1.8), the associated Euler-Lagrange equation holds, i.e. that there exists a selection $\sigma(x)$ from the subdifferential $\partial g(x, \overline{u}(x))$ such that, for every $\eta \in C_c^\infty(\Omega)$, we have

$$\int_\Omega \langle \nabla L(\nabla \overline{u}(x)), \nabla \eta(x) \rangle \, \mathrm{d}x = -\int_\Omega \sigma(x)\, \eta(x) \, \mathrm{d}x.$$

Our result generalizes the case considered by Degiovanni and Marzocchi: in fact, their functional $\varphi \in W^{-1,p'}(\Omega)$, appearing in (1.10), is replaced here by a more general map $u \mapsto \int g(x, u(x)) \, \mathrm{d}x$.
Functionals of this type, with the same concavity assumption, were considered by Cellina and Colombo, [15], but their purpose was to prove existence of solutions and the domain of integration was one dimensional.

## 1.2 Higher differentiability of solutions to variational problems

In Part II of the present work, we use some of the results of the previous sections to investigate the higher differentiability properties of minimizers of large classes of functionals.

In order to prove regularity for a solution $\overline{u}$ to a variational problem, e.g. local Lipschitzianity of $\overline{u}$ or local Hölderianity of $\nabla\overline{u}$, the first step is often that of estabilishing the (local) existence of weak second derivatives. To illustrate this process, let us consider the special functional

$$\int_\Omega F\left(\nabla u\left(x\right)\right)dx \tag{1.11}$$

and suppose that $\overline{u}$, a solution to the problem of minimizing (1.11), has weak second derivatives. Let us suppose $F$ is a smooth function satisfying

$$\left|\nabla F\left(\xi\right)\right| \leq K_2\left|\xi\right|$$

$$K_1\left|z\right|^2 \leq \sum_{i,j=1}^{N} F_{\xi_i\xi_j}\left(\xi\right)z_iz_j \leq K_2\left|z\right|^2$$

so that $\overline{u}$ satisfies the Euler-Lagrange equation

$$\int_\Omega \left\langle \nabla_\xi F\left(\nabla u\right), \nabla\varphi\right\rangle = 0$$

for all $\varphi \in W_0^{1,2}\left(\Omega\right)$. Since $\overline{u}$ admits weak second derivatives, we can take $\varphi = \frac{\partial}{\partial x_k}\psi$, with $\psi \in W_0^{2,2}\left(\Omega\right)$, and integrate by parts to obtain

$$\int_\Omega \left\langle H_F\left(\nabla\overline{u}\right)\cdot\left(\nabla\overline{u}\right)_{x_k}, \nabla\psi\right\rangle = 0,$$

where $H_F$ is the Hessian matrix of $F$. Recalling the assumptions on $F$, we have obtained that $w = \overline{u}_{x_k}$ satisfies an elliptic differential linear equation with bounded measurable coefficients. By De Giorgi Theorem [21], $w$ is locally Hölder continuous, then $\overline{u} \in C_{loc}^{1,\delta}\left(\Omega\right)$ for some $\delta$. Moreover, by induction, it is possible to prove that $F \in C^{m,\alpha}$ implies $\overline{u} \in C_{loc}^{m,\alpha}\left(\Omega\right)$. In particular, $F \in C^\infty$ implies $\overline{u} \in C^\infty$.

Classical results can be found in the book by Ladyzhenskaya and Uraltseva [28]. Here the authors consider Lagrangians that, beyond an ellipticity condition, have polynomial growth at infinity. More precisely, they consider smooth mappings $L$

satisfying the so called *natural growth condition*, that is there exist $K_i = K_i\left(|u|\right) > 0$, $i = 1, 2$, and $p > 1$ such that

$$K_1\left|\xi\right|^p \leq L\left(x, u, \xi\right) \leq K_2\left(1 + |\xi|\right)^p$$

$$K_1\left(1 + |\xi|\right)^{p-2}|z|^2 \leq \sum_{i,j=1}^{N} L_{\xi_i \xi_j}\left(x, u, \xi\right) z_i z_j \leq K_2\left(1 + |\xi|\right)^{p-2}|z|^2$$

for all $z \in \mathbb{R}^N$. They prove, among other results, that if $\overline{u}$ is a locally bounded solution to the problem of minimizing

$$\int_\Omega L(x, u(x), \nabla u(x))\,\mathrm{d}x \tag{1.12}$$

in a bounded region $\Omega$ of $\mathbb{R}^N$, it belongs to the class $W^{2,2}\left(\omega\right)$, where $\omega$ is an arbitrary interior subregion of $\Omega$.

Our aim is the investigation of the differentiability properties of solutions to variational problems having non-power growth of $L$ with respect to $|\xi|$.

In [31], [32] Marcellini introduces and investigates the regularity problem for solutions of minimizing problems of the type (1.11) satisfying the so called $p, q$-*growth condition*: there exist $K_1, K_2 > 0$ and $q \geq p \geq 2$ such that

$$K_1\left(1 + |\xi|\right)^{p-2}|z|^2 \leq \left\langle H_F\left(\xi\right) z, z\right\rangle \leq K_2\left(1 + |\xi|\right)^{q-2}|z|^2 \qquad \forall \xi, z \in \mathbb{R}^N.$$

By assuming that $p$, $q$ and $n$ satisfy the condition $\frac{q}{p} < \frac{n}{n-2}$, the author proves that any solution of (1.11) belonging to $W^{1,q}_{loc}\left(\Omega\right)$ has weak second derivatives and is locally Lipschitz-continuous in $\Omega$. $W^{2,2}$ regularity for solutions of systems satisfying $p, q$-growth condition is proved also in [5].

Anisotropic conditions have been the first non-standard growth conditions being investigated. Our results do not apply in these situations: we are interested in Lagrangians having growth faster than polynomial at infinity. These variational problems are considered in [29] and [33]. In [29] Lieberman gives a natural generalization of the natural conditions of Ladyzhenskaya and Uraltseva. In [33] Marcellini considers integrals of the type (1.12) where $L \in C^2\left(\Omega \times \mathbb{R} \times \mathbb{R}^N\right)$ satisfies, besides other assumptions, the *general growth condition*

$$K_1 g_1\left(|\xi|\right)|z|^2 \leq \sum_{i,j=1}^{N} L_{\xi_i \xi_j}\left(x, u, \xi\right) z_i z_j \leq K_2 g_2\left(|\xi|\right)|z|^2,$$

where $g_1$ and $g_2$ are real maps linked by a condition involving the dimension $N$. It is proved that if the minimizer $\overline{u}$ of (1.12) satisfies the corresponding Euler-Lagrange equation and

$$\int_\omega g_2\left(|\nabla \overline{u}|\right) \cdot \left(1 + |\nabla \overline{u}|^2\right) < \infty$$

for all $\omega \subset\subset \Omega$, then it belongs to $W^{2,2}_{loc}(\Omega)$ and it is locally Lipschitz continuous.

### 1.2.1 Higher differentiability of solutions to variational problems of faster growth

In Chapter 5 we prove local existence and properties of the weak second derivatives for solutions to variational problems of the kind (1.12). Here we assume that there exist constants $K_1, K_2 > 0$ and a comparison function $\Lambda : \mathbb{R} \to \mathbb{R}$, satisfying certain properties, such that

$$\Lambda'(|\xi|) \leq K_2 |\nabla_\xi L(x,u,\xi)|$$

$$K_1 \frac{\Lambda'(|\xi|)}{|\xi|} |z|^2 \leq \sum_{i,j=1}^{N} L_{\xi_i \xi_j}(x,u,\xi) z_i z_j \leq K_2 \Lambda''(|\xi|) |z|^2$$

for all $z \in \mathbb{R}^N$. Our assumptions on $\Lambda$ allow small perturbations of polynomial growth, e.g. $\Lambda(t) = \frac{1}{2} t^2 \log(t+e)$, as well as slow exponential growth, e.g. $\Lambda(t) \sim e^{t^\alpha}$, $\alpha < 1$.

The method is a variant of the well known method based on difference quotients

$$\delta^i_h u(x) = \frac{u(x+he_i) - u(x)}{h},$$

that gives a characterization of Sobolev spaces. The key point here is that we do not use as an admissible variation the map $\phi = \delta^i_{-h}\left(\eta^2 \cdot \delta^i_h u\right)$ as usual, but the function

$$\phi = \delta^i_{-h}\left(\eta^2 \cdot \gamma_\Lambda\left(\delta^i_h u\right)\right),$$

where $\gamma_\Lambda$ is the (only) solution of the differential equation

$$\gamma'(t) = \frac{\Lambda''(t)}{t \cdot \Lambda'(t)} \cdot \gamma^2(t)$$

such that $\lim_{t\to 0^+} \gamma(t) = 0$ and $\lim_{t\to\infty} \gamma(t) = +\infty$. This function turns out to be exactly $c \cdot t$ for the case $L$ is of power growth, but does exist also in the other cases covered by our assumptions.

### 1.2.2 Higher differentiability for minimizers of irregular functionals

Finally, Chapter 6 deals with regularity properties for solutions to irregular variational problems. This issue has been studied in [25]: here the authors consider functionals of the type

$$\int_\Omega F(\nabla u(x))\, dx, \tag{1.13}$$

where $F$ is a continuous function satisfying

$$K_1 \left|\xi\right|^p \leq F\left(\xi\right) \leq K_2 \left(1 + \left|\xi\right|\right)^p$$

and they prove that any minimizer of (1.13) is locally Lipschitz continuous. The main point of this theorem is that the authors do not need to assume further assumptions on $F$, with the exception of a condition concerning uniform convexity. This assumption does not depend on the existence of any kind of derivatives. Similar results have been generalized in [23] and [24] for functionals verifying the $p, q$-growth condition.

In these papers the lack of smoothness for $F$ is overcome by approximating it with a sequence of smooth functions $F_c$, uniformly converging to $F$ on compact sets: for the approximate minimizers $\overline{u}_c$ standard regularity theory holds. If one succeeds in proving estimates for $\overline{u}_c$, uniform with respect to $c$, then, passing to the limit, the same estimate for the minimizer of the original problem holds.

We consider functionals of the form

$$\int_\Omega \left[a\left(x\right) f\left(\left|\nabla u\left(x\right)\right|\right) + g\left(x, u\left(x\right)\right)\right] dx, \tag{1.14}$$

where $a$ is locally Lipschitz continuous and $g$ satisfies some growth conditions. We suppose $f$ is a convex real map having quadratic growth, but we let it be not differentiable in a finite number of points of $\mathbb{R}$, possibly including the origin 0. We prove that any minimizer of (1.14) admits weak second derivatives. As noticed above, we treat these irregular variational problems also in Chapter 2, where we prove the validity of a suitable form of the Euler-Lagrange equation: here we do not make use of it, since we prove the result through an approximating argument, and we have the validity of the Euler-Lagrange equation in the classical form for the approximating functionals.

# Part I

# Necessary conditions

# Chapter 2

# Necessary conditions without differentiability assumptions on the Lagrangian

This chapter is based on the paper *Necessary conditions for solutions to variational problems*, SIAM J. Control Optim. **48** (2009), no. 5, 2977-2983, joint work with A. Cellina.

In what follows, $B[0,1]$ denotes the closed unit ball of $\mathbb{R}^N$. We set $F(\xi) = f(\|\xi\|)$, $f$ being a convex function, and

$$\partial f^+(t) = \sup\{\lambda : \lambda \in \partial f(t)\},$$

$$\partial f^-(t) = \inf\{\lambda : \lambda \in \partial f(t)\},$$

where $\partial f(t)$ is the subdifferential of $f$ at the point $t \in \mathbb{R}$, defined by

$$\partial f(t) = \{s \in \mathbb{R} : f(x) \geq f(t) + \langle s, x - t\rangle, \ \forall x \in \mathbb{R}\}.$$

We consider mappings satisfying the following exponential growth condition.
**Assumption A**. The convex function $f$ is such that there exist $K$ and $t_0$ such that, for $t \geq t_0$,

$$\partial f^+(t) \leq K f(t).$$

**Theorem 1** *Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$. Let $f : \mathbb{R} \to \mathbb{R}$ be symmetric, convex, and satisfying the growth assumption A. Let $g(\cdot,\cdot)$ and $g_v(\cdot,\cdot)$ be Carathéodory functions and assume that for every $U$ there exists $\xi_U \in L^1_{loc}$ such* *that* $|v| \leq U$ *implies* $|g_v(x,v)| \leq \xi_U(x)$*. Let* $u$ *be a locally bounded solution to the problem of minimizing*

$$\int_\Omega [f(\|\nabla v(x)\|) + g(x, v(x))]\, dx \qquad on\ v_0 + W_0^{1,1}(\Omega). \tag{2.1}$$

*Then there exists* $p \in L^1(\Omega)$*, a selection from the map* $x \to \partial F(\nabla u(x))$*, such that*

$$div\ p(\cdot) = g_v(\cdot, u(\cdot))$$

*in the sense of distributions.*

**Remarks**:

**(i)** *Notice that, although* $p$ *has a weak divergence, there is no claim that it belongs to* $W^{1,1}(\Omega)$*.*

**(ii)** *The classical argument for the validity of the Euler-Lagrange equation, that is passing to the limit under the integral sign, fails when* $F$ *is not differentiable, since* $p$ *is not defined as a pointwise limit. The proof of the Theorem relies on a decomposition of the domain and the use of the Hahn-Banach and Riesz representation theorems.*

**(iii)** *As described in the Introduction, the condition stated in the Theorem is the equivalent of the normal form of the Pontryagin maximum principle for the multidimensional case.*

*Proof*: By the assumption of convexity, $f$ is not differentiable at most on a countable set, possibly containing 0. Set $k_0 = 0$ and call $k_i$ the other points of non differentiability for $f$. Set

$$A_i = \{x : \|\nabla u(x)\| = k_i\}$$

and $B = \Omega \setminus (\cup_i A_i)$. Fix $\eta \in C_0^1(\Omega)$, and set

$$A_i^+ = \{x \in A_i : \langle \nabla u, \nabla \eta \rangle \geq 0\}$$

and

$$A_i^- = \{x \in A_i : \langle \nabla u, \nabla \eta \rangle < 0\} :$$

this partition of $A_i$ depends on $\eta$. For $\varepsilon > 0$, we have

$$\frac{1}{\varepsilon}\left\{\int_\Omega [f(\|\nabla u + \varepsilon \nabla \eta\|) + g(x, u + \varepsilon\eta) - f(\|\nabla u\|) - g(x,u)]\right\} \geq 0. \tag{2.2}$$

Consider a compact set $O$ containing $\operatorname{supp}(\eta)$; set $D = \sup\{\|\nabla\eta\|\}$, $U = \sup\{|u|\}$ and $H = \sup\{|\eta|\}$ on $O$. Then

$$\left|\frac{g(x, u(x) + \varepsilon\eta(x)) - g(x, u(x))}{\varepsilon}\right| \leq H\xi_{U+\varepsilon H}(x)$$

so that, by dominated convergence,

$$\int_\Omega \frac{g(x, u(x) + \varepsilon\eta(x)) - g(x, u(x))}{\varepsilon}\, dx \longrightarrow \int_\Omega g_v(x, u(x))\, \eta(x)\, dx.$$

When $x \in B$, $f$ is differentiable at $\|\nabla u(x)\|$ and

$$\frac{f(\|\nabla u + \varepsilon\nabla\eta\|) - f(\|\nabla u\|)}{\varepsilon} \longrightarrow f'(\|\nabla u\|)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle.$$

On $A_0$, as $\varepsilon \to 0^+$, we have that

$$\frac{f(\|\nabla u + \varepsilon\nabla\eta\|) - f(\|\nabla u\|)}{\varepsilon} = \frac{f(\|\varepsilon\nabla\eta\|) - f(0)}{\varepsilon} \longrightarrow \partial f^+(0)\,\|\nabla\eta\|,$$

pointwise with respect to $x$, while, for $x \in A_i$, $i = 1, \ldots, \infty$, we have

$$\frac{f(\|\nabla u + \varepsilon\nabla\eta\|) - f(\|\nabla u\|)}{\varepsilon} \longrightarrow \partial f^-(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle,$$

when $\langle\nabla u, \nabla\eta\rangle < 0$, and

$$\frac{f(\|\nabla u + \varepsilon\nabla\eta\|) - f(\|\nabla u\|)}{\varepsilon} \longrightarrow \partial f^+(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle$$

otherwise.

Moreover, for any $x$, we have

$$\begin{aligned}\left|\frac{f(\|\nabla u + \varepsilon\nabla\eta\|) - f(\|\nabla u\|)}{\varepsilon}\right| &= \left|\frac{f(\|\nabla u\| + \theta(\varepsilon, x)) - f(\|\nabla u\|)}{\varepsilon}\right| \\ &= s(x)\,\frac{|\theta(\varepsilon, x)|}{\varepsilon},\end{aligned}$$

where $|\theta(\varepsilon, x)| \leq \varepsilon D$, and for some $s(x) \in \partial f(\xi(x))$, with $\xi(x)$ in the interval of extremes $\|\nabla u(x)\|$ and $\|\nabla u(x)\| + \theta(\varepsilon, x)$. Consider assumption A. Then, either

$$\max\{\|\nabla u(x)\|, \|\nabla u(x)\| + \theta(\varepsilon, x)\} \leq t_0 + D,$$

and in this case $s(x) \leq \partial f^+(t_0 + D)$, or

$$\max\{\|\nabla u(x)\|, \|\nabla u(x)\| + \theta(\varepsilon, x)\} > t_0 + D,$$

i.e. both $\|\nabla u(x)\|$ and $\xi(x)$ are $> t_0$, so that

$$f(\xi(x)) \leq f(\|\nabla u(x)\|)\, e^{K\varepsilon D} \leq f(\|\nabla u(x)\|)\, e^{KD}$$

and

$$\partial f^+(\xi(x)) \leq K f(\|\nabla u(x)\|)\, e^{KD}.$$

Hence

$$\left|\frac{f(\|\nabla u + \varepsilon \nabla \eta\|) - f(\|\nabla u\|)}{\varepsilon}\right| \leq \max\left\{D\partial f^+(t_0 + D), DKf(\|\nabla u\|)\, e^{KD}\right\},$$

an integrable function independent of $\varepsilon$. By dominated convergence, from (2.2), we obtain

$$\int_{A_0} \partial f^+(0)\, \|\nabla\eta\| + \sum_{i=1}^{\infty}\left[\int_{A_i^+} \partial f^+(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle + \int_{A_i^-} \partial f^-(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle\right]$$

$$+ \int_B f'(\|\nabla u\|)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle + \int_\Omega g_v(x,u)\,\eta \geq 0.$$

The same considerations, when applied to the variation $-\eta$, yield

$$\int_{A_0} \partial f^+(0)\, \|\nabla\eta\| - \sum_{i=1}^{\infty}\left[\int_{A_i^-} \partial f^+(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle + \int_{A_i^+} \partial f^-(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle\right]$$

$$- \int_B f'(\|\nabla u\|)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle - \int_\Omega g_v(x,u)\,\eta \geq 0.$$

From these two inequalities we obtain

$$-\int_{A_0} \partial f^+(0)\, \|\nabla\eta\| - \sum_{i=1}^{\infty}\left[\int_{A_i^+} \partial f^+(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle + \int_{A_i^-} \partial f^-(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle\right]$$

$$\leq \int_B f'(\|\nabla u\|)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle + \int_\Omega g_v(x,u)\,\eta \leq \int_{A_0} \partial f^+(0)\, \|\nabla\eta\| -$$

$$-\sum_{i=1}^{\infty}\left[\int_{A_i^-} \partial f^+(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle + \int_{A_i^+} \partial f^-(k_i)\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle\right].$$

Adding the term

$$\sum_{i=1}^{\infty}\int_{A_i} \frac{1}{2}\left[\partial f^+(k_i) + \partial f^-(k_i)\right]\left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta\right\rangle$$

to all sides, we have the following estimate, independent on the partition $A_i^+$ and $A_i^-$:

$$-\int_{A_0} \partial f^+(0)\,\|\nabla\eta\| - \sum_{i=1}^{\infty}\int_{A_i}\frac{1}{2}\left[\partial f^+(k_i)-\partial f^-(k_i)\right]\left|\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta\right\rangle\right| \tag{2.3}$$

$$\le \int_B f'(\|\nabla u\|)\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta\right\rangle + \sum_{i=1}^{\infty}\int_{A_i}\frac{1}{2}\left[\partial f^+(k_i)+\partial f^-(k_i)\right]\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta\right\rangle$$

$$+\int_\Omega g_v(x,u)\,\eta$$

$$\le \int_{A_0} \partial f^+(0)\,\|\nabla\eta\| + \sum_{i=1}^{\infty}\int_{A_i}\frac{1}{2}\left[\partial f^+(k_i)-\partial f^-(k_i)\right]\left|\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta\right\rangle\right|.$$

Set

$$X=\Bigg\{(v,w)\in L^1(A_0,\mathbb{R}^n)\times L^1\left(\bigcup_{i=1}^{\infty}A_i,\mathbb{R}\right):$$

$$\exists\eta\in C_0^1(\Omega)\ \text{ such that } v=\partial f^+(0)\,\nabla\eta|_{A_0},$$

$$w|_{A_i}=\left[\partial f^+(k_i)-\partial f^-(k_i)\right]\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta\right\rangle\Bigg|_{A_i},\, i=1\ldots,\infty\Bigg\}.$$

$X$ is a linear subspace of $L^1(A_0,\mathbb{R}^n)\times L^1\left(\bigcup_{i=1}^{\infty}A_i,\mathbb{R}\right)$.

Define the map $T: X\to\mathbb{R}$ as follows:

$$T(v,w)=-\int_B f'(\|\nabla u\|)\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta\right\rangle \tag{2.4}$$

$$-\sum_{i=1}^{\infty}\frac{1}{2}\int_{A_i}\left[\partial f^+(k_i)+\partial f^-(k_i)\right]\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta\right\rangle-\int_\Omega g_v(x,u)\,\eta.$$

We claim that $T$ is well defined and that it is a continuous linear functional on $X$.

In fact, consider $(v,w)$ in $X$, and assume that there exist $\eta_1$ and $\eta_2$ such that $v=\partial f^+(0)\,\nabla\eta_1|_{A_0}=\partial f^+(0)\,\nabla\eta_2|_{A_0}$ and

$$\begin{aligned} w|_{A_i} &= \left[\partial f^+(k_i)-\partial f^-(k_i)\right]\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta_1\right\rangle\Bigg|_{A_i}\\ &= \left[\partial f^+(k_i)-\partial f^-(k_i)\right]\left\langle \frac{\nabla u}{\|\nabla u\|},\nabla\eta_2\right\rangle\Bigg|_{A_i}. \end{aligned}$$

Applying the estimate (2.3) to the variation $\eta_1 - \eta_2$ we have

$$\begin{aligned}\left| \int_B f'\left(\|\nabla u\|\right) \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta_1 - \nabla\eta_2 \right\rangle \right. &+ \sum_{i=1}^{\infty} \frac{1}{2} \int_{A_i} \left[\partial f^+\left(k_i\right) + \partial f^-\left(k_i\right)\right] \\ \cdot \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta_1 - \nabla\eta_2 \right\rangle &+ \left. \int_\Omega g_v\left(x,u\right)\left[\eta_1 - \eta_2\right] \right| \\ \leq \int_{A_0} \partial f^+(0) \left\|\nabla\eta_1 - \nabla\eta_2\right\| &+ \sum_{i=1}^{\infty} \int_{A_i} \frac{1}{2} \left[\partial f^+\left(k_i\right) - \partial f^-\left(k_i\right)\right] \\ \cdot \left| \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta_1 - \nabla\eta_2 \right\rangle \right| &= 0,\end{aligned}$$

so that $T$ is well defined. It is clearly linear and, from

$$\left|T\left(v,w\right)\right| \leq \int_{A_0} \|v\| + \frac{1}{2} \int_{\bigcup A_i} |w| \quad \forall\left(v,w\right) \in X,$$

it is bounded. Hence, by the Hahn-Banach theorem, there exists $L$, a continuous linear functional on $L^1\left(A_0, \mathbb{R}^n\right) \times L^1\left(\cup_{i=1}^{\infty} A_i, \mathbb{R}\right)$, such that $L|_X \equiv T$ and

$$\left|L\left(v,w\right)\right| \leq \int_{A_0} \|v\| + \frac{1}{2} \int_{\cup A_i} |w|$$

for every $(v,w) \in L^1\left(A_0, \mathbb{R}^n\right) \times L^1\left(\bigcup_{i=1}^{\infty} A_i, \mathbb{R}\right)$.

Let us define $L^* : L^1\left(A_0, \mathbb{R}^n\right) \to \mathbb{R}$, setting

$$L^*\left(v\right) = L\left(v,0\right)$$

and $L^{**} : L^1\left(\bigcup_{i=1}^{\infty} A_i, \mathbb{R}\right) \to \mathbb{R}$, setting

$$L^{**}\left(w\right) = L\left(0,w\right).$$

We have that

$$\left|L^*\left(v\right)\right| \leq \int_{A_0} \|v\| \quad \forall v \in L^1\left(A_0, \mathbb{R}^n\right)$$

and

$$\left|L^{**}\left(w\right)\right| \leq \frac{1}{2} \int_{\bigcup_{i=1}^{\infty} A_i} |w| \quad \forall w \in L^1\left(\bigcup_{i=1}^{\infty} A_i, \mathbb{R}\right),$$

so that $\|L^*\| \leq 1$ and $\|L^{**}\| \leq \frac{1}{2}$.

By Riesz's Theorem, there exists $\alpha \in L^\infty\left(A_0, \mathbb{R}^n\right)$, supess $\|\alpha\| \leq 1$, such that, for every $v \in L^1\left(A_0, \mathbb{R}^n\right)$,

$$L^*\left(v\right) = \int_{A_0} \left\langle \alpha, v \right\rangle$$

and there exists $\beta \in L^{\infty}\left(\cup_{i=1}^{\infty} A_i, \mathbb{R}\right)$, with $|\beta| \leq \frac{1}{2}$ a.e., such that, for every $w \in L^1\left(\bigcup_{i=1}^{\infty} A_i, \mathbb{R}\right)$,

$$L^{**}(w) = \int_{\bigcup A_i} \beta w.$$

Hence, we can conclude that, for $\eta \in C_0^1(\Omega)$, we have

$$\begin{aligned}
&T\left(\partial f^{+}(0)\, \nabla\eta|_{A_0}, \left[\partial f^{+}(k_i) - \partial f^{-}(k_i)\right] \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta \right\rangle\Big|_{\bigcup A_i}\right) \\
&= L\left(\partial f^{+}(0)\, \nabla\eta|_{A_0}, \left[\partial f^{+}(k_i) - \partial f^{-}(k_i)\right] \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta \right\rangle\Big|_{\bigcup A_i}\right) \\
&= L\left(\partial f^{+}(0)\, \nabla\eta|_{A_0}, 0\right) + L\left(0, \left[\partial f^{+}(k_i) - \partial f^{-}(k_i)\right] \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta \right\rangle\Big|_{\bigcup A_i}\right) \\
&= L^{*}\left(\partial f^{+}(0)\, \nabla\eta|_{A_0}\right) + L^{**}\left(\left[\partial f^{+}(k_i) - \partial f^{-}(k_i)\right] \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta \right\rangle\Big|_{\bigcup A_i}\right) \\
&= \int_{A_0} \partial f^{+}(0) \langle \alpha, \nabla\eta \rangle + \int_{\cup A_i} \beta \left[\partial f^{+}(k_i) - \partial f^{-}(k_i)\right] \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta \right\rangle.
\end{aligned}$$

Equating the definition (2.4) to the equality above, we obtain

$$\begin{aligned}
&\int_{A_0} \partial f^{+}(0) \langle \alpha, \nabla\eta \rangle \\
&+ \sum_{i=1}^{\infty} \int_{A_i} \left[\frac{1}{2}\left[\partial f^{+}(k_i) + \partial f^{-}(k_i)\right] + \beta\left[\partial f^{+}(k_i) - \partial f^{-}(k_i)\right]\right] \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta \right\rangle \\
&+ \sum_{i=0}^{\infty} \int_{B_i} f'(\|\nabla u\|) \left\langle \frac{\nabla u}{\|\nabla u\|}, \nabla\eta \right\rangle + \int_{\Omega} g_v(x, u)\, \eta = 0.
\end{aligned}$$

Since

$$\partial F(\xi) = \begin{cases} \partial f^{+}(0)\, B[0,1] & \text{for } \xi = 0 \\ \{b\frac{\xi}{\|\xi\|} : \partial f^{-}(k_i) \leq b \leq \partial f^{+}(k_i)\} & \text{for } \|\xi\| = k_i \\ f'(\|\xi\|)\, \frac{\xi}{\|\xi\|} & \text{otherwise} \end{cases},$$

from the properties of $\alpha$ and $\beta$ we have that the map

$$\begin{aligned}
p(x) &= \partial f^{+}(0)\, \alpha(x)\, \chi_{A_0}(x) + \sum_{i=1}^{\infty} \left[\frac{1}{2}\left[\partial f^{+}(k_i) + \partial f^{-}(k_i)\right]\right. \\
&+ \left. \beta(x)\left[\partial f^{+}(k_i) - \partial f^{-}(k_i)\right]\right] \frac{\nabla u(x)}{\|\nabla u(x)\|} \chi_{A_i}(x) \\
&+ f'(\|\nabla u(x)\|)\, \frac{\nabla u(x)}{\|\nabla u(x)\|} \chi_B(x)
\end{aligned}$$

is a selection from $\partial F\left(\nabla u\left(x\right)\right)$ and

$$\int_{\Omega}\left[\left\langle p\left(x\right),\nabla\eta(x)\right\rangle + g_v\left(x,u\right)\eta\left(x\right)\right]\,dx = 0$$

for every $\eta \in C_c^1(\Omega)$. Moreover, from our assumptions on $g$ and the local boundedness of $u$, we have that $f\left(\left\|\nabla u\left(\cdot\right)\right\|\right) \in L^1\left(\Omega\right)$; then, from assumption A, we obtain that every selection from $\partial f\left(\left\|\nabla u\left(\cdot\right)\right\|\right)$ is integrable, thus proving the Theorem.

□

**Example**. Let $\Omega$ be a bounded open subset of $\mathbb{R}^2$, and consider the minimization problem (2.1) where $g\left(x,v\right) = v$ and

$$F\left(\xi\right) = f\left(\left\|\xi\right\|\right) = \begin{cases} \sqrt{2}\left\|\xi\right\| & \text{for } \left\|\xi\right\| \leq \sqrt{2} \\ 1+\frac{1}{2}\left\|\xi\right\|^2 & \text{for } \left\|\xi\right\| \geq \sqrt{2} \end{cases} \tag{2.5}$$

We have that $\partial F\left(0\right) = \sqrt{2}B\left[0,1\right]$. Then, as described in [14], the function

$$\tilde{u}\left(x\right) = \begin{cases} 0 & \text{for } \frac{\|x\|}{2} \leq \sqrt{2} \\ \left(\frac{\|x\|}{2}\right)^2 - 2 & \text{for } \frac{\|x\|}{2} \geq \sqrt{2} \end{cases} \tag{2.6}$$

is a solution to the minimization problem, among those functions $v$ satisfying the same values as $\tilde{u}$ on $\partial\Omega$, i.e., more precisely, for $v \in \tilde{u} + W_0^{1,1}\left(\Omega\right)$. We have

$$\nabla\tilde{u}\left(x\right) = \begin{cases} 0 & \text{for } \frac{\|x\|}{2} < \sqrt{2} \\ \frac{1}{2}x & \text{for } \frac{\|x\|}{2} > \sqrt{2} \end{cases} \tag{2.7}$$

Hence,

$$\partial F\left(\nabla\tilde{u}\left(x\right)\right) = \begin{cases} \sqrt{2}B\left[0,1\right] & \text{for } \frac{\|x\|}{2} < \sqrt{2} \\ \nabla F\left(\nabla\tilde{u}\left(x\right)\right) = \nabla\tilde{u}\left(x\right) = \frac{1}{2}x & \text{for } \frac{\|x\|}{2} > \sqrt{2} \end{cases}. \tag{2.8}$$

Then, although the function $\nabla\tilde{u}\left(x\right)$ is discontinuous, the vector function

$$p\left(x\right) = \frac{1}{2}x$$

is an everywhere smooth selection from the map $x \rightarrow \partial F\left(\nabla\tilde{u}\left(x\right)\right)$ and has everywhere divergence equal to 1.

# Chapter 3

# Higher integrability for solutions to variational problems

In this Chapter we prove some regularity results in the form of higher integrability properties for solutions to special variational problems. The proofs are independent on the validity of the Euler-Lagrange equation; this fact prompted us to try to use the higher integrability property to extend the validity of the Euler-Lagrange equation for a class of functionals with super-exponential growth Lagrangians. A result along these lines is presented in Chapter 4.

## 3.1 Statement of the result

Given a convex function $f : \mathbb{R}^N \to \mathbb{R}$, its subdifferential at the point $\xi \in \mathbb{R}^N$ is the set

$$\partial f(\xi) = \{s \in \mathbb{R}^N : f(x) \geq f(\xi) + \langle s, x - \xi \rangle \,,\ \forall x \in \mathbb{R}^N\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^N$.

Given $f$, a possibly extended valued function, by $f^*$ we denote the *polar* or *conjugate* of $f$

$$f^*(x^*) = \sup_x \{\langle x, x^* \rangle - f(x)\}$$

and by $Dom\,(f)$ its effective domain.

Suppose $L(x, u, \xi) \geq 0$ and the mapping $\xi \mapsto L(x, u, \xi)$ is convex for $\xi$ such that $\|\xi\| \geq 1$ and attains its minimum in 0 for every $(x, u)$.

**Assumption A**. Suppose $L(x,\cdot,\cdot) \in C^1(\mathbb{R}\times\mathbb{R}^n)$ for each fixed $x$ and it is such that, for every $\omega \subset\subset \Omega$ and $U$, there exist constants $M = M(\omega,U)$, $K = K(\omega,U)$ and $m_1 = m_1(\omega,U) > 0$ and a function $\alpha = \alpha_{\omega,U}$ in $L^1(\omega)$ such that for every $x \in \omega$, for every $|u| \leq U$, we have:

- for every $\xi \in \mathbb{R}^n$, $\left|\frac{\partial L(x,u,\xi)}{\partial u}\right| \leq KL(x,u,\xi)$;
- for every $\|\xi\| \geq 1$, $L(x,u,\xi) \geq m_1$;
- $\alpha_{\omega,U}(x) \geq \sup_{\{|u|\leq U;\|\xi\|\leq 1\}} \|\nabla_\xi L(x,u,\xi)\|$;
- $\langle \nabla_\xi L(x,u,\xi), \xi\rangle \geq M\|\nabla_\xi L(x,u,\xi)\|\,\|\xi\|$.

The higher integrability results will depend on the following Condition. In it, and for the remainder of the Chapter, for an open $O \subset\subset \Omega$ and $\delta > 0$, we set $O_\delta = O + B(0,\delta)$.

**Condition C**. For every open $O \subset\subset \Omega$, $\delta^0 > 0$ and $U$ there exist: $\delta \leq \delta^0$, such that $\overline{O_\delta}$ is in $\Omega$; a Lipschitzian function $\eta \in C_c(O_\delta)$, with $\eta(x) \geq 0$ and $\eta(x) = 1$ on $O$, and constants $\tilde{K} = \tilde{K}(U,O_\delta) \geq 0$ and $R = R(U,O_\delta)$, such that, for every $\xi$ with $\|\xi\| \geq R$, for every $u$ with $|u| \leq U$, for almost every $x \in O_\delta$, for every $\varepsilon > 0$ sufficiently small, we have:

$$\log L(x, u - \varepsilon\eta u, \xi(1-\varepsilon\eta) - \varepsilon u\nabla\eta) - \log L(x,u,\xi) \leq \varepsilon\tilde{K}. \tag{3.1}$$

Next Theorem infers the higher integrability result from the validity of Condition C.

**Theorem 2** *Let $L$ satisfy Assumption A and Condition C. Let $\tilde{u}$ be a locally bounded solution to the problem of minimizing*

$$\int_\Omega L(x,u(x),\nabla u(x))\ dx$$

*on $u^0 + W_0^{1,1}(\Omega)$. Then*

$$\|\nabla_\xi L(\cdot,\tilde{u}(\cdot),\nabla\tilde{u}(\cdot))\|\,\|\nabla\tilde{u}(\cdot)\| \in \mathrm{L}^1_{loc}(\Omega).$$

## 3.2 Validity of Condition C

We shall need the following preliminary result.

**Lemma 1** *Let* $G : \mathbb{R} \rightarrow 2^{\mathbb{R}}$ *be upper semicontinuous, strictly increasing and such that* $G(0) = \{0\}$*. Assume that, for a selection* $g$ *from* $G$*,*

$$\int^{\infty} g\left(\frac{1}{s}\right) ds < \infty. \tag{3.2}$$

*Then, the implicit Cauchy problem*

$$\begin{cases} x(t) \in G(x'(t)) \\ x(0) = 0 \end{cases}$$

*admits a solution* $\tilde{x}$*, positive on some interval* $(0, \tau)$*.*

Notice that the condition expressed by (3.2) is independent on the selection $g$; in fact, $G$ is multi-valued at most on countably many points, and the map $s \rightarrow \frac{1}{s}$ is strictly monotonic.

*Proof:* Set $\gamma = G^{-1}$: $\gamma$ is single-valued, continuous, defined on some interval $[0, \delta]$ and $\gamma(0) = 0$. Define

$$R = \left\{ (y, x) : 0 \leq y \leq z; \frac{1}{\gamma(z)} \leq x \leq \frac{1}{\gamma(y)} \right\} :$$

we claim that, for every $z > 0$,

$$\int_{(0,z)} \frac{1}{\gamma(y)} dy = z \frac{1}{\gamma(z)} + meas\,(R).$$

In fact, we have that $R$ can be described as

$$R = \left\{ (x, y) : 0 \leq y \leq \gamma^{-1}\left(\frac{1}{x}\right); \frac{1}{\gamma(z)} \leq x < \infty \right\},$$

so that

$$meas\,(R) = \int_{\frac{1}{\gamma(z)}}^{\infty} \gamma^{-1}\left(\frac{1}{y}\right) dy = \int_{\frac{1}{\gamma(z)}}^{\infty} g\left(\frac{1}{s}\right) ds,$$

that is finite by assumption.

Hence, the map $\Phi(x) = \int_0^x \frac{1}{\gamma(y)} dy$ is well defined, differentiable, positive for $x > 0$ and $\Phi(0) = 0$. Define $\tilde{x}(t)$ implicitely by

$$\Phi(\tilde{x}(t)) - t = 0;$$

then, $\tilde{x}$ is a differentiable function, $\tilde{x}(0) = 0$ and $x'(t) = \gamma(x(t))$.

$\square$

It is easy to show that the Lagrangeans of exponential growth satisfy Condition C. However, the following result shows that this Condition is satisfied by a substantially larger class of functions.

**Lemma 2** *Let $L$ satisfy Assumption A. In addition, assume that, for every $\omega \subset\subset \Omega$ and $U > 0$, either*

*i) $L$ satisfies the Exponential growth condition, that is: $\exists k = k(\omega, U)$ such that, for almost every $x \in \omega$, for every $|u| \leq U$, for every $\xi \in \mathbb{R}^n$, we have*

$$|\nabla_\xi L(x,u,\xi)| \leq k L(x,u,\xi)\,; \tag{3.3}$$

*or*

*ii) there exist a convex twice differentiable function $\Lambda$, with $\Lambda'(t) \geq 0$ for $t \geq 0$ and $\Lambda'(0) = 0$, and a positive constant $h_1$, such that $\|\nabla_\xi L(x,u,\xi)\| \leq h_1 \Lambda'(\|\xi\|)$, and*

$$\int^{\infty} \frac{\Lambda''(t)}{t\Lambda'(t)} dt < \infty; \tag{3.4}$$

*or*

*iii) for $\|\xi\| \geq 1$, the map $\xi \to \log L(x,u,\xi)$ is convex ; moreover there exist a constant $h_2$ and a twice differentiable function $\Lambda$, such that the map $\mathbb{L}(t) = \log \Lambda(t)$ is convex and such that, for almost every $x \in \omega$, for every $|u| \leq U$ and for every $\|\xi\| \geq 1$, we have*

$$\left\| \frac{\nabla_\xi L}{L}(x,u,\xi) \right\| \leq h_2 \frac{\Lambda'}{\Lambda}(\|\xi\|)\,.$$

*Finally, $\mathbb{L}$ satisfies*

$$\int^{\infty} \frac{\mathbb{L}''(t)}{t\mathbb{L}'(t)} dt < \infty.$$

*Then: Condition C is satisfied.*

**Examples**. Condition i) is classical and is widely used to prove the validity of the Euler Lagrange equation. The map $t \mapsto \exp(|t|^p)$ for $p > 1$ satisfies condition iii). The map $t \mapsto \exp(\exp(t))$ does not satisfy the assumptions of the Lemma.

In the construction that follows, we shall need the subdifferential $\partial(\Lambda)^*$ of the polar of a convex function $\Lambda$. Although $\Lambda$ will be assumed to be smooth, its polar need not be smooth.

*Proof*: Having fixed a variation $\eta$, we shall use the shorthand notation

$$\ell_\varepsilon (x, u, \xi) = \log L (x, u - \varepsilon\eta u, \xi (1 - \varepsilon\eta) - \varepsilon u \nabla\eta) .$$

Fix $O$, $\delta^0$ and $U$, let $\delta \leq \delta^0$ be such that $\overline{O_\delta}$ is in $\Omega$.

a) We claim that, in case i), to satisfy inequality (3.1) we can choose any Lipschitz continuous function $\eta \in C_c (O_\delta)$, with $\eta (x) \geq 0$ and $\eta (x) = 1$ on $O$.

In fact, since, for $s \in [0, 1]$, we have $|u - s\varepsilon\eta u| \leq U$, we obtain

$$\begin{aligned} &\ell_\varepsilon (x, u, \xi) - \log L (x, u, \xi) \\ &\qquad = \int_0^1 \frac{\partial \log L (x, u - s\varepsilon\eta u, \xi (1 - \varepsilon\eta) - \varepsilon u\nabla\eta)}{\partial u} (-\varepsilon\eta u)\, ds \\ &\qquad + \int_0^1 \langle \nabla_\xi \log L (x, u, \xi (1 - \varepsilon\eta) - s\varepsilon u\nabla\eta) , -\varepsilon u\nabla\eta\rangle\, ds \\ &\qquad + \log L (x, u, \xi (1 - \varepsilon\eta)) - \log L (x, u, \xi) . \end{aligned}$$

By assumption, $L$ is non-decreasing with respect to $t$ on each half-line $\{t\xi : t \geq 0\}$ for fixed $x$ and $u$, hence the third term at the right is non positive. Moreover,

$$\left| \frac{\partial \log L}{\partial u} \right| = \frac{\left|\frac{\partial L}{\partial u}\right|}{L} \leq K$$

so that the first term is bounded by $\varepsilon$ times a constant. By (3.3),

$$|\nabla_\xi \log L| = \frac{|\nabla_\xi L|}{L} \leq k,$$

hence, the same is true for the second term.

b) Consider case ii). From

$$\ell (\varepsilon) - \log L (x, u, \xi) = \varepsilon (-\eta u) \int_0^1 \left[ \frac{\frac{\partial L}{\partial u}}{L} \right] ds + \varepsilon \int_0^1 \left\langle \frac{\nabla_\xi L}{L}, -\eta\xi - u\nabla\eta \right\rangle ds,$$

where the first integrand is evaluated at $(x, u - s\varepsilon\eta u, \xi (1 - \varepsilon\eta) - \varepsilon u\nabla\eta)$ and the second at $(x, u, \xi (1 - s\varepsilon\eta) - s\varepsilon u\nabla\eta)$, we obtain

$$\ell (\varepsilon) - \log L (x, u, \xi) \leq \varepsilon \left[ KU + \left\langle \frac{\nabla_\xi L}{L} (x, u, \xi_\varepsilon) , -\eta\xi - u\nabla\eta \right\rangle \right], \tag{3.5}$$

where $\xi_\varepsilon = (1 - s_\varepsilon\varepsilon\eta) \xi - s_\varepsilon\varepsilon u\nabla\eta$, for some $0 \leq s_\varepsilon \leq 1$.

Consider the function

$$G (z) = z \frac{7U}{M\partial (\Lambda)^* \left(\frac{1}{z}\right)}, \tag{3.6}$$

so that

$$G\left(\frac{1}{z}\right)=\frac{1}{z}\frac{7U}{M\partial\left(\Lambda\right)^{*}\left(z\right)}.$$

Notice that, under our assumptions, $t>0$ implies that $0\notin\partial\left(\Lambda\right)^{*}\left(t\right)$: in fact, otherwise, $0\in\partial\left(\Lambda\right)^{*}\left(\tau\right)$ for every $0\leq\tau\leq t$ and $\Lambda'$ would be discontinuous at 0. Notice also that $G$ satisfies the assumptions of Lemma 1. In fact, set $z=\Lambda'\left(t\right)$ so that $t\to\infty$ as $z\to\infty$ and, by the change of variables formula,

$$\int^{\infty}G\left(\frac{1}{z}\right)dz=\int^{\infty}\frac{7U}{M\Lambda'\left(t\right)\partial\left(\Lambda\right)^{*}\left(\Lambda'\left(t\right)\right)}\Lambda''\left(t\right)dt=\int^{\infty}\frac{7U}{M\Lambda'\left(t\right)t}\Lambda''\left(t\right)dt,$$

so that the condition of Lemma 1 is satisfied by our assumption (3.4).

Consider $\tilde{x}$, the solution to $x\in G\left(x'\right)$, provided by Lemma 1. Define $\eta$ as follows: let $d\left(x\right)$ be the distance from a point $x\in O_{\delta}$ to $\partial O_{\delta}$ and set

$$\eta\left(x\right)=\inf\left\{\frac{1}{\tilde{x}\left(\delta\right)}\tilde{x}\left(d\left(x\right)\right),1\right\}$$

so that, in particular, $\eta=1$ on $O$. Almost everywhere, $d$ is differentiable with $\left\|\nabla d\right\|=1$ and, at a point of differentiability, we have

$$\nabla\eta\left(x\right)=\begin{cases}0 & \text{if } d\left(x\right)>\delta\\ \frac{1}{\tilde{x}(\delta)}\tilde{x}'\left(d\left(x\right)\right)\nabla d\left(x\right) & \text{if } d\left(x\right)<\delta\end{cases}.$$

Hence, a.e., we have that $\left\|\nabla\eta\right\|\leq\frac{1}{\tilde{x}(\delta)}\tilde{x}'\left(\delta\right)$ and that, either $\nabla\eta=0$, or

$$\eta\left(x\right)=\frac{1}{\tilde{x}\left(\delta\right)}\tilde{x}\left(d\left(x\right)\right)=\frac{1}{\tilde{x}\left(\delta\right)}\tilde{x}'\left(d\left(x\right)\right)\frac{7U}{M\partial\left(\Lambda^{*}\right)\left(\frac{1}{\tilde{x}'(d(x))}\right)}\tag{3.7}$$

$$=\left\|\nabla\eta\left(x\right)\right\|h\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\left(x\right)\right\|\right),$$

where we set

$$h\left(z\right)=\frac{7U}{M\partial\left(\Lambda\right)^{*}\left(\frac{1}{z}\right)},\tag{3.8}$$

an increasing function.

By the estimate (3.5), for those $x$ such that

$$\left\langle\nabla_{\xi}L\left(x,u,\xi_{\varepsilon}\right),-\eta\xi-u\nabla\eta\right\rangle\leq0,\tag{3.9}$$

any $\tilde{K}\geq KU$ will do to prove the result. Since the mapping $\xi\mapsto L\left(x,u,\xi\right)$ is convex and attains its minimum in 0, for $0\leq\varepsilon\leq1$, we have

$$\frac{d}{ds}L\left(x,u,\xi\left(1-s\varepsilon\eta\right)\right)\leq0,$$

i.e., $\langle \nabla_\xi L(x,u,\xi(1-s\varepsilon\eta)), -\eta\xi\rangle \le 0$, so that

$$\left\langle \frac{\nabla_\xi L}{L}(x,u,\xi(1-s\varepsilon\eta)), -\eta\xi\right\rangle \le 0;$$

we infer that, when $\nabla\eta(x)=0$, (3.9) holds.

Hence, we are left to consider those $x$ such that at once $\eta(x)=\|\nabla\eta(x)\|\cdot h(\tilde{x}(\delta)\|\nabla\eta(x)\|)$ and

$$\langle \nabla_\xi L(x,u,\xi_\varepsilon), -\eta\xi-u\nabla\eta\rangle \ge 0.$$

Given any $v,w\in\mathbb{R}^n$, from the convexity of $L(x,u,\cdot)$ we obtain that its gradient is monotonic, i.e., that

$$(s_1-s_2)\langle \nabla_\xi L(x,u,v+s_1w)-\nabla_\xi L(x,u,v+s_2w), w\rangle \ge 0,$$

i.e., that the mapping $s\mapsto\langle\nabla_\xi L(x,u,v+sw),w\rangle$ is non decreasing. Hence, from the inequality

$$\langle \nabla_\xi L(x,u,\xi_\varepsilon), -\eta\xi-u\nabla\eta\rangle \ge 0,$$

we obtain

$$\langle \nabla_\xi L(x,u,\overline{\xi}), -\eta\xi-u\nabla\eta\rangle \ge 0,$$

where $\overline{\xi}=(1-\varepsilon\eta)\xi-\varepsilon u\nabla\eta$. We infer that

$$\langle \nabla_\xi L(x,u,\overline{\xi}),\xi\rangle \le \left\langle \nabla_\xi L(x,u,\overline{\xi}), -u\frac{\nabla\eta}{\eta}\right\rangle \le \|\nabla_\xi L(x,u,\overline{\xi})\|\cdot U\frac{\|\nabla\eta\|}{\eta}.$$

Hence, recalling the assumptions on $\nabla_\xi L$, from the inequality

$$\begin{aligned}\langle \nabla_\xi L(x,u,\overline{\xi}),\xi\rangle &= \langle \nabla_\xi L(x,u,\overline{\xi}),\overline{\xi}\rangle-\varepsilon\langle \nabla_\xi L(x,u,\overline{\xi}),-\eta\xi-u\nabla\eta\rangle\\ &\ge \|\nabla_\xi L(x,u,\overline{\xi})\|\left[M\|\overline{\xi}\|-\varepsilon\eta\|\xi\|-\varepsilon U\|\nabla\eta\|\right],\end{aligned}$$

we obtain

$$\begin{aligned}U\frac{\|\nabla\eta\|}{\eta} &\ge M\|\overline{\xi}\|-\varepsilon\eta\|\xi\|-\varepsilon U\|\nabla\eta\|\\ &\ge M\left[(1-\varepsilon\eta)\|\xi\|-\varepsilon U\|\nabla\eta\|\right]-\varepsilon\eta\|\xi\|-\varepsilon U\|\nabla\eta\|,\end{aligned}$$

i.e.,

$$U\frac{\|\nabla\eta\|}{\eta}+\varepsilon\|\nabla\eta\|U[M+1]\ge\|\xi\|[M(1-\varepsilon\eta)-\varepsilon\eta].$$

We are free to assume $M<1$; taking $\varepsilon<\frac{M}{4}$, we finally have

$$3U\frac{\|\nabla\eta\|}{\eta}\ge U\frac{\|\nabla\eta\|}{\eta}+\varepsilon\|\nabla\eta\|U[M+1]\ge\|\xi\|[M(1-\varepsilon\eta)-\varepsilon\eta]\ge\frac{1}{2}M\|\xi\| \tag{3.10}$$

and, recalling (3.7), we obtain

$$\|\xi\| \leq \frac{6U}{Mh\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\left(x\right)\right\|\right)}. \tag{3.11}$$

Since $\|\nabla\eta\|$ is bounded, there exists $R$ such that, for all $\varepsilon$ sufficiently small, $\|\xi\| \geq R$ implies that $\|\xi_\varepsilon\| \geq 1$. Hence, from the assumptions on the existence and properties of $\Lambda$, we have

$$\left\langle \frac{\nabla_\xi L}{L}\left(x,u,\xi_\varepsilon\right), -\eta\xi - u\nabla\eta \right\rangle \leq \frac{h_1\Lambda'\left(\|\xi\| + \varepsilon U\left\|\nabla\eta\right\|\right)}{m_1}\left(\eta\left\|\xi\right\| + U\left\|\nabla\eta\right\|\right), \tag{3.12}$$

hence, recalling (3.5), from (3.10) we have

$$\begin{aligned}
&\ell\left(\varepsilon\right) - \log L\left(x,u,\xi\right) \\
&\quad\leq \varepsilon KU + \varepsilon\left\|\nabla\eta\right\|\frac{h_1}{m_1}\Lambda'\left(\frac{6U}{M\cdot h\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)} + \varepsilon\left\|\nabla\eta\right\|U\right)\left(\frac{6U}{M} + U\right).
\end{aligned}$$

There exists $\sigma$ such that, for $\|\nabla\eta\| < \sigma$, we have

$$\frac{6}{M\cdot h\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)} + \varepsilon\left\|\nabla\eta\right\| \leq \frac{7}{M\cdot h\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)};$$

hence, for those $x$ such that $\|\nabla\eta\left(x\right)\| < \sigma$, recalling (3.8), we obtain

$$\begin{aligned}
&\frac{h_1}{m_1}\left\|\nabla\eta\right\|\Lambda'\left(\frac{6U}{M\cdot h\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)} + \varepsilon\left\|\nabla\eta\right\|U\right) \\
&\qquad\leq \frac{h_1}{m_1}\left\|\nabla\eta\right\|\Lambda'\left(\frac{7U}{M\cdot h\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)}\right) \\
&\qquad= \frac{h_1}{m_1}\left\|\nabla\eta\right\|\Lambda'\left(\partial\left(\Lambda\right)^*\left(\frac{1}{\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|}\right)\right) \\
&\qquad= \frac{h_1}{m_1}\frac{1}{\tilde{x}\left(\delta\right)},
\end{aligned}$$

a constant independent on $\varepsilon$, thus proving the result in this case.

It is left to consider those $x$ such that $\|\nabla\eta\left(x\right)\| \geq \sigma$: in this case, from (3.11), we have

$$\|\xi\| \leq \frac{6U}{M\cdot h\left(\tilde{x}\left(\delta\right)\sigma\right)}$$

and, from (3.5), the result follows from the boundedness of $\|\nabla\eta\|$.

c) Case iii). Consider the function

$$\tilde{G}\left(z\right) = z\frac{7U}{M_1\partial\left(\mathbb{L}\right)^*\left(\frac{1}{z}\right)}, \tag{3.13}$$

so that

$$\tilde{G}\left(\frac{1}{z}\right)=\frac{7U}{z\partial\left(\mathbb{L}\right)^{*}\left(z\right)}.$$

As in the proof of point ii), the condition of Lemma 1 is satisfied. Let $\tilde{x}$ be the solution to $\tilde{x}\in\tilde{G}\left(\tilde{x}'\right)$, provided by Lemma 1 and define $\eta$ so that

$$\nabla\eta\left(x\right)=\begin{cases}0 & \text{if } d\left(x\right)>\delta\\ \frac{1}{\tilde{x}(\delta)}\tilde{x}'\left(d\left(x\right)\right)\nabla d\left(x\right) & \text{if } d\left(x\right)<\delta\end{cases},$$

hence either $\nabla\eta=0$, or $\eta\left(x\right)=\left\Vert \nabla\eta\left(x\right)\right\Vert \tilde{h}\left(\tilde{x}\left(\delta\right)\left\Vert \nabla\eta\left(x\right)\right\Vert \right)$, with

$$\tilde{h}\left(z\right)=\frac{7U}{M_{1}\partial\left(\mathbb{L}\right)^{*}\left(\frac{1}{z}\right)}. \tag{3.14}$$

For those $x$ such that

$$\left\langle \frac{\nabla_{\xi}L}{L}\left(x,u,\xi_{\varepsilon}\right),-\eta\xi-u\nabla\eta\right\rangle \leq0, \tag{3.15}$$

any $\tilde{K}\geq KU$ will do to prove the result. Since $\nabla\eta\left(x\right)=0$ implies

$$\left\langle \frac{\nabla_{\xi}L}{L}\left(x,u,\xi\left(1-s\varepsilon\eta\right)\right),-\eta\xi\right\rangle \leq0,$$

(3.15) holds in this case. So we consider the case where at once

$$\eta\left(x\right)=\left\Vert \nabla\eta\left(x\right)\right\Vert \tilde{h}\left(\tilde{x}\left(\delta\right)\left\Vert \nabla\eta\left(x\right)\right\Vert \right)$$

and $\left\langle \frac{\nabla_{\xi}L}{L}\left(x,u,\xi_{\varepsilon}\right),-\eta\xi-u\nabla\eta\right\rangle >0$. From the convexity of $\log L$ and the inequality

$$\left\langle \frac{\nabla_{\xi}L}{L}\left(x,u,\xi\left(1-s\varepsilon\eta\right)\right),-\eta\xi-u\nabla\eta\right\rangle >0,$$

setting $\overline{\xi}=\left(1-\varepsilon\eta\right)\xi-\varepsilon u\nabla\eta$, we obtain

$$\left\langle \frac{\nabla_{\xi}L}{L}\left(x,u,\overline{\xi}\right),-\eta\xi-u\nabla\eta\right\rangle >0.$$

Hence, from

$$\left\langle \frac{\nabla_{\xi}L}{L}\left(x,u,\overline{\xi}\right),\xi\right\rangle <\left\langle \frac{\nabla_{\xi}L}{L}\left(x,u,\overline{\xi}\right),-u\frac{\nabla\eta}{\eta}\right\rangle \leq\left\Vert \frac{\nabla_{\xi}L}{L}\left(x,u,\overline{\xi}\right)\right\Vert \cdot U\frac{\left\Vert \nabla\eta\right\Vert }{\eta}$$

we infer

$$\left\langle \frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),\xi\right\rangle = \left\langle \frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),\overline{\xi}\right\rangle - \varepsilon\left\langle \frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),-\eta\xi-u\nabla\eta\right\rangle$$

$$\geq \left\|\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right)\right\|\left[M_1\left\|\overline{\xi}\right\|-\varepsilon\eta\left\|\xi\right\|-\varepsilon U\left\|\nabla\eta\right\|\right],$$

so that

$$U\frac{\|\nabla\eta\|}{\eta} \geq M_1\left\|\overline{\xi}\right\|-\varepsilon\eta\left\|\xi\right\|-\varepsilon U\left\|\nabla\eta\right\|$$

$$\geq M_1\left[\left(1-\varepsilon\eta\right)\left\|\xi\right\|-\varepsilon U\left\|\nabla\eta\right\|\right]-\varepsilon\eta\left\|\xi\right\|-\varepsilon U\left\|\nabla\eta\right\|,$$

and, assuming $M_1 < 1$ and $\varepsilon < \frac{M_1}{4}$, we obtain

$$\eta\left\|\xi\right\| \leq \frac{6U\left\|\nabla\eta\right\|}{M_1}$$

and

$$\left\|\xi\right\| \leq \frac{6U}{M_1\tilde{h}\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\left(x\right)\right\|\right)}.$$

For $R$ sufficiently large, $\|\xi\| \geq R$ implies that $\|\xi_\varepsilon\| \geq 1$. We infer that

$$\left\langle \frac{\nabla_\xi L}{L}\left(x,u,\xi_\varepsilon\right),-\eta\xi-u\nabla\eta\right\rangle \leq \left(\eta\left\|\xi\right\|+U\left\|\nabla\eta\right\|\right)h_2\mathbb{L}'\left(\left\|\xi_\varepsilon\right\|\right),$$

that

$$\left\|\xi_\varepsilon\right\| \leq \frac{6U}{M_1\tilde{h}\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)}+\varepsilon\left\|\nabla\eta\right\|U$$

and that $\mathbb{L}'$ is non-decreasing, hence we obtain

$$\left\|\nabla\eta\right\|h_2\mathbb{L}'\left(\left\|\xi_\varepsilon\right\|\right) \leq \left\|\nabla\eta\right\|h_2\mathbb{L}'\left(\frac{6U}{M_1\tilde{h}\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)}+\varepsilon\left\|\nabla\eta\right\|U\right).$$

There exists $\sigma$ such that, for $\|\nabla\eta\| < \sigma$, we have

$$\frac{6}{\tilde{h}\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)}+\varepsilon\left\|\nabla\eta\right\| \leq \frac{7}{\tilde{h}\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)}.$$

For those $x$ such that $\|\nabla\eta(x)\| < \sigma$, recalling (3.14), we obtain

$$\left\|\nabla\eta\right\|h_2\left(\mathbb{L}\right)'\left(\frac{6U}{M_1\tilde{h}\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)}+\varepsilon\left\|\nabla\eta\right\|U\right)$$

$$\leq \left\|\nabla\eta\right\|h_2\left(\mathbb{L}\right)'\left(\frac{7U}{M_1\tilde{h}\left(\tilde{x}\left(\delta\right)\left\|\nabla\eta\right\|\right)}\right)$$

$$= \|\nabla\eta\| \, h_2 \left(\mathbb{L}\right)' \left(\partial \left(\mathbb{L}\right)^* \left(\frac{1}{\tilde{x}\left(\delta\right) \|\nabla\eta\|}\right)\right) = \frac{h_2}{\tilde{x}\left(\delta\right)}.$$

Hence, from the estimate (3.5), the result is true in this case.

For those $x$ such that $\|\nabla\eta\left(x\right)\| \geq \sigma$ we have

$$\|\xi\| \leq \frac{6U}{M_1 h\left(\tilde{x}\left(\delta\right)\sigma\right)}$$

and the result follows again from the boundedness of $\|\nabla\eta\|$.

□

## 3.3 Higher integrability of a solution

Now we can prove the higher integrability result.

*Proof of Theorem 2*: a) Fix $O \subset\subset \Omega$. It is enough to prove the existence of $H_1$ such that

$$\int_O \left\langle \nabla_\xi L\left(x, \tilde{u}\left(x\right), \nabla\tilde{u}\left(x\right)\right), \nabla\tilde{u}\left(x\right)\right\rangle dx \leq H_1.$$

In fact, taking $O$ to be $\omega$ in Assumption A, the fourth point proves the claim. Let $O_{\delta^0} \subset\subset \Omega$, let $U$ a bound for $|\tilde{u}|$ on $O_{\delta^0}$. Let $\delta$, $\eta$ and the constants $R$ and $\tilde{K}$ be provided by Condition C (we assume $R \geq 1$).

Since $\tilde{u}$ is a solution, for the variation $-\varepsilon\eta\tilde{u}$, with $\varepsilon > 0$, we obtain

$$0 \leq \frac{1}{\varepsilon} \int_\Omega \left[L\left(x, \tilde{u} - \varepsilon\eta\tilde{u}, \nabla\tilde{u}\left(1 - \varepsilon\eta\right) - \varepsilon\tilde{u}\nabla\eta\right) - L\left(x, \tilde{u}, \nabla\tilde{u}\right)\right] dx. \tag{3.16}$$

We have

$$\begin{aligned} & L\left(x, \tilde{u} - \varepsilon\eta\tilde{u}, \nabla\tilde{u}\left(1 - \varepsilon\eta\right) - \varepsilon\tilde{u}\nabla\eta\right) - L\left(x, \tilde{u}, \nabla\tilde{u}\right) \\ & = \varepsilon \int_0^1 \left[\frac{\partial L}{\partial u}\left(-\eta\tilde{u}\right) + \left\langle \nabla_\xi L, -\eta\nabla\tilde{u} - \tilde{u}\nabla\eta\right\rangle\right] ds, \end{aligned}$$

where $\frac{\partial L}{\partial u}$ and $\nabla_\xi L$ are computed at $\left(x, \tilde{u} - s\varepsilon\eta\tilde{u}, \nabla\tilde{u}\left(1 - s\varepsilon\eta\right) - s\varepsilon\tilde{u}\nabla\eta\right)$, hence, as $\varepsilon \to 0$, by the continuity of $L$ and of its partial derivatives,

$$\begin{aligned} & \frac{L\left(x, \tilde{u} - \varepsilon\eta\tilde{u}, \nabla\tilde{u}\left(1 - \varepsilon\eta\right) - \varepsilon\tilde{u}\nabla\eta\right) - L\left(x, u, \nabla\tilde{u}\right)}{\varepsilon} \\ & \qquad \to \frac{\partial L}{\partial u}\left(-\eta\tilde{u}\right) + \left\langle \nabla_\xi L, -\eta\nabla\tilde{u} - \tilde{u}\nabla\eta\right\rangle, \end{aligned} \tag{3.17}$$

pointwise w.r.t. $x$, and with the r.h.s. computed at $\left(x, \tilde{u}\left(x\right), \nabla\tilde{u}\left(x\right)\right)$. Set $O_\delta^- = \{x \in O_\delta : \|\nabla\tilde{u}\left(x\right)\| < R\}$ and $O_\delta^+ = \{x \in O_\delta : \|\nabla\tilde{u}\left(x\right)\| \geq R\}$: on $O_\delta^-$, the left hand side of (3.17) is uniformly bounded, so that we can say that for some $\tilde{M}$, we have, for every $\varepsilon$,

$$\left|\frac{1}{\varepsilon}\int_{\Omega^-}\left[L\left(x,\tilde{u}-\varepsilon\eta\tilde{u},\nabla\tilde{u}\left(1-\varepsilon\eta\right)-\varepsilon\tilde{u}\nabla\eta\right)-L\left(x,\tilde{u}\left(x\right),\nabla\tilde{u}\left(x\right)\right)\right]dx\right|\leq\tilde{M}.$$

b) On $O_\delta^+$, consider the constant $\tilde{K}$: setting

$$\tilde{\ell}_\varepsilon\left(x\right)=\log L\left(x,\tilde{u}\left(x\right)-\varepsilon\eta\left(x\right)\tilde{u}\left(x\right),\nabla\tilde{u}\left(x\right)\left(1-\varepsilon\eta\left(x\right)\right)-\varepsilon\tilde{u}\left(x\right)\nabla\eta\left(x\right)\right),$$

from (3.16), we have

$$\begin{aligned}-\tilde{M}&\leq\int_{\Omega^+}\left(\frac{e^{\tilde{\ell}_\varepsilon}-e^{\log L(x,\tilde{u}(x),\nabla\tilde{u}(x))}}{\varepsilon}\right)dx\\&=\int_{\Omega^+}\left(\frac{e^{\tilde{\ell}_\varepsilon-\varepsilon\tilde{K}+\varepsilon\tilde{K}}-e^{\log L(x,\tilde{u}(x),\nabla\tilde{u}(x))}}{\varepsilon}\right)dx\\&=\int_{\Omega^+}e^{\tilde{\ell}_\varepsilon-\varepsilon\tilde{K}}\left[\frac{e^{\varepsilon\tilde{K}}-1+1-e^{\log L(x,\tilde{u}(x),\nabla\tilde{u}(x))-\tilde{\ell}_\varepsilon+\varepsilon\tilde{K}}}{\varepsilon}\right]dx,\end{aligned}$$

i.e,

$$\begin{aligned}&\tilde{M}+\int_{\Omega^+}e^{\tilde{\ell}_\varepsilon-\varepsilon\tilde{K}}\left[\frac{e^{\varepsilon\tilde{K}}-1}{\varepsilon}\right]dx\\&\qquad\geq\int_{\Omega^+}e^{\tilde{\ell}_\varepsilon-\varepsilon\tilde{K}}\left[\frac{e^{\log L(x,\tilde{u}(x),\nabla\tilde{u}(x))-\tilde{\ell}_\varepsilon+\varepsilon\tilde{K}}-1}{\varepsilon}\right]dx.\end{aligned}\tag{3.18}$$

Since, on $O_\delta^+$, $\tilde{\ell}_\varepsilon-\varepsilon\tilde{K}\leq\log L\left(x,\tilde{u},\nabla\tilde{u}\right)$ and also $\frac{e^{\varepsilon\tilde{K}}-1}{\varepsilon}\leq\tilde{K}e^{\tilde{K}}$, the left hand side of (3.18) is bounded by

$$\tilde{M}+\tilde{K}e^{\tilde{K}}\int_\Omega L\left(x,\tilde{u},\nabla\tilde{u}\right)dx=H,$$

independent of $\varepsilon$.

c) Consider the right hand side. For fixed $x$ we have

$$\log L\left(x,u,\nabla\tilde{u}\right)-\tilde{\ell}_\varepsilon=-\varepsilon\left[\frac{\frac{\partial L}{\partial u}}{L}\left(-\eta\tilde{u}\right)+\frac{1}{L}\left\langle\nabla_\xi L,-\eta\nabla\tilde{u}-\tilde{u}\nabla\eta\right\rangle\right]+o\left(\varepsilon\right)$$

so that, as $\varepsilon\rightarrow0$, pointwise w.r.t. $x$,

$$\frac{e^{\log L(x,u,\nabla\tilde{u})-\tilde{\ell}_\varepsilon+\varepsilon\tilde{K}}-1}{\varepsilon}\rightarrow\tilde{K}+\frac{\frac{\partial L}{\partial u}}{L}\eta\tilde{u}+\frac{1}{L}\left\langle\nabla_\xi L,\eta\nabla\tilde{u}+\tilde{u}\nabla\eta\right\rangle.\tag{3.19}$$

In addition, by (3.1), $\log L(x,u,\nabla\tilde{u}) - \tilde{\ell}_\varepsilon + \varepsilon\tilde{K} \geq 0$, so that the left hand side of (3.19) is non negative and so is its limit,

$$\tilde{K} + \frac{\frac{\partial L}{\partial u}}{L}\eta\tilde{u} + \frac{1}{L}\left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle,$$

is also non negative. Finally, pointwise, $e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K}} \to e^{\log L(x,\tilde{u},\nabla\tilde{u})}$. Hence, applying Fatou's lemma, we obtain

$$\int_{O_\delta^+} L(x,\tilde{u},\nabla\tilde{u})\left[\tilde{K} + \frac{\frac{\partial L}{\partial u}}{L}\eta\tilde{u} + \frac{1}{L}\left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle\right] \leq H$$

i.e.,

$$\int_{O_\delta^+}\left[\tilde{K}L(x,\tilde{u},\nabla\tilde{u}) + \frac{\partial L}{\partial u}\eta\tilde{u} + \left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle\right] \leq H.$$

Since the integrand above is non-negative, we have obtained, in particular, that

$$\int_{O\cap O_\delta^+}\left[\tilde{K}L(x,\tilde{u},\nabla\tilde{u}) + \frac{\partial L}{\partial u}\eta\tilde{u} + \left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle\right] \leq H.$$

On $O$ we have that $\eta \equiv 1$, $\tilde{u}$ bounded and that, by Assumption A, there exists $K$ such that, for $x \in O$,

$$\left|\frac{\partial L(x,\tilde{u}(x),\nabla\tilde{u}(x))}{\partial u}\right| \leq KL(x,\tilde{u}(x),\nabla\tilde{u}(x))$$

hence there exists $H^+$ such that

$$\int_{O\cap O_\delta^+}\left\langle \nabla_\xi L(x,\tilde{u}(x),\nabla\tilde{u}(x)), \nabla\tilde{u}(x)\right\rangle dx \leq H^+. \tag{3.20}$$

On the other hand, we have that

$$\left|\left\langle \nabla_\xi L(x,\tilde{u}(x),\nabla\tilde{u}(x)), \nabla\tilde{u}(x)\right\rangle\right| \leq \alpha_{\omega,U}(x),$$

on $O\cap O_\delta^-$, hence we obtain that the integral

$$\int_O \left\langle \nabla_\xi L(x,\tilde{u}(x),\nabla\tilde{u}(x)), \nabla\tilde{u}(x)\right\rangle dx$$

is bounded, thus proving the theorem.

$\square$

## 3.4 Higher integrability for solutions to variational problems with fast growth

In this section we take into account minimum problems having simpler structure. They represent a particular case of the problems treated before: in this way the hypothesis of Theorem 2 become essentially one simple assumption. Moreover, this version of the higher integrability result is the one contained in the paper *Higher integrability for solutions to variational problems with fast growth*, Journal of Convex Analysis **18** (2011), no. 1, joint work with A. Cellina.

We consider the problem of minimizing an integral functional of the kind

$$\int_\Omega \left[ e^{f(\|\nabla u(x)\|)} + g\left(x, u\left(x\right)\right) \right] \, dx. \tag{3.21}$$

In general, in order to establish the validity of the Euler Lagrange equation for the solution to this problem , i.e., in order to prove that, for every admissible variation $\eta$, the equation

$$\int_\Omega \left\{ e^{f(\|\nabla \tilde{u}(x)\|)} f'\left(\|\nabla \tilde{u}\left(x\right)\|\right) \left\langle \frac{\nabla \tilde{u}\left(x\right)}{\|\nabla \tilde{u}\left(x\right)\|}, \nabla \eta\left(x\right) \right\rangle + g_u\left(x, \tilde{u}\left(x\right)\right) \eta\left(x\right) \right\} dx = 0 \tag{3.22}$$

holds, one has preliminarly to prove that the integrand is in $L^1$. Indeed, the difficulties come mainly from the first term: we prove

$$e^{f(\|\nabla \tilde{u}(\cdot)\|)} f'\left(\|\nabla \tilde{u}\left(\cdot\right)\|\right) \xi\left(\cdot\right) \in L^1_{loc}\left(\Omega\right),$$

for every function $\xi$ such that $\int_\Omega e^{f(\xi(x))} \, dx < \infty$.

In the Theorem that follows, we use the notation $\frac{1}{p\partial f^*(p)}$: we mean 0 when $p \notin Dom\left(\partial f^*\right)$ and, when $p \in Dom\left(\partial f^*\right)$, we mean any selection from the set-valued map $p \to \frac{1}{p\partial f^*(p)}$: since $\frac{1}{p\partial f^*(p)}$ is strictly decreasing, it is multi-valued at most on a countable set, and any two selections will differ only on a set of measure zero.

**Theorem 3** *Let $f : \mathbb{R} \to \mathbb{R}$ be convex, differentiable, symmetric, $f\left(0\right) = 0$ and assume that*

$$\int^\infty \frac{1}{p\partial f^*\left(p\right)} dp < \infty.$$

*Let $g$ be differentiable with respect to $u$, and let $g$ and $g_u$ be Carathéodory functions, and assume that for every $U$ there exists $\alpha_U \in L^1_{loc}$ such that $|v| \leq U$ implies $|g_u\left(x, v\right)| \leq \alpha_U\left(x\right)$. Let $\tilde{u} \in u^0 + W^{1,1}_0\left(\Omega\right)$ be a locally bounded solution to the* *problem of minimizing*

$$\int_{\Omega}\left[e^{f(\|\nabla u(x)\|)}+g\left(x,u\left(x\right)\right)\right]\,dx.$$

*Then, for every function* $\xi$ *such that* $\int_{\Omega}e^{f(\xi(x))}\,dx<\infty$*, we have that*

$$e^{f(\|\nabla\tilde{u}(\cdot)\|)}f'\left(\|\nabla\tilde{u}\left(\cdot\right)\|\right)\xi\left(\cdot\right)\in L^1_{loc}\left(\Omega\right).$$

**Examples.**

1. The map $f\left(s\right)=s-2\sqrt{s+1}+2$ is convex, differentiable and of linear growth. Its conjugate is the extended-valued function $f^*\left(p\right)=\frac{p^2}{1-|p|}$ for $|p|<1$, $=\infty$ elsewhere. The conditions of the theorem are satisfied.
2. Another map satisfying the assumption of the Theorem is $f\left(s\right)=2e^{s^{\frac{1}{2}}}\left(s^{\frac{1}{2}}-1\right)-s$; then $f^{*\prime}\left(p\right)=\left(\ln\left(p+1\right)\right)^2$ and $\int^{\infty}\frac{1}{p(\ln(p+1))^2}dp<\infty$.
3. A map $f$ that does not satisfy the assumption of the Theorem is $f\left(s\right)=\frac{1}{e}\left(e^s-s-1\right)$; in this case, we have $f^{*\prime}\left(p\right)=1+\ln\left(p+\frac{1}{e}\right)$.

**Remarks.**

**(i)** *The result applies, in particular, to the function* $\xi\left(x\right)=\|\nabla\tilde{u}\left(x\right)\|$*, so that we have*

$$e^{f(\|\nabla\tilde{u}(\cdot)\|)}f'\left(\|\nabla\tilde{u}\left(\cdot\right)\|\right)\|\nabla\tilde{u}\left(\cdot\right)\|\in L^1_{loc}\left(\Omega\right).$$

**(ii)** *This result is weaker than the Lipschitz regularity or the local Lipschitz regularity of* $\overline{u}$*, that are the results already known in the literature; however, it holds for a larger class of functionals.*

**(iii)** *We do not assume further regularity on* $f$ *besides its being convex and differentiable: in particular, we do not assume the existence of a second derivative of* $f$*, nor we assume its strict convexity.*

In Theorem 3 we assume the solution $\tilde{u}$ to be locally bounded. The validity of this assumption can be guaranteed:

i) when $g=0$, assuming that the boundary datum $u^0$ is in $L^{\infty}$, through a standard comparison result, noticing that, with the exception of the case $f\equiv 0$, $e^{f(\|v\|)}$ is a strictly convex function of $z$.

ii) in general, assuming that there exist $p \in \mathbb{R}^+$, $\alpha \in L^1(\Omega)$ and $\beta \in \mathbb{R}$ such that $u_0 \in W^{1,p}(\Omega)$ and

$$|g(x,u)| \leq \alpha(x) + \beta |u|^p .$$

In fact, with the exception of the case $f \equiv 0$, there are $A$ and $B > 0$ such that that $f(t) \geq A + Bt$; hence, fix $N^*$ larger than $\sup\{N, p\}$. For suitable constants, we have

$$\infty > \int_\Omega \left[ e^{f(\|\nabla \tilde{u}(x)\|)} + g(x, \tilde{u}(x)) \right] dx \geq \int_\Omega \left[ e^{A+B\|\nabla \tilde{u}(x)\|} - |\alpha(x)| - |\beta| \, |\tilde{u}(x)|^p \right] dx$$

$$\geq A_1 + B_1 \|\nabla \tilde{u}(x)\|_{L^{N^*}(\Omega)}^{N^*} - |\beta| \, \|\tilde{u}\|_{L^p(\Omega)}^p$$

$$\geq A_1 + B_1 \|\nabla \tilde{u}(x)\|_{L^{N^*}(\Omega)}^{N^*} - C_1 \|u_o\|_{L^p(\Omega)}^p - C_1 \|\tilde{u} - u_o\|_{L^p(\Omega)}^p .$$

By Poincaré's inequality,

$$\infty > A_2 + B_1 \|\nabla \tilde{u}(x)\|_{L^{N^*}(\Omega)}^{N^*} - C_2 \|\nabla \tilde{u} - \nabla u_0\|_{L^p(\Omega)}^p .$$

By Holder's inequality,

$$\infty > A_2 + B_1 \|\nabla \tilde{u}(x)\|_{L^{N^*}(\Omega)}^{N^*} - C_3 \|\nabla u_0\|_{L^p(\Omega)}^p - D \|\nabla \tilde{u}\|_{L^{N^*}(\Omega)}^p ,$$

so that there are positive constants $h$ and $k$ such that

$$\infty > -h + k \|\nabla \tilde{u}(x)\|_{L^{N^*}(\Omega)}^{N^*} .$$

Hence, $\tilde{u}$ belongs to $C_B(\Omega)$ ([1]).

It is sufficient to prove the higher integrability result for the special case where $\xi(\cdot) = \|\nabla \tilde{u}(\cdot)\|$. Indeed the following Lemma holds:

**Lemma 3** *Let $\psi$ non negative and such that*

$$\int_O \psi e^{f(\psi)} f'(\psi) \leq M.$$

*Then, for any $\xi$ such that $\int_O e^{f(\xi)}$ is bounded, we have that*

$$\int_O \xi e^{f(\psi)} f'(\psi)$$

*is bounded.*

*Proof*: a) Consider the strictly increasing function $z(t) = f'(t)\, e^{f(t)}$ and call $t = i(z)$ its inverse, so that we have

$$z = e^{f(i(z))} f'(i(z)). \tag{3.23}$$

We have that $i(v) \to \infty$ as $v \to \infty$. Define the function $\phi$ as $\phi(z) = i(z)\, z$, hence, in terms of $t$,

$$\phi\left(f' e^{f(t)}\right) = t f' e^{f(t)}. \tag{3.24}$$

b) We wish to compute the polar $g^*$ of the function $g(b) = e^{f(b)}$. Define $b_z$ implicitely, setting

$$z = g'(b_z) = e^{f(b_z)} f'(b_z),$$

and notice that the previous equality defines $b_z$ uniquely and we have $b_z = i(z)$, where $i$ is defined in a). Then

$$g^*(z) = \sup_b bz - g(b) = b_z e^{f(b_z)} f'(b_z) - e^{f(b_z)} = b_z z - e^{f(b_z)} = i(z)\, z - e^{f(i(z))}$$

so that, by (3.24) and (3.23),

$$g^*(z) \le \phi\left(f'(b_z)\, e^{f(b_z)}\right) = \phi\left(f'(i(z))\, e^{f(i(z))}\right) = \phi(z).$$

For any $t$ and $b$, we have

$$b f'(t)\, e^{f(t)} = b v(t) \le g^*(v(t)) + g(b) \le \phi(v(t)) + g(b).$$

Set, in the previous inequality, $t = \psi$ and $b = \xi$. From the definition of $\phi$, we obtain

$$\begin{aligned} \xi f'(\psi)\, e^{f(\psi)} &\le \phi\left(f'(\psi)\, e^{f(\psi)}\right) + e^{f(\xi)} \\ &= \psi f'(\psi)\, e^{f(\psi)} + e^{f(\xi)}. \end{aligned}$$

From the assumptions of the Lemma, the proof is completed.

□

The argument used to prove Theorem 3 is similar to the one of the proof of Theorem 2: we do not repeat it here. We only recall the special variation $\eta$ is defined as

$$\eta(x) = \inf\left\{\frac{1}{\tilde{x}(\delta)} \tilde{x}(d(x)), 1\right\},$$

where $\tilde{x}$ is the solution to $\tilde{x} \in G(\tilde{x}')$, provided by Lemma 1, and

$$G(z) = z \frac{2U}{\partial f^*\left(\frac{1}{z}\right)} \tag{3.25}$$

satisfies the assumptions of Lemma 1. In fact, $G(0) = \{0\}$ and $G$ is a strictly increasing multi-valued map (single-valued except on a countable set); we have

$$G\left(\frac{1}{x'}\right) = \frac{2U}{x' \partial f^*(x')}$$

so that, by the assumptions of Theorem 3, the condition of Lemma 1 is satisfied.

# Chapter 4

# On the validity of the Euler-Lagrange equation

This Chapter is devoted to the validity of the Euler-Lagrange equation related to some classes of smooth integral functionals, whose lagrangians could eventually have derivatives growing faster than the lagrangian itself. In section 4.1 we consider minimizers $\tilde{u}$ of the integral functional

$$\int_{\Omega} L\left(x, u\left(x\right), \nabla u\left(x\right)\right) dx.$$

Under the assumptions on $L$ stated in the previous chapter, we have obtained $\left|\nabla \tilde{u}\right| \cdot \left|\nabla L\left(\cdot, \tilde{u}, \nabla \tilde{u}\right)\right| \in L^1_{loc}\left(\Omega\right)$: we use this fact in order to prove that, when $L$ satisfies a simple growth assumption, the Euler-Lagrange equation holds. The remarkable point here is that faster than exponential growths for $L$, with respect to the third variable, are allowed.

In section 4.2 we consider the functional $I$ satisfying the following structure condition:

$$I\left(u\right) = \int_{\Omega} \left[L\left(\nabla u\left(x\right)\right) + g\left(x, u\left(x\right)\right)\right] dx.$$

Here $g\left(x, u\right)$ is a concave map with respect the variable $u$, but $L$ does not have to satisfy any growth assumptions.

## 4.1 On the validity of the Euler-Lagrange equation for a special class of functionals

Let $L\left(x, u, \xi\right)$ be as in Assumption A of section 3.1. Let us assume $\xi \mapsto L\left(x, u, \xi\right)$ has radial growth at infinity: more precisely, suppose that for every $U > 0$ there exists a comparison map $\Lambda \in C^1(\mathbb{R})$, $\Lambda$ convex, such that

$$K_1 \Lambda(|\xi|) \leq L(x,u,\xi) \leq K_2 \Lambda(|\xi|)$$

$$K_1 \Lambda'(|\xi|) \leq |\nabla_\xi L(x,u,\xi)| \leq K_2 \Lambda'(|\xi|)$$

a.e. $x \in \Omega$, $\forall u \leq U$.

**Theorem 4** *Set* $z(t) = \frac{\Lambda'(t)}{\Lambda(t)}$*: if there exists* $c > 0$ *such that either*

*i)* $z(t) \leq c$ *or*

*ii)* $z(t)$ *is non decreasing and, for every* $t \geq t_0$*,* $z(t) \leq c(1 + \log t)$*,*

*then the Euler-Lagrange equation for the functional*

$$\int_\Omega L(x, u(x), \nabla u(x))\, dx \qquad u \in u_0 + W_0^{1,1}(\Omega) \tag{4.1}$$

*holds.*

**Examples**. The hypothesis of Theorem 4 are satisfied by maps $L$ having less than exponential growths with respect to $\xi$. On the other hand, also faster maps $\Lambda$ are taken into account: for instance, the derivative of the map $\Lambda(t) = t^t$ has faster growth than $\Lambda$ itself; but, in this case, the assumptions of point ii) are satisfied and the Euler-Lagrange equation holds.

*Proof of the Theorem*: Let $\tilde{u}$ be a locally bounded minimizer of the functional (4.1) and let $\eta \in C_c^1(\Omega)$. We have

$$\int_\Omega \frac{L(x, \tilde{u}(x) + \varepsilon\eta(x), \nabla\tilde{u}(x) + \varepsilon\nabla\eta(x)) - L(x, \tilde{u}(x), \nabla\tilde{u}(x))}{\varepsilon} dx \geq 0. \tag{4.2}$$

and

$$\begin{aligned}
\left|\frac{L(x, \tilde{u} + \varepsilon\eta, \nabla\tilde{u} + \varepsilon\nabla\eta) - L(x, \tilde{u}, \nabla\tilde{u})}{\varepsilon}\right| &\leq \left|\frac{L(x, \tilde{u} + \varepsilon\eta, \nabla\tilde{u}) - L(x, \tilde{u}, \nabla\tilde{u})}{\varepsilon}\right| \\
&+ \left|\frac{L(x, \tilde{u} + \varepsilon\eta, \nabla\tilde{u} + \varepsilon\nabla\eta) - L(x, \tilde{u} + \varepsilon\eta, \nabla\tilde{u})}{\varepsilon}\right| \\
&\leq \left|\frac{\partial L}{\partial u}(x, \tilde{u} + \bar{s}\varepsilon\eta, \nabla\tilde{u}) \cdot \eta\right| + \left|\left\langle \nabla_\xi L\left(x, \tilde{u} + \varepsilon\eta, \nabla\tilde{u} + \bar{t}\varepsilon\nabla\eta\right), \nabla\eta \right\rangle\right| \\
&\leq K\ L(x, \tilde{u} + \bar{s}\varepsilon\eta, \nabla\tilde{u})|\eta| + K_2\ \Lambda'\left(\left|\nabla\tilde{u} + \bar{t}\varepsilon\nabla\eta\right|\right)|\nabla\eta| \\
&\leq K\frac{K_2}{K_1}\ L(x, \tilde{u}, \nabla\tilde{u})|\eta| + K_2\ \Lambda'(|\nabla\tilde{u}| + \varepsilon|\nabla\eta|)|\nabla\eta|,
\end{aligned} \tag{4.3}$$

where $0 < \overline{s}, \overline{t} < 1$.

In order to pass to the limit in (4.2) by dominated convergence and then to obtain the validity of the Euler-Lagrange equation we need to prove there exist $K, h_0 > 0$ such that

$$\Lambda'(t+h) \leq K\left[1 + \Lambda(t) + t\Lambda'(t)\right] \qquad \forall 0 < h < h_0. \tag{4.4}$$

Indeed in this case we obtain the integrability of the term (4.3) by Theorem 2: the assumptions of the Theorem are satisfied since the convexity of the map $\mathbb{L} : t \mapsto \log \Lambda(t)$ is equivalent to $z$ to be non decreasing and

$$\int^{\infty} \frac{\mathbb{L}''(t)}{t \cdot \mathbb{L}'(t)} dt = \int^{\infty} \frac{z'(t)}{t \cdot z(t)} dt = \left.\frac{\log z(t)}{t}\right|^{\infty} + \int^{\infty} \frac{\log z(t)}{t^2} dt < \infty$$

since $z(t) \leq c(1 + \log t)$ for every $t \geq t_0$.

Let us prove (4.4). We have

$$\begin{aligned} \Lambda'(t+h) &= z(t+h)\frac{\Lambda(t+h)}{\Lambda(t)}\Lambda(t) & (4.5)\\ &= z(t+h) \cdot \exp\left(\int_t^{t+h} z(s)\, ds\right) \cdot \Lambda(t) : & (4.6) \end{aligned}$$

in case i), for $h < h_0$, the term (4.6) is less than $K \cdot \Lambda(t)$; in case ii) if $t < t_0$ then $\Lambda'(t+h) \leq K$ since $\Lambda$ is convex, otherwise

$$\begin{aligned} (4.6) &\leq c\left[1 + \log(t+h_0)\right] \cdot \exp\left[c\int_t^{t+h_0} (1 + \log s)\, ds\right] \cdot \Lambda(t) \\ &\leq c\left(1 + \log t + \frac{h_0}{t}\right) \cdot (t+h_0)^{ch_0} \cdot \exp\left[c\, t\left(\log(t+h_0) - \log t\right)\right] \cdot \Lambda(t) \\ &\leq c_1(1 + \log t) \cdot t^{ch_0} e^{ch_0} \Lambda(t) \\ &\leq c_2 \cdot t\Lambda(t) \\ &\leq K \cdot t\Lambda'(t) \end{aligned}$$

since $z$ is non decreasing and $t \geq t_0$.

□

## 4.2 On the validity of the Euler-Lagrange equation without growth assumptions

This section is based on the paper *On the validity of the Euler-Lagrange equation in a nonlinear case* , Nonlinear Analysis: Theory, Methods & Applications **73** (2010), Issue 1, 266-269, joint work with G. Bonfanti.

In the following, we denote by $p'$ the dual conjugate exponent of $p$, i.e. $1/p + 1/p' = 1$, and by $p^*$ the Sobolev conjugate exponent of $p$, that is $p^* = \frac{Np}{N-p}$ if $p < N$. For the properties of Sobolev functions we refer to [1] and, for those of convex and concave functions, to [37]: we only recall here that, given a concave function $f : \mathbb{R}^N \to \mathbb{R}$, its subdifferential at the point $\xi \in \mathbb{R}^N$ is the set

$$\partial f\left(\xi\right) = \{s \in \mathbb{R}^N : f\left(x\right) \leq f\left(\xi\right) + \left\langle s, x - \xi\right\rangle,\ \forall x \in \mathbb{R}^N\},$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^N$.

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and $L : \mathbb{R}^N \to \mathbb{R}$ be a convex map. Let us consider the problem (P) of minimizing $I$ on the set $u_0 + W_0^{1,p}\left(\Omega\right)$, $1 \leq p < \infty$, where $I$ is the integral functional

$$I(u) = \int_\Omega \left[L\left(\nabla u\left(x\right)\right) + g\left(x, u\left(x\right)\right)\right]\ \mathrm{d}x$$

and the boundary datum $u_0$ is such that $I\left(u_0\right) < +\infty$ and $u_0 \in W^{1,p}\left(\Omega\right) \cap L^\infty_{loc}\left(\Omega\right)$. We observe that assuming $u_0$ to be locally bounded is non restrictive if the Lavrentiev phenomenon does not occur, as already remarked in [22].

In order to prove our result we need the following growth assumption on $g$:

**Assumption A**. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory map such that $u \mapsto g\left(x, u\right)$ is concave for almost every $x \in \Omega$. Moreover:

- if $p < N$, there exist $\alpha_1 \in L^{(p^*)'}\left(\Omega\right)$ and $\beta_1 \in L^\infty\left(\Omega\right)$ such that

$$\left|g\left(x, u\right)\right| \leq \alpha_1\left(x\right) + \beta_1\left(x\right)\left|u\right|^{p^*}, \quad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R};$$

- if $p = N$, there exist $r > 1$, $\alpha_2 \in L^{r'}\left(\Omega\right)$ and $\beta_2 \in L^\infty\left(\Omega\right)$ such that

$$\left|g\left(x, u\right)\right| \leq \alpha_2\left(x\right) + \beta_2\left(x\right)\left|u\right|^r, \quad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R};$$

- if $p > N$, there exist $\alpha_3 \in L^1\left(\Omega\right)$ and $\beta_3 : \mathbb{R} \to \mathbb{R}$ non-decreasing such that

$$\left|g\left(x, u\right)\right| \leq \alpha_3\left(x\right) + \beta_3\left(u\right), \quad \text{a.e. } x \in \Omega,\ \forall u \in \mathbb{R}.$$

We can now state our main result

**Theorem 5** *If $L$ is convex of class $C^1(\mathbb{R}^N)$ and $g$ satisfies Assumption A, then the Euler-Lagrange equation associated to problem (P) holds, i.e. if $\overline{u}$ is a minimizer for $I$, then there exists a selection $\sigma(\cdot)$ from the set-valued map $x \mapsto \partial g(x, \overline{u}(x))$ such that*

$$\int_{\Omega} \langle \nabla L(\nabla \overline{u}(x)), \nabla \eta(x) \rangle \, dx = -\int_{\Omega} \sigma(x) \eta(x) \, dx \quad \forall \eta \in C_c^{\infty}(\Omega).$$

*Remark*: We observe that Assumption A is sufficient in order to prove our main result, but it does not guarantee the existence of a minimum for problem (P) in the case $p > N$.

During the proof we will use some tools from the theory of multivalued functions and selections: for these topics we refer to [2]. We only give the notion of upper semicontinuity for a set-valued map $F : X \to 2^{\mathbb{R}^N}$, where $(X, d)$ is a metric space: $F$ is upper semicontinuous on $X$ if for every $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$ then

$$F(y) \subseteq \left\{ \xi \in \mathbb{R}^N : |\xi - \zeta| < \varepsilon, \text{ for some } \zeta \in F(x) \right\}.$$

We recall that if $F$ is closed graph and it has values all contained in a compact subset of $\mathbb{R}^N$, then it is upper semicontinuous.

For the sake of brevity, we will treat at the same time the cases $p < N$ and $p = N$, by assuming $g$ to satisfy the following general condition:

$$|g(x, u)| \leq \alpha(x) + \beta(x) |u|^r, \quad \text{a.e. } x \in \Omega, \ \forall u \in \mathbb{R}, \tag{4.7}$$

where $r = p^*$ if $p < N$, some $r \in (1, \infty)$ if $p = N$ and $\alpha \in L^{r'}(\Omega)$, $\beta \in L^{\infty}(\Omega)$.

*Proof of Theorem 5*: Given a set $X$, we denote

$$\|X\| = \max \{|x| : x \in X\}.$$

We prove the theorem in several steps.

*(a)* When $p \leq N$, we show that if $g$ satisfies (4.7), then there exist a constant $C > 0$ and a function $\beta' \in L^{\infty}(\Omega)$ such that

$$\|\partial g(x, u)\| \leq 2\alpha(x) + \beta'(x) |u|^{r-1} + C,$$

a.e. $x \in \Omega, \ \forall u \in \mathbb{R}$.

Fix $(x,u) \in \Omega \times \mathbb{R}$ and let $y \in \partial g(x,u)$. From the concavity of $u \mapsto g(x,u)$, we have

$$g(x,u+h) \leq g(x,u) + yh, \ \forall h \in \mathbb{R}.$$

Define the constant

$$h_0 = -\frac{y}{|y|}(1+|u|^r)^{1/r}.$$

By the choice of $h_0$, we have

$$\begin{aligned}
|yh_0| &\leq |g(x,u+h_0)| + |g(x,u)| \\
&\leq 2\alpha(x) + \beta(x)\left[|u+h_0|^r + |u|^r\right] \\
&\leq 2\alpha(x) + \beta'(x)\left[|h_0|^r + |u|^r\right] \\
&= 2\alpha(x) + \beta'(x)\left[1 + 2|u|^r\right].
\end{aligned}$$

Finally, using the inequality

$$(1+|\xi|^s)^{1-\frac{1}{s}} \leq 1 + |\xi|^{s-1}$$

and up to renaming $\beta'$, we obtain

$$\begin{aligned}
|y| &\leq \frac{2\alpha(x)}{(1+|u|^r)^{1/r}} + \beta'(x)(1+|u|^r)^{1-\frac{1}{r}} \\
&\leq 2\alpha(x) + \beta'(x)|u|^{r-1} + C.
\end{aligned}$$

*(b)* Recalling that $g$ is a Carathéodory map, we have that for any $u : \Omega \to \mathbb{R}$ measurable, the set valued map $x \mapsto \partial g(x,u(x))$ is also measurable (see Corollary 4.6 of [38]).

In our special case, the validity of this statement follows also from the proof of Lemma 1, pag. 100 of [15]. Indeed, for any $\varepsilon > 0$ by Lusin and Scorza Dragoni theorems there exists a closed subset $E_\varepsilon \subset \Omega$ such that $u|_{E_\varepsilon}$, $\alpha|_{E_\varepsilon}$ and $g|_{E_\varepsilon \times [-M,M]}$ are continuous, $m(\Omega \setminus E_\varepsilon) < \varepsilon$ and $|u(x)| < M$ for any $x$ in $E_\varepsilon$.

When $p \leq N$, from *(a)* we have

$$\|\partial g(x,u)\| \leq 2\alpha(x) + \beta'(x)|u|^r + C, \text{ a.e. } x \in \Omega, \ \forall u \in \mathbb{R}$$

so there exists $K > 0$ such that

$$\|\partial g(x,u)\| \leq K \quad \forall (x,u) \in E_\varepsilon \times [-M,M].$$

If $p > N$, we obtain the same result using the definition of subdifferential and the monotonicity of $\beta_1$.

The set valued map $(x,u) \mapsto \partial g(x,u)$ on $E_\varepsilon \times [-M,M]$ has values all contained in a compact set, and it can be easily seen that its graph is closed; then it is upper semicontinuous. From the boundedness of $u|_{E_\varepsilon}$, it follows that the map $x \mapsto \partial g(x,u(x))$ is upper semicontinuous on $E_\varepsilon$. Finally, Lusin's theorem yields the claim.

*(c)* Assume that $g$ satisfies Assumption A. We claim that for any $u \in L^r(\Omega)$ there exists a measurable selection $\sigma(\cdot)$ from the set valued map $x \mapsto \partial g(x,u(x))$ such that $\sigma$ belongs to $L^{r'}(\Omega)$ if $1 < r < \infty$ (that is case $p \leq N$) and to $L^1(\Omega)$ if $r = \infty$ (case $p > N$).
The Kuratowski and Ryll-Nardzewski theorem, [27], yields the existence of a measurable selection $\sigma(x)$ from the map $\partial g(x,u(x))$.
As for the integrability, by step *(a)*, we obtain that in the case $p \leq N$

$$|\sigma(x)| \leq \|\partial g(x,u(x))\| \leq 2\alpha(x) + \beta'(x)\,|u(x)|^{r-1} + C, \text{ a.e. } x \in \Omega.$$

Then, Assumption A let us conclude that $\sigma$ belongs to $L^{r'}(\Omega)$. When $p > N$, we use the boundedness of $u$ in order to prove that $\sigma$ is in $L^1(\Omega)$.

*(d)* Let $\overline{u}$ be a solution of problem (P) and let $\sigma(x) \in \partial g(x,\overline{u}(x))$ be the selection given by step *(c)*. For $u \in u_0 + W_0^{1,p}(\Omega)$, define the functional

$$J(u) = \int_\Omega [L(\nabla u(x)) + \sigma(x)\cdot(u(x) - \overline{u}(x)) + g(x,\overline{u}(x))]\,\mathrm{d}x.$$

We claim that $\overline{u}$ is a minimizer for $J$, too.

Indeed, let $v \in u_0 + W_0^{1,p}(\Omega)$; from the concavity of the map $u \mapsto g(x,u)$, we have

$$g(x,v(x)) \leq g(x,\overline{u}(x)) + \sigma(x)\cdot(v(x) - \overline{u}(x)) \quad \text{a.e. } x \in \Omega;$$

then

$$J(\overline{u}) = I(\overline{u}) \leq I(v) \leq J(v).$$

*(e)* Consider the functional $\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}$ such that

$$\varphi \mapsto \int_\Omega \sigma(x)\,\varphi(x)\ \mathrm{d}x.$$

Since, by step *(c)*, $\sigma$ belongs to $L^{(p^*)'}(\Omega)$ if $p < N$, to $L^{r'}(\Omega)$ if $p = N$ and to $L^1(\Omega)$ if $p > N$, Sobolev embedding theorem let us conclude that $\Phi$ is continuous in all three cases.

Finally, Theorem 1.1 in [22] yields that the minimizer $\overline{u}$ of $J$ satisfies the Euler-Lagrange equation, that is

$$\int_\Omega \langle \nabla L\left(\nabla \overline{u}\left(x\right)\right), \nabla \eta\left(x\right)\rangle \,\mathrm{d}x = -\Phi\left(\eta\right) \qquad \forall \eta \in \mathcal{C}_0^\infty\left(\Omega\right),$$

hence our claim.

□

# Part II

# Higher differentiability

# Chapter 5

# Higher differentiability of solutions to variational problems of faster growth

Chapters 5 and 6 concern the local existence and the properties of the weak second derivatives for solutions to some classes of variational problems. In Chapter 5 we consider smooth lagrangians whose growth could possibly be more than polynomial. In Chapter 6 we extend the result to the case of irregular lagrangians, that is, lagrangians that could possibly have no derivatives.

In what follows we are concerned with the higher differentiability properties of a solution $u$ to the problem of minimizing

$$I\left(v\right)=\int_{\Omega}f\left(x,v\left(x\right),\nabla v\left(x\right)\right)dx \tag{5.1}$$

on $u_0+W^{1,1}\left(\Omega\right)$, where $f\left(x,u,\xi\right)$ is a smooth function, convex in the variable $\xi$ and satisfying further regularity and growth assumptions, that will be described below. In particular, under our conditions, it will be equivalent to define $u$ as a solution to the corresponding Euler Lagrange equation, i.e., a solution to

$$\int_{\Omega}\left[\left\langle \nabla_{\xi}f\left(x,u\left(x\right),\nabla u\left(x\right)\right),\nabla\eta\left(x\right)\right\rangle +f_u\left(x,u\left(x\right),\nabla u\left(x\right)\right)\eta\left(x\right)\right]dx=0$$

for every suitable variation $\eta$. The regularity that we seek are the local existence and the properties of the weak second derivatives. Similar results are often deduced, in the theory of partial differential equations or systems, through a standard difference quotient argument. The technique is shown in next section for a model case; here we also present the difficulties arising when the natural growth conditions are not satisfied, that is the context in which we look for regularity of minimizers. Of course, we are not the first ones to consider problems of faster growth: regularity results for this class of functionals have been already considered in [29], [33] and [35], with techniques, assumptions and results that are different from ours.

## 5.1 Natural growth conditions and general growth conditions

Let us consider, for the moment, the solution $u$ for the problem of minimizing

$$\int_\Omega F\left(\nabla v\left(x\right)\right)dx \tag{5.2}$$

on $u_0 + W_0^{1,1}\left(\Omega\right)$. In the classical case of a function $F$ growing as $\left|\xi\right|^2$, the local existence of the second derivative of $u$ is well known and one can proceed as follows.

The function $u$ satisfies the Euler-Lagrange equation

$$\int_\Omega \left\langle \nabla F\left(\nabla u\left(x\right)\right), \nabla\phi\left(x\right)\right\rangle dx = 0$$

for every suitable variation $\phi$. Fix $\omega \subset\subset \Omega$ and let $\eta$ be a smooth map, equal to 1 on $\omega$ and such that $\mathrm{spt}\left(\eta\right) \subset\subset \Omega$ and consider $\phi = \delta^i_{-h}\left(\eta^2\delta^i_h u\right)$, where by $\delta^i_h u$ we mean the difference quotient $\frac{u(x+he_i)-u(x)}{h}$: $\phi$ is, morally, $-\frac{\partial}{\partial x_i}\left(\eta^2\frac{\partial}{\partial x_i}u\right)$, but these derivatives could not exist.

By integrating by parts the difference quotients, we obtain

$$0 = \int_\Omega \left\langle \delta^i_h \nabla F\left(\nabla u\right), \nabla\left(\eta^2\delta^i_h u\right)\right\rangle,$$

that is,

$$\begin{aligned} 0 &= \int_\Omega \left\langle \delta^i_h\nabla u, 2\eta\nabla\eta\delta^i_h u + \eta^2\delta^i_h\nabla u\right\rangle \\ &= \int_\Omega \left[2\eta\delta^i_h u\left\langle\delta^i_h\nabla u, \nabla\eta\right\rangle + \eta^2\left|\delta^i_h\nabla u\right|^2\right], \end{aligned}$$

that can be written as

$$\int_{\mathrm{spt}\ \eta}\left(\left|\nabla\eta\right|\delta^i_h u\right)^2 = \int_{\mathrm{spt}\ \eta}\left|\nabla\eta\delta^i_h u + \eta\delta^i_h\nabla u\right|^2.$$

The first term is bounded by $\int_\Omega |\nabla\eta|^2 \left(\frac{\partial u}{\partial x_i}\right)^2$. Then, the right hand side is bounded by a constant independent of $h$: recalling that the integrand reduces to $\left|\delta_h^i \nabla u\right|^2$ on $\omega$, we can deduce that $u \in W^{2,2}_{loc}$.

More generally, similar arguments can be used to treat minimizers of integral funtionals satisfying the natural growth conditions and involving also the variables $u$ and $x$. In the context of general growth conditions (see for example [33]), one needs to use a class of variations, *nonlinear* in the difference quotient $\delta_h^i u$, that is wider than the class of variations, *linear* in $\delta_h^i u$, that has been used for the standard problems. More precisely, one uses variations of the kind $\phi = \delta_{-h}^i \left(\eta^2 \cdot \gamma\left(\delta_h^i u\right)\right)$, so that the Euler-Lagrange equation becomes, in the case of functional (5.2),

$$0 = \int_\Omega \left\langle \delta_h^i \nabla F(\nabla u), 2\eta\nabla\eta\gamma\left(\delta_h^i u\right) + \eta^2\gamma'\left(\delta_h^i u\right)\delta_h^i \nabla u\right\rangle,$$

i.e.,

$$\begin{aligned} &\int_{\text{spt }\eta} \frac{\left[\left\langle \delta_h^i \nabla F(\nabla u), \nabla\eta\right\rangle \gamma\left(\delta_h^i u\right)\right]^2}{\left\langle \delta_h^i \nabla F(\nabla u), \delta_h^i \nabla u\right\rangle \gamma'\left(\delta_h^i u\right)} \\ &= \int_{\text{spt }\eta} \left| \frac{\left\langle \delta_h^i \nabla F(\nabla u), \nabla\eta\right\rangle \gamma\left(\delta_h^i u\right)}{\left[\left\langle \delta_h^i \nabla F(\nabla u), \delta_h^i \nabla u\right\rangle \gamma'\left(\delta_h^i u\right)\right]^{\frac{1}{2}}} + \eta\left[\left\langle \delta_h^i \nabla F(\nabla u), \delta_h^i \nabla u\right\rangle \gamma'\left(\delta_h^i u\right)\right]^{\frac{1}{2}} \right|^2. \end{aligned} \tag{5.3}$$

In order to handle the left hand side of (5.3) and prove the integrability of

$$\frac{\left[\left\langle \delta_h^i \nabla F(\nabla u), \nabla\eta\right\rangle \gamma\left(\delta_h^i u\right)\right]^2}{\left\langle \delta_h^i \nabla F(\nabla u), \delta_h^i \nabla u\right\rangle \gamma'\left(\delta_h^i u\right)}, \tag{5.4}$$

we have to compare it with an integrable function, independent of $h$: for $F(\xi)$ growing as a function $\Lambda(|\xi|)$, we have, from Chapter 3, that, under slight assumptions, the map $|\nabla u| \cdot \Lambda'(|\nabla u|)$ is locally integrable: we shall reduce the term (5.4) to this function. In order to do that, we look for solutions to the differential equation

$$\gamma'(t) = \phi(t) \cdot \gamma^2(t) = \frac{\Lambda''(t)}{t\Lambda'(t)} \cdot \gamma^2(t). \tag{5.5}$$

Under the simple conditions

$$\int_0^{t_0} \phi_\Lambda(t) = +\infty, \qquad \int_{t_0}^{\infty} \phi_\Lambda(t) < +\infty, \tag{5.6}$$

there exists one and only one solution $\gamma_\Lambda$ of (5.5), defined on $(0, \infty)$ and such that $\lim_{t \to 0^+} \gamma(t) = 0$ and $\lim_{t \to \infty} \gamma(t) = +\infty$: we can write it as

$$\gamma_\Lambda(t) = \frac{1}{\int_t^\infty \phi(s)\, ds}.$$

In such a way, the variation reduces to the standard variation in the cases considered by Ladyzhenskaya and Ural'tseva; that is, the function $\gamma_\Lambda(t)$ turns out to be exactly $c \cdot t$ for the case $\Lambda(t) = t^p$. In addition, the hypothesis (5.6) are satisfied also by maps $\Lambda$ having faster than polynomial growth. They are not satisfied by $\Lambda(t) = e^t$: in this case, $\phi(t) = \frac{1}{t}$ and $\int^\infty \phi$ diverges. If the differential equation (5.5) admits a solution, we succeed in controlling the left hand side of (5.3) with a term of the type

$$\int_{\text{spt } \eta} |\nabla u(x)| \cdot \Lambda'(|\nabla u(x)|)\, dx.$$

Hence, we obtain an estimate for the right hand side of (5.3), independent of $h$, and, with some efforts, the local existence of the weak second derivatives of $u$ and their properties.

When we consider general problems of the type (5.1), we have to handle also terms involving mixed derivatives of $f$ with respect to its variables. We will reduce them to integrable term $|\nabla u| \cdot \Lambda'(|\nabla u|)$ using the properties of polar functions. Of course, we shall need some assumptions on the derivatives of $\Lambda$ and a lot of comparison conditions involving $f$ and $\Lambda$. We list them in next section.

## 5.2 Basic assumptions

For a given function $g$, in this section we shall use the notation $\phi_g$ to denote the function

$$\phi_g = \frac{g''(t)}{t \cdot g'(t)}.$$

We shall assume the following properties on the integrand $f$ and on the comparison real function $\Lambda$.

**Assumption A**. The function $\Lambda$ is symmetric with respect to 0, three times differentiable and satisfies:

1. $\Lambda'(0) = 0$ and, for $t > 0$, $\Lambda'(t) > 0$, $\frac{\Lambda'(t)}{t} \geq c > 0$ and $\Lambda'''(t) \geq 0$; hence, in particular, $\Lambda'(t)$ is convex.

2. For $t > 0$, the map $\phi_\Lambda(t) = \dfrac{\Lambda''(t)}{t \cdot \Lambda'(t)}$ is such that $\frac{d}{dt}\phi_\Lambda(t) < 0$; equivalently, $t\Lambda'\Lambda''' - \Lambda''(\Lambda' + t\Lambda'') < 0$;

3. $\int_0^{t_0} \phi_\Lambda(t) = +\infty$, $\int_{t_0}^{\infty} \phi_\Lambda(t) < +\infty$;

4. The map $\frac{\phi_\Lambda(t)}{\left(\int_t^\infty \phi_\Lambda(s)\right)^2}$ is bounded.

The above point 3) in Assumption A is independent of the choice of $t_0 > 0$. In the case of $\Lambda(t) = \frac{1}{p}t^p$, we have $\int_t^\infty \phi(s) = \frac{(p-1)}{t}$; for $\Lambda(t) = e^t$, we have $\phi(t) = \frac{1}{t}$, so that the requirement of Assumption A, that $\int^\infty \phi$ be convergent, is not verified by $e^t$. A sufficient condition for the validity of (4) is the following: there exists $K$ such that $-\phi'_\Lambda \leq K\phi_\Lambda^{\frac{3}{2}}$. In fact, if this is the case, $2\phi(t)^{\frac{1}{2}} \leq K\int_t^\infty \phi(s)\,ds$.

The map $\Lambda$ is connected to $f$ through the following comparison assumptions. In it, and in the sequel, $H_f(x,u,\xi)$ is the Hessian matrix of $f$ with respect to the variable $\xi$.

**Assumption B**. For every $\omega \subset\subset \Omega$ and positive $U$, there exist positive constants $K_1$, $K_2$, $K$, and a function $\tau \in L^2(\omega)$, such that, for $i = 1, \ldots n$, a.e. $x \in \omega$ and $|u| \leq U$, we have, $\forall \xi \in \mathbb{R}^n$,

i) $\Lambda'(|\xi|) \leq K_2 |\nabla_\xi f(x,u,\xi)|$;

ii) $\lambda$ eigenvalue of $H_f(x,u,\xi)$ implies $K_1 \frac{\Lambda'(|\xi|)}{|\xi|} \leq \lambda \leq K_2 \Lambda''(|\xi|)$;

iii) $\left|\frac{\partial}{\partial x_i}\nabla_\xi f(x,u,\xi)\right| \leq \tau(x) + K\Lambda'(|\xi|)$;

iv) $\left|\frac{\partial}{\partial u}\nabla_\xi f(x,u,\xi)\right| \leq K\frac{\Lambda'(|\xi|)}{|\xi|}$;

v) $|f_{uu}(x,u,\xi)| \leq K\frac{\Lambda'(|\xi|)}{|\xi|}$;

vi) $\left|\frac{\partial}{\partial x_i} f_u(x,u,\xi)\right| \leq \tau(x) + K\left(|\xi|\,\Lambda'(|\xi|)\,\Lambda''(|\xi|)\right)^{\frac{1}{2}}$.

The regularity properties of a solution, that we are going to prove, are based on two facts: first, the validity of the Euler-Lagrange equation for a solution $u$; second, the integrability of the term $|\nabla u| \cdot |\nabla f(x,u,\nabla u)|$. We shall assume their validity in next assumption.

**Assumption C**.

i) Let $u$ be a solution to the problem of minimizing (5.1). Then, for every variation $\phi \in W_0^{1,1}(\Omega)$, we have

$$\int_\Omega [\langle \nabla_\xi f(x,u(x),\nabla u(x)), \nabla\phi(x)\rangle + f'_u(x,u(x),\nabla u(x))\phi(x)]dx = 0.$$

ii) For a solution $u$, for every $\omega \subset\subset \Omega$,

$$\int_{\omega} |\nabla u(x)| \cdot |\nabla f\left(x, u(x), \nabla u(x)\right)| dx < \infty.$$

Point iii) of Assumption B rules out the occurrence of the phenomenon of non-validity of the Euler-Lagrange equation described in [3].

Point ii) of Assumption C is verified whenever, for some $p > 1$ and $M$, we have $f\left(x, u, \xi\right) \leq M\left(1 + |\xi|^p\right)$, since in this case, by the convexity of $f$, $|\nabla_{\xi} f\left(x, u, \xi\right)| \leq M^* \left(1 + |\xi|^{p-1}\right)$. It is also verified by a larger class of functions $f$, including functions of exponential growth and of growth faster than exponential, as shown in [17] and [9].

## 5.3 Properties of $f$ and of $\Lambda$

**Proposition 1** *Under Assumption A (3), the map*

$$\gamma_{\Lambda}\left(t\right) = \frac{1}{\int_t^{\infty} \phi_{\Lambda}\left(s\right) ds}$$

*is the only solution to the differential equation*

$$\gamma'\left(t\right) = \phi_{\Lambda}\left(t\right) \cdot \gamma^2\left(t\right) = \frac{\Lambda''\left(t\right)}{t \cdot \Lambda'\left(t\right)} \cdot \gamma^2\left(t\right),$$

*defined on* $(0, +\infty)$ *and such that* $\lim_{t \to 0} \gamma\left(t\right) = 0$ *and* $\lim_{t \to \infty} \gamma\left(t\right) = +\infty$. *In addition, Assumption A (1) implies that* $\gamma_{\Lambda}\left(t\right) \leq t$ *and Assumption A (4), that* $\gamma'_{\Lambda}\left(t\right)$ *is bounded.*

*Proof*: From the convexity of $\Lambda'$ we infer $0 = \Lambda'\left(0\right) \geq \Lambda'\left(t\right) + \Lambda''\left(t\right)\left(0 - t\right)$, hence $\phi_{\Lambda}\left(t\right) \geq \frac{1}{t^2}$, so that $\gamma_{\Lambda}\left(t\right) \leq t$. Since $\gamma'_{\Lambda}\left(t\right) = \frac{\phi(t)}{\left(\int_t^{\infty} \phi_{\Lambda}(s) ds\right)^2}$, we obtain that $\gamma'_{\Lambda}$ is bounded.

□

**Examples**. For $\Lambda\left(t\right) = \frac{1}{2}t^2$, we have $\phi_{\Lambda}\left(t\right) = \frac{1}{t^2}$ and $\gamma_{\Lambda}\left(t\right) = t$. In the case of $\Lambda\left(t\right) = \frac{1}{p}t^p$, we have $\phi_{\Lambda}\left(t\right) = \left(p - 1\right)\frac{1}{t^2}$, $\gamma_{\Lambda}\left(t\right) = \frac{t}{(p-1)}$ and $\gamma'_{\Lambda} \equiv \frac{1}{(p-1)}$. However, the condition

$$\int_{t_0}^{\infty} \phi_{\Lambda}\left(t\right) < +\infty$$

is satisfied, for instance, by functions $\Lambda$ such that (for $t \geq 1$)

$$\frac{\Lambda''\left(t\right)}{\Lambda'\left(t\right)} = \frac{1}{\left(\log\left(t\right)\right)^2}$$

or

$$\frac{\Lambda''(t)}{\Lambda'(t)} = \frac{1}{(\log(t))(\log(\log(t)))^2};$$

in these cases, the corresponding solutions $\gamma_\Lambda$ are $\gamma_\Lambda(t) = \log(t)$ and $\gamma_\Lambda(t) = \log(\log(t))$.

In what follows, it will be convenient to assume that $\gamma_\Lambda$ has been extended to the whole real axis $(-\infty, +\infty)$, setting $\gamma_\Lambda(-t) = -\gamma_\Lambda(t)$. Notice that, from he properties of $\gamma_\Lambda$ we have that $v \in W^{1,1}$ implies that $\gamma_\Lambda(v) \in W^{1,1}$.

**Lemma 4** *Let* $g : [0,\infty) \to [0,\infty)$ *be continuous and differentiable on* $(0,\infty)$, *with* $g' > 0$, *and such that* $g(0) = 0$ *and* $\lim_{a\to 0} \frac{a}{g(a)} = k \geq 0$. *Assume, further, that the map* $G(t) = \frac{g^{-1}(t)}{t}$ *is non-decreasing. Then, for every* $a, b \geq 0$,

$$g(b)\frac{a}{g(a)} \leq a + b.$$

*Proof*: Define $\Phi(t) = \int_0^t G(s)\,ds$: $\Phi$ is convex with minimum 0 for $t = 0$, defined for every $t \geq 0$ and such that

$$\Phi(g(b)) = \int_0^{g(b)} G(s)\,ds = \int_0^b G(g(t))\,g'(t)\,dt = \int_0^b \frac{t}{g(t)} g'(t)\,dt.$$

Since $G(g(t))$ is non decreasing, $\frac{d}{dt}G(g(t)) \geq 0$, hence $\frac{t}{g(t)}g'(t) \leq 1$, that implies $\Phi(g(b)) \leq b$.

Consider the polar $\Phi^*$: we obtain that $\Phi^*(0) = 0$; moreover, from

$$\lim_{z\to 0^+} \Phi'(z) = \lim_{z\to 0^+} G(z) = \lim_{a\to 0^+} G(g(a)) = \lim_{a\to 0} \frac{a}{g(a)} = k,$$

we infer $\Phi^*(k) = 0$. Set

$$t^* = \begin{cases} 0 & \text{if } t > 0 \implies \Phi^*\left(\frac{t}{g(t)}\right) > 0 \\ \sup\left\{t : \Phi^*\left(\frac{t}{g(t)}\right) = 0\right\} & \text{otherwise.} \end{cases}$$

Fix any $a \geq 0$. From $\frac{a}{g(a)} = G(g(a)) = \Phi'(g(a))$ we infer that $\frac{a}{g(a)} \in \mathrm{Dom}(\Phi^*)$. For $t \in \left(0, \frac{a}{g(a)}\right)$, set $(\Phi^*)'_-(t) = \inf\{y : y \in \partial\Phi^*(t)\}$, so that $(\Phi^*)'_-$ coincides a.e. with the derivative $(\Phi^*)'$ of the locally Lipschitian function $\Phi^*$. The function $(\Phi^*)'$, hence the function $(\Phi^*)'_-$, is integrable on $\left(0, \frac{a}{g(a)}\right)$, since $\Phi^*\left(\frac{a}{g(a)}\right)$ is finite; moreover, the map $z(s) = \frac{s}{g(s)}$ is differentiable on $\left(0, \frac{a}{g(a)}\right)$. Hence, by the change of variables formula ([40]), we obtain

$$\begin{aligned} \Phi^*\left(\frac{a}{g(a)}\right) &= \Phi^*\left(\frac{a}{g(a)}\right) - \Phi^*\left(\frac{t^*}{g(t^*)}\right) \\ &= \int_{\frac{t^*}{g(t^*)}}^{\frac{a}{g(a)}} (\Phi^*)'_-(t)\,dt = \int_{t^*}^{a} (\Phi^*)'_-\left(\frac{s}{g(s)}\right)\frac{g - sg'}{g^2}ds. \end{aligned}$$

From the definition of $G$ we have $\frac{s}{g(s)} = \Phi'(g(s))$, so that $g(s) \in \partial\Phi^*\left(\frac{s}{g(s)}\right)$, hence $g(s) \geq (\Phi^*)'_-\left(\frac{s}{g(s)}\right)$: we obtain

$$\Phi^*\left(\frac{a}{g(a)}\right) = \int_{t^*}^{a} (\Phi^*)'_-\left(\frac{s}{g(s)}\right)\frac{g - sg'}{g^2}ds \leq \int_{t^*}^{a}\frac{g - sg'}{g}ds \leq a - t^*.$$

Finally, by polarity, we have

$$g(b)\frac{a}{g(a)} \leq \Phi(g(b)) + \Phi^*\left(\frac{a}{g(a)}\right) \leq b + a.$$

$\square$

The inequality in the previous Lemma is sharp: when $g(a) = a$, we obtain: for every $a, b \geq 0$, $b \leq a + b$.

The derivatives of $f$ and of $\nabla f$ with respect to their variables, that have to be considered in the proof of the higher differentiability result, yield, through the comparison assumptions, several different combinations of derivatives of $\Lambda$, and these combinations have to be estimated. In the case of a polynomial $\Lambda$, all these different combinations of derivatives turn out to be the same expression, apart from multiplicative constants. This is clearly not the case in general; however, next Corollary shows that Assumption 2 is sufficient to provide a common upper estimate for all these different expressions.

**Corollary 1** *Let $\Lambda$ satisfy Assumption A. Then, for $a, b > 0$,*

*a)*

$$\Lambda'(a)\ \ b \leq \Lambda'(a)a + \Lambda'(b)b$$

*b)*

$$\frac{\Lambda'(a)}{a}b^2 \leq \Lambda'(a)a + \Lambda'(b)b$$

*c)*

$$\Lambda''(a)\frac{b\Lambda'(b)}{\Lambda''(b)} \leq \Lambda'(a)a + \Lambda'(b)b$$

*and*

*d)*

$$\left(a\Lambda'(a)\Lambda''(a)\right)^{\frac{1}{2}}\left(\frac{\Lambda'(b)\ \ b}{\Lambda''(b)}\right)^{\frac{1}{2}} \leq \Lambda'(a)a + \Lambda'(b)b$$

*Proof*: a) Set $\alpha = \Lambda'(a)\,a$, $\beta = \Lambda'(b)\,b$, and define the map $g$ setting $g(\alpha) = g(\Lambda'(a)\,a) = \Lambda'(a)$, so that $\frac{\alpha}{g(\alpha)} = a$ and $\lim_{\alpha\to 0}\frac{\alpha}{g(\alpha)} = \lim_{a\to 0} a = 0$. Moreover, differentiating with respect to $a$ the identity $G(g(\alpha)) = \frac{\alpha}{g(\alpha)}$, we obtain $G'(g(\alpha))\,\Lambda''(a) = 1$ that implies that $G$ is non-decreasing. Lemma 4 can be applied.

b) Set $\alpha = \Lambda'(a)\,a$, and define $g$ by $g(\Lambda'(a)\,a) = a^2$, so that $\frac{\alpha}{g(\alpha)} = \frac{\Lambda'(a)}{a}$ and $G'(g(\alpha))\,2a = \frac{a\Lambda''(a)-\Lambda'(a)}{a^2}$. Recalling that $a\Lambda''(a) - \Lambda'(a) \geq 0$, we obtain that $G'$ is non-negative.

c) Set $\alpha = \Lambda'(a)\,a$, and define $g$ by $g(a\Lambda'(a)) = \frac{a\Lambda'(a)}{\Lambda''(a)}$. Differentiating $g$, we obtain

$$g'\left(a\Lambda'(a)\right)\left(a\Lambda'' + \Lambda'\right) = \frac{\Lambda''\Lambda' + a\left(\Lambda''\right)^2 - a\Lambda'\Lambda'''}{\left(\Lambda''\right)^2} > 0$$

by Assumption A (2), so that $g' > 0$. In addition, we have that, as $\alpha \to 0$, $a \to 0$, so that $\lim_{\alpha\to 0}\frac{\alpha}{g(\alpha)} = \lim_{a\to 0}\Lambda''(a) \geq c$ and, from $G(g(\alpha)) = \frac{\alpha}{g(\alpha)}$ we obtain

$$G'(g(\alpha))\,\frac{\Lambda''\Lambda' + a\left(\Lambda''\right)^2 - a\Lambda'\Lambda'''}{\left(\Lambda''\right)^2} = \Lambda''',$$

that implies that $G$ is non-decreasing, by Assumptions A (1) and (2).

d) Set $\alpha = \Lambda'(a)\,a$, and define $g$ by $g(a\Lambda'(a)) = \left(\frac{a\Lambda'(a)}{\Lambda''(a)}\right)^{\frac{1}{2}}$, so that $\frac{\alpha}{g(\alpha)} = \left(a\Lambda'(a)\,\Lambda''(a)\right)^{\frac{1}{2}}$. We obtain $G\left(\left(\frac{a\Lambda'(a)}{\Lambda''(a)}\right)^{\frac{1}{2}}\right) = \left(a\Lambda'(a)\,\Lambda''(a)\right)^{\frac{1}{2}}$ and

$$G'(g(\alpha))\,\frac{\Lambda''\Lambda' + a\left(\Lambda''\right)^2 - a\Lambda'\Lambda'''}{2\left(\Lambda''\right)^{\frac{3}{2}} a^{\frac{1}{2}} \left(\Lambda'\right)^{\frac{1}{2}}} = \frac{1}{2}\,\frac{\Lambda'\Lambda'' + a\left(\Lambda''\right)^2 + a\Lambda'\Lambda'''}{\left(a\Lambda'\Lambda''\right)^{\frac{1}{2}}},$$

so that $G'$ is positive by Assumptions A (1) and (2).

$\square$

## 5.4 Higher differentiability of a solution

For any real $h \neq 0$ and $i = 1, \ldots, n$, we set $\delta_h^i u$ to be the difference quotient of the function $u$, defined by $\delta_h^i u(x) = \frac{u(x+he_i)-u(x)}{h}$, where $e_i$ is the unit vector in the direction $x_i$. Here we list the main properties of the difference quotients that will be used in the following proofs (for more details see, for example, [26]):

i) Let $u \in W^{1,p}(\Omega)$. Then $\delta_h^i u \in L^p(\omega)$ for any $\omega \subset\subset \Omega$ satisfying $h < \operatorname{dist}(\omega, \partial\Omega)$, and we have

$$\left\|\delta_h^i u\right\|_{L^p(\omega)} \leq \left\|u_{x_i}\right\|_{L^p(\Omega)}.$$

ii) Let $u \in L^p(\Omega)$, $1 < p < \infty$, and suppose there exists a constant $k$ such that $\delta_h^i u \in L^p(\omega)$ and $\left\|\delta_h^i u\right\|_{L^p(\omega)} \leq K$ for all $h > 0$ and $\omega \subset\subset \Omega$ satisfying $h < \operatorname{dist}(\omega, \partial\Omega)$. Then the weak derivative $u_{x_i}$ exists and satisfies $\|u_{x_i}\|_{L^p(\Omega)} \leq K$.

iii) Integration by parts formula holds: suppose $\operatorname{spt}(v) \subset\subset \Omega$, then

$$\int_\Omega u(x)\,\delta_h^i v(x)\,dx = -\int_\Omega v(x)\,\delta_{-h}^i u(x)\,dx,$$

with $h$ small enough.

iv)

$$\delta_h^i(u \cdot v)(x) = u(x + he_i)\,\delta_h^i v(x) + v(x)\,\delta_h^i u(x).$$

v) Let $u \in W^{1,p}(\Omega)$, $1 < p < \infty$. Then $\delta_h^i u \to u_{x_i}$ in $L^p(\omega)$ for any $\omega \subset\subset \Omega$ satisfying $h < \operatorname{dist}(\omega, \partial\Omega)$.

The following Lemmas are basic to the proof of the regularity of a solution.

**Lemma 5** *Let $\Lambda$ satisfy Assumptions A; let $u$ be such that*

$$\int_{\omega'} |\nabla u(x)| \cdot \Lambda'(|\nabla u(x)|)\,dx \leq H_\Lambda.$$

*Then, for every $\omega \subset\subset \omega'$, for every $h$ sufficiently small,*

$$\int_\omega \left|\delta_h^i u(x)\right| \cdot \Lambda'\left(\left|\delta_h^i u(x)\right|\right)dx \leq H_\Lambda.$$

*Proof*: Let $h$ be so small that $\{\omega + th : 0 \leq t \leq 1\} \subset\subset \omega'$. The function $\alpha(t) = t \cdot \Lambda'(t)$ is increasing and convex by Assumption A; then,

$$\int_\omega \alpha\left(\delta_h^i u\right) = \int_\omega \alpha\left(\frac{1}{h}\int_0^1 \langle \nabla u(x + she_i), h\rangle\,ds\right)dx$$

$$\leq \int_\omega \alpha\left(\int_0^1 |\nabla u(x + she_i)|\,ds\right)dx \leq \int_\omega \left(\int_0^1 \alpha(|\nabla u(x + she_i)|)\,ds\right)dx$$

$$= \int_0^1 \left(\int_\omega \alpha(|\nabla u(x + she_i)|)\,dx\right)ds \leq \int_{\omega'} |\nabla u(x)| \cdot \Lambda'(|\nabla u(x)|)\,dx.$$

□

**Lemma 6** *Under Assumptions A, B and C, let $u \in W^{1,1}(\Omega)$ be a locally bounded solution to the problem of minimizing*

$$\int_\Omega f(x, v(x), \nabla v(x))\, dx.$$

*Then there exists a constant $H$ such that, for every $\omega \subset\subset \Omega$, for every $i$ and $j$ and for every $h$ sufficiently small,*

$$\int_\omega \left| \frac{u_{x_j}(x + he_i) - u_{x_j}(x)}{h} \right|^2 \gamma'_\Lambda \left(\delta^i_h u\right)$$

$$\cdot \int_0^1 \frac{\Lambda'\left(\left|(1-s)\nabla u(x) + (s)\nabla u(x+he_i)\right|\right)}{\left|(1-s)\nabla u(x) + (s)\nabla u(x+he_i)\right|} ds\, dx \le H.$$

*Proof*: a) The Euler Lagrange equation is: $\forall \phi \in W^{1,1}_0(\Omega)$,

$$0 = \int_\Omega \left[\langle \nabla_\xi f(x, u(x), \nabla u(x)), \nabla\phi(x)\rangle + f_u(x, u(x), \nabla u(x))\,\phi(x)\right] dx = A + B. \tag{5.7}$$

Fix $\omega \subset\subset \Omega$; let $\eta \in C^1_c(\Omega)$ be such that $\eta \equiv 1$ on $\omega$; set $\tilde{h} = \frac{1}{2}\mathrm{dist}(\mathrm{spt}\eta, C\Omega)$ and consider $h$ such that $|h| \le \tilde{h}$; there exists $\omega' \subset\subset \Omega$ such that, for $|v| = 1$, $\mathrm{spt}\eta + hv \subset\subset \omega'$. Choose, as a variation, $\phi = \delta^i_{-h}\left(\eta^2 \cdot \gamma_\Lambda\left(\delta^i_h u\right)\right)$. Consider $A$, the first term in the Euler Lagrange equation. Calling $b(x) = \nabla u(x + he_i)$ and $a(x) = \nabla u(x)$, we have

$$A = \int_\Omega \left\langle \nabla_\xi f(x, u, \nabla u), \delta^i_{-h}\left(2\eta\nabla\eta\gamma_\Lambda\left(\delta^i_h u\right) + \eta^2\gamma'_\Lambda\left(\delta^i_h u\right)\frac{b-a}{h}\right)\right\rangle$$

$$= \int_\Omega \left\langle \delta^i_h\left(\nabla_\xi f(x, u, \nabla u)\right), 2\eta\nabla\eta\gamma_\Lambda\left(\delta^i_h u\right) + \eta^2\gamma'_\Lambda\left(\delta^i_h u\right)\frac{b-a}{h}\right\rangle$$

$$= \int_\Omega \frac{1}{h}\left\langle \nabla_\xi f(x + he_i, u(x + he_i), \nabla u(x + he_i)) - \nabla_\xi f(x, u(x), \nabla u(x)),\right.$$

$$\left. 2\eta\nabla\eta\gamma_\Lambda\left(\delta^i_h u\right) + \eta^2\gamma'_\Lambda\left(\delta^i_h u\right)\frac{b-a}{h}\right\rangle.$$

Set

$$x^t = x + the_i, \quad u^t = u(x) + t\left(u(x + the_i) - u(x)\right),$$

$$\nabla u^t = \nabla u(x) + t\left(\nabla u(x + he_i) - \nabla u(x)\right),$$

to obtain

$$
\begin{aligned}
A = & \int_{\mathrm{spt}\eta} \left\langle \int_0^1 \left[ \frac{\partial}{\partial x_i} \nabla_\xi f\left(x^t, u^t, \nabla u^t\right) + \frac{\partial}{\partial u} \nabla_\xi f\left(x^t, u^t, \nabla u^t\right) \frac{u\left(x+he_i\right) - u\left(x\right)}{h} \right.\right. \\
& \left.\left. + \frac{b-a}{h} H_f\left(x^t, u^t, \nabla u^t\right) \right] dt, 2\eta \nabla\eta \gamma_\Lambda\left(\delta_h^i u\right) + \eta^2 \gamma'_\Lambda\left(\delta_h^i u\right) \frac{b-a}{h} \right\rangle dx \\
= & A_1 + A_2 + A_3
\end{aligned}
$$

It is convenient to set $H_\nabla = \sup\left\{|\nabla\eta|\right\}$, $H_\Lambda = \int_{\omega'} \Lambda'\left(|\nabla u|\right)|\nabla u|$ and $H_\gamma$ such that $\left|\gamma'_\Lambda\left(t\right)\right| \leq H_\gamma$ . Consider $A_1$: we have

$$
\begin{aligned}
|A_1| \leq & \int_{\mathrm{spt}\eta} \tau 2\eta \left|\nabla\eta\right| \left|\gamma_\Lambda\left(\delta_h^i u\right)\right| dx + K \int_{\mathrm{spt}\eta} \int_0^1 \Lambda'\left(\left|\nabla u^t\right|\right) 2\eta \left|\nabla\eta\right| \left|\gamma_\Lambda\left(\delta_h^i u\right)\right| dt dx \\
& + K \int_{\mathrm{spt}\ \eta} \int_0^1 \left(\left|\nabla u^t\right|\right)^{\frac{1}{2}} \left(\Lambda'\left(\left|\nabla u^t\right|\right)\right)^{\frac{1}{2}} \eta^2 \gamma'_\Lambda\left(\delta_h^i u\right)^{\frac{1}{2}} \gamma'_\Lambda\left(\delta_h^i u\right)^{\frac{1}{2}} \left(\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\right)^{\frac{1}{2}} \left|\frac{b-a}{h}\right| dt dx \\
& + \int_{\mathrm{spt}\eta} \tau \eta^2 \gamma'_\Lambda\left(\delta_h^i u\right) \left|\frac{b-a}{h}\right| dx.
\end{aligned}
\tag{5.8}
$$

Recall that $\left|\gamma_\Lambda\left(\delta_h^i u\right)\right| \leq \left|\delta_h^i u\right|$: Corollary 1 a) and Lemma 5 can be applied to yield that the sum of the first two terms at the right hand side of (5.8), for all sufficiently small $h$, is bounded by a constant independent on $h$. Moreover, the third term is bounded by

$$
\int_{\mathrm{spt}\ \eta} \left( \int_0^1 \frac{1}{\varepsilon} \left|\nabla u^t\right| \Lambda'\left(\left|\nabla u^t\right|\right) \eta^2 \gamma'_\Lambda\left(\delta_h^i u\right) dt + \varepsilon \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} dt \eta^2 \gamma'_\Lambda\left(\delta_h^i u\right) \left|\frac{b-a}{h}\right|^2 \right) dx
$$

and, recalling that $\int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} dt \geq c$, the fourth is bounded by

$$
\int_{\mathrm{spt}\ \eta} \left( \frac{1}{\varepsilon}\frac{1}{c} \tau^2 \eta^2 \gamma'_\Lambda\left(\delta_h^i u\right) + \varepsilon \eta^2 \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} dt \gamma'_\Lambda\left(\delta_h^i u\right) \left|\frac{b-a}{h}\right|^2 \right) dx
$$

and we obtain the existence of $C_1$ such that, for $h$ sufficiently small,

$$
|A_1| \leq C_1 \left(1 + \frac{1}{\varepsilon}\right) + \varepsilon \int_{\mathrm{spt}\ \eta} \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} dt \eta^2 \gamma'_\Lambda\left(\delta_h^i u\right) \left|\frac{b-a}{h}\right|^2 dx.
$$

Consider $A_2$. Recalling Assumption B iv), we have

$$
|A_2| \leq \int_{\mathrm{spt}\ \eta} K \int_0^1 \left(\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\right) dt \left|\delta_h^i u\right| \ 2\eta \ \left|\nabla\eta\right| \left|\gamma_\Lambda\left(\delta_h^i u\right)\right| dx \tag{5.9}
$$

$$+K\int_{\text{spt }\eta}\int_0^1\left(\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\right)^{\frac{1}{2}}\left|\delta_h^i u\right|\eta\left(\gamma_\Lambda'\left(\delta_h^i u\right)\right)^{\frac{1}{2}}$$

$$\cdot\left(\gamma_\Lambda'\left(\delta_h^i u\right)\right)^{\frac{1}{2}}\eta\left(\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\right)^{\frac{1}{2}}\left|\frac{b-a}{h}\right|dtdx.$$

and Corollary 1 b) and Lemma 5 can be applied, to yield that, for $h$ sufficiently small, the first term at the right hand side of (5.9) is bounded by $4KH_\nabla H_\Lambda$. Moreover

$$\int_0^1\left(\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\right)^{\frac{1}{2}}\left|\delta_h^i u\right|\eta\left(\gamma_\Lambda'\left(\delta_h^i u\right)\right)^{\frac{1}{2}}\left(\gamma_\Lambda'\left(\delta_h^i u\right)\right)^{\frac{1}{2}}\eta\left(\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\right)^{\frac{1}{2}}\left|\frac{b-a}{h}\right|dt$$

$$\leq\frac{1}{\varepsilon}H_\gamma\int_0^1\left(\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\right)\left|\delta_h^i u\right|^2dt+\varepsilon\int_0^1\gamma_\Lambda'\left(\delta_h^i u\right)\eta^2\left(\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\right)\left|\frac{b-a}{h}\right|^2dt,$$

and Corollary 1 b) can be applied, to yield the existence of a constant $C_2$ such that, for $h$ sufficiently small,

$$A_2\leq C_2\left(1+\frac{1}{\varepsilon}\right)+\varepsilon\int_{\text{spt }\eta}\int_0^1\frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}dt\eta^2\gamma_\Lambda'\left(\delta_h^i u\right)\left|\frac{b-a}{h}\right|^2dx.$$

Consider

$$A_3=\int_{\text{spt}\eta}\left\langle\int_0^1\left[\frac{b-a}{h}H_f\left(x^t,u^t,\nabla u^t\right)\right]dt,2\eta\nabla\eta\gamma_\Lambda\left(\delta_h^i u\right)+\eta^2\gamma_\Lambda'\left(\delta_h^i u\right)\frac{b-a}{h}\right\rangle dx$$

$$=\int_{\text{spt}\eta}\int_0^1\left[2\eta\left\langle\frac{b-a}{h},H_f\left(x^t,u^t,\nabla u^t\right)\frac{b-a}{h}\right\rangle^{\frac{1}{2}}\frac{\left(\gamma_\Lambda'\left(\delta_h^i u\right)\right)^{\frac{1}{2}}}{\left(\gamma_\Lambda'\left(\delta_h^i u\right)\right)^{\frac{1}{2}}}\right.$$

$$\cdot\frac{\left\langle\frac{b-a}{h},H_f\left(x^t,u^t,\nabla u^t\right)\nabla\eta\right\rangle\gamma_\Lambda\left(\delta_h^i u\right)}{\left\langle\frac{b-a}{h},H_f\left(x^t,u^t,\nabla u^t\right)\frac{b-a}{h}\right\rangle^{\frac{1}{2}}}$$

$$\left.+\left\langle\frac{b-a}{h},H_f\left(x^t,u^t,\nabla u^t\right)\eta^2\gamma_\Lambda'\left(\delta_h^i u\right)\frac{b-a}{h}\right\rangle\right]dtdx.$$

For a symmetric matrix $H_f$, we have that $\lambda_{min}\leq\frac{\left|H_f z\right|^2}{\langle z,H_f z\rangle}\leq\lambda_{max}$; in particular, by Assumption B, we obtain

$$\frac{\left|H_f\left(x^t,u^t,\nabla u^t\right)\frac{b-a}{h}\right|}{\left\langle\frac{b-a}{h},H_f\left(x^t,u^t,\nabla u^t\right)\frac{b-a}{h}\right\rangle^{\frac{1}{2}}}\leq\left(K_2\Lambda''\left(\left|\nabla u^t\right|\right)\right)^{\frac{1}{2}},$$

so that

$$A_3 \geq \int_\Omega \int_0^1 \left[ -2\eta \left(\gamma_\Lambda' \left(\delta_h^i u\right)\right)^{\frac{1}{2}} \left\langle \frac{b-a}{h}, H_f\left(x^t, u^t, \nabla u^t\right) \frac{b-a}{h} \right\rangle^{\frac{1}{2}} \frac{\left|\gamma_\Lambda \left(\delta_h^i u\right)\right|}{\left(\gamma_\Lambda' \left(\delta_h^i u\right)\right)^{\frac{1}{2}}} \right.$$

$$\left. \cdot \left(K_2 \Lambda'' \left(\left|\nabla u^t\right|\right)\right)^{\frac{1}{2}} |\nabla \eta| + \eta^2 \gamma_\Lambda' \left(\delta_h^i u\right) \left\langle \frac{b-a}{h}, H_f\left(x^t, u^t, \nabla u^t\right) \frac{b-a}{h} \right\rangle \right] dt dx$$

$$= \int_\Omega \left[ -2\alpha_h \cdot \beta_h + \left(\alpha_h\right)^2 \right] dx.$$

Consider

$$\beta_h = K_2^{\frac{1}{2}} |\nabla \eta| \int_0^1 \left( \Lambda''\left(\left|\nabla u^t\right|\right) \frac{\Lambda'\left(\left|\delta_h^i u\right|\right) \left|\delta_h^i u\right|}{\Lambda''\left(\left|\delta_h^i u\right|\right)} \right)^{\frac{1}{2}} dt;$$

applying Corollary 1 c), we obtain, for all $h$ sufficiently small,

$$\int_{\text{spt } \eta} \left(\beta_h\right)^2 \leq 2K_2 H_\nabla^2 H_\Lambda = C_\beta,$$

with $C_\beta$ independent of $h$; hence we have that

$$\begin{aligned} A_3 + C_\beta &\geq \int_{\text{spt } \eta} \left|\beta_h - a_h\right|^2 \\ &\geq \int_{\text{spt } \eta} \left(\beta_h\right)^2 - \frac{1}{\varepsilon} \int_{\text{spt } \eta} \left(\beta_h\right)^2 - \varepsilon \int_{\text{spt } \eta} \left(\alpha_h\right)^2 + \int_{\text{spt } \eta} \left(\alpha_h\right)^2 \end{aligned}$$

and we infer

$$\int_{\text{spt } \eta} \left| \eta \left(\gamma_\Lambda' \left(\delta_h^i u\right)\right)^{\frac{1}{2}} \left\langle \frac{b-a}{h}, \int_0^1 H_f\left(x^t, u^t, \nabla u^t\right) \frac{b-a}{h} \right\rangle^{\frac{1}{2}} \right|^2 (1-\varepsilon) \leq A_3 + \frac{1}{\varepsilon} C_\beta.$$

A fortiori, by Assumption B, for each $t$, we have

$$\left\langle \frac{b-a}{h}, H_f\left(x^t, u^t, \nabla u^t\right) \frac{b-a}{h} \right\rangle \geq \lambda_{min}(t) \left| \frac{b-a}{h} \right|^2 \geq \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} \left| \frac{b-a}{h} \right|^2,$$

and we obtain

$$(1-\varepsilon) \int_{\text{spt}\eta} \eta^2 \gamma_\Lambda' \left(\delta_h^i u\right) \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} \left| \frac{b-a}{h} \right|^2 \leq A_3 + \frac{1}{\varepsilon} C_\beta. \tag{5.10}$$

We are left to estimate the term

$$|B| = \left| \int_{\text{spt } \eta} f_u(x, u, \nabla u)\, \delta_{-h}^i \left(\eta^2 \cdot \gamma_\Lambda \left(\delta_h^i u\right)\right) \right|$$

$$= \left| \int_{\mathrm{spt}\ \eta} \delta_h^i \left[ f_u \left( x, u, \nabla u \right) \right] \cdot \eta^2 \gamma_\Lambda (\delta_h^i u) \right|$$

$$\leq \int_{\mathrm{spt}\ \eta} \left( \int_0^1 \left| \frac{\partial}{\partial x_i} f_u \left( x^t, u^t, \nabla u^t \right) \right| + \left| \frac{\partial}{\partial u} f_u \left( x^t, u^t, \nabla u^t \right) \right| \left| \delta_h^i u \right| \right) \left| \gamma_\Lambda \left( \delta_h^i u \right) \right| dx$$

$$+ \int_{\mathrm{spt}\ \eta} \left( \int_0^1 \left| \left\langle \nabla_\xi f_u \left( x^t, u^t, \nabla u^t \right), \frac{b-a}{h} \right\rangle \right| dt \cdot \eta^2 \left| \gamma_\Lambda \left( \delta_h^i u \right) \right| \right) dx$$

$$= B_1 + B_2 + B_3.$$

Recalling Assumption B *vi*) and Corollary 1 d), we obtain, for every $h$ sufficiently small,

$$B_1 \leq \int_{\mathrm{spt}\ \eta} \left[ \tau \left| \delta_h^i u \right| + K \int_0^1 \left( \left| \nabla u^t \right| \Lambda' \left( \left| \nabla u^t \right| \right) \Lambda'' \left( \left| \nabla u^t \right| \right) \right)^{\frac{1}{2}} \cdot \left( \gamma_\Lambda' \left( \left| \delta_h^i u \right| \right) \right)^{\frac{1}{2}} \right.$$

$$\left. \cdot \left( \frac{\left| \delta_h^i u \right| \Lambda' \left( \left| \delta_h^i u \right| \right)}{\Lambda'' \left( \left| \delta_h^i u \right| \right)} \right)^{\frac{1}{2}} \right] dx$$

$$\leq K_\tau + K H_\gamma \left[ \int_0^1 \int_{\mathrm{spt}\ \eta} \Lambda' \left( \left| \nabla u^t \right| \right) \left| \nabla u^t \right| dx dt + \int_{\mathrm{spt}\ \eta} \Lambda' \left( \left| \delta_h^i u \right| \right) \left| \delta_h^i u \right| dx \right]$$

$$\leq K_\tau + 2 K H_\gamma H_\Lambda.$$

Analogously, from Assumption B $v$) and recalling that $\Lambda'(t) \leq t \Lambda''(t)$, from Corollary 1 b), we obtain

$$B_2 \leq K \int_{\mathrm{spt}\ \eta} \int_0^1 \frac{\Lambda' \left( \left| \nabla u^t \right| \right)}{\left| \nabla u^t \right|} \left| \delta_h^i u \right| \cdot \gamma_\Lambda' \left( \left| \delta_h^i u \right| \right)^{\frac{1}{2}} \left( \frac{\left| \delta_h^i u \right| \Lambda' \left( \left| \delta_h^i u \right| \right)}{\Lambda'' \left( \left| \delta_h^i u \right| \right)} \right)^{\frac{1}{2}} dx$$

$$\leq K H_\gamma \left[ \int_0^1 \int_{\mathrm{spt}\ \eta} \Lambda' \left( \left| \nabla u^t \right| \right) \left| \nabla u^t \right| dx dt + \int_{\mathrm{spt}\ \eta} \Lambda' \left( \left| \delta_h^i u \right| \right) \left| \delta_h^i u \right| dx \right].$$

Finally,

$$B_3 \leq K \int_{\mathrm{spt}\ \eta} \int_0^1 \frac{\Lambda' \left( \left| \nabla u^t \right| \right)}{\left| \nabla u^t \right|} \left| \frac{b-a}{h} \right| \eta^2 \left| \gamma_\Lambda \left( \delta_h^i u \right) \right| dt\ dx$$

$$= K \int_{\mathrm{spt}\ \eta} \int_0^1 \left( \frac{\Lambda' \left( \left| \nabla u^t \right| \right)}{\left| \nabla u^t \right|} \right)^{\frac{1}{2}} \eta \frac{\left| \gamma_\Lambda \left( \delta_h^i u \right) \right|}{\left| \gamma_\Lambda' \left( \delta_h^i u \right) \right|^{\frac{1}{2}}} \left( \frac{\Lambda' \left( \left| \nabla u^t \right| \right)}{\left| \nabla u^t \right|} \right)^{\frac{1}{2}} \eta \left| \gamma_\Lambda' \left( \delta_h^i u \right) \right|^{\frac{1}{2}} \left| \frac{b-a}{h} \right|$$

$$\leq K \int_{\mathrm{spt}\ \eta} \left( \frac{1}{\varepsilon} \int_0^1 \left| \delta_h^i u \right|^2 \frac{\Lambda' \left( \left| \nabla u^t \right| \right)}{\left| \nabla u^t \right|} \eta^2 + \varepsilon \int_0^1 \frac{\Lambda' \left( \left| \nabla u^t \right| \right)}{\left| \nabla u^t \right|} \eta^2 \gamma_\Lambda' \left( \delta_h^i u \right) \left| \frac{b-a}{h} \right|^2 \right).$$

From the above estimates we infer the existence of a constant $C^B$ such that, for $h$ sufficiently small,

$$|B| \leq C^B \left(1+\frac{1}{\varepsilon}\right)+\varepsilon \int_{\text{spt }\eta} \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} dt \eta^2 \gamma'_\Lambda \left(\delta_h^i u\right)\left|\frac{b-a}{h}\right|^2 dx.$$

From (5.10) and (5.7) we obtain

$$(1-\varepsilon) \int_{\text{spt}\eta} \eta^2 \gamma'_\Lambda \left(\delta_h^i u\right) \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\left|\frac{b-a}{h}\right|^2 \leq A_3+\frac{1}{\varepsilon} C_\beta=-A_1-A_2-B+\frac{1}{\varepsilon} C_\beta$$

$$\leq\left(\left(C^1+C^2+C^B\right)\left(1+\frac{1}{\varepsilon}\right)+3\varepsilon \int_{\text{spt }\eta} \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} dt \eta^2 \gamma'_\Lambda \left(\delta_h^i u\right)\left|\frac{b-a}{h}\right|^2 dx\right)+\frac{1}{\varepsilon} C_\beta$$

hence,

$$(1-4\varepsilon) \int_{\text{spt}\eta} \eta^2 \gamma'_\Lambda \left(\delta_h^i u\right) \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|}\left|\frac{b-a}{h}\right|^2 \leq\left(C^1+C^2+C^B\right)\left(1+\frac{1}{\varepsilon}\right)+\frac{1}{\varepsilon} C_\beta .$$

Recalling that $\eta=1$ on $\omega$, we obtain, for all sufficiently small $h$,

$$\int_\omega \frac{|b-a|^2}{|h|^2} \gamma'_\Lambda \left(\delta_h^i u\right) \int_0^1 \frac{\Lambda'\left(\left|\nabla u^t\right|\right)}{\left|\nabla u^t\right|} dt dx \leq H \tag{5.11}$$

hence, the result.

□

To state the higher differentiability theorem, let us set

$$G(t)=\int^t\left[\frac{\Lambda'(s)}{s} \gamma'_\Lambda (s)\right]^{\frac{1}{2}} ds,$$

so that we have $\left(G'(t)\right)^2=\frac{\Lambda'(t)}{t} \gamma'_\Lambda (t)$.

**Theorem 6** *Besides assumptions A and B, assume that there exist* $s^*$ *and* $\varepsilon>0$*, such that, for* $s \geq s^*$*, we have* $\Lambda''(s) \geq\left(s \Lambda'(s)\right)^{\varepsilon}$*. Then:*

*i) for some* $p>1$, $u_{x_j} \in W^{1,p}(\omega)$ *and*

*ii)* $G\left(\left|u_{x_j}\right|\right) \in W^{1,2}$.

*Proof*: a) Assuming $\varepsilon < 1$, set $p = \frac{2}{2-\varepsilon}$, so that $1 < p < 2$ and $\frac{p}{2-p}(1-\varepsilon) = 1$. We have

$$\begin{aligned}\left(\frac{|b-a|}{|h|}\right)^p &= \left(\frac{|b-a|}{|h|}\right)^p \left(\gamma'_\Lambda(|\delta_h u|)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}\right)^{\frac{p}{2}} \frac{1}{\left(\gamma'_\Lambda(|\delta_h u|)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}\right)^{\frac{p}{2}}} \qquad (5.12)\\ &\le \frac{p}{2}\left[\left(\frac{|b-a|}{|h|}\right)^p \left(\gamma'_\Lambda(|\delta_h u|)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}\\ &\quad + \frac{2-p}{2}\left[\frac{1}{\left(\left(\gamma'_\Lambda(|\delta_h u|)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}\right)\right)^{\frac{p}{2}}}\right]^{\frac{2}{2-p}}\end{aligned}$$

Since $\gamma'_\Lambda$ is positive and continuous and $\gamma_\Lambda$ is non-decreasing, there exist $\Gamma$ and $\Gamma'$ such that: for $z \le z^*$ we have $\gamma'_\Lambda(z) \ge \Gamma'$ and, for $z \ge z^*$, $\gamma_\Lambda \ge \Gamma$. Moreover, $\frac{\Lambda'}{t} \ge c$.

From (5.11) we have

$$\begin{aligned} H &\ge \int_\omega \frac{|b-a|^2}{|h|^2}\gamma'_\Lambda(\delta_h u)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}dtdx\\ &= \int_\omega Z = \int_{\omega\cap\{x:|\delta_h u|<z^*\}} Z + \int_{\omega\cap\{x:|\delta_h u|\ge z^*\}} Z\end{aligned}$$

so that both

$$\int_{\omega\cap\{x:|\delta_h u|<z^*\}} \frac{|b-a|^2}{|h|^2} \le \frac{H}{c\Gamma'}$$

and

$$\int_{\omega\cap\{x:|\delta_h u|\ge z^*\}} \frac{|b-a|^2}{|h|^2}\gamma'_\Lambda(\delta_h u)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{(|\nabla u^t|)}dtdx \le H \qquad (5.13)$$

hold.

Consider (5.12) to obtain

$$\begin{aligned} &\int_{\omega\cap\{x:|\delta_h u|\ge z^*\}} \left(\frac{|b-a|}{|h|}\right)^p \qquad (5.14)\\ &\le \int_{\omega\cap\{x:|\delta_h u|\ge z^*\}} \frac{p}{2}\left[\left(\frac{|b-a|}{|h|}\right)^p \left(\gamma'_\Lambda(|\delta_h u|)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}\right)^{\frac{p}{2}}\right]^{\frac{2}{p}}\\ &\quad + \int_{\omega\cap\{x:|\delta_h u|\ge z^*\}} \frac{2-p}{2}\left[\frac{1}{\left(\gamma'_\Lambda(|\delta_h u|)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}\right)^{\frac{p}{2}}}\right]^{\frac{2}{2-p}}.\end{aligned}$$

For $|\delta_h u| \geq z^*$, we have

$$\begin{aligned}\left(\frac{1}{\gamma'_\Lambda\left(|\delta_h u|\right)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}}\right)^{\frac{p}{2-p}} &= \left(\frac{|\delta_h u|\,\Lambda'\left(|\delta_h u|\right)}{\gamma^2_\Lambda\left(|\delta_h u|\right)\Lambda''\left(|\delta_h u|\right)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}}\right)^{\frac{p}{2-p}} \\ &\leq \left(\frac{|\delta_h u|\,\Lambda'\left(|\delta_h u|\right)}{\Gamma^2 c\left(|\delta_h u|\,\Lambda'\left(|\delta_h u|\right)\right)^{\varepsilon}}\right)^{\frac{p}{2-p}} \\ &= \left(\frac{1}{\Gamma^2 c}\right)^{\frac{p}{2-p}} |\delta_h u|\,\Lambda'\left(|\delta_h u|\right)\end{aligned}$$

hence,

$$\int_{\omega\cap\{x:|\delta_h u|\geq z^*\}}\left[\frac{1}{\left(\gamma'_\Lambda\left(|\delta_h u|\right)\int_0^1 \frac{\Lambda'(|\nabla u^t|)}{|\nabla u^t|}\right)^{\frac{p}{2}}}\right]^{\frac{2}{2-p}} \leq C\int_\omega |\delta_h u|\,\Lambda'\left(|\delta_h u|\right)$$

and the second integral at the r.h.s. of (5.14) is bounded. The first integral equals

$$\frac{p}{2}\int_{\omega\cap\{x:|\delta_h u|\geq z^*\}}\left(\frac{|b-a|}{|h|}\right)^2 \gamma'_\Lambda\left(|\delta_h u|\right)\int_0^1 \frac{\Lambda'\left(|\nabla u^t|\right)}{|\nabla u^t|}$$

bounded by (5.11). We have shown the existence of $H_1$ such that, for all sufficiently small $h$

$$\int_\omega \left|\frac{u_{x_j}\left(x+he_i\right)-u_{x_j}\left(x\right)}{h}\right|^p \leq H_1$$

that gives

$$u_{x_j} \in W^{1,p}\left(\omega\right). \tag{5.15}$$

b) To prove the Theorem, notice that, as $|h| \to 0$, $\nabla u\left(\cdot + he_i\right) \to \nabla u\left(\cdot\right)$ in $L^1$, hence pointwise along the subsequence $(h_n)$, so that

$$\int_0^1 \frac{\Lambda'\left(\left|\left(1-s\right)\nabla u\left(x\right)+\left(s\right)\nabla u\left(x+h_n e_i\right)\right|\right)}{\left|\left(1-s\right)\nabla u\left(x\right)+\left(s\right)\nabla u\left(x+h_n e_i\right)\right|}ds \to \frac{\Lambda'\left(\left|\nabla u\left(x\right)\right|\right)}{\left|\nabla u\left(x\right)\right|}.$$

Moreover, $\delta_{h_n}\left(u\right)$ converges to $u_{x_i}$ in $L^p$, hence, taking subsequences, $\gamma'_L\left(|\delta_{h_n} u|\right) \to \gamma'_\Lambda\left(|u_{x_i}|\right)$ pointwise. Finally, the subsequence $\left(\frac{u_{x_j}(x+h_n e_i)-u_{x_j}(x)}{h_n}\right)_n$ is contained in a bounded set of $L^p\left(\omega\right)$, with $p > 1$, hence a subsequence (still called $h_n$) converges weakly and it is classical to show that the weak limit is $u_{x_j x_i}$. A subsequence of convex combinations, $c_n = \sum_k \lambda^n_k \delta_{h_n}\left(u_{x_j}\right)$, converges pointwise to $u_{x_j x_i}$.

Apply Fatou's Lemma and inequality (5.11): we have

$$
\begin{aligned}
&\int_\omega \frac{\Lambda'\left(\left|u_{x_i}(x)\right|\right)}{\left|u_{x_i}(x)\right|}\left|u_{x_ix_j}(x)\right|^2\gamma'_\Lambda\left(\left|u_{x_i}(x)\right|\right)dx \\
&\qquad\qquad \le \int_\omega \frac{\Lambda'\left(\left|\nabla u(x)\right|\right)}{\left|\nabla u(x)\right|}\left|u_{x_ix_j}(x)\right|^2\gamma'_\Lambda\left(\left|u_{x_i}(x)\right|\right)dx \\
&\qquad\qquad \le \liminf\int_\omega\left[\int_0^1\frac{\Lambda'\left(\left|(1-s)\nabla u(x)+(s)\nabla u(x+h_ne_i)\right|\right)}{\left|(1-s)\nabla u(x)+(s)\nabla u(x+h_ne_i)\right|}ds\right] \\
&\qquad\qquad\qquad\qquad \cdot\left(\sum_k\lambda_k^n\left|\delta_{h_n}\left(u_{x_j}\right)\right|\right)^2\gamma'_\Lambda\left(\left|\delta_{h_n}u\right|\right) \\
&\qquad\qquad \le \liminf\int_\omega\left[\int_0^1\frac{\Lambda'\left(\left|(1-s)\nabla u(x)+(s)\nabla u(x+h_ne_i)\right|\right)}{\left|(1-s)\nabla u(x)+(s)\nabla u(x+h_ne_i)\right|}ds\right] \\
&\qquad\qquad\qquad\qquad \cdot\left(\sum_k\lambda_k^n\left|\delta_{h_n}\left(u_{x_j}\right)\right|^2\right)\gamma'_\Lambda\left(\left|\delta_{h_n}u\right|\right) \\
&\qquad\qquad = \liminf\sum_k\lambda_k^n\int_\omega\left[\int_0^1\frac{\Lambda'\left(\left|(1-s)\nabla u(x)+(s)\nabla u(x+h_ne_i)\right|\right)}{\left|(1-s)\nabla u(x)+(s)\nabla u(x+h_ne_i)\right|}ds\right] \\
&\qquad\qquad\qquad\qquad \cdot\left|\delta_{h_n}\left(u_{x_j}\right)\right|^2\gamma'_L\left(\left|\delta_h u\right|\right) \\
&\qquad\qquad \le H.
\end{aligned}
$$

Hence, the map $G\left(\left|u_{x_j}\right|\right)$ is in $W^{1,2}$.

□

# Chapter 6

# Higher differentiability for minimizers of irregular functionals

In Chapter 5 we looked for local existence and properties of weak second derivatives for solutions to smooth variational problems. In what follows, we discuss the same topic without assuming everywhere differentiability of the functional being minimized.

The problem of regularity for solutions to variational problems of the type

$$\int_{\Omega} F\left(\nabla v\left(x\right)\right) dx,$$

with $F$ satisfying only an uniform convexity condition, has been treated in [25], [23] and [24]. Here, we consider the problem of minimizing the functional

$$I\left[v\right] = \int_{\Omega} \left[a\left(x\right) F\left(\nabla v(x)\right) + g\left(x, v\left(x\right)\right)\right] dx$$

on $u_0 + W_0^{1,2}\left(\Omega\right)$, where $\Omega$ is a bounded subset of $\mathbb{R}^n$, $a \in W_{loc}^{1,\infty}\left(\Omega\right)$, $a\left(x\right) \geq \alpha > 0$, and there exist $K > 0$ and $\tau \in L_{loc}^2\left(\Omega\right)$ such that

i) $g_{vv}\left(x, v\right) \leq K$;

ii) $g_{vx_i}\left(x, v\right) \leq \tau\left(x\right)$.

Assume $F$ is such that $F\left(\xi\right) = f\left(\left|\xi\right|\right)$ for every $\xi \in \mathbb{R}^n$, $f : \mathbb{R} \to \mathbb{R}$ being a convex even function. By convexity, $f$ is not differentiable at most on a countable set; let us assume $f$ is not differentiable on a finite number of points in $\mathbb{R}$: call $a_1 = 0$ and $a_k$, $k = 2, \cdots, K$, the other points of nondifferentiability for $f$. Set

$\Delta_k = \sup \{\lambda : \lambda \in \partial f (a_k)\} - \inf \{\lambda : \lambda \in \partial f (a_k)\}$ and assume $0 < c_0 \leq f'' (t) \leq c_1$ $\forall t \neq a_k$.

**Theorem 7** *If the functional $I$ satisfies the assumptions stated above and $u \in u_0 + W_0^{1,2} (\Omega)$ is a minimizer of $I$, then $u \in W_{loc}^{2,2} (\Omega)$.*

*Proof*: For $c \in \mathbb{R}^+$, set

$$E_{c,k} = \left\{ t \in \mathbb{R} : a_k - \frac{\Delta_k}{c} \leq t \leq a_k + \frac{\Delta_k}{c} \right\}.$$

For $c$ large enough $\{E_{c,k}\}_k$ is a family of disjoint sets; then, for every $c$ large enough, there exists a map $f_c \in C^2 (\mathbb{R})$ such that $f_c(t) = f (t)$ for $t \in \mathbb{R} \setminus \cup_k E_{c,k}$ and

$$c_0 \leq f_c''(t) \leq c \qquad \forall t \in E_{c,k}.$$

Let $I_c$ be the functional

$$I_c [v] = \int_\Omega [a (x) F_c (\nabla v(x)) + g (x, v (x))] \, dx$$

where $F_c (\xi) = f_c (|\xi|)$ and let $u_c$ be a locally bounded minimizer of $I_c$ on $u_0 + W_0^{1,2} (\Omega)$: as $c \to \infty$, $\{u_c\}$ is a minimizing sequence for the functional $I$. In fact, setting $A_{c,k} = \{x : |\nabla u_c (x)| \in E_{c,k}\}$ and $B_c = \Omega \setminus \cup_k A_{c,k}$, we obtain

$$\begin{aligned}
0 \leq I[u_c] - I[u] &= \int_{\cup_k A_{c,k}} a (x) F (\nabla u_c) + \int_{B_c} a (x) F (\nabla u_c) + \int_\Omega g (x, u_c) - I [u] \\
&= \int_{\cup_k A_{c,k}} a (x) F (\nabla u_c) + \int_{B_c} a (x) F_c (\nabla u_c) + \int_\Omega g (x, u_c) - I [u] \\
&= \int_{\cup_k A_{c,k}} a (x) [F (\nabla u_c) - F_c (\nabla u_c)] + \int_\Omega [a (x) F_c (\nabla u_c) + g (x, u_c)] - I [u] \\
&\leq \int_{\cup_k A_{c,k}} a (x) [F (\nabla u_c) - F_c (\nabla u_c)] + \int_\Omega a (x) [F_c (\nabla u) - F (\nabla u)] \\
&= \int_{\cup_k A_{c,k}} a (x) [F (\nabla u_c) - F_c (\nabla u_c) + F_c (\nabla u) - F (\nabla u)] .
\end{aligned}$$

From the convexity of $f$ and $f_c$ and the definition of $E_{c,k}$, it is possible to prove that there exists $\Gamma > 0$ such that if $t \in E_{c,k}$ then

$$|f_c (t) - f (t)| \leq \frac{\Gamma}{c};$$

so we obtain
$$0 \leq I[u_c] - I[u] \leq \frac{2\Gamma}{c} \int_\Omega a \longrightarrow 0$$
as $c \to \infty$.

Let us introduce the absolutely continuous map $v_c : \mathbb{R}^+ \to \mathbb{R}$ such that $v_c\,(0) = 1$ and $v_c'\,(t) = c$ if $t \in E_{c,k}$ for some $k$, $v_c'\,(t) = 0$ otherwise. It is clear that $v_c$ is bounded by a constant $V$ independent of $c$.

From the regularity of $I_c$ we have that the minimizer $u_c$ satisfies the Euler Lagrange equation
$$0 = \int_\Omega \langle a\,(x)\,\nabla_\xi F_c\,(\nabla u_c)\,, \nabla \phi \rangle + \int_\Omega g_v\,(x, u_c) \cdot \phi$$
for all $\phi \in W_0^{1,2}\,(\Omega)$; moreover, we have $u_c \in W_{loc}^{2,2}\,(\Omega)$: for instance, in Chapter 5, $\Lambda\,(t) = \frac{1}{2}t^2$ satisfies Assumptions A and B, then Lemma 6 implies $u_c \in W_{loc}^{2,2}\,(\Omega)$.

Having fixed $\omega \subset\subset \omega_1 \subset\subset \omega_2 \subset\subset \Omega$, let $\eta \in C_c^1(\omega_2)$ be such that $\eta \equiv 1$ on $\omega_1$ and, for $|h| < \dfrac{1}{2}\mathrm{dist}\,(\partial\omega_2, \mathrm{spt}\;\eta)$, let $\phi = \delta_{-h}^i\left(\eta^2 \cdot \delta_h^i u_c \cdot V_c\right)$, where $\delta_h^i u_c\,(x) = \frac{u_c(x+he_i)-u_c(x)}{h}$, $\delta_h u_c = \left(\delta_h^1 u_c, \ldots, \delta_h^n u_c\right)^{tr}$ and $V_c = v_c\,(|\delta_h u_c|)$. In order to prove that $\phi \in W_0^{1,2}\,(\Omega)$, we have in particular to show that $\eta^2 \delta_h^i u_c \nabla V_c \in L^2$: setting $A_{c,h,k} = \{x : |\delta_h u_c\,(x)\,| \in E_{c,k}\}$, we have
$$\begin{aligned}
\int_\Omega \eta^4 \left|\delta_h^i u_c\right|^2 \left|(V_c)_{x_j}\right|^2 &\leq \int_{\mathrm{spt}\eta} \left|\delta_h^i u_c\right|^2 \left| v_c'\,(|\delta_h u_c|)\,\frac{\delta_h u_c \cdot \delta_h\,(u_c)_{x_j}}{|\delta_h u_c|}\right|^2 \\
&\leq \int_{\mathrm{spt}\eta \cap \cup_k A_{c,h,k}} \left|\delta_h^i u_c\right|^2 c^2 \left|\delta_h\,(u_c)_{x_j}\right|^2 \\
&\leq Hc^2 \int_{\mathrm{spt}\eta} \left|\delta_h\,(u_c)_{x_j}\right|^2 \\
&\leq Hc^2 \int_{\omega_2} \left|(\nabla u_c)_{x_j}\right|^2
\end{aligned}$$

Using the properties of difference quotients we obtain
$$0 = \int_\Omega \left\langle a\,(x)\,\delta_h^i \nabla_\xi F_c\,(\nabla u_c)\,, 2\eta\nabla\eta\delta_h^i u_c V_c + \eta^2 \delta_h^i \nabla u_c V_c + \eta^2 \delta_h^i u_c \nabla V_c \right\rangle \tag{6.1}$$
$$+ \int_\Omega \left\langle \delta_h^i a\,(x) \cdot \nabla_\xi F_c\,(\nabla u_c\,(x + he_i))\,, 2\eta\nabla\eta\delta_h^i u_c V_c + \eta^2 \delta_h^i \nabla u_c V_c + \eta^2 \delta_h^i u_c \nabla V_c \right\rangle$$
$$+ \int_\Omega \delta_h^i g_v\,(x, u_c) \cdot \eta^2 \cdot \delta_h^i u_c \cdot V_c.$$

Recalling the assumptions on $g$, one can prove that
$$\left|\delta_h^i g_v\,(x, u_c)\right| \leq K \left|\delta_h^i u_c\right| + \int_0^1 \tau\,(x + the_i)\,dt.$$

Since

$$\int_{\mathrm{spt}\eta}\left(\int_0^1 \tau\left(x+the_i\right)dt\right)^2 dx \le \int_0^1\int_{\mathrm{spt}\eta}\tau^2\left(x+the_i\right)dxdt \le \int_{\omega_2}\tau^2\left(x\right)dx,$$

we obtain

$$\begin{aligned}\left|\int_\Omega \delta_h^i g_v\left(x,u_c\right)\cdot\eta^2\cdot\delta_h^i u_c\cdot V_c\right| &\le V\int_{\mathrm{spt}\eta}\left[K\left|\delta_h^i u_c\right|+\int_0^1\tau\left(x+the_i\right)dt\right]\left|\delta_h^i u_c\right| \\ &\le \overline{K},\end{aligned}$$

with $\overline{K}$ independent of $h$ and $c$.

Considering the second term of (6.1), we have

$$\begin{aligned}&\left|\int_\Omega\left\langle\delta_h^i a\left(x\right)\cdot\nabla_\xi F_c\left(\nabla u_c\left(x+he_i\right)\right),2\eta\nabla\eta\delta_h^i u_c V_c+\eta^2\delta_h^i\nabla u_c V_c+\eta^2\delta_h^i u_c\nabla V_c\right\rangle\right| \\ &\qquad\le\left\|a\right\|_{W^{1,\infty}(\mathrm{spt}\eta)}\cdot\left\|\nabla_\xi F_c\left(\nabla u_c\right)\right\|_{L^2(\omega_2)}\cdot\left\|\phi\right\|_{W^{1,2}(\mathrm{spt}\eta)};\end{aligned} \tag{6.2}$$

since

$$\int_{\omega_2}\left|f_c'\left(\left|\nabla u_c\right|\right)\right|^2\le\int_{\omega_2}\left[\sum_k\Delta_k+c_1\left|\nabla u_c\right|\right]^2\le\mathrm{cost}\cdot I\left[u_c\right]\to\mathrm{cost}\cdot I\left[u\right]$$

as $c\to\infty$, we obtain that also the right hand side of (6.2) is bounded by a constant independent of $c$ and $h$.

Now we can write

$$\overline{K}\ge\int_\Omega\left\langle a\left(x\right)\delta_h^i\nabla_\xi F_c\left(\nabla u_c\right),2\eta\nabla\eta\delta_h^i u_c V_c+\eta^2\delta_h^i\nabla u_c V_c+\eta^2\delta_h^i u_c\nabla V_c\right\rangle;$$

summing over the indices $i$ we have

$$n\overline{K}\ge\liminf_{h\to0}\sum_i\int_\Omega\left\langle a\left(x\right)\delta_h^i\nabla_\xi F_c\left(\nabla u_c\right),2\eta\nabla\eta\delta_h^i u_c V_c\right\rangle \tag{6.3}$$

$$+\liminf_{h\to0}\sum_i\int_\Omega\left\langle a\left(x\right)\delta_h^i\nabla_\xi F_c\left(\nabla u_c\right),\eta^2\delta_h^i\nabla u_c V_c\right\rangle \tag{6.4}$$

$$+\liminf_{h\to0}\sum_i\int_\Omega\left\langle a\left(x\right)\delta_h^i\nabla_\xi F_c\left(\nabla u_c\right),\eta^2\delta_h^i u_c\nabla V_c\right\rangle \tag{6.5}$$

Setting $H_{F_c}$ the Hessian matrix of $F_c$, we write

$$\begin{aligned}\delta_h^i\nabla_\xi F_c\left(\nabla u_c\right) &= \frac{1}{h}\nabla_\xi F_c\left(\nabla u_c\left(x\right)+t\left[\nabla u_c\left(x+he_i\right)-\nabla u_c\left(x\right)\right]\right)\Big|_0^1 \\ &= \int_0^1 H_{F_c}\left(\nabla u_c\left(x\right)+t\left[\nabla u_c\left(x+he_i\right)-\nabla u_c\left(x\right)\right]\right)\cdot\delta_h^i\nabla u_c\left(x\right) \\ &= \int_0^1 H_{F_c}\left(\nabla u_c^{t,i}\right)\cdot\delta_h^i\nabla u_c.\end{aligned}$$

Consider (6.4) and recall that there exist $K_1, K_2 > 0$ such that

$$K_1 \frac{f_c'(|\xi|)}{|\xi|} |z|^2 \leq \langle H_{F_c}(\xi) z, z \rangle \leq K_2 f_c''(|\xi|) |z|^2$$

for every $\xi, z \in \mathbb{R}^n$; we have

$$\liminf_{h \to 0} \sum_i \int_\Omega \left\langle a(x) \delta_h^i \nabla_\xi F_c(\nabla u_c), \eta^2 \delta_h^i \nabla u_c V_c \right\rangle$$

$$= \liminf_{h \to 0} \sum_i \int_\Omega a \eta^2 V_c \left\langle \int_0^1 H_{F_c}\left(\nabla u_c^{t,i}\right) \cdot \delta_h^i \nabla u_c, \delta_h^i \nabla u_c \right\rangle$$

$$\geq \alpha K_1 \liminf_{h \to 0} \sum_i \int_\Omega \eta^2 \frac{f_c'\left(\left|\nabla u_c^{t,i}\right|\right)}{\left|\nabla u_c^{t,i}\right|} \left|\delta_h^i \nabla u_c\right|^2$$

$$\geq \alpha c_0 K_1 \liminf_{h \to 0} \sum_i \int_\Omega \eta^2 \left|\delta_h^i \nabla u_c\right|^2$$

$$\geq \alpha c_0 K_1 \sum_i \int_\Omega \eta^2 \left|(\nabla u_c)_{x_i}\right|^2$$

by Fatou's Lemma.

Consider the term (6.5): we obtain

$$\liminf_{h \to 0} \sum_i \int_\Omega \left\langle a(x) \delta_h^i \nabla_\xi F_c(\nabla u_c), \eta^2 \delta_h^i u_c \nabla V_c \right\rangle$$

$$= \liminf_{h \to 0} \int_\Omega \frac{a \eta^2 c}{|\delta_h u_c|} \left\langle \left( \int_0^1 H_{F_c} \right) \delta_h u_c \cdot \delta_h \nabla u_c, \delta_h u_c \cdot \delta_h \nabla u_c \right\rangle \chi_{\cup_k A_{c,h,k}}$$

$$\geq \alpha c \int_{\cup_k A_{c,k}} \frac{\eta^2}{|\nabla u_c|} \langle H_{F_c}(\nabla u_c) \nabla u_c \cdot H_{u_c}, \nabla u_c \cdot H_{u_c} \rangle$$

$$\geq \frac{\alpha}{K_2} \int_{\cup_k A_{c,k}} \eta^2 \frac{|H_{F_c}(\nabla u_c) \nabla u_c \cdot H_{u_c}|^2}{|\nabla u_c|}.$$

Finally consider (6.3):

$$\liminf_{h \to 0} \sum_i \int_\Omega \left\langle a(x) \delta_h^i \nabla_\xi F_c(\nabla u_c), 2\eta \nabla \eta \delta_h^i u_c V_c \right\rangle$$

$$\geq -2 \|a\|_{L^\infty(\mathrm{spt}\eta)} \sup\{|\nabla \eta|\} V \liminf_{h \to 0} \int_\Omega \eta \left| \sum_i \delta_h^i u_c \cdot \delta_h^i \nabla_\xi F_c(\nabla u_c) \right|.$$

In order to apply Vitali's Convergence Theorem, we just need to show that $\eta \cdot \left|\sum_i \delta_h^i u_c \cdot \delta_h^i \nabla_\xi F_c(\nabla u_c)\right|$ is an equi-integrable sequence. Fix $\varepsilon > 0$ and let $\delta > 0$ be such that for any $E \subset \Omega$: $m(E) < \delta$ we have

$$\int_E \left|(u_c)_{x_i}\right|^2 < \frac{\varepsilon}{n}, \quad \int_E \left|(\nabla u_c)_{x_i}\right|^2 < \frac{\varepsilon}{nK_2^2 c^2}.$$

$\forall i$. Fix such an $E$, then

$$\int_E \eta \left|\sum_i \delta_h^i u_c \cdot \delta_h^i \nabla_\xi F_c(\nabla u_c)\right| \leq \sum_i \int_E \eta \frac{1}{2}\left[\left|\delta_h^i u_c\right|^2 + \left|\delta_h^i \nabla_\xi F_c(\nabla u_c)\right|^2\right]$$

$$\leq \sum_i \int_E \frac{1}{2}\eta \left[\left|\int_0^1 (u_c)_{x_i}(x + the_i)\,dt\right|^2 + \left|\int_0^1 H_{F_c}(\nabla u_c(x + the_i)) \cdot (\nabla u_c)_{x_i}(x + the_i)\,dt\right|^2\right]$$

$$\leq \sum_i \frac{1}{2}\int_0^1 dt \int_{E \cap \operatorname{spt}\eta} \left[\left|(u_c)_{x_i}(x + the_i)\right|^2 + \left|H_{F_c}(\nabla u_c(x + the_i)) \cdot (\nabla u_c)_{x_i}(x + the_i)\right|^2\right]$$

$$\leq \sum_i \frac{1}{2}\int_0^1 dt \int_{(E \cap \operatorname{spt}\eta) + the_i} \left[\left|(u_c)_{x_i}(x)\right|^2 + K_2^2\left[f_c''(|\nabla u_c(x)|)\right]^2 \left|(\nabla u_c)_{x_i}(x + the_i)\right|^2\right] < \varepsilon$$

since $m((E \cap \operatorname{spt}\eta) + the_i) \leq m(E)$.

Leaving out several constants, we found $\overline{K}$ independent of $c$ such that

$$\begin{aligned}
\overline{K} \geq \; & - \; 2\int_\Omega \eta \left|\sum_i H_{F_c}(\nabla u_c) \cdot (u_c)_{x_i} \cdot (\nabla u_c)_{x_i}\right| + \sum_i \int_\Omega \eta^2 \left|(\nabla u_c)_{x_i}\right|^2 \\
& + \int_{\cup_k A_{c,k}} \eta^2 \frac{\left|H_{F_c}(\nabla u_c) \cdot \nabla u_c \cdot H_{u_c}\right|^2}{|\nabla u_c|} \\
\geq \; & \textstyle\sum_i \displaystyle\int_{B_c} \left[-2\eta\left|(u_c)_{x_i}\right| \cdot \left|(\nabla u_c)_{x_i}\right| + \eta^2 \left|(\nabla u_c)_{x_i}\right|^2\right] \\
& + \sum_i \int_{\cup_k A_{c,k}} \eta^2 \left|(\nabla u_c)_{x_i}\right|^2 \\
& + \int_{\cup_k A_{c,k}} \left[-2\eta\left|H_{F_c}(\nabla u_c) \cdot \nabla u_c \cdot H_{u_c}\right| + \eta^2 \cdot \frac{\left|H_{F_c}(\nabla u_c) \cdot \nabla u_c \cdot H_{u_c}\right|^2}{|\nabla u_c|}\right].
\end{aligned}$$

From this inequality we obtain

$$\begin{aligned}
\overline{K} + \; & \textstyle\sum_i \displaystyle\int_{B_c} \left|(u_c)_{x_i}\right|^2 + \int_{\cup_k A_{c,k}} |\nabla u_c| \\
\geq \; & \textstyle\sum_i \displaystyle\int_{B_c} \left[\left|(u_c)_{x_i}\right| - \eta\left|(\nabla u_c)_{x_i}\right|\right]^2 \\
& + \sum_i \int_{\cup_k A_{c,k}} \eta^2 \left|(\nabla u_c)_{x_i}\right|^2 \\
& + \int_{\cup_k A_{c,k}} \left[|\nabla u_c|^{\frac{1}{2}} - \eta \cdot \frac{\left|H_{F_c} \cdot \nabla u_c \cdot H_{u_c}\right|}{|\nabla u_c|^{\frac{1}{2}}}\right]^2;
\end{aligned}$$

the left hand side is bounded by a constant $\overline{\overline{K}}$ independent of $c$, then

$$\overline{\overline{K}} \geq \int_{\cup_k A_{c,k}} \eta^2 \left|(\nabla u_c)_{x_i}\right|^2$$

and

$$\overline{\overline{K}} \geq \int_{B_c} \left[\left|(u_c)_{x_i}\right| - \eta \left|(\nabla u_c)_{x_i}\right|\right]^2$$

$\forall i$. Since $\int_{B_c} \left|(u_c)_{x_i}\right|^2$ is bounded by a constant independent of $c$, the same holds for $\int_{B_c} \eta^2 \left|(\nabla u_c)_{x_i}\right|^2$. Then we can find a constant $M$ such that

$$M \geq \int_{\Omega} \eta^2 \left|(\nabla u_c)_{x_i}\right|^2 \geq \int_{\omega_1} \left|(\nabla u_c)_{x_i}\right|^2 \quad \forall c, \forall i.$$

Since for every $h, c > 0$

$$\int_{\omega} |\delta_h^i (\nabla u_c)|^2 \leq \int_{\omega_1} |(\nabla u_c)_{x_i}|^2,$$

there exists a sequence $\left\{u_{c_j}\right\}$ such that $\delta_h^i \left(\nabla u_{c_j}\right) \rightharpoonup \delta_h^i (\nabla u)$ in $L^2(\omega)$ and we have $\int_{\omega} |\delta_h^i (\nabla u)|^2 \leq M$ for all $h$. Then $u \in W_{loc}^{2,2}(\Omega)$.

$\square$

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