Mathematische Annalen (2026) 395:96
https://doi.org/10.1007/s00208-026-03523-4

Mathematische Annalen

Quantitative spectral stability for the Robin Laplacian

Veronica Felli[1] · **Prasun Roychowdhury**[2] · **Giovanni Siclari**[3]

Received: 10 May 2025 / Revised: 11 March 2026 / Accepted: 14 June 2026
© The Author(s) 2026

Abstract
This paper deals with eigenelements of the Laplacian in bounded domains, under Robin boundary conditions, without any assumption on the sign of the Robin parameter. We quantify the asymptotics of the variation of simple eigenvalues under the singular perturbation produced by removing a shrinking set and imposing the same Robin condition on its boundary. We also study the convergence rate of the corresponding eigenfunctions.

Mathematics Subject Classification 35P15 · 35B40 · 35B44 · 35B25

1 Introduction

Let $N \geq 2$ and $\Omega \subset \mathbb{R}^N$ be a bounded, connected, and Lipschitz open set. We deal with Robin eigenvalue problems of the type

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

✉ Veronica Felli
veronica.felli@unimib.it

Prasun Roychowdhury
prasunroychowdhury1994@gmail.com

Giovanni Siclari
giovanni.siclari@sns.it

[1] Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano–Bicocca, Via Cozzi 55, 20125 Milan, Italy

[2] Graduate School of Advanced Science and Engineering, Waseda University, 3-4-1 Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

[3] Centro di Ricerca Matematica Ennio De Giorgi, Scuola Normale Superiore di Pisa, Piazza dei Cavalieri 3, 56126 Pisa, Italy

Published online: 30 June 2026

where $\alpha \in \mathbb{R}$ and ν denotes the outer unit normal vector to $\partial\Omega$. Let $x_0 \in \Omega$, $r_0 > 0$, and let $\Sigma \subset \mathbb{R}^N$ be an open, bounded, and Lipschitz set such that

$$\overline{B_{r_0}(x_0)} \subset \Omega, \quad \overline{\Sigma} \subset B_{r_0}, \quad \text{and} \quad B_{r_0} \setminus \overline{\Sigma} \text{ is connected}, \tag{1.2}$$

where $B_{r_0} = \{y \in \mathbb{R}^N : |y| < r_0\}$ and $B_{r_0}(x_0) = x_0 + B_{r_0}$. We note that, for our purposes, the first two conditions in (1.2) are not restrictive since Ω is open and Σ is bounded; in fact, as Ω is open, there exists r_0 such that $\overline{B_{r_0}(x_0)} \subset \Omega$ and, as Σ is bounded, up to replacing it with a suitable scaling, we can assume that its closure is contained within the ball B_{r_0} - which is not restrictive given that we will work with small scalings of Σ, see (1.3). For notational convenience, we will then require the first two conditions in (1.2), noting however that one can easily reduce to the situation where they hold, starting simply from any $x_0 \in \Omega$ and a bounded Σ. On the other hand, the validity of the third condition in (1.2) is essential, as shown by the counterexample presented in Remark B.2.

For every $\varepsilon \in (0, 1]$ we consider the scaled set

$$\Sigma_\varepsilon := x_0 + \varepsilon\Sigma. \tag{1.3}$$

The aim of the present paper is to investigate the stability of the spectrum of problem (1.1) under the removal of the set $\overline{\Sigma_\varepsilon}$ from Ω, as the parameter $\varepsilon \in (0, 1]$ goes to 0 and the hole $\overline{\Sigma_\varepsilon}$ consequently disappears. More precisely, we study the asymptotic behavior, as $\varepsilon \to 0^+$, of the eigenvalues of the problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega \setminus \overline{\Sigma_\varepsilon}, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial\Sigma_\varepsilon, \end{cases} \tag{1.4}$$

proving, as a first step, that they converge to the eigenvalues of problem (1.1). While some results in this direction are available in the literature, in this paper we are interested in quantitative estimates, with the aim of computing the vanishing order of the eigenvalue variation. We will focus on the case

$$\alpha \neq 0 \tag{1.5}$$

since, while the choice $\alpha = 0$ could fall within our analysis as a special case, it has already been analyzed in detail in [19].

In heat conduction theory, Robin boundary conditions serve to interpolate between a perfectly insulating boundary, described by Neumann boundary conditions $\alpha = 0$, and a temperature-fixing boundary, represented by Dirichlet boundary conditions corresponding to $\alpha = +\infty$. Notably, the Robin boundary condition is the most commonly used in optical imaging, see e.g. [7, 8]. Previous research has largely focused on the initial few Robin eigenvalues, with particular emphasis on applications in shape optimization, related isoperimetric inequalities, and the asymptotic behavior of the lowest eigenvalues, see e.g. [12]. The aim of the present paper is instead to explore the stability

of the spectrum of the Robin Laplacian amidst singular perturbations, specifically the excision of small holes from a bounded domain. Given the significance of eigenvalues and eigenvectors of differential operators in the theory of partial differential equations, it is of interest to understand how eigenelements are affected by small perturbations, such as changes in the domain: this has relevance in various fields, including quantum mechanics, materials science, heat conduction, climate modeling, and acoustics.

Since the pioneering paper [41], a vast literature about quantitative spectral stability for elliptic operators has flourished. Geometric perturbations through domain perforation for the Dirichlet Laplacian have been studied in [1, 4, 21, 36] in Euclidean spaces and in [16] in the general setting of Riemannian manifolds. The same kind of perturbations have been investigated in [2] for the fractional Laplacian under Dirichlet boundary conditions and in [26] for polyharmonic operators, while the case of a generic Dirichlet form has been addressed in [11, 34]. As discussed in [1], in dimension $N = 2$ a relevant application of the study of the Dirichlet Laplacian under removal of small slits can be found, through some isospectrality property, in Aharonov-Bohm operators with moving poles, see e.g. [23, 24] for some recent quantitative spectral stability results. Regarding spectral stability under Neumann conditions on the boundary of small holes, we refer to [19, 29–31, 37, 38], in addition to [41].

Eigenvalue problems in perforated domains have significant analogies with problems with mixed varying boundary conditions, as highlighted in [21], where a homogeneous Neumann eigenvalue problem for the Laplacian is perturbed by prescribing zero Dirichlet conditions on a small subset of the boundary. We refer to [6, 22] for the complementary problem, i.e. the perturbation of the Dirichlet Laplacian by imposing a homogeneous Neumann condition on a vanishing portion of the boundary. Furthermore, such problems exhibit the same singular nature and partially share the difficulties and methodological approaches even of Dirichlet eigenvalue problems in domains with a thin tube attached to the boundary, see [3, 6, 25].

More closely related to the Robin Laplacian is the study of the dependence of the spectrum on α and the asymptotic behavior of eigenvalues as either $\alpha \to +\infty$ or $\alpha \to -\infty$. There are many results in this direction, see for example [14, 27, 33, 35] and [12] for a comprehensive overview. Regarding small perturbations produced by perforation, qualitative spectral stability in dimension 2, under Dirichlet or Neumann conditions on $\partial\Omega$ and Robin conditions on the hole's boundary, has been established in [9], in the case $\alpha > 0$ (possibly depending on ε). Under Dirichlet conditions on the external boundary of a planar domain and Robin conditions on the boundary of a removed shrinking disk, asymptotic expansions have been obtained in [39, 40], letting the parameter α varying as $\kappa\varepsilon^{\sigma}$, with $\kappa > 0$ and $\sigma \in \mathbb{R}$. The case of a large positive parameter $\alpha = \kappa\varepsilon^{-1}$, under Robin boundary conditions on both external and internal boundaries, has been treated in [43, 44], for a removed small subdomain of radius $O(\varepsilon)$. Spectral stability for elliptic operators in perturbed domains subject to Robin boundary conditions has been investigated in [17], for a wide class of domain perturbations, including cutting of small holes. Finally, we mention [13] for spectral stability for the Laplacian under very general Robin conditions involving measures on the boundary, under some weak notion of convergence of sets.

In the present paper, we first observe that the spectrum of the Robin problem is stable under the removal of small holes, i.e. the eigenvalues of the perturbed problem

(1.4) converge to those of the unperturbed one (1.1) as $\varepsilon \to 0^+$, see Theorem 2.1. Although this result is well-known for $\alpha > 0$, see e.g. [17] and [13, Corollary 3.5], to the best of our knowledge it is new for $\alpha < 0$. Once it is understood that the eigenvalue variation due to the domain perturbation goes to zero as the hole disappears, we address the problem of quantifying its vanishing order for simple eigenvalues. In Theorem 2.2, we establish that such an order is $N - 1$ if the point x_0, where the holes are concentrating, does not lie in the nodal set of the limit eigenfunction u_n^0. On the other hand, if u_n^0 vanishes at x_0, the rate of convergence of the eigenvalues depends on the vanishing order of u_n^0 at x_0. This is proved in Theorem 2.3 for $N \geq 3$, exploiting the blow-up result provided by Theorem 5.4, which allows us to obtain a sharp asymptotic expansion for the eigenvalue variation. In dimension $N = 2$, the blow-up analysis fails to be effective; consequently, in Theorem 2.4 we are able to provide only an estimate for the vanishing rate of the eigenvalue variation, depending on the vanishing order of the limit eigenfunction. Nevertheless, in the case of a spherical hole we can derive a sharp expansion in any dimension $N \geq 2$, providing a more explicit expression for its coefficients, see Theorems 2.7 and 2.8. Some results about the convergence of eigenfunctions are also obtained in Theorems 2.5 and 2.6.

The paper is organized as follows. In Sect. 2 we state and describe our main results. In Sect. 3 we introduce the functional setting and discuss some preliminary trace and extension results; furthermore, we prove Theorem 2.1 establishing qualitative spectral stability. In Sect. 4 an asymptotic expansion for the eigenvalue variation is obtained by applying the abstract result stated in Appendix A. This analysis is then refined in Sect. 5 by means of a blow up procedure. Furthermore, Sect. 6 is focused on the case of round holes. Finally, in Appendix B we give the details of the extension results presented in Sect. 3 pointing out the relevance of the connectedness assumptions.

2 Main results

By classical spectral theory, see also Sect. 3, problem (1.1) admits a non-decreasing, diverging sequence of eigenvalues $\{\lambda_j^0\}_{j\in\mathbb{N}\setminus\{0\}}$, repeated according to their multiplicity. Similarly, for every $\varepsilon \in (0, 1]$, (1.4) admits a non-decreasing, diverging sequence of eigenvalues $\{\lambda_j^\varepsilon\}_{j\in\mathbb{N}\setminus\{0\}}$, counted with their multiplicity.

The following spectral stability result serves as the starting point of our analysis.

Theorem 2.1 *For every* $j \in \mathbb{N} \setminus \{0\}$

$$\lim_{\varepsilon\to 0^+} \lambda_j^\varepsilon = \lambda_j^0. \tag{2.1}$$

Once the above spectral stability is achieved, our main goal is to quantify, for simple eigenvalues, the convergence rate of λ_j^ε to λ_j^0. Let us fix $n \in \mathbb{N} \setminus \{0\}$ such that

$$\lambda_n^0 \text{ is simple.} \tag{2.2}$$

Let $u_n^0 \in H^1(\Omega)$ be an associated eigenfunction with

$$\int_\Omega |u_n^0(x)|^2 \, \mathrm{d}x = 1. \tag{2.3}$$

By classical regularity theory, u_n^0 is smooth in Ω. For every $\varepsilon \in (0, 1]$ we choose an eigenfunction u_n^ε associated with λ_n^ε such that

$$\int_{\Omega\setminus\overline{\Sigma_\varepsilon}} |u_n^\varepsilon(x)|^2 \,\mathrm{d}x = 1 \quad \text{and} \quad \int_{\Omega\setminus\overline{\Sigma_\varepsilon}} u_n^0(x) u_n^\varepsilon(x) \,\mathrm{d}x \geq 0. \tag{2.4}$$

Different behaviors are observed in the two cases $u_n^0(x_0) \neq 0$ and $u_n^0(x_0) = 0$.

Theorem 2.2 *Let $N \geq 2$ and $u_n^0(x_0) \neq 0$. Then*

$$\lambda_n^\varepsilon - \lambda_n^0 = \alpha\, \mathcal{H}^{N-1}(\partial\Sigma)\, |u_n^0(x_0)|^2 \varepsilon^{N-1} + o(\varepsilon^{N-1}) \quad \text{as } \varepsilon \to 0^+,$$

where $\mathcal{H}^{N-1}$ is the $(N-1)$-dimensional Hausdorff measure.

An immediate consequence of the above theorem is that, if $u_n^0(x_0) \neq 0$, for small ε the variation $\lambda_n^\varepsilon - \lambda_n^0$ has the same sign as the parameter α. In particular, if $\alpha > 0$, we have $\lambda_n^\varepsilon > \lambda_n^0$ provided ε is sufficiently small. This is consistent with the fact that, as $\alpha \to +\infty$, the Robin problem tends to be a Dirichlet problem, whose eigenvalues are monotone with respect to domain inclusion, in the sense that each Dirichlet eigenvalue increases as the domain gets smaller. It is worth noting that, in the presence of small holes, the Robin problem exhibits an order of convergence $N-1$ for the perturbed eigenvalues which is intermediate between $N-2$, which is the order obtained in [1] (see also [21, Theorem 2.14] and [5, Appendix A]) for the Dirichlet problem, and N, which is the one observed for the Neumann problem in [19]. This behavior is naturally expected, given the nature of the Robin operator as an intermediary between the Dirichlet and the Neumann Laplacians.

In the case $u_n^0(x_0) = 0$, the situation is more intricate, since other quantities, that vanish faster than ε^{N-1}, come into play. The vanishing order of the eigenvalue variation $\lambda_n^\varepsilon - \lambda_n^0$ can be quantified in terms of the asymptotic behavior of the eigenfunctions at x_0. For every $u \in C^\infty(\Omega)$ and $i \in \mathbb{N}$, we consider the polynomial

$$P_i^{u,x_0}(x) := \sum_{\beta\in\mathbb{N}^N,\, |\beta|=i} \frac{1}{\beta!} D^\beta u(x_0) x^\beta, \quad x \in \mathbb{R}^N, \tag{2.5}$$

where $|\beta| = \sum_{j=1}^N \beta_j$ denotes the length of the multi-index $\beta = (\beta_1, \ldots, \beta_N) \in \mathbb{N}^N$. In particular, for $i = 0$ we have $P_0^{u,x_0}(x) := u(x_0)$. We say that u vanishes of order $j \in \mathbb{N}$ at x_0 if

$$P_j^{u,x_0} \not\equiv 0 \quad \text{and} \quad P_i^{u,x_0} \equiv 0 \quad \text{for all } 0 \leq i < j.$$

In particular, if $u(x_0) \neq 0$, the vanishing order of u at x_0 is 0.

From now on, we will denote by $k \geq 0$ the vanishing order of u_n^0 at x_0. We define

$$h := \max\{k, 1\}. \tag{2.6}$$

In dimension $N \geq 3$, we consider the unique finite energy weak solution $\widetilde{V}$ to the problem

$$\begin{cases} -\Delta \widetilde{V} = 0, & \text{in } \mathbb{R}^N \setminus \overline{\Sigma}, \\ \partial_\nu \widetilde{V} = \partial_\nu P_h^{u_n^0, x_0} + \alpha\, u_n^0(x_0), & \text{on } \partial\Sigma, \end{cases} \tag{2.7}$$

see Proposition 5.3. The related quantity

$$\widetilde{\mathcal{T}} := \int_{\mathbb{R}^N \setminus \Sigma} |\nabla \widetilde{V}|^2 \, \mathrm{d}x = \int_{\partial\Sigma} \left(\partial_\nu P_h^{u_n^0, x_0} + \alpha\, u_n^0(x_0) \right) \widetilde{V} \, \mathrm{d}\sigma \tag{2.8}$$

plays a crucial role in the asymptotic expansion of perturbed eigenelements.

The following theorem shows that, if $u_n^0(x_0) = 0$ and $N \geq 3$, the quantity $\widetilde{\mathcal{T}}$ appears in the first significant term of the asymptotic expansion of the eigenvalue variation.

Theorem 2.3 *Let $N \geq 3$. If $u_n^0(x_0) = 0$ and $k \geq 1$ is the vanishing order of u_n^0 at x_0, then*

$$\lambda_n^\varepsilon - \lambda_n^0 = -\varepsilon^{N+2k-2} \left[\widetilde{\mathcal{T}} + \int_\Sigma |\nabla P_k^{u_n^0, x_0}|^2 \, \mathrm{d}x \right] + o(\varepsilon^{N+2k-2}) \quad \text{as } \varepsilon \to 0^+.$$

Under the hypotheses of Theorem 2.3, in particular under the condition $u_n^0(x_0) = 0$, we have $h = k$ and formula (2.8) defining $\widetilde{\mathcal{T}}$ simplifies to

$$\widetilde{\mathcal{T}} = \int_{\partial\Sigma} \left(\partial_\nu P_k^{u_n^0, x_0} \right) \widetilde{V} \, \mathrm{d}\sigma.$$

In particular, in this case, the dominant term of the asymptotic expansion is the same as obtained in [19] for the Neumann problem. Anyway, it is convenient to consider the more general definition of $\widetilde{\mathcal{T}}$ given in (2.8), which also contemplates the case $u_n^0(x_0) \neq 0$, since this will come into play in the description of the variation of eigenfunctions even if $u_n^0(x_0) \neq 0$, see Theorem 2.5 below.

Theorem 2.3 implies that, if $N \geq 3$, $u_n^0(x_0) = 0$, and ε is sufficiently small, the eigenvalue variation $\lambda_n^\varepsilon - \lambda_n^0$ has a negative sign, regardless of the sign of the parameter α and in contrast to what happens in the Dirichlet case.

The proof of Theorem 2.3 is based on a blow-up analysis which does not seem to have a direct generalization to the 2-dimensional case; indeed, a Hardy type inequality is used to identify the concrete functional space containing the limit profile $\widetilde{V}$, and this is not available in dimension $N = 2$. However, if $N = 2$, we can still provide a rougher estimate of the eigenvalue variation.

Theorem 2.4 *Let $N = 2$. If $u_n^0(x_0) = 0$ and $k \geq 1$ is the vanishing order of u_n^0 at x_0, then*

$$\lambda_n^\varepsilon - \lambda_n^0 = O(\varepsilon^{2k-\delta}) \quad \text{as } \varepsilon \to 0^+,$$

for every $\delta > 0$.

From the blow-up analysis, it is also possible to obtain precise information on the behavior of the perturbed eigenfunctions, as stated in the following theorem.

Theorem 2.5 *Let $N \geq 3$. Let*

$$\Phi_\varepsilon(x) := \frac{u_n^\varepsilon(x_0 + \varepsilon x) - u_n^0(x_0)}{\varepsilon^h},$$

where u_n^0 and u_n^ε are as in (2.3) and (2.4) respectively, and h is defined in (2.6). Then, for every $R > 0$ such that $\overline{\Sigma} \subset B_R$,

$$\Phi_\varepsilon \to P_h^{u_n^0, x_0} - \widetilde{V} \quad \text{in } H^1(B_R \setminus \overline{\Sigma}) \quad \text{as } \varepsilon \to 0^+, \tag{2.9}$$

where $\widetilde{V}$ is the unique finite energy weak solution of (2.7) (in the sense clarified in (5.1)) and $P_h^{u_n^0, x_0}$ is defined in (2.5). Furthermore,

$$\lim_{\varepsilon \to 0} \varepsilon^{-(N+2h-2)} \|u_n^\varepsilon - u_n^0\|^2_{H^1(\Omega \setminus \overline{\Sigma_\varepsilon})} = \widetilde{\mathcal{T}}, \tag{2.10}$$

where $\widetilde{\mathcal{T}}$ is defined in (2.8).

In dimension $N = 2$, we have the following rougher result.

Theorem 2.6 *Let $N = 2$. If u_n^0 and u_n^ε are as in (2.3) and (2.4) respectively, and h is defined in (2.6), then*

$$\|u_n^\varepsilon - u_n^0\|^2_{H^1(\Omega \setminus \overline{\Sigma_\varepsilon})} = O(\varepsilon^{2h-\delta}) \quad \text{as } \varepsilon \to 0^+,$$

for any $\delta > 0$.

In any dimension $N \geq 2$, more explicit expansions of the eigenvalue variation can be derived for round holes, i.e. for

$$\Sigma = B_{r_1}, \tag{2.11}$$

for some $0 < r_1 < r_0$.

Theorem 2.7 *Let $N \geq 3$ and Σ be as in (2.11). If $u_n^0(x_0) = 0$ and $k \geq 1$ is the vanishing order of u_n^0 at x_0, then*

$$\lambda_n^\varepsilon - \lambda_n^0 = -\frac{k(N+2k-2)}{N+k-2} r_1^{N+2k-2} \left(\int_{\partial B_1} |P_k^{u_n^0, x_0}|^2 \, \mathrm{d}\sigma \right) \varepsilon^{N+2k-2} + o(\varepsilon^{N+2k-2}) \quad \text{as } \varepsilon \to 0^+,$$

where $P_k^{u_n^0, x_0}$ is defined in (2.5).

In dimension 2, using the arguments developed in [19, Section 6], we obtain the following expansion for round holes.

Theorem 2.8 *Let $N = 2$ and Σ be as in (2.11). If $u_n^0(x_0) = 0$ and $k \geq 1$ is the vanishing order of u_n^0 at x_0, then*

$$\lambda_n^\varepsilon - \lambda_n^0 = -2k\pi r_1^{2k} \left(\left| \frac{\partial^k u_n^0}{\partial x_1^k}(x_0) \right|^2 + \frac{1}{k^2} \left| \frac{\partial^k u_n^0}{\partial x_1^{k-1} \partial x_2}(x_0) \right|^2 \right) \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0^+.$$

Remark 2.9 In this paper, we have assumed α to be constant for clarity of exposition. However, our results can be generalized to the case where α is a function over $\overline{\Omega}$. More precisely, let $\alpha \in C^0(\overline{\Omega} \setminus \{x_0\})$ be such that

$$\alpha_0 := \lim_{x \to x_0} \frac{\alpha(x)}{|x - x_0|^\sigma} \text{ exists, is finite and } \alpha_0 \neq 0,$$

for some $\sigma \in (-1, +\infty)$. For example, we expect that, if $u_0^n(x_0) \neq 0$, by following the proof of Theorem 2.2, one could prove the expansion

$$\lambda_n^\varepsilon - \lambda_n^0 = \alpha_0 \mathcal{H}^{N-1}(\partial\Sigma)\, |u_n^0(x_0)|^2 \varepsilon^{N+\sigma-1} + o(\varepsilon^{N+\sigma-1}) \quad \text{as } \varepsilon \to 0^+;$$

similarly, all the other results presented in Sect. 2 are expected to be generalizable accordingly.

Remark 2.10 In this paper, we have primarily focused on asymptotic expansions of simple eigenvalues, leaving the multiple case for future research. In particular, we expect that the tools introduced herein can be combined with the classical *Lemma on small eigenvalues* by Y. Colin de Verdière [15] (see also [10]) to produce asymptotic expansions of eigenbranches bifurcating from multiple eigenvalues, in the spirit of [4, 6]. Furthermore, we intend to investigate whether—and in what form—the splitting phenomena, described in the recent paper [20] for the Neumann problem and in [28] for Dirichlet eigenvalues, can also be observed in the Robin setting.

3 Preliminaries

3.1 Functional setting

For any open set $E \subset \mathbb{R}^N$, we denote with $L^2(E)$ and $H^1(E)$ the usual Lebesgue and Sobolev spaces, endowed with the scalar products $(u, v)_{L^2(E)} := \int_E uv \, \mathrm{d}x$ and $(u, v)_{H^1(E)} := \int_E \big(\nabla u \cdot \nabla v + uv\big) \, \mathrm{d}x$, respectively.

The following proposition provides a uniform extension result, due to [42]. For completeness, we provide a proof in Appendix B.

Proposition 3.1 *For $N \geq 2$, let $\Omega \subset \mathbb{R}^N$ and $\Sigma \subset \mathbb{R}^N$ be open, bounded, and Lipschitz sets such that Ω is connected and (1.2) is satisfied for some $x_0 \in \Omega$ and $r_0 > 0$. For every $\varepsilon \in (0, 1]$, let Σ_ε be defined in (1.3). Then, for every $\varepsilon \in (0, 1]$ there exists an (inner) extension operator*

$$\mathsf{E}_\varepsilon : H^1(\Omega \setminus \overline{\Sigma}_\varepsilon) \to H^1(\Omega)$$

such that, for every $u \in H^1(\Omega \setminus \overline{\Sigma}_\varepsilon)$ and $\varepsilon \in (0, 1]$,

$$(\mathsf{E}_\varepsilon u)\big|_{\Omega \setminus \overline{\Sigma}_\varepsilon} = u \tag{3.1}$$

and

$$\|\mathsf{E}_\varepsilon u\|^2_{H^1(\Omega)} \le K\|u\|^2_{H^1(\Omega\setminus\overline{\Sigma}_\varepsilon)}, \tag{3.2}$$

for some constant $K > 0$ which is independent of ε.

For clarity of exposition, in the following we will assume x_0 to be the origin. This is not a restrictive choice, as it can be easily verified by translation. For every $\varepsilon \in (0, 1]$ we define

$$\Omega_\varepsilon := \Omega \setminus \overline{\Sigma}_\varepsilon. \tag{3.3}$$

To study the variation of the eigenvalues as $\varepsilon \to 0^+$, it is important to understand how the constants of the trace operators on the hole's boundary depend on the parameter ε.

Proposition 3.2 *Let Σ_ε and Ω_ε be as in (1.3) (with $x_0 = 0$) and (3.3), respectively.*

(i) *If $N \ge 3$, there exists a constant $C > 0$, which does not depend on ε, such that, for every $\varepsilon \in (0, 1]$ and $u \in H^1(\Omega_\varepsilon)$,*

$$\int_{\partial\Sigma_\varepsilon} u^2 \,\mathrm{d}\sigma \le C\varepsilon \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2)\,\mathrm{d}x. \tag{3.4}$$

(ii) *If $N = 2$, for every $\delta \in (0, 1)$ there exists a constant $C_\delta > 0$, depending on δ but independent of ε, such that, for every $\varepsilon \in (0, 1]$ and $u \in H^1(\Omega_\varepsilon)$,*

$$\int_{\partial\Sigma_\varepsilon} u^2 \,\mathrm{d}\sigma \le C_\delta \varepsilon^{1-\delta} \int_{\Omega_\varepsilon} (|\nabla u|^2 + u^2)\,\mathrm{d}x. \tag{3.5}$$

Proof Let $u \in H^1(\Omega_\varepsilon)$ and $u_\varepsilon(x) := (\mathsf{E}_\varepsilon u)(\varepsilon x)$. Then $u_\varepsilon \in H^1(\Sigma)$. By the Hölder inequality, a change of variables, and classical Sobolev trace theory

$$\begin{aligned}
\int_{\partial\Sigma_\varepsilon} u^2 \,\mathrm{d}\sigma &= \varepsilon^{N-1} \int_{\partial\Sigma} u_\varepsilon^2 \,\mathrm{d}\sigma \le C_1 \varepsilon^{N-1} \int_\Sigma (|\nabla u_\varepsilon|^2 + u_\varepsilon^2)\,\mathrm{d}x \\
&= C_1 \varepsilon^{N-1} \int_\Sigma (\varepsilon^2 |\nabla(\mathsf{E}_\varepsilon u)(\varepsilon x)|^2 + |(\mathsf{E}_\varepsilon u)(\varepsilon x)|^2)\,\mathrm{d}x \\
&= C_1 \varepsilon^{-1} \int_{\Sigma_\varepsilon} (\varepsilon^2 |\nabla(\mathsf{E}_\varepsilon u)|^2 + |\mathsf{E}_\varepsilon u|^2)\,\mathrm{d}x,
\end{aligned}$$

for some positive constant $C_1 > 0$ independent of ε. Furthermore, if $N \ge 3$, by the Hölder and Sobolev inequalities and (3.2)

$$\int_{\Sigma_\varepsilon} |\mathsf{E}_\varepsilon u|^2 \,\mathrm{d}x \le |\Sigma|^{2/N} \varepsilon^2 \left(\int_\Omega |\mathsf{E}_\varepsilon u|^{\frac{2N}{N-2}} \,\mathrm{d}x \right)^{\frac{N-2}{N}} \le C_2 \varepsilon^2 \|\mathsf{E}_\varepsilon u\|^2_{H^1(\Omega)} \le C_2 K \varepsilon^2 \|u\|^2_{H^1(\Omega_\varepsilon)},$$

for some constant $C_2 > 0$ independent of ε. On the other hand, if $N = 2$, for every $\delta \in (0, 1)$ we have

$$\int_{\Sigma_\varepsilon} |\mathsf{E}_\varepsilon u|^2 \, \mathrm{d}x \le |\Sigma|^{\frac{p-2}{p}} \varepsilon^{2-\delta} \left(\int_\Omega |\mathsf{E}_\varepsilon u|^p \, \mathrm{d}x \right)^{\frac{2}{p}} \le C_3 \varepsilon^{2-\delta} \|\mathsf{E}_\varepsilon u\|^2_{H^1(\Omega)} \le C_3 K \varepsilon^{2-\delta} \|u\|^2_{H^1(\Omega_\varepsilon)},$$

where $p = \frac{4}{\delta}$ and $C_3 > 0$ is a positive constant depending on δ. Hence (3.4) and (3.5) are proved. □

Remark 3.3 By the classical Sobolev trace inequality on the domain $\Omega \setminus \overline{B_{r_0}}$, there exists a constant $C_{r_0} > 0$, depending on Ω and r_0, such that

$$\left(\int_{\partial\Omega} v^2 \, \mathrm{d}\sigma \right)^{\frac{1}{2}} \le C_{r_0} \|v\|_{H^1(\Omega \setminus \overline{B_{r_0}})} \le C_{r_0} \|v\|_{H^1(\Omega_\varepsilon)}$$

for every $\varepsilon \in (0, 1]$ and $v \in H^1(\Omega_\varepsilon)$.

For every $\alpha \in \mathbb{R}$ and $c \ge 0$, we consider the eigenvalue problem

$$\begin{cases} -\Delta u + cu = \mu u, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

which is equivalent to (1.1) in the sense that it has the same eigenfunctions and eigenvalues that are shifted by an amount c. For every $\varepsilon \in (0, 1]$, the corresponding perturbed problem is

$$\begin{cases} -\Delta u + cu = \mu u, & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial\Sigma_\varepsilon, \end{cases} \tag{3.7}$$

where ν is the unit outer normal vector of Ω_ε on $\partial\Omega_\varepsilon$. For every $\alpha \in \mathbb{R}$, $c \ge 0$, and $\varepsilon \in [0, 1]$, we define the symmetric bilinear form

$$\begin{aligned} &q_{\varepsilon,c,\alpha} : H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}, \\ &q_{\varepsilon,c,\alpha}(v, w) = \int_{\Omega_\varepsilon} (\nabla v \cdot \nabla w + cvw) \, \mathrm{d}x + \alpha \int_{\partial\Omega_\varepsilon} vw \, \mathrm{d}\sigma, \end{aligned} \tag{3.8}$$

where $\Omega_0 := \Omega$. We say that $u \in H^1(\Omega_\varepsilon)$ is an eigenfunction of problem (3.7) associated to the eigenvalue $\mu \in \mathbb{R}$ if

$$q_{\varepsilon,c,\alpha}(u, v) = \mu \int_{\Omega_\varepsilon} uv \, \mathrm{d}x \quad \text{for every } v \in H^1(\Omega_\varepsilon),$$

while $u \in H^1(\Omega)$ is an eigenfunction of problem (3.6) associated to the eigenvalue μ if

$$q_{0,c,\alpha}(u, v) = \mu \int_\Omega uv \, dx \quad \text{for every } v \in H^1(\Omega).$$

By [32, Theorem 18.1] the bilinear form $q_{\varepsilon,c,\alpha}$ is semibounded from below. By classical spectral theory, the spectrum of problem (3.7) is a diverging sequence $\{\mu_{j,\varepsilon,c,\alpha}\}_{j\in\mathbb{N}\setminus\{0\}}$. It is clear that

$$\mu_{j,\varepsilon,c,\alpha} = c + \lambda_j^\varepsilon \quad \text{for every } c \in \mathbb{R} \text{ and } j \in \mathbb{N} \setminus \{0\},$$

where $\{\lambda_j^\varepsilon\}_{j\in\mathbb{N}\setminus\{0\}}$ are the eigenvalues of (1.4). Hence, studying the eigenvalue variation for problems (1.1) and (1.4) is equivalent to studying the eigenvalue variation for problems (3.6) and (3.7) for any $c \in \mathbb{R}$. Furthermore, by [32, Theorem 18.1] and Proposition 3.2, for any $\alpha \in \mathbb{R}$ there exist $c_\alpha > 0$, $C_\alpha > 0$, and $\varepsilon_\alpha \in (0, 1)$, such that, for all $\varepsilon \in [0, \varepsilon_\alpha)$,

$$\mu_{1,\varepsilon,c_\alpha,\alpha} > 1$$

and

$$\|v\|^2_{H^1(\Omega_\varepsilon)} \le C_\alpha q_{\varepsilon,c_\alpha,\alpha}(v, v) \quad \text{for every } v \in H^1(\Omega_\varepsilon). \tag{3.9}$$

In view of Remark 3.3 and (3.4)–(3.5), there exists a constant $\kappa_\alpha > 0$ which depends only on Ω, r_0, Σ, and α such that

$$\|v\|^2_{H^1(\Omega_\varepsilon)} \ge \kappa_\alpha q_{\varepsilon,c_\alpha,\alpha}(v, v) \quad \text{for every } v \in H^1(\Omega_\varepsilon). \tag{3.10}$$

We will study the eigenvalue variation for problems (3.6) and (3.7) with $c = c_\alpha$. For any $\alpha \in \mathbb{R}$ fixed (and c_α chosen accordingly as explained above), let us define

$$q_\varepsilon := q_{\varepsilon,c_\alpha,\alpha} \quad \text{for every } \varepsilon \in [0, 1], \tag{3.11}$$

and

$$\mu_j^\varepsilon := \mu_{j,\varepsilon,c_\alpha,\alpha} \quad \text{for every } j \in \mathbb{N} \setminus \{0\} \text{ and } \varepsilon \in [0, 1]. \tag{3.12}$$

By classical spectral theory, there exists an orthonormal basis $\{u_j^\varepsilon\}_{j\in\mathbb{N}\setminus\{0\}}$ of $L^2(\Omega_\varepsilon)$ such that u_j^ε is an eigenfunction associated to the eigenvalue μ_j^ε; consequently, it is also an eigenfunction of (1.4) associated to

$$\lambda_j^\varepsilon = \mu_j^\varepsilon - c_\alpha. \tag{3.13}$$

If $j = n$, u_n^0 and u_n^ε are chosen to satisfy (2.3) and (2.4) respectively.

3.2 Spectral stability

In this subsection, we prove that $\mu_j^\varepsilon \to \mu_j^0$ (and hence $\lambda_j^\varepsilon \to \lambda_j^0$) as $\varepsilon \to 0^+$ for every $j \in \mathbb{N} \setminus \{0\}$. To this end, for every $\varepsilon \in (0, \varepsilon_\alpha)$, let R_ε be the inverse of the

operator $-\Delta + c_\alpha$ on Ω_ε under Robin boundary conditions. More precisely, for every $\varepsilon \in (0, \varepsilon_\alpha)$,

$$R_\varepsilon : L^2(\Omega_\varepsilon) \to H^1(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon),$$

and, for every $f \in L^2(\Omega_\varepsilon)$, $R_\varepsilon f$ is the unique weak solution to the problem

$$\begin{cases} -\Delta u + c_\alpha u = f, & \text{in } \Omega_\varepsilon, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u = 0, & \text{on } \partial\Sigma_\varepsilon, \end{cases}$$

i.e.

$$q_\varepsilon(R_\varepsilon f, v) = \int_{\Omega_\varepsilon} f v \, \mathrm{d}x \quad \text{for all } v \in H^1(\Omega_\varepsilon). \tag{3.14}$$

Similarly, we can define R_0 as the inverse of the operator $-\Delta + c_\alpha$ on Ω under Robin boundary conditions. For every $\varepsilon \in [0, \varepsilon_\alpha)$, the operator R_ε is linear and bounded as an operator from $L^2(\Omega_\varepsilon)$ into $H^1(\Omega_\varepsilon)$; furthermore, R_ε, understood as an operator from $L^2(\Omega_\varepsilon)$ into $L^2(\Omega_\varepsilon)$, is compact and self-adjoint. Moreover, since $\mu_1^\varepsilon > 1$, we have $\|R_\varepsilon\|_{\mathcal{L}(L^2(\Omega_\varepsilon))} \le 1$, where $\|R_\varepsilon\|_{\mathcal{L}(L^2(\Omega_\varepsilon))}$ is the operator norm.

For every $\varepsilon \in (0, 1]$, we define the linear and continuous operators

$$P_\varepsilon : L^2(\Omega) \to L^2(\Omega_\varepsilon), \quad P_\varepsilon u := u_{|\Omega_\varepsilon},$$

and

$$\widetilde{R}_\varepsilon : L^2(\Omega) \to L^2(\Omega), \quad \widetilde{R}_\varepsilon u := \mathsf{E}_\varepsilon R_\varepsilon P_\varepsilon u.$$

Proposition 3.4 *For every $u \in L^2(\Omega)$*

$$\lim_{\varepsilon \to 0^+} \left\| \widetilde{R}_\varepsilon u - R_0 u \right\|_{L^2(\Omega)} = 0.$$

Proof Let $u \in L^2(\Omega)$. Then by (3.2), (3.9), and (3.14), and the fact that $\|R_\varepsilon\|_{\mathcal{L}(L^2(\Omega_\varepsilon))} \le 1$,

$$\begin{aligned} \|\widetilde{R}_\varepsilon u\|^2_{H^1(\Omega)} &\le K \|R_\varepsilon P_\varepsilon u\|^2_{H^1(\Omega_\varepsilon)} \le K C_\alpha q_\varepsilon(R_\varepsilon P_\varepsilon u, R_\varepsilon P_\varepsilon u) \\ &= K C_\alpha (P_\varepsilon u, R_\varepsilon P_\varepsilon u)_{L^2(\Omega_\varepsilon)} \le K C_\alpha \|u\|^2_{L^2(\Omega)} \end{aligned} \tag{3.15}$$

for every $\varepsilon \in (0, \varepsilon_\alpha)$. In particular, the family $\{\widetilde{R}_\varepsilon u\}_{\varepsilon \in (0, \varepsilon_\alpha)}$ is bounded in $H^1(\Omega)$. Hence there exist $v \in H^1(\Omega)$ and a sequence $\varepsilon_j \to 0^+$ such that $\widetilde{R}_{\varepsilon_j} u \rightharpoonup v$ weakly in $H^1(\Omega)$ as $j \to \infty$. Furthermore, for every $w \in H^1(\Omega)$ and $j \in \mathbb{N}$ we have

$$q_{\varepsilon_j}(R_{\varepsilon_j} P_{\varepsilon_j} u, w) = (P_{\varepsilon_j} u, w)_{L^2(\Omega_{\varepsilon_j})}. \tag{3.16}$$

We would like to pass to the limit as $j \to \infty$ in (3.16). By (3.11)

$$\begin{aligned} q_{\varepsilon_j}(R_{\varepsilon_j} P_{\varepsilon_j} u, w) &= \int_{\Omega_{\varepsilon_j}} \big(\nabla(R_{\varepsilon_j} P_{\varepsilon_j} u) \cdot \nabla w + c_\alpha (R_{\varepsilon_j} P_{\varepsilon_j} u) w\big)\, \mathrm{d}x + \alpha \int_{\partial\Omega_{\varepsilon_j}} (R_{\varepsilon_j} P_{\varepsilon_j} u) w \, \mathrm{d}\sigma \\ &= \int_{\Omega} \big(\nabla(\widetilde{R}_{\varepsilon_j} u) \cdot \nabla w + c_\alpha (\widetilde{R}_{\varepsilon_j} u) w\big)\, \mathrm{d}x + \alpha \int_{\partial\Omega} (R_{\varepsilon_j} P_{\varepsilon_j} u) w \, \mathrm{d}\sigma \\ &\quad - \int_{\Sigma_{\varepsilon_j}} \big(\nabla(\widetilde{R}_{\varepsilon_j} u) \cdot \nabla w + c_\alpha (\widetilde{R}_{\varepsilon_j} u) w\big)\, \mathrm{d}x + \alpha \int_{\partial\Sigma_{\varepsilon_j}} (R_{\varepsilon_j} P_{\varepsilon_j} u) w \, \mathrm{d}\sigma \end{aligned}$$

and, by the Cauchy–Schwarz inequality,

$$\left| \int_{\Sigma_\varepsilon} \big(\nabla(\widetilde{R}_\varepsilon u) \cdot \nabla w + c_\alpha (\widetilde{R}_\varepsilon u) w\big)\, \mathrm{d}x \right| \le \max\{1, c_\alpha\} \|\widetilde{R}_\varepsilon u\|_{H^1(\Omega)} \|w\|_{H^1(\Sigma_\varepsilon)} \to 0$$

as $\varepsilon \to 0^+$, thanks to the absolute continuity of integrals, since $|\Sigma_\varepsilon| = \varepsilon^N |\Sigma| \to 0^+$. Hence, thanks to (3.5) if $N = 2$ or (3.4) if $N \ge 3$, we may pass to the limit as $j \to \infty$ in (3.16), thus obtaining

$$q_0(v, w) = (u, w)_{L^2(\Omega)} \quad \text{for all } w \in H^1(\Omega).$$

Hence $v = R_0 u$. By the Urysohn Subsequence principle and the Rellich–Kondrachov Theorem, we conclude that $\widetilde{R}_\varepsilon u \to R_0 u$ strongly in $L^2(\Omega)$. □

Based on Proposition 3.4, we are now able to prove the stability result stated in Theorem 2.1.

Proof of Theorem 2.1 We claim that

$$\lim_{\varepsilon \to 0^+} \|\widetilde{R}_\varepsilon - R_0\|_{\mathcal{L}(L^2(\Omega))} = 0. \tag{3.17}$$

By compactness of the operator $\widetilde{R}_\varepsilon - R_0$, for every $\varepsilon \in (0, \varepsilon_\alpha)$ there exists some $f_\varepsilon \in L^2(\Omega)$ with $\|f_\varepsilon\|_{L^2(\Omega)} = 1$ such that

$$\|R_0 - \widetilde{R}_\varepsilon\|_{\mathcal{L}(L^2(\Omega))} = \|R_0 f_\varepsilon - \widetilde{R}_\varepsilon f_\varepsilon\|_{L^2(\Omega)}.$$

Since $\{f_\varepsilon\}_\varepsilon$ is bounded in $L^2(\Omega)$, there exist a sequence $\varepsilon_j \to 0^+$ and some $f \in L^2(\Omega)$ such that $f_{\varepsilon_j} \rightharpoonup f$ weakly in $L^2(\Omega)$ as $j \to \infty$. By compactness of the operator R_0, $R_0 f_{\varepsilon_j} \to R_0 f$ strongly in $L^2(\Omega)$, as $j \to \infty$. Furthermore, thanks to (3.15), up to a further subsequence, $\widetilde{R}_{\varepsilon_j} f_{\varepsilon_j} \rightharpoonup g$ weakly in $H^1(\Omega)$ for some $g \in H^1(\Omega)$. For every $v \in L^2(\Omega)$, since R_ε is self-adjoint we have

$$(\widetilde{R}_{\varepsilon_j} f_{\varepsilon_j}, v)_{L^2(\Omega)} = (P_{\varepsilon_j} f_{\varepsilon_j}, R_{\varepsilon_j} P_{\varepsilon_j} v)_{L^2(\Omega_{\varepsilon_j})} + (\widetilde{R}_{\varepsilon_j} f_{\varepsilon_j}, v)_{L^2(\Sigma_{\varepsilon_j})}.$$

By (3.15) and the absolute continuity of integrals, we have

$$(\widetilde{R}_{\varepsilon_j} f_{\varepsilon_j}, v)_{L^2(\Sigma_{\varepsilon_j})} \to 0^+ \quad \text{as } j \to \infty.$$

Furthermore,

$$(P_{\varepsilon_j} f_{\varepsilon_j}, R_{\varepsilon_j} P_{\varepsilon_j} v)_{L^2(\Omega_{\varepsilon_j})} = (f_{\varepsilon_j}, \widetilde{R}_{\varepsilon_j} v)_{L^2(\Omega)} - (f_{\varepsilon_j}, \widetilde{R}_{\varepsilon_j} v)_{L^2(\Sigma_{\varepsilon_j})}$$

and so, by Proposition 3.4 and the absolute continuity of integrals,

$$(g, v)_{L^2(\Omega)} = \lim_{k\to\infty} (\widetilde{R}_{\varepsilon_j} f_{\varepsilon_j}, v)_{L^2(\Omega)} = (f, R_0 v)_{L^2(\Omega)} = (R_0 f, v)_{L^2(\Omega)}.$$

Hence $g = R_0 f$. We conclude that $\widetilde{R}_{\varepsilon_j} f_{\varepsilon_j} \to R_0 f$ strongly in $L^2(\Omega)$ as $j \to \infty$. By the Urysohn Subsequence principle and compactness of R_0, we deduce (3.17).

It is easy to verify that $\sigma(\widetilde{R}_\varepsilon) = \left\{\frac{1}{\mu_j^\varepsilon}\right\}_{j\in\mathbb{N}\setminus\{0\}} \cup \{0\}$, where we are denoting with $\sigma(\widetilde{R}_\varepsilon)$ the spectrum of $\widetilde{R}_\varepsilon$. Hence (2.1) follows from (3.17), [18, Chapter XI 9.5, Lemma 5, Page 1091], and (3.13). □

4 Eigenvalue variation

For any $\alpha \in \mathbb{R}$ fixed, let $c_\alpha > 0$ and $\varepsilon_\alpha \in (0, 1)$ be as in Sect. 3. For every $\varepsilon \in (0, \varepsilon_\alpha)$, we consider the bilinear form $q_\varepsilon = q_{\varepsilon,c_\alpha,\alpha}$ defined in (3.8) and (3.11). Let $n \in \mathbb{N}\setminus\{0\}$ be such that (2.2) is satisfied; hence, by (3.13), also the eigenvalue μ_n^0, defined as in (3.12), is simple. Let u_n^0 be an associated eigenfunction satisfying (2.3).

For every $\varepsilon \in (0, \varepsilon_\alpha)$, we define the linear functional $L_\varepsilon : H^1(\Omega_\varepsilon) \to \mathbb{R}$ as

$$L_\varepsilon(v) := q_\varepsilon(u_n^0, v) - \mu_n^0 \int_{\Omega_\varepsilon} u_n^0 v \, dx = \int_{\partial\Sigma_\varepsilon} \left(\frac{\partial u_n^0}{\partial \nu} + \alpha u_n^0\right) v \, d\sigma, \tag{4.1}$$

where ν is the outer normal unit vector to Ω_ε on $\partial\Sigma_\varepsilon$. We also consider the quadratic functional

$$J_\varepsilon : H^1(\Omega_\varepsilon) \to \mathbb{R}, \quad J_\varepsilon(v) := \frac{1}{2} q_\varepsilon(v, v) - L_\varepsilon(v).$$

We omit the proof of the following proposition, since it is standard. We refer to [19, Proposition 3.4] and [11, Proposition 2.4] for analogous results in a different setting.

Proposition 4.1 *For every $\varepsilon \in (0, \varepsilon_\alpha)$, there exists a unique function $V_\varepsilon \in H^1(\Omega_\varepsilon)$ such that*

$$J_\varepsilon(V_\varepsilon) = \min\{J_\varepsilon(v) : v \in H^1(\Omega_\varepsilon)\}.$$

Furthermore, V_ε is the unique weak solution to the problem

$$\begin{cases} -\Delta V_\varepsilon + c_\alpha V_\varepsilon = 0, & \text{in } \Omega_\varepsilon, \\ \partial_\nu V_\varepsilon + \alpha V_\varepsilon = 0, & \text{on } \partial\Omega, \\ \partial_\nu V_\varepsilon + \alpha V_\varepsilon = \partial_\nu u_n^0 + \alpha u_n^0, & \text{on } \partial\Sigma_\varepsilon, \end{cases}$$

that is,

$$q_\varepsilon(V_\varepsilon, v) = L_\varepsilon(v) \quad \text{for every } v \in H^1(\Omega_\varepsilon). \tag{4.2}$$

For every $\varepsilon \in (0, \varepsilon_\alpha)$, we consider the quantity

$$\mathcal{T}_\varepsilon := -2\left(\frac{1}{2}\, q_\varepsilon(V_\varepsilon, V_\varepsilon) - L_\varepsilon(V_\varepsilon)\right) = q_\varepsilon(V_\varepsilon, V_\varepsilon). \tag{4.3}$$

The quantity $\mathcal{T}_\varepsilon$, which we refer to as *torsional rigidity*, turns out to be an important ingredient in the asymptotic expansion of the eigenvalue variation. An alternative characterization of $\mathcal{T}_\varepsilon$ is given in the following lemma, whose proof is omitted, as it is similar to the ones given in [23, Proposition 6.5] and [19, Lemma 3.1] in analogous situations.

Lemma 4.2 *For every $\varepsilon \in (0, \varepsilon_\alpha)$,*

$$\mathcal{T}_\varepsilon = \sup_{u \in H^1(\Omega_\varepsilon)\setminus\{0\}} \frac{(L_\varepsilon(u))^2}{q_\varepsilon(u,u)}. \tag{4.4}$$

Thanks to the above lemma, we obtain the following quantitative estimate of the vanishing order of $\mathcal{T}_\varepsilon$ as $\varepsilon \to 0^+$.

Proposition 4.3 *Let $k \geq 0$ be the vanishing order of u_n^0 at $x_0 = 0$ and $h := \max\{k, 1\}$. Then*

$$\mathcal{T}_\varepsilon = \begin{cases} O(\varepsilon^{N+2h-2}), & \text{if } N \geq 3, \\ O(\varepsilon^{2h-\delta}), & \text{if } N = 2, \end{cases} \quad \text{as } \varepsilon \to 0^+, \tag{4.5}$$

for every $\delta \in (0, 1)$. In particular,

$$\lim_{\varepsilon\to 0^+} q_\varepsilon(V_\varepsilon, V_\varepsilon) = \lim_{\varepsilon\to 0^+} \|V_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} = \lim_{\varepsilon\to 0^+} \mathcal{T}_\varepsilon = 0. \tag{4.6}$$

Proof If $N \geq 3$, by (3.4) and (3.9),

$$\begin{aligned} \int_{\partial\Sigma_\varepsilon} \left[\frac{\partial u_n^0}{\partial \nu} + \alpha u_n^0\right] u \,\mathrm{d}\sigma &\leq \sqrt{C}\,\varepsilon^{\frac{1}{2}} \left\| \frac{\partial u_n^0}{\partial \nu} + \alpha u_n^0 \right\|_{L^2(\partial\Sigma_\varepsilon)} \|u\|_{H^1(\Omega_\varepsilon)} \\ &\leq \sqrt{C}\,\sqrt{C_\alpha}\varepsilon^{\frac{1}{2}} \left\| \frac{\partial u_n^0}{\partial \nu} + \alpha u_n^0 \right\|_{L^2(\partial\Sigma_\varepsilon)} \sqrt{q_\varepsilon(u,u)} \end{aligned}$$

for every $u \in H^1(\Omega_\varepsilon)$. Hence, in view of (4.4),

$$\mathcal{T}_\varepsilon \leq C C_\alpha \varepsilon \int_{\partial\Sigma_\varepsilon} \left(\frac{\partial u_n^0}{\partial \nu} + \alpha u_n^0\right)^2 \mathrm{d}\sigma. \tag{4.7}$$

If $u_n^0(0) = 0$, then $k = h \geq 1$, $u_n^0(x) = O(|x|^k)$ and $|\nabla u_n^0(x)| = O(|x|^{k-1})$ as $x \to 0$, so that (4.7) yields $\mathcal{T}_\varepsilon = O(\varepsilon^{N+2k-2}) = O(\varepsilon^{N+2h-2})$ as $\varepsilon \to 0^+$. If $u_n^0(0) \neq 0$, then $h = 1$ and, by boundedness of u_n^0 and ∇u_n^0 in a neighbourhood of 0, (4.7) directly implies that $\mathcal{T}_\varepsilon = O(\varepsilon^N) = O(\varepsilon^{N+2h-2})$ as $\varepsilon \to 0^+$. Estimate (4.5) is thereby proved in the case $N \geq 3$.

The proof of (4.5) in the case $N = 2$ proceeds similarly, using (3.5) instead of (3.4). □

Proposition 4.4 *Let V_ε be as in Proposition 4.1. Then*

$$\|V_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = o(\mathcal{T}_\varepsilon) \quad \text{as } \varepsilon \to 0^+, \tag{4.8}$$

and

$$\|V_\varepsilon\|^2_{L^2(\partial\Omega)} = o(\mathcal{T}_\varepsilon) \quad \text{as } \varepsilon \to 0^+. \tag{4.9}$$

Proof In view of (3.9), to prove (4.8) it is enough to show that

$$\|V_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = o\big(\|V_\varepsilon\|^2_{H^1(\Omega_\varepsilon)}\big) \quad \text{as } \varepsilon \to 0^+.$$

We argue by contradiction. Suppose that there exist a positive constant $c > 0$ and a sequence $\varepsilon_j \to 0^+$ such that

$$\|V_{\varepsilon_j}\|^2_{L^2(\Omega_{\varepsilon_j})} \geq c\|V_{\varepsilon_j}\|^2_{H^1(\Omega_{\varepsilon_j})} \quad \text{for every } j \in \mathbb{N}.$$

For every $j \in \mathbb{N}$, we define

$$W_j := \frac{\mathsf{E}_{\varepsilon_j}(V_{\varepsilon_j})}{\|\mathsf{E}_{\varepsilon_j}(V_{\varepsilon_j})\|_{L^2(\Omega)}},$$

where $\mathsf{E}_{\varepsilon_j}$ is the extension operator introduced in Proposition 3.1. Then $\{W_j\}_{j\in\mathbb{N}}$ is bounded in $H^1(\Omega)$ by Proposition 3.1. It follows that, up to a subsequence, $W_j \rightharpoonup W$ weakly in $H^1(\Omega)$ as $j \to +\infty$, for some $W \in H^1(\Omega)$. The compactness of the embedding $H^1(\Omega) \subset L^2(\Omega)$ implies that $\|W\|_{L^2(\Omega)} = 1$.

Let $\varphi \in C^\infty_c(\overline{\Omega} \setminus \{0\})$. Then there exists $j_0 \in \mathbb{N}$ such that $\varphi \in C^\infty_c(\overline{\Omega} \setminus \overline{\Sigma_{\varepsilon_j}})$ for all $j \geq j_0$. Then, by (3.1) and (4.2),

$$\int_\Omega \big(\nabla W_j \cdot \nabla\varphi + c_\alpha W_j\varphi\big)\,\mathrm{d}x + \alpha \int_{\partial\Omega} W_j\varphi\,\mathrm{d}\sigma = 0. \tag{4.10}$$

Passing to the limit as $j \to +\infty$, we conclude that

$$\int_\Omega \big(\nabla W \cdot \nabla\varphi + c_\alpha W\varphi\big)\,\mathrm{d}x + \alpha \int_{\partial\Omega} W\varphi\,\mathrm{d}\sigma = 0 \quad \text{for every } \varphi \in C^\infty_c(\overline{\Omega}\setminus\{0\}). \tag{4.11}$$

Since $C^\infty_c(\overline{\Omega} \setminus \{0\})$ is dense in $H^1(\Omega)$, we may test the above equation with W and conclude that $W = 0$ in view of (3.9) with $\varepsilon = 0$. This is a contradiction because $\|W\|_{L^2(\Omega)} = 1$. Estimate (4.8) is thereby proved.

To prove (4.9) we argue in a similar way, and assume, by contradiction, that there exist $c > 0$ and a sequence $\varepsilon_j \to 0^+$ such that

$$\|V_{\varepsilon_j}\|^2_{L^2(\partial\Omega)} \geq c\|V_{\varepsilon_j}\|^2_{H^1(\Omega_{\varepsilon_j})} \quad \text{for every } j \in \mathbb{N}.$$

For every $j \in \mathbb{N}$, we define $Z_j := \|V_{\varepsilon_j}\|^{-1}_{L^2(\partial\Omega)} \mathsf{E}_{\varepsilon_j}(V_{\varepsilon_j})$. By Proposition 3.1, it follows that $\{Z_j\}_{j\in\mathbb{N}}$ is bounded in $H^1(\Omega)$, hence there exists $Z \in H^1(\Omega)$ such that $Z_j \rightharpoonup Z$ weakly in $H^1(\Omega)$, up to a subsequence. The compactness of the trace map $H^1(\Omega) \to L^2(\partial\Omega)$ implies that $\|Z\|_{L^2(\partial\Omega)} = 1$.

Letting $\varphi \in C^\infty_c(\overline{\Omega} \setminus \{0\})$, we have that Z_j satisfies (4.10) for all j sufficiently large, so that, passing to the limit, Z satisfies (4.11) for every $\varphi \in C^\infty_c(\overline{\Omega} \setminus \{0\})$, and hence, by density of $C^\infty_c(\overline{\Omega} \setminus \{0\})$ in $H^1(\Omega)$, for every $\varphi \in H^1(\Omega)$. As above, this would imply that $Z = 0$ in view of (3.9), giving rise to a contradiction, because $\|Z\|_{L^2(\partial\Omega)} = 1$. Estimate (4.9) is thereby proved. □

The following theorem applies a known perturbative theory for eigenvalue problems in perforated domains to our context. Originally developed by Courtois in [16] for the Dirichlet problem, this theory is based on the *Lemma on small eigenvalues* by Y. Colin de Verdière [15]. It was later revisited in [1] for simple eigenvalues and in [4] for multiple ones, whereas a recent extension to Neumann problem is due to [19]. A generalization to a broad class of abstract perturbative problems – including the aforementioned ones – is provided in [11]; a simplified version of this, adapted to our case, is presented in Appendix A and will now be applied to problem (1.4).

Theorem 4.5 *Let $n \in \mathbb{N} \setminus \{0\}$ be such that the n-th eigenvalue λ^0_n of (1.1) satisfies (2.2), with an associated eigenfunction u^0_n as in (2.3); for every $\varepsilon \in (0,1)$, let λ^ε_n be the n-th eigenvalue of (1.4). Then*

$$\lambda^\varepsilon_n - \lambda^0_n = -\mathcal{T}_\varepsilon - \int_{\Sigma_\varepsilon} \left(|\nabla u^0_n|^2 - \lambda^0_n |u^0_n|^2\right) \mathrm{d}x + \alpha \int_{\partial\Sigma_\varepsilon} |u^0_n|^2 \,\mathrm{d}\sigma + o(\mathcal{T}_\varepsilon) + o\left(\int_{\Sigma_\varepsilon} \left(|\nabla u^0_n|^2 - \lambda^0_n |u^0_n|^2\right) \mathrm{d}x\right) + o\left(\int_{\partial\Sigma_\varepsilon} |u^0_n|^2 \,\mathrm{d}\sigma\right) \quad (4.12)$$

as $\varepsilon \to 0^+$. Moreover, if u^ε_n is the eigenfunction of (1.4) associated with the eigenvalue λ^ε_n chosen as in (2.4) for every $\varepsilon \in (0,1)$, then

$$\|u^\varepsilon_n - (u^0_n - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} = o(\mathcal{T}_\varepsilon) + O\big(\|u^0_n\|^4_{L^2(\Sigma_\varepsilon)}\big), \quad (4.13)$$

$$\|u^\varepsilon_n - u^0_n\|^2_{H^1(\Omega_\varepsilon)} = \mathcal{T}_\varepsilon + o(\mathcal{T}_\varepsilon) + O\big(\|u^0_n\|^4_{L^2(\Sigma_\varepsilon)}\big) + O\left(\mathcal{T}^{1/2}_\varepsilon \|u^0_n\|^2_{L^2(\Sigma_\varepsilon)}\right), \quad (4.14)$$

as $\varepsilon \to 0^+$.

Proof In view of (1.3),

$$\lim_{\varepsilon\to 0^+} |\overline{\Sigma_\varepsilon}|_N = 0, \quad (4.15)$$

where $|\overline{\Sigma_\varepsilon}|_N$ denotes the N-dimensional Lebesgue measure of $\overline{\Sigma_\varepsilon}$. This, together with (2.2), Theorem 2.1, (3.13), (3.9), and (4.6), ensures that the assumptions of Theorem A.1 are satisfied. Hence, Theorem A.1 yields

$$\lambda^\varepsilon_n - \lambda^0_n = \mu^\varepsilon_n - \mu^0_n = \frac{\mu^0_n \int_{\Omega_\varepsilon} u^0_n V_\varepsilon \,\mathrm{d}x + O\big(\|V_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\big)}{\int_{\Omega_\varepsilon} |u^0_n|^2 \,\mathrm{d}x + O\big(\|V_\varepsilon\|_{L^2(\Omega_\varepsilon)}\big)} \quad \text{as } \varepsilon \to 0^+. \quad (4.16)$$

To clarify the above expansion further, we first observe, by (2.3), (4.15), and (4.6), that

$$\frac{1}{\int_{\Omega_\varepsilon} |u_n^0|^2\,\mathrm{d}x + O(\|V_\varepsilon\|_{L^2(\Omega_\varepsilon)})} = \frac{1}{1+o(1)} = 1 + o(1) \quad \text{as } \varepsilon \to 0^+. \tag{4.17}$$

Furthermore,

$$O(\|V_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}) = o(\mathcal{T}_\varepsilon) \quad \text{as } \varepsilon \to 0^+, \tag{4.18}$$

in view of Proposition 4.4. Finally, by (4.1) tested with V_ε, (4.2) tested with V_ε, (4.2) tested with u_n^0, (1.1), and an integration by part,

$$\begin{aligned}
\mu_n^0 \int_{\Omega_\varepsilon} u_n^0 V_\varepsilon\,\mathrm{d}x &= q_\varepsilon(V_\varepsilon, u_n^0) - L_\varepsilon(V_\varepsilon) = L_\varepsilon(u_n^0) - \mathcal{T}_\varepsilon \\
&= \int_{\partial\Sigma_\varepsilon} \frac{\partial u_n^0}{\partial \nu} u_n^0\,\mathrm{d}\sigma + \alpha \int_{\partial\Sigma_\varepsilon} |u_n^0|^2\,\mathrm{d}\sigma - \mathcal{T}_\varepsilon \\
&= -\int_{\Sigma_\varepsilon} \left(|\nabla u_n^0|^2 + (c_\alpha - \mu_n^0)|u_n^0|^2\right)\mathrm{d}x + \alpha \int_{\partial\Sigma_\varepsilon} |u_n^0|^2\,\mathrm{d}\sigma - \mathcal{T}_\varepsilon,
\end{aligned} \tag{4.19}$$

where ν is the outer normal derivative to Ω_ε on $\partial\Sigma_\varepsilon$. Plugging (4.17), (4.18), and (4.19) into (4.16), and taking into account that $c_\alpha - \mu_n^0 = -\lambda_n^0$, we obtain (4.12).

For every $\varepsilon \in (0,1)$, let u_n^ε be the eigenfunction of (1.4), associated with the eigenvalue λ_n^ε, chosen as in (2.4). We consider the orthogonal projection onto the eigenspace associated with the eigenvalue λ_n^ε, i.e.

$$\Pi_\varepsilon(v) := (v, u_n^\varepsilon)_{L^2(\Omega_\varepsilon)} u_n^\varepsilon \quad \text{for every } v \in L^2(\Omega_\varepsilon).$$

By Theorem A.1, (3.9), (4.3), and Proposition 4.4

$$\|u_n^0 - V_\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} = o\left(\mathcal{T}_\varepsilon\right) \quad \text{as } \varepsilon \to 0^+, \tag{4.20}$$

$$\|u_n^0 - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} = O\left(\mathcal{T}_\varepsilon\right) \quad \text{as } \varepsilon \to 0^+, \tag{4.21}$$

$$\|u_n^0 - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} = o\left(\mathcal{T}_\varepsilon\right) \quad \text{as } \varepsilon \to 0^+. \tag{4.22}$$

In particular, by (4.21) and Proposition 4.3,

$$\lim_{\varepsilon \to 0^+} \|u_n^0 - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{H^1(\Omega_\varepsilon)} = 0. \tag{4.23}$$

It follows that, thanks to (2.4) and the simplicity of λ_n^ε,

$$u_n^\varepsilon = \frac{\Pi_\varepsilon(u_n^0 - V_\varepsilon)}{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}} \tag{4.24}$$

for all $\varepsilon > 0$ sufficiently small. Hence

$$\|u_n^\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} = \frac{\left|1 - \|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}\right|^2}{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)}} \|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)}. \tag{4.25}$$

Furthermore, in view of (4.6), (4.22), and the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} \\
&\quad = 1 - \int_{\Sigma_\varepsilon} |u_n^0|^2 \,\mathrm{d}x + \|u_n^0 - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} - 2(u_n^0 - \Pi_\varepsilon(u_n^0 - V_\varepsilon), u_n^0)_{L^2(\Omega_\varepsilon)} \\
&\quad = 1 - \int_{\Sigma_\varepsilon} |u_n^0|^2 \,\mathrm{d}x + o(\mathcal{T}_\varepsilon^{1/2}).
\end{aligned} \tag{4.26}$$

Hence, plugging (4.26) in (4.25) we obtain

$$\begin{aligned}
\|u_n^\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} &= o(\mathcal{T}_\varepsilon) + O(\|u_n^0\|^4_{L^2(\Sigma_\varepsilon)}) + o\left(\sqrt{\mathcal{T}_\varepsilon}\|u_n^0\|^2_{L^2(\Sigma_\varepsilon)}\right) \\
&= o(\mathcal{T}_\varepsilon) + O(\|u_n^0\|^4_{L^2(\Sigma_\varepsilon)})
\end{aligned}$$

as $\varepsilon \to 0^+$, since $\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} = \|u_n^0\|^2_{H^1(\Omega)} + o(1)$ in view of (4.23). By the Cauchy–Schwarz inequality and (4.20), it then follows that

$$\begin{aligned}
\|u_n^\varepsilon - (u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} &= \|u_n^0 - V_\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} + \|u_n^\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} \\
&\quad - 2\big(u_n^\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon), u_n^0 - V_\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\big)_{H^1(\Omega_\varepsilon)} \\
&= o(\mathcal{T}_\varepsilon) + O\big(\|u_n^0\|^4_{L^2(\Sigma_\varepsilon)}\big) \quad \text{as } \varepsilon \to 0^+,
\end{aligned} \tag{4.27}$$

so that (4.13) is proved.

To prove (4.14), we observe that, by (3.9), (4.3), Proposition 3.2, and Proposition 4.4,

$$\|V_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} = \mathcal{T}_\varepsilon + (1 - c_\alpha)\|V_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} - \alpha \int_{\partial\Omega} V_\varepsilon^2 \,\mathrm{d}\sigma - \alpha \int_{\partial\Sigma_\varepsilon} V_\varepsilon^2 \,\mathrm{d}\sigma = \mathcal{T}_\varepsilon + o(\mathcal{T}_\varepsilon) \tag{4.28}$$

as $\varepsilon \to 0^+$. Moreover, from (4.20), (4.24), and (4.26), it follows that

$$\begin{aligned}
&(u_n^\varepsilon - (u_n^0 - V_\varepsilon), V_\varepsilon)_{H^1(\Omega_\varepsilon)} \\
&= \frac{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)} - 1}{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}} \|V_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} \\
&\quad + \frac{\left(\Pi_\varepsilon(u_n^0 - V_\varepsilon) - (u_n^0 - V_\varepsilon) + \left(1 - \|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}\right) u_n^0, V_\varepsilon\right)_{H^1(\Omega_\varepsilon)}}{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}} \\
&= \frac{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)} - 1}{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}} \left(\mathcal{T}_\varepsilon + o(\mathcal{T}_\varepsilon)\right) + \frac{\left(\Pi_\varepsilon(u_n^0 - V_\varepsilon) - (u_n^0 - V_\varepsilon), V_\varepsilon\right)_{H^1(\Omega_\varepsilon)}}{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}} \\
&\quad + \frac{1 - \|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}}{\|\Pi_\varepsilon(u_n^0 - V_\varepsilon)\|_{L^2(\Omega_\varepsilon)}} (u_n^0, V_\varepsilon)_{H^1(\Omega_\varepsilon)} \\
&= o(\mathcal{T}_\varepsilon) + O\left(\mathcal{T}_\varepsilon^{1/2} \|u_n^0\|^2_{L^2(\Sigma_\varepsilon)}\right) \quad \text{as } \varepsilon \to 0^+. \qquad (4.29)
\end{aligned}$$

By (4.27), (4.28), and (4.29) we finally conclude that

$$\begin{aligned}
\|u_n^\varepsilon - u_n^0\|^2_{H^1(\Omega_\varepsilon)} &= \|u_n^\varepsilon - (u_n^0 - V_\varepsilon)\|^2_{H^1(\Omega_\varepsilon)} + \|V_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} - 2(u_n^\varepsilon - (u_n^0 - V_\varepsilon), V_\varepsilon)_{H^1(\Omega_\varepsilon)} \\
&= \mathcal{T}_\varepsilon + o(\mathcal{T}_\varepsilon) + O\left(\left\|u_n^0\right\|^4_{L^2(\Sigma_\varepsilon)}\right) + O\left(\mathcal{T}_\varepsilon^{1/2} \|u_n^0\|^2_{L^2(\Sigma_\varepsilon)}\right) \quad \text{as } \varepsilon \to 0^+,
\end{aligned}$$

thus completing the proof of (4.14). □

An asymptotic expansion for some of the terms appearing in (4.12) is provided in the following proposition, enabling us to derive Theorems 2.2 and 2.4 directly from expansion (4.12).

Proposition 4.6 *Let $\ell \geq 1$ be the vanishing order of $u_n^0 - u_n^0(0)$ at 0. Then, as $\varepsilon \to 0^+$,*

$$\begin{aligned}
\int_{\Sigma_\varepsilon} |\nabla u_n^0|^2 \, \mathrm{d}x &= \varepsilon^{N+2\ell-2} \int_\Sigma |\nabla P_\ell^{u_n^0,0}|^2 \, \mathrm{d}x + O(\varepsilon^{N+2\ell-1}), \\
\int_{\Sigma_\varepsilon} |u_n^0|^2 \, \mathrm{d}x &= \varepsilon^N |u_n^0(0)|^2 \, |\Sigma|_N + \varepsilon^{N+2\ell} \int_\Sigma |P_\ell^{u_n^0,0}|^2 \, \mathrm{d}x + 2\varepsilon^{N+\ell} \, u_n^0(0) \int_\Sigma P_\ell^{u_n^0,0} \, \mathrm{d}x \\
&\quad + u_n^0(0) \, O(\varepsilon^{\ell+1+N}) + O(\varepsilon^{2\ell+1+N}), \\
\int_{\partial\Sigma_\varepsilon} |u_n^0|^2 \, \mathrm{d}\sigma &= \varepsilon^{N-1} |u_n^0(0)|^2 \mathcal{H}^{N-1}(\partial\Sigma) + \varepsilon^{N+2\ell-1} \int_{\partial\Sigma} |P_\ell^{u_n^0,0}|^2 \, \mathrm{d}\sigma \\
&\quad + 2\varepsilon^{N+\ell-1} \, u_n^0(0) \int_{\partial\Sigma} P_\ell^{u_n^0,0} \, \mathrm{d}\sigma + u_n^0(0) \, O(\varepsilon^{\ell+N}) + O(\varepsilon^{2\ell+N}).
\end{aligned}$$

Proof By standard elliptic regularity theory, u_n^0 is a smooth function; hence

$$u_n^0(x) - u_n^0(0) = P_\ell^{u_n^0,0}(x) + O(|x|^{\ell+1}) \quad \text{and} \quad \nabla u_n^0(x) = \nabla P_\ell^{u_n^0,0}(x) + O(|x|^\ell) \quad \text{as } |x| \to 0.$$

Then, the desired estimates follow directly from a change of variables. □

Proof of Theorems 2.2 and 2.4 Theorems 2.2 and 2.4 are now a simple consequence of Theorem 4.5, Proposition 4.3, and Proposition 4.6. □

5 Blow-up analysis for $N \geq 3$

In this section, we use a blow-up analysis to derive the asymptotics of $\mathcal{T}_\varepsilon$, as $\varepsilon \to 0^+$, for $N \geq 3$.

The following proposition describes the asymptotic behavior of u_n^0 near $x_0 = 0 \in \Omega$; it directly follows from analyticity of u_n^0.

Proposition 5.1 *If u_n^0 vanishes of order $k \geq 0$ at $x_0 = 0 \in \Omega$, then, for every $R > 0$,*

$$r^{-k} u_n^0(rx) \to P_k^{u_n^0,0}(x) \quad \text{uniformly in } \overline{B_R} \text{ and in } H^1(B_R)$$

as $r \to 0^+$; furthermore, $P_k^{u_n^0,0}$ is a harmonic and homogeneous polynomial of degree k.

If $u_n^0 - u_n^0(0)$ vanishes of order $\ell \geq 1$ at $x_0 = 0 \in \Omega$, then, for every $R > 0$,

$$\frac{u_n^0(rx) - u_n^0(0)}{r^\ell} \to P_\ell^{u_n^0,0}(x) \quad \text{uniformly in } \overline{B_R} \text{ and in } H^1(B_R)$$

and

$$r^{-\ell+1} \nabla u_n^0(rx) \to \nabla P_\ell^{u_n^0,0}(x) \quad \text{uniformly in } \overline{B_R} \text{ and in } H^1(B_R; \mathbb{R}^N)$$

as $r \to 0^+$.

The following Hardy-type inequality serves as an essential tool for conducting blow-up analysis, especially in the identification of limit profiles. We refer to [19, Lemma 5.3] for a proof.

Lemma 5.2 *Let $N \geq 3$ and E be an open Lipschitz subset of $\mathbb{R}^N$ such that $\overline{E} \subset B_{R_0}$ and $B_{R_0} \setminus \overline{E}$ is connected, for some $R_0 > 0$. Then there exists $C_H = C_H(N, E) > 0$ such that*

$$\int_{B_R \setminus E} \frac{u^2}{|x|^2} \, \mathrm{d}x \leq C_H \left(\int_{B_R \setminus E} |\nabla u|^2 \, \mathrm{d}x + \frac{1}{R^2} \int_{B_R \setminus E} u^2 \, \mathrm{d}x \right)$$

for all $u \in H^1(B_R \setminus \overline{E})$ and $R > 2R_0$. Moreover, for all $u \in C_c^\infty(\mathbb{R}^N \setminus E)$, there holds

$$\int_{\mathbb{R}^N \setminus E} \frac{u^2}{|x|^2} \, \mathrm{d}x \leq C_H \int_{\mathbb{R}^N \setminus E} |\nabla u|^2 \, \mathrm{d}x.$$

For every open Lipschitz set $E \subset \mathbb{R}^N$ such that $\overline{E} \subset B_{R_0}$ and $B_{R_0} \setminus \overline{E}$ is connected for some $R_0 > 0$, let us consider the functional space $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus E)$, defined as the completion of $C_c^\infty(\mathbb{R}^N \setminus E)$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}(\mathbb{R}^N\setminus E)} := \left(\int_{\mathbb{R}^N\setminus E} |\nabla u|^2 \,\mathrm{d}x\right)^{\frac{1}{2}}.$$

The Hardy inequality provided by the above lemma implies that

$$u \mapsto \left(\int_{\mathbb{R}^N\setminus E} \left(|\nabla u|^2 + \frac{u^2}{|x|^2}\right) \mathrm{d}x\right)^{\frac{1}{2}},$$

is an equivalent norm on $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus E)$, which can be characterized also as

$$\mathcal{D}^{1,2}(\mathbb{R}^N \setminus E) = \left\{u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N \setminus E) : \int_{\mathbb{R}^N\setminus E} \left(|\nabla u|^2 + \frac{u^2}{|x|^2}\right) \mathrm{d}x < \infty\right\}.$$

Let h and $P_h^{u_n^0,0}$ be as in (2.6) and (2.5), respectively, with $x_0 = 0 \in \Omega$. We define the functional $\widetilde{J} : \mathcal{D}^{1,2}(\mathbb{R}^N\setminus\Sigma) \to \mathbb{R}$ as

$$\widetilde{J}(v) := \frac{1}{2}\int_{\mathbb{R}^N\setminus\Sigma} |\nabla v|^2 \,\mathrm{d}x - \int_{\partial\Sigma} \left(\partial_\nu P_h^{u_n^0,0} + \alpha\, u_n^0(0)\right) v \,\mathrm{d}\sigma.$$

Proposition 5.3 *There exists a unique $\widetilde{V} \in \mathcal{D}^{1,2}(\mathbb{R}^N\setminus\Sigma)$ such that*

$$\widetilde{J}(\widetilde{V}) = \min\{\widetilde{J}(v) : v \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma)\}.$$

Furthermore, $\widetilde{V} \not\equiv 0$ and $\widetilde{V}$ is the unique $\mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma)$-function satisfying

$$\int_{\mathbb{R}^N\setminus\Sigma} \nabla\widetilde{V}\cdot\nabla\varphi \,\mathrm{d}x = \int_{\partial\Sigma} \left(\partial_\nu P_h^{u_n^0,0} + \alpha\, u_n^0(0)\right)\varphi \,\mathrm{d}\sigma \tag{5.1}$$

for all $\varphi \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma)$.

Proof The proof of the existence of a unique minimizer $\widetilde{V}$ can be obtained by standard variational methods. To verify that $\widetilde{V} \not\equiv 0$, we argue by contradiction. If $\widetilde{V} \equiv 0$, (5.1) would imply that $\partial_\nu P_h^{u_n^0,0} + \alpha u_n^0(0) = 0$ on $\partial\Sigma$. If $u_n^0(0) = 0$, then $\Delta P_h^{u_n^0,0} = 0$ and

$$0 = \int_{\partial\Sigma} (\partial_\nu P_h^{u_n^0,0}) P_h^{u_n^0,0} \,\mathrm{d}\sigma = -\int_\Sigma |\nabla P_h^{u_n^0,0}|^2 \,\mathrm{d}x;$$

this would imply that $P_h^{u_n^0,0}$ is constant, contradicting the fact that $P_h^{u_n^0,0}$ is a polynomial of degree $h \geq 1$. On the other hand, if $u_n^0(0) \neq 0$, then $h = 1$ and

$$P_h^{u_n^0,0}(x) = \nabla u_n^0(0)\cdot x;$$

hence, by the Divergence Theorem,

$$\alpha u_n^0(0)\mathcal{H}^{N-1}(\partial\Sigma) = -\int_{\partial\Sigma} \nabla u_n^0(0)\cdot\nu\,\mathrm{d}\sigma = 0,$$

giving rise to a contradiction in view of (1.5). □

We observe that (5.1) is the weak formulation of (2.7). We are now ready to state the main result of this section.

Theorem 5.4 *Let h and $P_h^{u_n^0,0}$ be as in (2.6) and (2.5), respectively, with $x_0 = 0 \in \Omega$. If $N \geq 3$ and $\widetilde{V}$ is the function introduced in Proposition 5.3, then*

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-N-2h+2}\mathcal{T}_\varepsilon = \widetilde{\mathcal{T}} = \int_{\partial\Sigma} \big(\partial_\nu P_h^{u_n^0,0} + \alpha\, u_n^0(0)\big)\widetilde{V}\,\mathrm{d}\sigma,$$

where $\mathcal{T}_\varepsilon$ is defined in (4.3). Furthermore, if $V_\varepsilon \in H^1(\Omega_\varepsilon)$ is as in Proposition 4.1 and

$$\widetilde{V}_\varepsilon(x) := \varepsilon^{-h} V_\varepsilon(\varepsilon x), \quad x \in \big(\tfrac{1}{\varepsilon}\Omega\big)\setminus\Sigma, \tag{5.2}$$

then

$$\widetilde{V}_\varepsilon \to \widetilde{V} \quad \text{in } H^1(B_R\setminus\overline{\Sigma}) \quad \text{as } \varepsilon\to 0^+,$$

for all $R > 0$ such that $\overline{\Sigma} \subset B_R$.

Proof Let $r_0 > 0$ be as in (1.2) with $x_0 = 0$. Let $R > r_0$ and $\varepsilon < \frac{r_0}{2R}$. As $\frac{r_0}{\varepsilon} > 2r_0$ and $R < \frac{r_0}{2\varepsilon} < \frac{r_0}{\varepsilon}$, applying Lemma 5.2 to the function $\widetilde{V}_\varepsilon$, by a change of variable we have

$$\begin{aligned}
\int_{B_R\setminus\Sigma}\left(|\nabla\widetilde{V}_\varepsilon|^2 + \frac{\widetilde{V}_\varepsilon^2}{|x|^2}\right)\mathrm{d}x &\leq \int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma}\left(|\nabla\widetilde{V}_\varepsilon|^2 + \frac{\widetilde{V}_\varepsilon^2}{|x|^2}\right)\mathrm{d}x\\
&\leq \int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma}|\nabla\widetilde{V}_\varepsilon|^2\,\mathrm{d}x + C_H\left(\int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma}|\nabla\widetilde{V}_\varepsilon|^2\,\mathrm{d}x + \frac{\varepsilon^2}{r_0^2}\int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma}\widetilde{V}_\varepsilon^2\,\mathrm{d}x\right)\\
&= \varepsilon^{-N-2h+2}\left(\int_{B_{r_0}\setminus\varepsilon\Sigma}|\nabla V_\varepsilon|^2\,\mathrm{d}x + C_H\int_{B_{r_0}\setminus\varepsilon\Sigma}\left(|\nabla V_\varepsilon|^2 + r_0^{-2}V_\varepsilon^2\right)\mathrm{d}x\right)\\
&\leq C_1\varepsilon^{-N-2h+2}\left(\int_{\Omega_\varepsilon}|\nabla V_\varepsilon|^2\,\mathrm{d}x + \int_{\Omega_\varepsilon}V_\varepsilon^2\,\mathrm{d}x\right)\\
&= C_1\varepsilon^{-N-2h+2}||V_\varepsilon||^2_{H^1(\Omega_\varepsilon)},
\end{aligned}$$

for some $C_1 > 0$ independent of R and ε. Therefore, by (3.9), (4.3), and Proposition 4.3, we deduce that, for every $0 < \varepsilon < \frac{r_0}{2R}$,

$$\int_{B_R\setminus\Sigma}\left(|\nabla\widetilde{V}_\varepsilon|^2 + \frac{\widetilde{V}_\varepsilon^2}{|x|^2}\right)\mathrm{d}x \leq C, \tag{5.3}$$

where the above constant C is independent of R and ε.

This implies that, for every $R > r_0$, there exists $C_R > 0$ such that

$$||\widetilde{V}_\varepsilon||_{H^1(B_R\setminus\overline{\Sigma})} < C_R, \quad \text{for all } \varepsilon \in \left(0, \tfrac{r_0}{2R}\right). \tag{5.4}$$

Hence, by a diagonal argument, for every sequence $\varepsilon_j \to 0^+$, there exist a subsequence (still denoted as $\{\varepsilon_j\}_{j\in\mathbb{N}}$) and $W \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N \setminus \overline{\Sigma})$ such that, for every $R > r_0$, $W \in H^1(B_R \setminus \overline{\Sigma})$ and

$$\widetilde{V}_{\varepsilon_j} \rightharpoonup W \quad \text{weakly in } H^1(B_R \setminus \overline{\Sigma}) \quad \text{as } j \to +\infty. \tag{5.5}$$

Therefore, by the Rellich–Kondrachov Theorem and the compactness of the trace operator into $L^2(\partial\Sigma)$, we have, for all $R > r_0$,

$$\widetilde{V}_{\varepsilon_j} \to W \quad \text{in } L^2(B_R \setminus \overline{\Sigma}) \quad \text{as } j \to +\infty, \tag{5.6}$$

$$\widetilde{V}_{\varepsilon_j} \to W \quad \text{in } L^2(\partial\Sigma) \quad \text{as } j \to +\infty. \tag{5.7}$$

Hence, by (5.3) and the weak lower semicontinuity of norms, we have

$$\int_{B_R\setminus\Sigma} \left(|\nabla W|^2 + \frac{W^2}{|x|^2}\right) \mathrm{d}x \le \liminf_{j\to+\infty} \int_{B_R\setminus\Sigma} \left(|\nabla \widetilde{V}_{\varepsilon_j}|^2 + \frac{\widetilde{V}^2_{\varepsilon_j}}{|x|^2}\right) \mathrm{d}x \le C$$

for every $R > r_0$; hence, being C independent of R, we conclude that

$$\int_{\mathbb{R}^N\setminus\Sigma} \left(|\nabla W|^2 + \frac{W^2}{|x|^2}\right) \mathrm{d}x \le C,$$

so that $W \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma)$.

Let $v \in C^\infty_c(\mathbb{R}^N \setminus \Sigma)$. There exist $R_v > r_0$ and $j_v \in \mathbb{N}$ such that

$$\operatorname{supp} v \subset B_{R_v} \setminus \Sigma \subset \tfrac{1}{\varepsilon_j}\Omega \setminus \Sigma \quad \text{for all } j \ge j_v.$$

Since V_ε satisfies (4.2), by the change of variable $x = \varepsilon_j y$ we obtain

$$\begin{aligned}\int_{B_{R_v}\setminus\Sigma} &\left(\nabla \widetilde{V}_{\varepsilon_j} \cdot \nabla v + \varepsilon_j^2 c_\alpha \widetilde{V}_{\varepsilon_j} v\right) \mathrm{d}x \\ &= -\alpha\varepsilon_j \int_{\partial\Sigma} \widetilde{V}_{\varepsilon_j} v \,\mathrm{d}\sigma + \int_{\partial\Sigma} \left(\frac{\nabla u^0_n(\varepsilon_j x)}{\varepsilon_j^{h-1}} \cdot \nu + \alpha\,\varepsilon_j \frac{u^0_n(\varepsilon_j x)}{\varepsilon_j^h}\right) v \,\mathrm{d}\sigma.\end{aligned} \tag{5.8}$$

In both cases, $u^0_n(0) = 0$ and $u^0_n(0) \neq 0$, by Proposition 5.1 we have the uniform convergence

$$\frac{\nabla u^0_n(\varepsilon x)}{\varepsilon^{h-1}} \cdot \nu + \varepsilon\alpha \frac{u^0_n(\varepsilon x)}{\varepsilon^h} \to \nabla P_h^{u^0_n,0}(x) \cdot \nu + \alpha\, u^0_n(0), \quad \text{in } \partial\Sigma, \tag{5.9}$$

as $\varepsilon \to 0^+$. Consequently

$$\int_{\partial\Sigma} \left(\frac{\nabla u_n^0(\varepsilon_j x)}{\varepsilon_j^{h-1}} \cdot \nu + \alpha\, \varepsilon_j \frac{u_n^0(\varepsilon_j x)}{\varepsilon_j^h} \right) v \, \mathrm{d}\sigma \to \int_{\partial\Sigma} \left(\partial_\nu P_h^{u_n^0,0} + \alpha\, u_n^0(0) \right) v \, \mathrm{d}\sigma, \quad (5.10)$$

as $j \to +\infty$. Hence, applying (5.5), (5.6), (5.7), and (5.10), we can pass to the limit in (5.8), thus obtaining

$$\int_{\mathbb{R}^N\setminus\Sigma} \nabla W \cdot \nabla v \, \mathrm{d}x = \int_{\partial\Sigma} \left(\partial_\nu P_h^{u_n^0,0} + \alpha\, u_n^0(0) \right) v \, \mathrm{d}\sigma \quad \text{for all } v \in \mathcal{D}^{1,2}(\mathbb{R}^N \setminus \Sigma).$$

By uniqueness of the solution to problem (2.7), we conclude that

$$W = \widetilde{V}.$$

By Urysohn's subsequence principle, we conclude that the convergences (5.5), (5.6), and (5.7) hold as $\varepsilon \to 0$, and not just along the sequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$.

Moreover, by a change of variable, (4.2) tested with V_ε, (4.3), (5.7), and (5.9),

$$\begin{aligned} \varepsilon^{-N-2h+2}\mathcal{T}_\varepsilon &= \int_{\partial\Sigma_\varepsilon} \varepsilon^{-N-2h+2} \left(\partial_\nu u_n^0 + \alpha\, u_n^0 \right) V_\varepsilon \, \mathrm{d}\sigma \\ &= \int_{\partial\Sigma} \varepsilon^{-h+1} \left(\partial_\nu u_n^0(\varepsilon x) + \alpha\, u_n^0(\varepsilon x) \right) \widetilde{V}_\varepsilon \, \mathrm{d}\sigma \\ &\to \int_{\partial\Sigma} \left(\partial_\nu P_h^{u_n^0,0} + \alpha\, u_n^0(0) \right) \widetilde{V} \, \mathrm{d}\sigma \quad \text{as } \varepsilon \to 0^+. \end{aligned}$$

To conclude, it only remains to prove the strong H^1 convergence. By scaling, $\widetilde{V}_\varepsilon$ weakly solves

$$\begin{cases} -\Delta \widetilde{V}_\varepsilon + c_\alpha \varepsilon^2 \widetilde{V}_\varepsilon = 0, & \text{in } \frac{1}{\varepsilon}\Omega \setminus \overline{\Sigma}, \\ \partial_\nu \widetilde{V}_\varepsilon + \alpha\varepsilon \widetilde{V}_\varepsilon = 0, & \text{on } \partial\left(\frac{\Omega}{\varepsilon}\right), \\ \partial_\nu \widetilde{V}_\varepsilon + \alpha\varepsilon \widetilde{V}_\varepsilon = \varepsilon^{-h+1} \partial_\nu u_n^0(\varepsilon x) + \varepsilon^{-h+1} \alpha u_n^0(\varepsilon x), & \text{on } \partial\Sigma. \end{cases}$$

For $R > r_0$ fixed, let $\varepsilon \in (0, \frac{r_0}{2R})$, so that $B_R \setminus \Sigma \subset \frac{1}{\varepsilon}\Omega \setminus \Sigma$. Then, testing the difference between the above equation and (2.7) with $\widetilde{V}_\varepsilon - \widetilde{V}$ we obtain

$$\begin{aligned} &\int_{B_R\setminus\Sigma} |\nabla(\widetilde{V}_\varepsilon - \widetilde{V})|^2 \, \mathrm{d}x \\ &= -\varepsilon^2 c_\alpha \int_{B_R\setminus\Sigma} \widetilde{V}_\varepsilon (\widetilde{V}_\varepsilon - \widetilde{V}) \, \mathrm{d}x + \int_{\partial B_R} (\widetilde{V}_\varepsilon - \widetilde{V})(\partial_\nu \widetilde{V}_\varepsilon - \partial_\nu \widetilde{V}) \, \mathrm{d}\sigma \\ &\quad + \int_{\partial\Sigma} (\widetilde{V}_\varepsilon - \widetilde{V}) \left(-\alpha\varepsilon \widetilde{V}_\varepsilon + \frac{\nabla u_n^0(\varepsilon x)}{\varepsilon^{h-1}} \cdot \nu + \varepsilon\alpha \frac{u_n^0(\varepsilon x)}{\varepsilon^h} - \partial_\nu P_h^{u_n^0,0} - \alpha u_n^0(0) \right) \mathrm{d}\sigma. \end{aligned} \quad (5.11)$$

We notice that, if $0 < \varepsilon < \frac{r_0}{2R}$, (1.2) ensures that $B_{2R}\setminus B_{r_0} \subset \frac{1}{\varepsilon}\Omega\setminus\Sigma$. Hence, $\widetilde{V}_\varepsilon$ weakly solves $-\Delta\widetilde{V}_\varepsilon = -c_\alpha\varepsilon^2\widetilde{V}_\varepsilon$ in $B_{2R}\setminus B_{r_0}$. Since the family $\{\varepsilon^2\widetilde{V}_\varepsilon\}_{\varepsilon\in(0,\frac{r_0}{2R})}$ is bounded in $L^2(B_{2R}\setminus B_{r_0})$ by (5.4), by classical elliptic regularity theory we have $\{\widetilde{V}_\varepsilon\}_{0<\varepsilon<\frac{r_0}{2R}}$ is bounded in $H^2\big(B_{\frac{3}{2}R}\setminus B_{\frac{r_0+R}{2}}\big)$ uniformly with respect to ε. Therefore, passing to traces, $\{\partial_\nu\widetilde{V}_\varepsilon\}_{0<\varepsilon<\frac{r_0}{2R}}$ is bounded in $L^2(\partial B_R)$. Hence, using this, (5.9), and the weak convergence $\widetilde{V}_\varepsilon\rightharpoonup\widetilde{V}$ in $H^1(B_R\setminus\overline{\Sigma})$, which implies strong $L^2(B_R\setminus\overline{\Sigma})$-convergence and strong convergence of traces in $L^2(\partial B_R)$ and in $L^2(\partial\Sigma)$, the right hand side of (5.11) vanishes in the limit as $\varepsilon\to 0^+$. This proves that $\nabla\widetilde{V}_\varepsilon\to\nabla\widetilde{V}$ in $L^2(B_R\setminus\overline{\Sigma})$. This fact, along with (5.6), gives $\widetilde{V}_\varepsilon\to\widetilde{V}$ in $H^1(B_R\setminus\overline{\Sigma})$. This concludes the proof. □

Proof of Theorem 2.3 By translation, it is not restrictive to assume $x_0 = 0$. The result is a direct consequence of Theorem 4.5, Proposition 4.6, and Theorem 5.4. □

Now, we study the rate of convergence of perturbed eigenfunctions, proving Theorems 2.5 and 2.6.

Proof of Theorems 2.5 and 2.6 By translation, it is not restrictive to assume $x_0 = 0$. Let us consider the function $\varphi_\varepsilon := \Phi_\varepsilon - \Phi^0_\varepsilon + \widetilde{V}_\varepsilon$, where $\widetilde{V}_\varepsilon$ is defined in (5.2) and

$$\Phi^0_\varepsilon(x) := \frac{u^0_n(x_0+\varepsilon x) - u^0_n(x_0)}{\varepsilon^h}.$$

By (4.13) and a change of variable,

$$\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma}|\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \varepsilon^2\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma}\varphi_\varepsilon^2\,\mathrm{d}x = o\left(\varepsilon^{-N-2h+2}\mathcal{T}_\varepsilon\right) + O\left(\varepsilon^{-N-2h+2}\left\|u^0_n\right\|^4_{L^2(\Sigma_\varepsilon)}\right), \tag{5.12}$$

as $\varepsilon\to 0^+$. If $u^0_n(0) = 0$, then $h = k$, where k is the vanishing order of u^0_n at 0. Therefore, $u^0_n(x) = O(|x|^k)$ as $|x|\to 0$, hence

$$\|u^0_n\|^4_{L^2(\Sigma_\varepsilon)} = O(\varepsilon^{2N+4k}) = o(\varepsilon^{N+2h-2})$$

as $\varepsilon\to 0^+$. On the other hand, in the case $u^0_n(0)\neq 0$, we have $h = 1$ and $u^0_n(x) = u^0_n(0) + O(|x|)$ as $|x|\to 0^+$; hence

$$\|u^0_n\|^4_{L^2(\Sigma_\varepsilon)} = O(\varepsilon^{2N}) = o(\varepsilon^N) = o(\varepsilon^{N+2h-2})$$

as $\varepsilon\to 0^+$. In both cases, we conclude that

$$\|u^0_n\|^4_{L^2(\Sigma_\varepsilon)} = o(\varepsilon^{N+2h-2}) \quad \text{as} \quad \varepsilon\to 0^+. \tag{5.13}$$

In particular, we have proved Theorem 2.6 in view of Theorem 4.5 and Proposition 4.3.

Furthermore, by (5.12), Theorem 5.4, and (5.13),

$$\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \varepsilon^2 \int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} \varphi_\varepsilon^2\,\mathrm{d}x \to 0 \quad \text{as } \varepsilon \to 0^+. \tag{5.14}$$

Let $r_0 > 0$ be as in (1.2). Choose $R > r_0$ and $\varepsilon < \frac{r_0}{2R}$. If $N \geq 3$, we can apply Lemma 5.2 to obtain

$$\begin{aligned}
\int_{B_R\setminus\Sigma} \left(|\nabla\varphi_\varepsilon|^2 + \frac{\varphi_\varepsilon^2}{|x|^2}\right)\mathrm{d}x &\leq \int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma} \left(|\nabla\varphi_\varepsilon|^2 + \frac{\varphi_\varepsilon^2}{|x|^2}\right)\mathrm{d}x \\
&\leq \int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma} |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + C_H\left(\int_{B_{\frac{r_0}{\varepsilon}}\setminus\Sigma} |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \frac{\varepsilon^2}{r_0^2}\int_{B_{\frac{r_0}{\epsilon}}\setminus\Sigma} \varphi_\varepsilon^2\,\mathrm{d}x\right) \\
&\leq (C_H+1)\left(\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} |\nabla\varphi_\varepsilon|^2\,\mathrm{d}x + \frac{\varepsilon^2}{r_0^2}\int_{\frac{1}{\varepsilon}\Omega\setminus\Sigma} \varphi_\varepsilon^2\,\mathrm{d}x\right).
\end{aligned}$$

From this and (5.14) it follows that

$$\int_{B_R\setminus\Sigma} \left(|\nabla\varphi_\varepsilon|^2 + \frac{\varphi_\varepsilon^2}{|x|^2}\right)\mathrm{d}x \to 0 \quad \text{as } \varepsilon \to 0^+,$$

for every $R > 0$ such that $\overline{\Sigma} \subset B_R$. Hence

$$\varphi_\varepsilon \to 0 \quad \text{strongly in } H^1(B_R \setminus \overline{\Sigma}) \text{ as } \varepsilon \to 0^+. \tag{5.15}$$

If $u_n^0(0) = 0$, then h is the vanishing order of u_n^0 at 0 and, by Proposition 5.1, $\Phi_\varepsilon^0 \to P_h^{u_n^0,0}$ as $\varepsilon \to 0^+$ in $H^1(B_R)$ for any $R > 0$.

If $u_n^0(0) \neq 0$, then $h = 1$ and

$$\Phi_\varepsilon^0(x) = \varepsilon^{-1}(u_n^0(\varepsilon x) - u_n^0(0)) \to \nabla u_n^0(0)\cdot x = P_1^{u_n^0,0}(x)$$

as $\varepsilon \to 0^+$ in $H^1(B_R)$ for any $R > 0$.

This, together with (5.15) and Theorem 5.4, establishes (2.9).

Finally, using (5.13) and Theorem 5.4 in (4.14), we obtain

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-(N+2h-2)} \|u_n^\varepsilon - u_n^0\|^2_{H^1(\Omega_\varepsilon)} = \widetilde{\mathcal{T}},$$

thus proving (2.10). □

6 The case of a round hole

In this section, we focus on the case where

$$\Sigma = B_{r_1},$$

for some $0 < r_1 < r_0$. In this case, we can compute $\widetilde{\mathcal{T}}$ explicitly if $N \geq 3$, while we determine the exact asymptotic behavior of $\mathcal{T}_\varepsilon$ if $N = 2$. We do this only in the case $u_n^0(0) = 0$, since the case $u_n^0(0) \neq 0$ is already completely covered by Theorem 2.2.

6.1 Dimension $N \geq 3$

If $N \geq 3$ and $u_n^0(0) = 0$, the quantity $\widetilde{\mathcal{T}}$ defined in (2.8) coincides with the one that appears in the blow-up analysis carried out in [19] for the Neumann problem; in the case of spherical holes, it can be explicitly computed as below.

Proposition 6.1 *If $u_n^0(0) = 0$, $N \geq 3$, and k is the vanishing order of u_n^0 at 0, then*

$$\widetilde{\mathcal{T}} = \frac{k^2 r_1^{N+2k-2}}{N+k-2} \int_{\partial B_1} |P_k^{u_n^0,0}|^2 \, \mathrm{d}\sigma,$$

where $\widetilde{\mathcal{T}}$ is as in (2.8).

Proof The proof is contained in [19, Lemma 6.1], up to a rescaling. □

We are now in a position to prove Theorem 2.7.

Proof of Theorem 2.7 Without loss of generality, we assume $x_0 = 0$. By the divergence theorem, since $P_k^{u_n^0,0}$ is harmonic by Proposition 5.1,

$$\begin{aligned}\int_{B_{r_1}} |\nabla P_k^{u_n^0,0}|^2 \, \mathrm{d}x &= \int_{B_{r_1}} \mathrm{div}\left(P_k^{u_n^0,0} \nabla P_k^{u_n^0,0}\right) \mathrm{d}x \\ &= \int_{\partial B_{r_1}} P_k^{u_n^0,0} \partial_\nu P_k^{u_n^0,0} \, \mathrm{d}\sigma = k r_1^{N+2k-2} \int_{\partial B_1} |P_k^{u_n^0}|^2 \, \mathrm{d}\sigma.\end{aligned}$$

Therefore Theorem 2.7 follows from Theorem 2.3 and Proposition 6.1. □

6.2 Dimension $N = 2$

In dimension $N = 2$, we can instead determine an asymptotic expansion of $\mathcal{T}_\varepsilon$ as $\varepsilon \to 0^+$. To this aim, we proceed as in [19, Section 6.2].

Let $N = 2$, $\Sigma = B_{r_1}$ for some $0 < r_1 < r_0$, $x_0 = 0 \in \Omega$, $u_n^0(0) = 0$ with $k \geq 1$ being the vanishing order of u_n^0 at 0. Let $P_k^{u_n^0,0}$ be as in (2.5).

For every $R > r_1$, $R \neq -k/\alpha$, and $\varepsilon > 0$ sufficiently small, the function $W_{\varepsilon,R} : B_R \setminus \varepsilon\overline{B_{r_1}} \to \mathbb{R}$, given in polar coordinates by

$$W_{\varepsilon,R}(r,\theta) = r_1^{2k} \frac{(\alpha R - k) r^k - (\alpha R + k) R^{2k} r^{-k}}{(\alpha R + k) R^{2k} + (\alpha R - k) r_1^{2k} \varepsilon^{2k}} P_k^{u_n^0,0}(\theta) \varepsilon^{2k}, \tag{6.1}$$

is well-defined. Moreover, $W_{\varepsilon,R} \in C^\infty(\overline{B_R \setminus \varepsilon B_{r_1}})$. Since the polynomial $P_k^{u_n^0,0}$ is harmonic and k-homogeneous, we have $-\Delta_{\mathbb{S}^1} P_k^{u_n^0,0} = k^2 P_k^{u_n^0,0}$ on $\mathbb{S}^1$; hence $W_{\varepsilon,R}$ is a classical solution to the problem

$$\begin{cases} -\Delta W_{\varepsilon,R} = 0, & \text{in } B_R \setminus \varepsilon\overline{B_{r_1}}, \\ \partial_\nu W_{\varepsilon,R} + \alpha W_{\varepsilon,R} = 0, & \text{on } \partial B_R, \\ \partial_\nu W_{\varepsilon,R} = \partial_\nu P_k^{u_n^0,0}, & \text{on } \varepsilon \partial B_{r_1}. \end{cases}$$

In particular, we also have $W_{\varepsilon,R} \in H^1(B_R \setminus \varepsilon\overline{B_{r_1}})$ and

$$\int_{B_R \setminus \varepsilon B_{r_1}} \nabla W_{\varepsilon,R} \cdot \nabla v \, \mathrm{d}x + \alpha \int_{\partial B_R} W_{\varepsilon,R} v \, \mathrm{d}\sigma = \int_{\partial(\varepsilon B_{r_1})} (\partial_\nu P_k^{u_n^0,0}) v \, \mathrm{d}\sigma \tag{6.2}$$

for every $v \in H^1(B_R \setminus \varepsilon\overline{B_{r_1}})$. Let us define

$$\mathcal{T}_\varepsilon^R := \int_{\partial(\varepsilon B_{r_1})} (\partial_\nu P_k^{u_n^0,0}) W_{\varepsilon,R} \, \mathrm{d}\sigma = \int_{B_R \setminus \varepsilon B_{r_1}} |\nabla W_{\varepsilon,R}|^2 \, \mathrm{d}x + \alpha \int_{\partial B_R} W_{\varepsilon,R}^2 \, \mathrm{d}\sigma. \tag{6.3}$$

Remark 6.2 We observe that, in view of (6.1),

$$W_{\varepsilon,R}(\varepsilon r_1, \theta) = -r_1^k P_k^{u_n^0,0}(\theta) \varepsilon^k + o(\varepsilon^k) \quad \text{as } \varepsilon \to 0^+, \tag{6.4}$$

uniformly with respect to $\theta \in \mathbb{S}^1$.

Proposition 6.3 *Let $u_n^0(0) = 0$ and $k \geq 1$ be the vanishing order of u_n^0 at 0. Then, for every $R > r_1$, $R \neq -k/\alpha$, we have*

$$\mathcal{T}_\varepsilon^R = k\pi r_1^{2k} \left(\left| \frac{\partial^k u_n^0}{\partial x_1^k}(0) \right|^2 + \frac{1}{k^2} \left| \frac{\partial^k u_n^0}{\partial x_1^{k-1} \partial x_2}(0) \right|^2 \right) \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0^+. \tag{6.5}$$

Furthermore,

$$\left\| W_{\varepsilon,R} \right\|_{L^2(B_R \setminus \varepsilon\overline{B_{r_1}})} = O(\varepsilon^{k+1}) \quad \text{as } \varepsilon \to 0^+, \tag{6.6}$$

$$\left\| W_{\varepsilon,R} \right\|_{H^1(B_R \setminus \varepsilon\overline{B_{r_1}})} = O(\varepsilon^k) \quad \text{as } \varepsilon \to 0^+. \tag{6.7}$$

Proof Since the polynomial $P_k^{u_n^0,0}$ is harmonic and k-homogeneous, we have necessarily

$$P_k^{u_n^0,0}(r\cos t, r\sin t) = r^k\left(\frac{\partial^k u_n^0}{\partial x_1^k}(0)\cos(kt) + \frac{1}{k}\frac{\partial^k u_n^0}{\partial x_1^{k-1}\partial x_2}(0)\sin(kt)\right).$$

By (6.4) and a change of variables

$$\begin{aligned}\int_{\partial B_{\varepsilon r_1}} \partial_\nu P_k^{u_n^0,0} W_{\varepsilon,R}\,\mathrm{d}\sigma &= -k\int_{\partial B_1} P_k^{u_n^0,0}(\varepsilon r_1\theta) W_{\varepsilon,R}(\varepsilon r_1\theta)\,\mathrm{d}\sigma\\ &= kr_1^{2k}\varepsilon^{2k}\int_{\partial B_1}|P_k^{u_n^0,0}(\theta)|^2\,\mathrm{d}\sigma + o(\varepsilon^{2k})\\ &= k\pi r_1^{2k}\varepsilon^{2k}\left(\left|\frac{\partial^k u_n^0}{\partial x_1^k}(0)\right|^2 + \frac{1}{k^2}\left|\frac{\partial^k u_n^0}{\partial x_1^{k-1}\partial x_2}(0)\right|^2\right) + o(\varepsilon^{2k})\end{aligned}$$

as $\varepsilon\to 0^+$, thus obtaining (6.5). Furthermore, (6.6) and (6.7) follow from (6.1). □

Let us now consider the problem as in Proposition 4.1, for the domain $\Omega = B_R$ and $\Sigma = B_{r_1}$ with $R > r_1$, replacing the boundary Robin datum u_n^0 with its approximation $P_k^{u_n^0,0}$. More precisely, let $Z_{\varepsilon,R}\in H^1(B_R\setminus\varepsilon\overline{B_{r_1}})$ be the unique solution of the problem

$$\begin{cases} -\Delta Z_{\varepsilon,R} + c_\alpha Z_{\varepsilon,R} = 0, & \text{in } B_R\setminus\varepsilon\overline{B_{r_1}},\\ \partial_\nu Z_{\varepsilon,R} + \alpha Z_{\varepsilon,R} = 0, & \text{on } \partial B_R,\\ \partial_\nu Z_{\varepsilon,R} + \alpha Z_{\varepsilon,R} = \partial_\nu P_k^{u_n^0,0} + \alpha P_k^{u_n^0,0}, & \text{on } \varepsilon\partial B_{r_1},\end{cases}$$

that is, for every $\varphi\in H^1(B_R\setminus\varepsilon\overline{B_{r_1}})$,

$$\begin{aligned}\int_{B_R\setminus B_{\varepsilon r_1}}(\nabla Z_{\varepsilon,R}\cdot\nabla\varphi + c_\alpha Z_{\varepsilon,R}\varphi)\,\mathrm{d}x + \alpha\int_{\partial(B_R\setminus B_{\varepsilon r_1})} Z_{\varepsilon,R}\,\varphi\,\mathrm{d}\sigma\\ = \int_{\partial B_{\varepsilon r_1}}\left(\partial_\nu P_k^{u_n^0,0} + \alpha P_k^{u_n^0,0}\right)\varphi\,\mathrm{d}\sigma. \quad (6.8)\end{aligned}$$

Let

$$\begin{aligned}\widetilde{\mathcal{T}}_{\varepsilon,R} &= \int_{\partial B_{\varepsilon r_1}}\left(\partial_\nu P_k^{u_n^0,0} + \alpha P_k^{u_n^0,0}\right) Z_{\varepsilon,R}\,\mathrm{d}\sigma\\ &= \int_{B_R\setminus B_{\varepsilon r_1}}(|\nabla Z_{\varepsilon,R}|^2 + c_\alpha Z_{\varepsilon,R}^2)\,\mathrm{d}x + \alpha\int_{\partial(B_R\setminus B_{\varepsilon r_1})} Z_{\varepsilon,R}^2\,\mathrm{d}\sigma. \quad (6.9)\end{aligned}$$

Arguing as in Proposition 4.3 and using (3.9), it is easy to verify that, for any $\delta\in(0,1)$,

$$\|Z_{\varepsilon,R}\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} = O(\varepsilon^{k-\frac{\delta}{2}}) \quad \text{as } \varepsilon\to 0^+, \quad (6.10)$$

where $k \geq 1$ is the vanishing order of u_n^0 at 0.

Proposition 6.4 *Let $N = 2$, $u_n^0(0) = 0$ and k be the vanishing order of u_n^0 at 0. For every $R > r_1$, $R \neq -k/\alpha$, and $\varepsilon > 0$ sufficiently small, let $\widetilde{\mathcal{T}}_{\varepsilon,R}$ and $\mathcal{T}_\varepsilon^R$ be as in (6.9) and (6.3), respectively. Then*

$$\widetilde{\mathcal{T}}_{\varepsilon,R} = \mathcal{T}_\varepsilon^R + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0^+. \tag{6.11}$$

Proof Let us choose $\varphi = W_{\varepsilon,R}$ in (6.8). By (6.3) we have

$$\int_{B_R \setminus B_{\varepsilon r_1}} (\nabla Z_{\varepsilon,R} \cdot \nabla W_{\varepsilon,R} + c_\alpha Z_{\varepsilon,R} W_{\varepsilon,R})\, \mathrm{d}x + \alpha \int_{\partial(B_R \setminus B_{\varepsilon r_1})} Z_{\varepsilon,R} W_{\varepsilon,R}\, \mathrm{d}\sigma - \alpha \int_{\partial B_{\varepsilon r_1}} P_k^{u_n^0,0} W_{\varepsilon,R}\, \mathrm{d}\sigma = \mathcal{T}_\varepsilon^R.$$

On the other hand, choosing $v = Z_{\varepsilon,R}$ in (6.2), in view of (6.9) we obtain

$$\int_{B_R \setminus B_{\varepsilon r_1}} \nabla W_{\varepsilon,R} \cdot \nabla Z_{\varepsilon,R}\, \mathrm{d}x + \alpha \int_{\partial B_R} W_{\varepsilon,R} Z_{\varepsilon,R}\, \mathrm{d}\sigma + \alpha \int_{\partial B_{\varepsilon r_1}} P_k^{u_n^0,0} Z_{\varepsilon,R}\, \mathrm{d}\sigma = \widetilde{\mathcal{T}}_{\varepsilon,R}.$$

Hence

$$\widetilde{\mathcal{T}}_{\varepsilon,R} - \mathcal{T}_\varepsilon^R = \alpha \int_{\partial B_{\varepsilon r_1}} P_k^{u_n^0,0} \left(Z_{\varepsilon,R} + W_{\varepsilon,R}\right) \mathrm{d}\sigma - \alpha \int_{\partial B_{\varepsilon r_1}} Z_{\varepsilon,R} W_{\varepsilon,R}\, \mathrm{d}\sigma - c_\alpha \int_{B_R \setminus B_{\varepsilon r_1}} Z_{\varepsilon,R} W_{\varepsilon,R}\, \mathrm{d}x.$$

By the Cauchy–Schwarz inequality, (3.5), (6.7) and (6.10), there exists a constant $C > 0$, which does not depend on ε, such that

$$\left| \int_{\partial B_{\varepsilon r_1}} P_k^{u_n^0,0} \left(Z_{\varepsilon,R} + W_{\varepsilon,R}\right) \mathrm{d}\sigma \right| \leq C\varepsilon^{k+\frac{1}{2}} \left(\int_{\partial B_{\varepsilon r_1}} \left(Z_{\varepsilon,R} + W_{\varepsilon,R}\right)^2 \mathrm{d}\sigma \right)^{\frac{1}{2}}$$
$$\leq C\sqrt{C_\delta}\varepsilon^{k+1-\frac{\delta}{2}} \left(\|Z_{\varepsilon,R}\|_{H^1(B_R \setminus \varepsilon\overline{B_{r_1}})} + \|W_{\varepsilon,R}\|_{H^1(B_R \setminus \varepsilon\overline{B_{r_1}})} \right) = O(\varepsilon^{2k+1-\delta}) \quad \text{as } \varepsilon \to 0^+,$$

for any $\delta \in (0, 1)$. Similarly, by the Cauchy–Schwarz inequality, (3.5), (6.7) and (6.10),

$$\int_{\partial B_{\varepsilon r_1}} Z_{\varepsilon,R} W_{\varepsilon,R}\, \mathrm{d}\sigma = O(\varepsilon^{2k+1-\frac{3}{2}\delta}) \quad \text{as } \varepsilon \to 0^+,$$

for any $\delta \in (0, 1)$. Furthermore, by the Cauchy–Schwarz inequality, (6.6) and (6.10),

$$\int_{B_R \setminus B_{\varepsilon r_1}} Z_{\varepsilon,R} W_{\varepsilon,R}\, \mathrm{d}x = O(\varepsilon^{2k+1-\frac{\delta}{2}}) \quad \text{as } \varepsilon \to 0^+,$$

for any $\delta \in (0, 1)$. Choosing $\delta < \frac{2}{3}$, we thereby prove (6.11). □

Let us now consider the problem as in Proposition 4.1 for $\Omega = B_R$ and $\Sigma = B_{r_1}$, for some $R > r_1$. Let $V_{\varepsilon,R} \in H^1(B_R \setminus \varepsilon\overline{B_{r_1}})$ be the unique solution of the problem

$$\begin{cases} -\Delta V_{\varepsilon,R} + c_\alpha V_{\varepsilon,R} = 0, & \text{in } B_R \setminus \varepsilon\overline{B_{r_1}}, \\ \partial_\nu V_{\varepsilon,R} + \alpha V_{\varepsilon,R} = 0, & \text{on } \partial B_R, \\ \partial_\nu V_{\varepsilon,R} + \alpha V_{\varepsilon,R} = \partial_\nu u_n^0 + \alpha u_n^0, & \text{on } \varepsilon\partial B_{r_1}, \end{cases}$$

that is, for every $\varphi \in H^1(B_R \setminus \varepsilon\overline{B_{r_1}})$,

$$\int_{B_R\setminus B_{\varepsilon r_1}} (\nabla V_{\varepsilon,R}\cdot\nabla\varphi + c_\alpha V_{\varepsilon,R}\varphi)\,\mathrm{d}x + \alpha\int_{\partial(B_R\setminus B_{\varepsilon r_1})} V_{\varepsilon,R}\,\varphi\,\mathrm{d}\sigma = \int_{\partial B_{\varepsilon r_1}} \left(\partial_\nu u_n^0 + \alpha u_n^0\right)\varphi\,\mathrm{d}\sigma.$$

Also, we denote the corresponding torsional rigidity as $\mathcal{T}_{\varepsilon,R}$, so that

$$\begin{aligned} \mathcal{T}_{\varepsilon,R} &= \int_{\partial B_{\varepsilon r_1}} \left(\partial_\nu u_n^0 + \alpha u_n^0\right) V_{\varepsilon,R}\,\mathrm{d}\sigma \\ &= \int_{B_R\setminus B_{\varepsilon r_1}} (|\nabla V_{\varepsilon,R}|^2 + c_\alpha V_{\varepsilon,R}^2)\,\mathrm{d}x + \alpha\int_{\partial(B_R\setminus B_{\varepsilon r_1})} V_{\varepsilon,R}^2\,\mathrm{d}\sigma. \end{aligned} \tag{6.12}$$

Proposition 6.5 *Let $N = 2$, $u_n^0(0) = 0$, and $k \geq 1$ be the vanishing order of u_n^0 at 0. For every $R > r_1$, we have*

$$\mathcal{T}_{\varepsilon,R} = \widetilde{\mathcal{T}}_{\varepsilon,R} + O(\varepsilon^{2k+1-\frac{\delta}{2}}) \quad \text{as } \varepsilon \to 0^+. \tag{6.13}$$

Proof Let $P_k^{u_n^0,0}$, $P_{k+1}^{u_n^0,0}$ be as in (2.5). Let us denote with $Y_{\varepsilon,R}$ the unique weak solution of the problem

$$\begin{cases} -\Delta Y_{\varepsilon,R} + c_\alpha Y_{\varepsilon,R} = 0, & \text{in } B_R \setminus \varepsilon\overline{B_{r_1}}, \\ \partial_\nu Y_{\varepsilon,R} + \alpha Y_{\varepsilon,R} = 0, & \text{on } \partial B_R, \\ \partial_\nu Y_{\varepsilon,R} + \alpha Y_{\varepsilon,R} = \partial_\nu P_{k+1}^{u_n^0,0} + \alpha P_{k+1}^{u_n^0,0}, & \text{on } \varepsilon\partial B_{r_1}. \end{cases}$$

Let $\Psi_\varepsilon = V_{\varepsilon,R} - Z_{\varepsilon,R} - Y_{\varepsilon,R} \in H^1(B_R\setminus\varepsilon\overline{B_{r_1}})$. Then, for every $\varphi \in H^1(B_R \setminus \varepsilon\overline{B_{r_1}})$,

$$\begin{aligned} &\int_{B_R\setminus B_{\varepsilon r_1}} (\nabla\Psi_\varepsilon\cdot\nabla\varphi + c_\alpha\Psi_\varepsilon\varphi)\,\mathrm{d}x + \alpha\int_{\partial(B_R\setminus B_{\varepsilon r_1})} \Psi_\varepsilon\,\varphi\,\mathrm{d}\sigma \\ &\qquad = \int_{\partial B_{\varepsilon r_1}} \left(\partial_\nu\big(u_n^0 - P_k^{u_n^0,0} - P_{k+1}^{u_n^0,0}\big) + \alpha\big(u_n^0 - P_k^{u_n^0,0} - P_{k+1}^{u_n^0,0}\big)\right)\varphi\,\mathrm{d}\sigma. \end{aligned}$$

In particular, testing with $\varphi = \Psi_\varepsilon$, we have

$$C_\alpha^{-1}\|\Psi_\varepsilon\|^2_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} \le \int_{B_R\setminus B_{\varepsilon r_1}} (|\nabla\Psi_\varepsilon|^2 + c_\alpha\Psi_\varepsilon^2)\,\mathrm{d}x + \alpha\int_{\partial(B_R\setminus B_{\varepsilon r_1})}\Psi_\varepsilon^2\,\mathrm{d}\sigma$$
$$= \int_{\partial B_{\varepsilon r_1}}\left(\partial_\nu\big(u_n^0 - P_k^{u_n^0,0} - P_{k+1}^{u_n^0,0}\big) + \alpha\big(u_n^0 - P_k^{u_n^0,0} - P_{k+1}^{u_n^0,0}\big)\right)\Psi_\varepsilon\,\mathrm{d}\sigma$$

by (3.9). Thanks to the Cauchy–Schwarz inequality, the Taylor expansion of u_n^0 at 0, and (3.5), there exists a constant $C > 0$, independent of ε, such that

$$\int_{\partial B_{\varepsilon r_1}}\left(\partial_\nu\big(u_n^0 - P_k^{u_n^0,0} - P_{k+1}^{u_n^0,0}\big) + \alpha\big(u_n^0 - P_k^{u_n^0,0} - P_{k+1}^{u_n^0,0}\big)\right)\Psi_\varepsilon\,\mathrm{d}\sigma$$
$$\le C\varepsilon^{k+\frac{3}{2}}\left(\int_{\partial B_{\varepsilon r_1}}\Psi_\varepsilon^2\,\mathrm{d}\sigma\right)^{\frac{1}{2}} \le C\sqrt{C_\delta}\varepsilon^{k+2-\frac{\delta}{2}}\|\Psi_\varepsilon\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})}$$

for any $\delta \in (0,1)$. Hence,

$$\|\Psi_\varepsilon\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} = O(\varepsilon^{k+2-\frac{\delta}{2}}) \quad \text{as } \varepsilon \to 0^+. \tag{6.14}$$

Arguing as in Proposition 4.3, by (3.9) we obtain

$$\|Y_{\varepsilon,R}\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} = O(\varepsilon^{k+1-\frac{\delta}{2}}) \quad \text{as } \varepsilon \to 0^+, \tag{6.15}$$

for any $\delta \in (0,1)$. By (3.9) and (6.9), we have

$$\|Z_{\varepsilon,R}\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} = O(\widetilde{\mathcal{T}}_{\varepsilon,R}^{1/2}) \quad \text{as } \varepsilon \to 0^+.$$

Hence, by (6.5) and Proposition 6.4, it follows that

$$\|Z_{\varepsilon,R}\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} = O(\varepsilon^k) \quad \text{as } \varepsilon \to 0^+. \tag{6.16}$$

Therefore, by (6.14), (6.15) and (6.16), we have

$$\|V_{\varepsilon,R}\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} = O(\varepsilon^k) \quad \text{as } \varepsilon \to 0^+. \tag{6.17}$$

Furthermore, by the Cauchy–Schwarz inequality

$$\|V_{\varepsilon,R} - Z_{\varepsilon,R}\|^2_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} - \|Y_{\varepsilon,R}\|^2_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})}$$
$$= \big(\Psi_\varepsilon, V_{\varepsilon,R} - Z_{\varepsilon,R} + Y_{\varepsilon,R}\big)_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})}$$
$$\le \|\Psi_\varepsilon\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})}\left(\|V_{\varepsilon,R}\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} + \|Z_{\varepsilon,R}\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} + \|Y_{\varepsilon,R},\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})}\right),$$

so that, in view of (6.14), (6.15), (6.16), and (6.17),

$$\|V_{\varepsilon,R} - Z_{\varepsilon,R}\|^2_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} - \|Y_{\varepsilon,R}\|^2_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} = O(\varepsilon^{2k+2-\frac{\delta}{2}}) \quad \text{as } \varepsilon \to 0^+, \tag{6.18}$$

for any $\delta \in (0, 1)$. From (6.15) and (6.18) it follows that

$$\|V_{\varepsilon,R} - Z_{\varepsilon,R}\|_{H^1(B_R\setminus\varepsilon\overline{B_{r_1}})} = O(\varepsilon^{k+1-\frac{\delta}{2}}) \quad \text{as } \varepsilon \to 0^+, \tag{6.19}$$

for any $\delta \in (0, 1)$.

By the Cauchy–Schwarz inequality for q_ε, (3.10) for $\Omega = B_R$, (6.19), (6.16), and (6.17)

$$\begin{aligned} q_\varepsilon(V_{\varepsilon,R} - Z_{\varepsilon,R}, V_{\varepsilon,R} + Z_{\varepsilon,R}) &\le \sqrt{q_\varepsilon(V_{\varepsilon,R} - Z_{\varepsilon,R}, V_{\varepsilon,R} - Z_{\varepsilon,R})}\sqrt{q_\varepsilon(V_{\varepsilon,R} + Z_{\varepsilon,R}, V_{\varepsilon,R} + Z_{\varepsilon,R})} \\ &= O(\varepsilon^{2k+1-\frac{\delta}{2}}) \quad \text{as } \varepsilon \to 0^+, \end{aligned}$$

for any $\delta \in (0, 1)$, where q_ε is as in (3.11) for the domain $B_R \setminus \varepsilon\overline{B_{r_1}}$. Hence, by (6.9) and (6.12),

$$|\widetilde{\mathcal{T}}_{\varepsilon,R} - \mathcal{T}_{\varepsilon,R}| = |q_\varepsilon(Z_{\varepsilon,R}, Z_{\varepsilon,R}) - q_\varepsilon(V_{\varepsilon,R}, V_{\varepsilon,R})| = |q_\varepsilon(V_{\varepsilon,R} - Z_{\varepsilon,R}, V_{\varepsilon,R} + Z_{\varepsilon,R})| = O(\varepsilon^{2k+1-\frac{\delta}{2}})$$

as $\varepsilon \to 0^+$, for any $\delta \in (0, 1)$, thus proving (6.13). □

In order to complete the proof of Theorem 2.8, we need the following general lemma concerning the comparison of quadratic forms with boundary terms. For any open and connected Lipschitz set $\Lambda \subset \mathbb{R}^N$ and any constant $C > 0$, we define

$$q_{\Lambda,C} : H^1(\Lambda) \to \mathbb{R}, \quad q_{\Lambda,C}(u) := \int_\Lambda \left(|\nabla u|^2 + Cu^2\right) \mathrm{d}x + \alpha \int_{\partial\Lambda} u^2 \,\mathrm{d}\sigma.$$

Lemma 6.6 *Let Λ_1, Λ_2 be open and connected Lipschitz sets in $\mathbb{R}^N$ such that $\overline{\Lambda_1} \subset \Lambda_2$. For every $\alpha > 0$ there exists a constant $d = d(\alpha, \Lambda_1, \Lambda_2) > 0$, which depends only on Λ_1, Λ_2, and α, such that*

$$q_{\Lambda_1,C}(u) \le q_{\Lambda_2,C}(u) \tag{6.20}$$

for every $u \in H^1(\Lambda_2)$ and $C \ge d$.

Proof Inequality (6.20) can be rewritten as

$$\int_{\Lambda_2\setminus\Lambda_1} \left(|\nabla u|^2 + Cu^2\right) \mathrm{d}x + \alpha \int_{\partial\Lambda_2} u^2 \,\mathrm{d}\sigma - \alpha \int_{\partial\Lambda_1} u^2 \,\mathrm{d}\sigma \ge 0$$

for any $u \in H^1(\Lambda_2)$. Hence, to conclude it is enough to show that there exists a constant $d > 0$, depending only on Λ_1, Λ_2, and α, such that

$$\int_{\Lambda_2\setminus\Lambda_1} \left(|\nabla u|^2 + du^2\right) \mathrm{d}x \geq |\alpha| \left(\int_{\partial\Lambda_2} u^2 \,\mathrm{d}\sigma + \int_{\partial\Lambda_1} u^2 \,\mathrm{d}\sigma\right)$$

for every $u \in H^1(\Lambda_2)$. The above inequality holds in view of [32, Theorem 18.1]. □

Proposition 6.7 *Let $N = 2$, $u_n^0(0) = 0$ and k be the vanishing order of u_n^0 at 0. Then*

$$\mathcal{T}_\varepsilon = k\pi r_1^{2k} \left(\left| \frac{\partial^k u_n^0}{\partial x_1^k}(0) \right|^2 + \frac{1}{k^2} \left| \frac{\partial^k u_n^0}{\partial x_1^{k-1}\partial x_2}(0) \right|^2 \right) \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0^+. \tag{6.21}$$

Proof For any $R > r_1$ with $R \neq -k/\alpha$, by Propositions 6.5 and 6.4 we have

$$\mathcal{T}_{\varepsilon,R} = \widetilde{\mathcal{T}}_{\varepsilon,R} + O(\varepsilon^{2k+1-\frac{\delta}{2}}) = \mathcal{T}_\varepsilon^R + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0^+.$$

Hence, by Proposition 6.3, for any $R > r_1$ with $R \neq -k/\alpha$ we have

$$\mathcal{T}_{\varepsilon,R} = k\pi r_1^{2k} \left(\left| \frac{\partial^k u_n^0}{\partial x_1^k}(0) \right|^2 + \frac{1}{k^2} \left| \frac{\partial^k u_n^0}{\partial x_1^{k-1}\partial x_2}(0) \right|^2 \right) \varepsilon^{2k} + o(\varepsilon^{2k}) \quad \text{as } \varepsilon \to 0^+. \tag{6.22}$$

Let $R_2 > R_1 > r_0 > r_1$ be such that $\overline{B_{R_1}} \subset \Omega$ and $\overline{\Omega} \subset B_{R_2}$ and $R_1, R_2 \neq -k/\alpha$. Using the characterization of $\mathcal{T}_{\varepsilon,R_2}, \mathcal{T}_{\varepsilon,R_1}, \mathcal{T}_\varepsilon$ given in (4.4) and choosing c_α larger than the constant d given by Lemma 6.6 applied first with $\Lambda_1 = B_{R_1}$ and $\Lambda_2 = \Omega$, and then with $\Lambda_1 = \Omega$ and $\Lambda_2 = B_{R_2}$, we obtain that

$$\mathcal{T}_{\varepsilon,R_2} \leq \mathcal{T}_\varepsilon \leq \mathcal{T}_{\varepsilon,R_1}.$$

Estimate (6.21) finally follows from (6.22) by comparison. □

Proof of Theorem 2.8 Without loss of generality, we assume $x_0 = 0$. We can proceed as in the proof of Theorem 2.7 to obtain that

$$\int_{B_{\varepsilon r_1}} |\nabla P_k^{u_n^0,0}|^2 \,\mathrm{d}x = k r_1^{2k} \varepsilon^{2k} \int_{\partial B_1} |P_k^{u_n^0,0}|^2 \,\mathrm{d}\sigma.$$

Then Theorem 2.8 follows from Theorem 4.5, Proposition 4.6, and Proposition 6.7. □

Appendix A.

In this appendix we present a simplified version of [11, Theorem 3.1], adapted to our case, specifically the removal of small sets in some bounded domain $\Omega \subset \mathbb{R}^N$. An alternative proof of this can also be obtained quite directly by applying [19, Lemma A.1], which revisits the well-known *Lemma on small eigenvalues* by Y. Colin de Verdière [15] in the context of simple eigenvalues.

Theorem A.1 *Let $\Omega \subset \mathbb{R}^N$ be a bounded open Lipschitz set. Let $\{K_\varepsilon\}_{\varepsilon\in[0,1]}$ be a family of compact sets such that*

$$K_\varepsilon \subset \Omega \ \text{ for every } \varepsilon \in [0,1], \quad K_0 = \emptyset, \quad \lim_{\varepsilon\to 0^+} |K_\varepsilon|_N = 0,$$

and

$$\text{the embedding } H^1(\Omega_\varepsilon) \subset L^2(\Omega_\varepsilon) \text{ is compact for every } \varepsilon \in [0,1], \tag{A.1}$$

where $\Omega_\varepsilon := \Omega \setminus K_\varepsilon$. For every $\varepsilon \in [0,1]$, let $q_\varepsilon : H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \to \mathbb{R}$ be a symmetric and continuous bilinear form such that

$$q_\varepsilon(v,v) \geq \eta \, \|v\|^2_{H^1(\Omega_\varepsilon)} \text{ for all } v \in H^1(\Omega_\varepsilon),$$

for some constant $\eta > 0$ that is independent of $\varepsilon \in [0,1]$. Letting $\{\rho_j^\varepsilon\}_{j\in\mathbb{N}\setminus\{0\}}$ be the eigenvalues of q_ε, repeated according to their multiplicity, we assume that

$$\lim_{\varepsilon\to 0} \rho_j^\varepsilon = \rho_j^0$$

for every $j \in \mathbb{N} \setminus \{0\}$. Suppose that, for some $n \in \mathbb{N} \setminus \{0\}$,

$$\rho_n^0 \text{ is simple},$$

and let u_n^0 be an associated eigenfunction such that $\int_\Omega |u_n^0|^2 \, \mathrm{d}x = 1$. For every $\varepsilon \in (0,1]$, let

$$L_\varepsilon : H^1(\Omega_\varepsilon) \to \mathbb{R}, \quad L_\varepsilon(v) := q_\varepsilon(u_n^0, v) - \rho_n^0 (u_n^0, v)_{L^2(\Omega_\varepsilon)}$$

and

$$J_\varepsilon : H^1(\Omega_\varepsilon) \to \mathbb{R}, \quad J_\varepsilon(v) := \frac{1}{2} q_\varepsilon(v,v) - L_\varepsilon(v).$$

Assume that

$$\lim_{\varepsilon\to 0^+} q_\varepsilon(V_\varepsilon, V_\varepsilon) = 0,$$

where V_ε is the unique solution to the minimization problem

$$J_\varepsilon(V_\varepsilon) = \min_{v\in H^1(\Omega_\varepsilon)} J_\varepsilon(v).$$

Then

$$\rho_n^\varepsilon - \rho_n^0 = \frac{\rho_n^0 \int_{\Omega_\varepsilon} V_\varepsilon u_n^0 \,\mathrm{d}x + O(\|V_\varepsilon\|^2_{L^2(\Omega_\varepsilon)})}{\int_{\Omega_\varepsilon} |u_n^0|^2 \,\mathrm{d}x + O(\|V_\varepsilon\|_{L^2(\Omega_\varepsilon)})} \quad \text{as } \varepsilon \to 0^+.$$

Finally, as $\varepsilon \to 0^+$,

$$\begin{aligned}
&q_\varepsilon\big(u_n^0 - V_\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon), u_n^0 - V_\varepsilon - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\big) = O\big(\|V_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\big),\\
&q_\varepsilon(u_n^0 - \Pi_\varepsilon(u_n^0 - V_\varepsilon), u_n^0 - \Pi_\varepsilon(u_n^0 - V_\varepsilon)) = q_\varepsilon(V_\varepsilon, V_\varepsilon) + O\big(\|V_\varepsilon\|_{L^2(\Omega_\varepsilon)} (q_\varepsilon(V_\varepsilon, V_\varepsilon))^{\frac{1}{2}}\big),\\
&\|u_n^0 - \Pi_\varepsilon(u_n^0 - V_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} = O\big(\|V_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}\big),
\end{aligned}$$

where, for every $\varepsilon \in (0, 1]$, $\Pi_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon)$ *is the projection map onto the eigenspace of* q_ε *associated to the eigenvalue* ρ_n^ε.

We observe that condition (A.1) is satisfied if K_ε is the closure of an open Lipschitz set; in particular, it holds for $K_\varepsilon = \overline{\Sigma}_\varepsilon$, with Σ_ε satisfying (1.3) for some Lipschitz bounded open set Σ.

Appendix B.

In this appendix, we present a proof of Proposition 3.1. The following lemma states that the Sobolev extension operator from a connected domain into an interior hole undergoes an estimate not only of the complete H^1-norms, but also of the L^2-norms of gradients.

Lemma B.1 *Let* $\Sigma \subset \mathbb{R}^N$ *be an open, bounded, and Lipschitz set such that*

$$\overline{\Sigma} \subset B_{r_0} \quad \text{and} \quad B_{r_0} \setminus \overline{\Sigma} \text{ is connected},$$

for some $r_0 > 0$. *There exist a linear bounded extension operator*

$$\mathsf{E} : H^1(B_{r_0} \setminus \overline{\Sigma}) \to H^1(B_{r_0})$$

and a constant $C > 0$ *such that, for every* $u \in H^1(B_{r_0} \setminus \overline{\Sigma})$,

$$(\mathsf{E}u)\big|_{B_{r_0}\setminus\overline{\Sigma}} = u, \tag{B.1}$$

$$\|\mathsf{E}u\|^2_{H^1(B_{r_0})} \le C\|u\|^2_{H^1(B_{r_0}\setminus\overline{\Sigma})}, \tag{B.2}$$

$$\|\nabla(\mathsf{E}u)\|^2_{L^2(B_{r_0})} \le C\|\nabla u\|^2_{L^2(B_{r_0}\setminus\overline{\Sigma})}. \tag{B.3}$$

Proof Let $\mathsf{E} : H^1(B_{r_0}\setminus\overline{\Sigma}) \to H^1(B_{r_0})$ be the linear and bounded extension operator defined as

$$\mathsf{E}u = \begin{cases} u, & \text{in } B_{r_0} \setminus \overline{\Sigma},\\ v, & \text{in } \Sigma, \end{cases}$$

where $v \in H^1(\Sigma)$ is the unique harmonic function inside Σ whose trace on $\partial\Sigma$ coincides with the trace of u. In particular, E satisfies (B.1) and (B.2). Moreover, $\mathsf{E}u \equiv \gamma$ in B_{r_0} if $u \equiv \gamma$ in $B_{r_0}\setminus\overline{\Sigma}$ for some constant $\gamma \in \mathbb{R}$.

Assume by contradiction that (B.3) does not hold for any choice of the constant C. Then there exists a sequence $\{u_n\}_{n\in\mathbb{N}\setminus\{0\}} \subset H^1(B_{r_0}\setminus\overline{\Sigma})$ such that

$$\|\nabla(\mathsf{E}u_n)\|^2_{L^2(B_{r_0})} > n\|\nabla u_n\|^2_{L^2(B_{r_0}\setminus\overline{\Sigma})} \quad \text{for all } n \in \mathbb{N}\setminus\{0\}.$$

Let us define

$$w_n := \frac{u_n}{\|\nabla(\mathsf{E}u_n)\|_{L^2(B_{r_0})}}.$$

Then

$$\|\nabla w_n\|^2_{L^2(B_{r_0}\setminus\overline{\Sigma})} < \frac{1}{n},$$

while

$$\|\nabla(\mathsf{E}w_n)\|_{L^2(B_{r_0})} = 1$$

for every $n \in \mathbb{N}\setminus\{0\}$. By connectedness of $B_{r_0}\setminus\overline{\Sigma}$ and the Poincaré-Wirtinger inequality, see for example [32, Theorem 13.27],

$$\int_{B_{r_0}\setminus\overline{\Sigma}} |w_n - \gamma_n|^2\,\mathrm{d}x \le \kappa \int_{B_{r_0}\setminus\overline{\Sigma}} |\nabla w_n|^2\,\mathrm{d}x,$$

for some $\kappa > 0$ depending only on B_{r_0} and Σ, where

$$\gamma_n := \frac{1}{|B_{r_0}\setminus\overline{\Sigma}|_N}\int_{B_{r_0}\setminus\overline{\Sigma}} w_n\,\mathrm{d}x.$$

Since $\nabla(\mathsf{E}\gamma_n) \equiv 0$ in B_{r_0}, it follows that, for every $n \in \mathbb{N}\setminus\{0\}$,

$$\begin{aligned} 1 = \|\nabla(\mathsf{E}w_n)\|^2_{L^2(B_{r_0})} &= \|\nabla(\mathsf{E}(w_n-\gamma_n))\|^2_{L^2(B_{r_0})} \le \|\mathsf{E}(w_n-\gamma_n)\|^2_{H^1(B_{r_0})} \\ &\le C\|w_n-\gamma_n\|^2_{H^1(B_{r_0}\setminus\overline{\Sigma})} \le C(1+\kappa)\int_{B_{r_0}\setminus\overline{\Sigma}} |\nabla w_n|^2\,\mathrm{d}x \le \frac{C(1+\kappa)}{n}, \end{aligned}$$

a contradiction. □

Proof of Proposition 3.1 Thanks to an easy translation argument, it is not restrictive to assume that $x_0 = 0$, so that $\overline{B_{r_0}} \subset \Omega$, $\overline{\Sigma} \subset B_{r_0}$, and $\Sigma_\varepsilon = \varepsilon\Sigma$. For every $\varepsilon \in (0,1]$ and $u \in H^1(\Omega\setminus\overline{\Sigma}_\varepsilon)$ we define

$$\mathsf{E}_\varepsilon u(x) = \begin{cases} \mathsf{E}\big(u(\varepsilon\cdot)\big|_{B_{r_0}\setminus\overline{\Sigma}}\big)(x/\varepsilon), & \text{if } x \in \varepsilon B_{r_0}, \\ u(x), & \text{if } x \in \Omega\setminus\varepsilon\overline{B}_{r_0}, \end{cases}$$

where $\mathsf{E} : H^1(B_{r_0}\setminus\overline{\Sigma}) \to H^1(B_{r_0})$ is the extension operator provided in Lemma B.1. Hence, for every $\varepsilon \in (0,1]$, $\mathsf{E}_\varepsilon : H^1(\Omega\setminus\overline{\Sigma}_\varepsilon) \to H^1(\Omega)$ is a well-defined linear

operator satisfying (3.1) in view of (B.1). For every $u \in H^1(\Omega \setminus \overline{\Sigma}_\varepsilon)$, two changes of variables and (B.2) yield

$$\begin{aligned}
\|\mathsf{E}_\varepsilon u\|^2_{L^2(\Omega)} &= \int_{\Omega\setminus\varepsilon\overline{B_{r_0}}} |u|^2 \,\mathrm{d}x + \int_{\varepsilon B_{r_0}} |\mathsf{E}_\varepsilon u|^2 \,\mathrm{d}x \\
&= \int_{\Omega\setminus\varepsilon\overline{B_{r_0}}} |u|^2 \,\mathrm{d}x + \varepsilon^N \int_{B_{r_0}} |\mathsf{E}u_\varepsilon|^2 \,\mathrm{d}x \\
&\le \int_{\Omega\setminus\varepsilon\overline{B_{r_0}}} |u|^2 \,\mathrm{d}x + \varepsilon^N C \int_{B_{r_0}\setminus\overline{\Sigma}} \left(|\nabla u_\varepsilon|^2 + u_\varepsilon^2\right) \mathrm{d}x \\
&= \int_{\Omega\setminus\varepsilon\overline{B_{r_0}}} |u|^2 \,\mathrm{d}x + C \int_{\varepsilon B_{r_0}\setminus\overline{\Sigma_\varepsilon}} \left(\varepsilon^2|\nabla u|^2 + u^2\right) \mathrm{d}x \le (C+1)\|u\|^2_{H^1(\Omega\setminus\overline{\Sigma_\varepsilon})}
\end{aligned} \tag{B.4}$$

where $u_\varepsilon(x) := u(\varepsilon x)$. Furthermore, by two change of variables and (B.3),

$$\begin{aligned}
\|\nabla(\mathsf{E}_\varepsilon u)\|^2_{L^2(\Omega)} &= \int_{\Omega\setminus\varepsilon\overline{B_{r_0}}} |\nabla u|^2 \,\mathrm{d}x + \int_{\varepsilon B_{r_0}} |\nabla(\mathsf{E}_\varepsilon u)|^2 \,\mathrm{d}x \\
&= \int_{\Omega\setminus\varepsilon\overline{B_{r_0}}} |\nabla u|^2 \,\mathrm{d}x + \varepsilon^{N-2} \int_{B_{r_0}} |\nabla(\mathsf{E}u_\varepsilon)|^2 \,\mathrm{d}x \\
&\le \int_{\Omega\setminus\varepsilon\overline{B_{r_0}}} |\nabla u|^2 \,\mathrm{d}x + C\varepsilon^{N-2} \int_{B_{r_0}\setminus\overline{\Sigma}} |\nabla u_\varepsilon|^2 \,\mathrm{d}x \\
&= \int_{\Omega\setminus\varepsilon\overline{B_{r_0}}} |\nabla u|^2 \,\mathrm{d}x + C \int_{\varepsilon B_{r_0}\setminus\overline{\Sigma_\varepsilon}} |\nabla u|^2 \,\mathrm{d}x \le (C+1)\|\nabla u\|^2_{L^2(\Omega\setminus\overline{\Sigma_\varepsilon})}.
\end{aligned} \tag{B.5}$$

The uniform estimate (3.2) then follows from (B.4)–(B.5). □

Remark B.2 The assumption that the set $B_{r_0} \setminus \overline{\Sigma}$ is connected is necessary for the validity of Proposition 3.1, as shown in the following example. Let $x_0 = 0 \in \Omega$ and $r_0 > 0$ be such that $\overline{B_{r_0}} \subset \Omega$. Let us consider an annular hole $\Sigma := B_{r_0/2} \setminus \overline{B_{r_0/4}}$, which clearly violates the connectedness condition of $B_{r_0} \setminus \overline{\Sigma}$. Let us assume, for the sake of contradiction, that there exists a family of uniformly bounded extension operators $\mathsf{E}_\varepsilon : H^1(\Omega \setminus \overline{\Sigma}_\varepsilon) \to H^1(\Omega)$ satisfying (3.1) and (3.2) for every $\varepsilon \in (0, 1]$, where $\Sigma_\varepsilon = \varepsilon\Sigma$. Let

$$u_\varepsilon(x) := \begin{cases} 1, & \text{if } x \in B_{r_0\varepsilon/4}, \\ 0, & \text{if } x \in \Omega \setminus \overline{B_{r_0\varepsilon/2}}. \end{cases}$$

Then, letting ω_N be the Lebesgue measure of the N-dimensional unit ball,

$$\begin{aligned}\frac{r_0^N\omega_N}{4^N}\varepsilon^N &= |B_{r_0\varepsilon/4}|_N \\ &= \|u_\varepsilon\|^2_{H^1(\Omega\setminus\overline{\Sigma}_\varepsilon)} \geq \frac{1}{K}\int_{B_{r_0\varepsilon/2}} |\nabla(\mathsf{E}_\varepsilon u_\varepsilon)|^2\,\mathrm{d}x \geq \frac{1}{K}\operatorname{Cap}_{B_{r_0\varepsilon/2}}\left(\overline{B_{r_0\varepsilon/4}}\right),\end{aligned} \tag{B.6}$$

for every $\varepsilon \in (0,1]$, where $\operatorname{Cap}_{B_{r_0\varepsilon/2}}\left(\overline{B_{r_0\varepsilon/4}}\right)$ is the classical condenser capacity of $\overline{B_{r_0\varepsilon/4}}$ in $B_{r_0\varepsilon/2}$. A direct computation yields

$$\operatorname*{Cap}_{B_{r_0\varepsilon/2}}\left(\overline{B_{r_0\varepsilon/4}}\right) = \begin{cases} \dfrac{N(N-2)\omega_N r_0^{N-2}}{4^{N-2}-2^{N-2}}\,\varepsilon^{N-2}, & \text{if } N \geq 3, \\ \dfrac{2\pi}{\log 2}, & \text{if } N = 2, \end{cases}$$

thus contradicting (B.6).

Acknowledgements The authors are supported by the MUR-PRIN project no. 20227HX33Z "Pattern formation in nonlinear phenomena" granted by the European Union - Next Generation EU. V. Felli and G. Siclari are partially supported by the 2026 GNAMPA-INdAM project n. CUP E53C25002010001 "Asymptotic analysis of variational problems".

Funding Open access funding provided by Università degli Studi di Milano - Bicocca within the CRUI-CARE Agreement.

Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Abatangelo, L., Felli, V., Hillairet, L., Léna, C.: Spectral stability under removal of small capacity sets and applications to Aharonov–Bohm operators. J. Spectr. Theory **9**(2), 379–427 (2019)
2. Abatangelo, L., Felli, V., Noris, B.: On simple eigenvalues of the fractional Laplacian under removal of small fractional capacity sets. Commun. Contemp. Math. **22**(8), 1950071 (2020)

3. Abatangelo, L., Felli, V., Terracini, S.: On the sharp effect of attaching a thin handle on the spectral rate of convergence. J. Funct. Anal. **266**(6), 3632–3684 (2014)
4. Abatangelo, L., Léna, C., Musolino, P.: Ramification of multiple eigenvalues for the Dirichlet–Laplacian in perforated domains. J. Funct. Anal. **283**(12), 109718 (2022)
5. Abatangelo, L., Léna, C., Musolino, P.: Asymptotic behavior of generalized capacities with applications to eigenvalue perturbations: the higher dimensional case. Nonlinear Anal. **238**, 34 (2024)
6. Abatangelo, L., Ognibene, R.: Sharp behavior of Dirichlet–Laplacian eigenvalues for a class of singularly perturbed problems. SIAM J. Math. Anal. **56**(1), 474–500 (2024)
7. Aronson, R.: Boundary conditions for diffusion of light. J. Opt. Soc. Am. A **12**(11), 2532–2539 (1995)
8. Arridge, S.R., Kaipio, J.P., Kolehmainen, V., Tarvainen, T.: Optical imaging. In: Handbook of Mathematical Methods in Imaging: Volume 1, 2nd edn, pp. 1033–1079. Springer New York (2015)
9. Barseghyan, D., Schneider, B.: Spectral convergence of the Laplace operator with Robin boundary conditions on a small hole. Mediterr. J. Math. **20**(6), 304 (2023)
10. Bisterzo, A., Ognibene, R., Roychowdhury, P., Siclari, G.: Quantitative spectral stability for multiple eigenvalues of compact operators. (**In preparation**)
11. Bisterzo, A., Siclari, G.: Quantitative spectral stability for compact operators. Calc. Var. Partial. Differ. Equ. **64**(6), 178 (2025)
12. Bucur, D., Freitas, P., Kennedy, J.: The Robin problem. In: Shape Optimization and Spectral Theory, pp. 78–119. De Gruyter Open, Warsaw (2017)
13. Bucur, D., Giacomini, A., Trebeschi, P.: The Robin–Laplacian problem on varying domains. Calc. Var. Partial. Differ. Equ. **55**(6), 133 (2016)
14. Cakoni, F., Chaulet, N., Haddar, H.: On the asymptotics of a Robin eigenvalue problem. C. R. Math. Acad. Sci. Paris **351**(13–14), 517–521 (2013)
15. Colin de Verdière, Y.: Sur la multiplicité de la première valeur propre non nulle du laplacien. Comment. Math. Helv. **61**(2), 254–270 (1986)
16. Courtois, G.: Spectrum of manifolds with holes. J. Funct. Anal. **134**(1), 194–221 (1995)
17. Dancer, E.N., Daners, D.: Domain perturbation for elliptic equations subject to Robin boundary conditions. J. Differ. Equ. **138**(1), 86–132 (1997)
18. Dunford, N., Schwartz, J.T.: Linear Operators. Part II. Wiley Classics Library. Wiley, New York (1988). Spectral theory. Selfadjoint operators in Hilbert space, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1963 original, A Wiley-Interscience Publication
19. Felli, V., Liverani, L., Ognibene, R.: Quantitative spectral stability for the Neumann Laplacian in domains with small holes. J. Funct. Anal. **288**(6), 110817 (2025)
20. Felli, V., Liverani, L., Ognibene, R.: On the splitting of Neumann eigenvalues in perforated domains. (2026). Preprint arXiv:2601.13129
21. Felli, V., Noris, B., Ognibene, R.: Eigenvalues of the Laplacian with moving mixed boundary conditions: the case of disappearing Dirichlet region. Calc. Var. Partial. Differ. Equ. **60**(1), 12 (2021)
22. Felli, V., Noris, B., Ognibene, R.: Eigenvalues of the Laplacian with moving mixed boundary conditions: the case of disappearing Neumann region. J. Differ. Equ. **320**, 247–315 (2022)
23. Felli, V., Noris, B., Ognibene, R., Siclari, G.: Quantitative spectral stability for Aharonov–Bohm operators with many coalescing poles. J. Eur. Math. Soc. (2025). (**published online**)
24. Felli, V., Noris, B., Siclari, G.: On Aharonov–Bohm operators with multiple colliding poles of any circulation. Nonlinear Anal. **258**, 113813 (2025)
25. Felli, V., Ognibene, R.: Sharp convergence rate of eigenvalues in a domain with a shrinking tube. J. Differ. Equ. **269**(1), 713–763 (2020)
26. Felli, V., Romani, G.: Perturbed eigenvalues of polyharmonic operators in domains with small holes. Calc. Var. Partial. Differ. Equ. **62**(4), 128 (2023)
27. Filinovskiy, A.V.: On the asymptotic behavior of eigenvalues and eigenfunctions of the Robin problem with large parameter. Math. Model. Anal. **22**(1), 37–51 (2017)
28. Flucher, M.: Approximation of Dirichlet eigenvalues on domains with small holes. J. Math. Anal. Appl. **193**(1), 169–199 (1995)
29. Jimbo, S.: Eigenvalues of the Laplacian in a domain with a thin tubular hole. J. Elliptic Parabol. Equ. **1**(1), 137–174 (2015)
30. Lanza de Cristoforis, M.: Simple Neumann eigenvalues for the Laplace operator in a domain with a small hole. Rev. Mat. Complut. **25**(2), 369–412 (2012)
31. Laurain, A., Nazarov, S., Sokolowski, J.: Singular perturbations of curved boundaries in three dimensions. The spectrum of the Neumann Laplacian. Z. Anal. Anwend. **30**(2), 145–180 (2011)

32. Leoni, G.: A First Course in Sobolev Spaces, Second Ed., Vol. 181 of Graduate Studies in Mathematics. American Mathematical Society, Providence (2017)
33. Levitin, M., Parnovski, L.: On the principal eigenvalue of a Robin problem with a large parameter. Math. Nachr. **281**(2), 272–281 (2008)
34. McGillivray, I.: Capacitary asymptotic expansion of the groundstate to second order. Commun. Partial Differ. Equ. **23**(11–12), 2219–2252 (1998)
35. Ognibene, R.: On asymptotics of Robin eigenvalues in the Dirichlet limit. Commun. Partial Differ. Equ. **50**(9), 1174–1210 (2025)
36. Ozawa, S.: Singular variation of domains and eigenvalues of the Laplacian. Duke Math. J. **48**(4), 767–778 (1981)
37. Ozawa, S.: Spectra of domains with small spherical Neumann boundary. J. Fac. Sci. Univ. Tokyo Sect. I A **30**, 259–277 (1983)
38. Ozawa, S.: Asymptotic property of an eigenfunction of the Laplacian under singular variation of domains—the Neumann condition. Osaka J. Math. **22**, 639–655 (1985)
39. Ozawa, S.: Singular variation of domain and spectra of the Laplacian with small Robin conditional boundary. I. Osaka J. Math. **29**(4), 837–850 (1992)
40. Ozawa, S., Roppongi, S.: Singular variation of domain and spectra of the Laplacian with small Robin conditional boundary. II. Kodai Math. J. **15**(3), 403–429 (1992)
41. Rauch, J., Taylor, M.: Potential and scattering theory on wildly perturbed domains. J. Funct. Anal. **18**, 27–59 (1975)
42. Sauter, S.A., Warnke, R.: Extension operators and approximation on domains containing small geometric details. East-West J. Numer. Math. **7**(1), 61–77 (1999)
43. Ward, M.J., Henshaw, W.D., Keller, J.B.: Summing logarithmic expansions for singularly perturbed eigenvalue problems. SIAM J. Appl. Math. **53**(3), 799–828 (1993)
44. Ward, M.J., Keller, J.B.: Strong localized perturbations of eigenvalue problems. SIAM J. Appl. Math. **53**(3), 770–798 (1993)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.