Stochastic Processes and their Applications 181 (2025) 104525

Contents lists available at ScienceDirect

Stochastic Processes and their Applications

journal homepage: www.elsevier.com/locate/spa

Correlation structure and resonant pairs for arithmetic random waves

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ARTICLE INFO

MSC:
60G60
60D05
58J50
35P20

Keywords:
Random eigenfunctions
Arithmetic random waves
Nodal length
Nodal intersections
Boundary length
Resonant pairs

ABSTRACT

The geometry of Arithmetic Random Waves has been extensively investigated in the last fifteen years, starting from the seminal papers (Rudnick and Wigman, 2008; Oravecz et al., 2008). In this paper we study the correlation structure among different functionals such as nodal length, boundary length of excursion sets, and the number of intersection of nodal sets with deterministic curves in different classes; the amount of correlation depends in a subtle fashion from the values of the thresholds considered and the symmetry properties of the deterministic curves. In particular, we prove the existence of *resonant pairs* of threshold values where the asymptotic correlation is full, that is, at such values one functional can be perfectly predicted from the other in the high energy limit. We focus mainly on the 2-dimensional case but we discuss some specific extensions to dimension 3.

1. Introduction

The geometry of nodal sets for Gaussian random eigenfunctions has been the object of a considerable amount of attention over the last 15 years. Most papers have focussed on the 2-dimensional case, either in Euclidean settings (Berry's random waves, see e.g. [1–4]), or on compact manifolds, most notably the sphere $\mathbb{S}^2$ (see e.g. [5–7]) and the torus $\mathbb{T}^2$ (see e.g. [8–12]), among others. The derivation of the expected value for the nodal length is now standard thanks to the Gaussian Kinematic Formula by Adler and Taylor (see [13]); the analysis of the variance is more challenging, and goes back to [5] for the case of the sphere (random spherical harmonics) and to [10] for the torus (arithmetic random waves); in the physics literature, the variance for the nodal length of planar eigenfunctions (Berry's random waves) was earlier given in [1].

The analysis of the asymptotic distribution for the nodal length of random eigenfunctions was basically started in [11]. In that paper, the nodal length of arithmetic random waves is expanded into orthogonal terms corresponding to so-called Wiener chaos components; it is then shown that the behaviour of nodal length is dominated (in the L^2 sense, as the eigenvalues diverge) by the fourth-order chaos, whose limiting distribution is nonGaussian in the toroidal case. A similar phenomenon takes place in the planar and spherical cases, see [3,6], respectively, although in both these cases the limiting behaviour is Gaussian.

The expansion into Wiener chaoses has allowed to provide an interpretation to the so-called Berry cancellation phenomenon: namely, the fact that the asymptotic variance of nodal length is an order of magnitude smaller than the variance for the measure

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https://doi.org/10.1016/j.spa.2024.104525
Received 11 April 2024; Received in revised form 2 October 2024; Accepted 12 November 2024
Available online 20 November 2024
0304-4149/

of boundary curves at any non-zero thresholds. As argued extensively elsewhere, see [3,6,11,14], this cancellation corresponds to the disappearance of the second-order chaos term, which in turn corresponds to the random L^2 norms of eigenfunctions and plays no role in the fluctuations of the nodal lines. Because this random norm has variance which is larger than all the other terms in the orthogonal decomposition, its disappearance in the nodal case fully explains the Berry's cancellation phenomenon.

The domination of this second order chaotic term yields another remarkable consequence, which was already derived, by a different argument, in [15]. In particular, the correlation between boundary lengths is asymptotically equal to one (in absolute value) as the eigenvalues diverge. This follows easily by the fact that the random sequences corresponding to boundary lengths at different level are all asymptotically proportional (up to different scaling constants) to the same sequence of random variables, namely the random L^2 norms of the eigenfunctions. Asymptotic correlation has been noted in the case of random spherical harmonics [15,16], but it obviously holds with exactly the same argument both for planar random waves and for toroidal eigenfunctions. For the same argument, it is also immediate to notice that nodal lengths and the boundary of level curves have asymptotically correlation zero ; indeed nodal length is dominated by terms in the fourth-order chaos, which are by construction orthogonal to the random norm, which belongs to the second-order chaos.

For the reasons we mentioned above, it is clear that these correlation/uncorrelation phenomena are in some sense an artifact due to the random fluctuations in the L^2 norms; for many applications this could sound meaningless (for instance, in a quantum mechanics framework random norms should be normalized to unity). In [16], a different issue was addressed for random spherical harmonics, namely the existence of correlation *after the effect of the random L^2- norm has been removed* (usually called partial correlation in mathematical statistics). Surprisingly, it was shown that partial correlation among boundary lengths at different levels persists, and indeed the asymptotic correlation with nodal lengths switches from zero to unity: namely, it is asymptotically possible to predict the boundary length of level curves at every threshold u, once the confounding effect of the random norm has been removed.

The purpose of this paper is to investigate an analogous question, in the case of arithmetic random waves. We focus mainly on the 2-dimensional case, but we discuss some specific extensions to dimension 3.

In particular, for planar and spherical random eigenfunctions it has been shown in [16,17] that the nodal case is fully uncorrelated with the behaviour at non-zero levels, but the two become completely correlated when the effect of the second-order chaos is removed. Investigating the same question in the case of Arithmetic Random Waves [18–21] we found a rather different picture. Indeed it is no longer the case that full correlation exists between the length of level curves or other geometric functionals, when the effect of the second-order chaos is removed; on the contrary, we observed the existence of specific sets of points, which we labelled *resonant pairs*, where this correlation is indeed full in the high-energy limit, whereas for generic pairs the correlation can take arbitrary values between -1 and 1. These pairs include the nodal case, where one of the two levels corresponds to $u = 0$. The existence of these resonant pairs, along an algebraic curve (whose equation we write down explicitly), characterizes not only the length of level curves, but also random intersections, a previously unknown phenomenon.

1.1. The physical meaning of getting rid of the second-order chaos

In terms of applications, random eigenfunctions can be viewed as the wave functions of quantum particles (as discussed in any quantum theory textbook). It is well-known that the squared modulus of these wave functions represent the probability density to find the particle in any given location after its measurement — as such, these wave functions should be exactly normalized to have squared integral equal to one. A properly normalized Gaussian random function has an L^2 norm that converges to unity in the high-energy limit (i.e., when the eigenvalues diverge), but for any finite value of the eigenvalue there will be small fluctuations away from unity. These fluctuations do not have any meaning in a quantum framework: it is very important to note, however, that the deviations from the unity norm are indeed proportional to the second-order chaos of the geometric functionals considered in this paper, as noted earlier for the torus in [11,20], and similarly for the sphere in [3,6]. In our view, it is hence more meaningful, at least from the point of view of quantum theory applications, to consider the asymptotic behaviour and correlations of arithmetic random waves after the effect of the second chaos has been removed, as we did here when computing partial correlations.

It is also of interest to understand what is the physical meaning of static curves. Our intuition is the following: when a curve is static, the intersection it has with random eigenfunctions are isotropic and hence independent from the distribution of the energy along different directions $k = (k_1, k_2)$. Under these circumstances, the second chaos depends only on the L^2 norm of the eigenfunction, which can be omitted for the reasons explained in the previous paragraph. For curves that are not static the distribution of the energies along different directions becomes relevant and the second chaos can no longer be simply identified with the L^2 energy of the eigenfunction: as a consequence, it does not disappear in the nodal case and the correlation structure is radically different.

1.2. Plan of the paper

In Section 2 we introduce Arithmetic Random Waves and some relevant known results on the geometry of their level sets, we also introduce Wiener chaos expansion of the geometric functionals of level sets. Section 3 contains the statement of our main results. The proofs of the results concerning the 2-dimensional correlation structure of Arithmetic Random Waves are included in Section 4, whereas those for partial correlation are given in Section 5. Section 6 is devoted to the arguments related to nodal surfaces. Appendix collects the proofs for the technical lemmas that we exploited to characterize static and doubly static curves.

2. Background and notation

2.1. Arithmetic random waves

In order to formulate our results more precisely, we now need to introduce more notation and definitions, which have all become standard in the last decade. We start by recalling the (by now standard) definition of Arithmetic Random Waves, first introduced in [8,9]. Let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ be the standard d-dimensional flat torus and Δ the Laplacian on $\mathbb{T}^d$. We are interested in the (totally discrete) spectrum of Δ i.e., eigenvalues $E > 0$ of the Helmholtz equation

$$\Delta f + E f = 0. \tag{2.1}$$

Let

$$S := \{n \in \mathbb{Z} : n = \lambda_1^2 + \cdots + \lambda_d^2 \ \text{ for some } \lambda_1, \dots \lambda_d \in \mathbb{Z}\}, \tag{2.2}$$

be the collection of all numbers expressible as a sum of two squares. Then, the eigenvalues of (2.1) (also called *energy levels* of the torus) are all numbers of the form $E_n = 4\pi^2 n$, with $n \in S$.

In order to describe the Laplace eigenspace corresponding to E_n, denote by Λ_n the set of *frequencies*:

$$\Lambda_n := \{\lambda \in \mathbb{Z}^d : \|\lambda\|^2 = n\}$$

whose cardinality

$$\mathcal{N}_n := |\Lambda_n| \tag{2.3}$$

equals the number of ways to express n as a sum of d squares. (Geometrically, Λ_n is the collection of all standard lattice points lying on the centred circle with radius $\sqrt{n}$.) For $\lambda \in \Lambda_n$ denote the complex exponential associated to the frequency λ

$$e_\lambda(x) = \exp(2\pi i \langle \lambda, x\rangle)$$

with $x \in \mathbb{T}^d$. Of course, the collection $\{e_\lambda(x)\}_{\lambda\in\Lambda_n}$ of the complex exponentials corresponding to the frequencies $\lambda \in \Lambda_n$, is an L^2-orthonormal basis of the eigenspace $\mathcal{E}_n$ of Δ corresponding to the eigenvalue E_n. In view of (2.3), we have $\dim \mathcal{E}_n = \mathcal{N}_n = |\Lambda_n|$; the fluctuations in the number $\mathcal{N}_n$ have been very widely studied starting from [22], see for instance [10] and the references therein.

Following [8–10], we define the *Arithmetic Random Waves* (also called *random Gaussian toral Laplace eigenfunctions*) to be the random fields

$$T_n(x) = \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda\in\Lambda_n} a_\lambda e_\lambda(x), \quad x \in \mathbb{T}^d, \tag{2.4}$$

where the coefficients a_λ are standard complex-Gaussian random variables[1] verifying the following properties: a_λ is stochastically independent of a_γ whenever $\gamma \notin \{\lambda, -\lambda\}$, and $a_{-\lambda} = \overline{a}_\lambda$ (ensuring that T_n is real-valued). By the definition (2.4), T_n is a stationary, i.e. the law of T_n is invariant under all the translations $f(\cdot) \mapsto f(x+\cdot)$, $x \in \mathbb{T}^d$, centred Gaussian random field with covariance function

$$r_n(x,y) = r_n(x-y) := \mathbb{E}[T_n(x)\overline{T_n(y)}] = \frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} e_\lambda(x-y) = \frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} \cos\left(2\pi\langle x-y, \lambda\rangle\right),$$

$x, y \in \mathbb{T}^d$ (by the standard abuse of notation for stationary fields). Note that $r_n(0) = 1$, i.e. T_n has unit variance.

The set Λ_n induces a discrete probability measure μ_n on the unit sphere $\mathbb{S}^{d-1}$

$$\mu_n = \frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} \delta_{\frac{\lambda}{\sqrt{n}}}. \tag{2.5}$$

It turns out that the behaviour of $\{\mu_n\}_n$ strongly depends on the dimension. Indeed, if $d = 2$, $\mathcal{N}_n$ is subject to large and erratic fluctuations, it grows on average, over integers which are sums of two squares, as $\text{const} \cdot \sqrt{\log n}$, but can be as small as 8 for an infinite sequence of prime numbers. These erratic fluctuations are mirrored by the behaviour of $\{\mu_n\}_n$: indeed, let us denote by

$$\widehat{\mu}_n(k) = \int_{\mathbb{S}^1} z^{-k}\, d\mu_n(z), z \in \mathbb{Z},$$

the Fourier coefficients of μ_n. (We note that $\widehat{\mu}_n(4) \in \mathbb{R}$ since μ_n is invariant under the transformations $z \to \bar{z}$ and $z \to i \cdot z$, and that $|\widehat{\mu}_n(4)| \le 1$ since μ_n is a probability measure.) Remarkably, [10,23] showed that the set of adherent points of $\{\widehat{\mu}_n(4)\}_{n\in S}$ is all of $[-1,1]$. It is known that for a density-1 sequence of eigenvalues, the sequence $\{\mu_n\}_n$ converge towards the uniform probability measure on the circle; weak-* limits of the sequence $\{\mu_n\}_n$ are partially classified in [23].

In dimension $d = 3$ instead, we have

$$n^{\frac{1}{2}-o(1)} \ll \mathcal{N}_n \ll n^{\frac{1}{2}+o(1)},$$

and the lattice points $\Lambda_n/\sqrt{n}$ become equidistributed with respect to the normalized Lebesgue measure on $\mathbb{S}^2$, as $n \to +\infty$ s.t. $n \not\equiv 0, 4, 7 (\text{mod} 8)$ [24].

[1] From now on, we assume that every random object considered in this paper is defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\mathbb{E}$ denoting mathematical expectation with respect to $\mathbb{P}$.

2.2. Geometry of level sets

We are interested in geometric properties of Arithmetic Random Waves, in particular we study the distribution of their level sets

$$\{x \in \mathbb{T}^d : T_n(x) = u\}, \quad u \in \mathbb{R}.$$

Recall that level sets are a.s. smooth submanifolds of codimension 1, this follows exploiting the a.s. Morse property of ARW, or equivalently Sard's lemma. We mainly focus on dimension $d = 2$ and investigate two types of functionals: (i) the 1-dimensional Hausdorff measure, and (ii) the number of intersection points between nodal sets and a fixed reference curve. More precisely, we shall focus on the following functionals:

- boundary length at level $u \neq 0$

$$\mathcal{L}_n^u = \mathcal{H}^1\{x \in \mathbb{T}^2 : T_n(x) = u\},$$

- nodal length

$$\mathcal{L}_n = \mathcal{H}^1\{x \in \mathbb{T}^2 : T_n(x) = 0\} =: \mathcal{L}_n^0,$$

- number of intersections of the nodal lines with smooth curves $\mathcal{C} \subset \mathbb{T}^2$ with no-where zero curvature:

$$\mathcal{Z}_n(\mathcal{C}) = \mathcal{H}^0\{x \in \mathbb{T}^2 : T_n(x) = 0,\ x \in \mathcal{C}\}.$$

In [10,11], the variance of the boundary length have been investigated; in particular, for $u \neq 0$, as $n \to +\infty$ s.t. $\mathcal{N}_n \to +\infty$,

$$\mathrm{Var}(\mathcal{L}_n^u) \sim \frac{1}{32} u^4 e^{-u^2} \frac{E_n}{\mathcal{N}_n}, \tag{2.6}$$

in the nodal case, the variance is of smaller order, indeed

$$\mathrm{Var}(\mathcal{L}_n) \sim \frac{1 + \hat{\mu}_n(4)^2}{512} \frac{E_n}{\mathcal{N}_n^2}, \tag{2.7}$$

with μ_n the measure (2.5). As for nodal intersections with a smooth curve, under the assumption that the curve has no-where zero curvature, we have [18]: as $n \to +\infty$ s.t. $\mathcal{N}_n \to +\infty$,

$$\mathrm{Var}(\mathcal{Z}_n(\mathcal{C})) = (4\mathcal{B}_{\mathcal{C}}(\mu_n) - L^2)\frac{n}{\mathcal{N}_n} + O\left(\frac{n}{\mathcal{N}_n^{\frac{3}{2}}}\right), \tag{2.8}$$

where L is the length of the curve, and, for a probability measure ν on the unit circle $\mathbb{S}^1$, we define

$$\mathcal{B}_{\mathcal{C}}(\nu) := \int_{\mathbb{S}^1} \left(\int_0^L \langle \theta, \dot{\gamma}(t)\rangle^2 dt\right)^2 d\nu(\theta),$$

$\gamma = (\gamma_1, \gamma_2) : [0, L] \to \mathbb{T}^2$ being the arc-length parametrization of $\mathcal{C}$, μ_n is the discrete probability measure introduced in (2.5). There are special curves for which the leading term in the variance (2.8) vanishes, as noted in [18].

Definition 2.1 (*[25, Definition 1]*)**.** A smooth curve $\mathcal{C} \subset \mathbb{T}^2$ with nowhere zero curvature and total length L is *static* if

$$\mathcal{B}_{\mathcal{C}}(\nu) := \int_{\mathbb{S}^1} \left(\int_0^L \langle \theta, \dot{\gamma}(t)\rangle^2 dt\right)^2 d\nu(\theta) = \frac{L^2}{4},$$

for every probability measure ν on the unit circle $\mathbb{S}^1$.

From [18, Corollary 7.2], a curve is static if and only if $\mathcal{B}_{\mathcal{C}}\left(\frac{d\theta}{2\pi}\right) = L^2/4$, where $\frac{d\theta}{2\pi}$ denotes the uniform probability measure on $\mathbb{S}^1$. As we shall see later in Lemma 3.5, equivalently a curve $\mathcal{C}$ of length L is static if and only if $\int_0^L \dot{\gamma}_1(t)^2 dt = \int_0^L \dot{\gamma}_2(t)^2 dt = L/2$ and $\int_0^L \dot{\gamma}_1(t)\dot{\gamma}_2(t) dt = 0$.

The variance of nodal intersections with static curves has been investigated in [25]. We first need some more notation: let $\delta > 0$, a sequence of energy levels is called δ-separated [26] if

$$\min_{\lambda \neq \lambda' \in \Lambda_n} \|\lambda - \lambda'\| \gg n^{1/4+\delta}.$$

In [26, Lemma 5], it was shown that in fact "most" n (density 1 subsequence) satisfy the well separatedness property for every $0 < \delta < 1/4$.

For a static curve (that we denote $\mathcal{C}'$ to avoid confusion), for δ-separated sequence of energy levels such that $\mathcal{N}_n \to +\infty$

$$\mathrm{Var}(\mathcal{Z}_n(\mathcal{C}')) \sim \frac{n}{4\mathcal{N}_n^2}(16\mathcal{A}_{\mathcal{C}'}(\mu_n) - L^2), \tag{2.9}$$

where for a smooth curve $\mathcal{C} \subset \mathbb{T}^2$ and a probability measure ν on the unit circle $\mathbb{S}^1$

$$\mathcal{A}_C(v) := \int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\left(\int_0^L \langle\theta,\dot\gamma(t)\rangle^2\langle\theta',\dot\gamma(t)\rangle^2 dt\right)^2 dv(\theta)v(\theta').$$

Regarding higher dimensions, see Section 3.3 and Section 6.

2.3. Chaos expansion

The celebrated Wiener chaos expansion concerns the representation of square integrable random variables in terms of an infinite orthogonal sum. In this section we recall briefly some basic facts on Wiener chaotic expansion for non-linear functionals of Gaussian fields. Denote by $\{H_k\}_{k\geq 0}$ the Hermite polynomials on $\mathbb{R}$, defined as follows

$$H_0 = 1, \qquad H_k(t) = (-1)^k\gamma^{-1}(t)\frac{d^k}{dt^k}\gamma(t), \quad k \geq 1, \tag{2.10}$$

where $\gamma(t) = e^{-t^2/2}/\sqrt{2\pi}$ is the standard Gaussian density on the real line; $\mathbb{H} = \{H_k/\sqrt{k!} : k \geq 0\}$ is a complete orthogonal system in

$$L^2(\gamma) = L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \gamma(t)dt).$$

The random eigenfunctions defined in (2.4) are a byproduct of the family of complex-valued, Gaussian random variables $\{a_\lambda\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define the space $\mathbf{A}$ to be the closure in $L^2(\mathbb{P})$ generated by all real, finite, linear combinations of random variables of the form $za_\lambda + \bar{z}a_{-\lambda}$, $z \in \mathbb{C}$; the space $\mathbf{A}$ is a real, centred, Gaussian Hilbert subspace of $L^2(\mathbb{P})$.

For each integer $q \geq 0$, the qth *Wiener chaos* $\mathcal{H}_q$ associated with $\mathbf{A}$ is the closed linear subspace of $L^2(\mathbb{P})$ generated by all real, finite, linear combinations of random variables of the form

$$H_{q_1}(a_1)\cdot H_{q_2}(a_2)\cdots H_{q_k}(a_k)$$

for $k \geq 1$, where the integers $q_1, q_2, \dots, q_k \geq 0$ satisfy $q_1 + q_2 + \cdots + q_k = q$ and $(a_1, a_2, \dots, a_k)$ is a real, standard, Gaussian vector extracted from $\mathbf{A}$. In particular $\mathcal{H}_0 = \mathbb{R}$.

As well-known Wiener chaoses $\{\mathcal{H}_q, q = 0, 1, 2, \dots\}$ are orthogonal, i.e., $\mathcal{H}_q \perp \mathcal{H}_p$ for $p \neq q$ (the orthogonality holds in the sense of $L^2(\mathbb{P})$) and the following decomposition holds: every real-valued function $F \in \mathbf{A}$ admits a unique expansion of the type

$$F = \sum_{q=0}^{\infty} F[q],$$

where the projections $F[q] \in \mathcal{H}_q$ for every $q = 0, 1, 2, \dots$ and the series converges in $L^2(\mathbb{P})$. Note that $F[0] = \mathbb{E}[F]$.

For $u \neq 0$, [27, Theorem 2.4] proves that the boundary length $\mathcal{L}_n^u$ is dominated by the second order chaos:

$$\mathcal{L}_n^u[2] = \sqrt{\frac{\pi}{8}}u^2\phi(u)\sqrt{2\pi^2 n}\frac{1}{\mathcal{N}_n/2}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2 - 1) + \mathcal{R}_1(n,u)$$

where, under [27, Condition 2.2], $\mathbb{E}[\mathcal{R}_1(n,u)^2] = O(4\pi^2 n/\mathcal{N}_n^2)$, and Λ_n^+ is the following subset of the set of frequencies: if n is not a square

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \lambda_2) \in \Lambda_n : \lambda_2 > 0\},$$

otherwise

$$\Lambda_n^+ = \{\lambda = (\lambda_1, \lambda_2) \in \Lambda_n : \lambda_2 > 0\} \cup \{(\sqrt{n}, 0)\}.$$

Note that for every $n \in S$, $|\Lambda_n^+| = \mathcal{N}_n/2$.

Also, inspired by [11, Lemma 4.2], we are able to show that the fourth chaotic component $\mathcal{L}_n^u[4]$ of the length of u-level curves can be written as

$$\mathcal{L}_n^u[4] = \phi(u)\sqrt{\frac{\pi}{2}}\frac{\sqrt{E_n/2}}{\mathcal{N}_n}\left[a(u)W_1(n)^2 - \frac{1}{4}W_2(n)^2 - \frac{1}{4}W_3(n)^2 - \frac{1}{2}W_4(n)^2 - \left(a(u) - \frac{1}{4}\right) + o_{\mathbb{P}}(1)\right],$$

where

$$W_1(n) = \frac{1}{n\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2 - 1)n, \tag{2.11}$$

$$W_2(n) = \frac{1}{n\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2 - 1)\lambda_1^2, \quad W_3(n) = \frac{1}{n\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2 - 1)\lambda_2^2, \tag{2.12}$$

$$W_4(n) = \frac{1}{n\sqrt{\mathcal{N}_n/2}}\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2 - 1)\lambda_1\lambda_2, \tag{2.13}$$

and

$$a(u) = \frac{1}{4}H_4(u) + \frac{1}{2}H_2(u) - \frac{1}{8},$$

and $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to zero in probability. Equivalently,

$$\mathcal{L}_n^u[4] = \phi(u)\sqrt{\frac{\pi}{2}}\frac{\sqrt{E_n/2}}{\mathcal{N}_n}\Big[\frac{1}{8}\frac{1}{\mathcal{N}_n/2}\sum_{\lambda,\lambda'\in\Lambda_n^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)\left(8a(u) - 2\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2\right) - \left(a(u)-\frac{1}{4}\right) + o_{\mathbb{P}}(1)\Big].$$

It is known, see [11, Section 1.4 and Lemma 4.2], that $\mathcal{L}_n^0$ is dominated by the fourth chaotic projection:

$$\mathcal{L}_n^0[4] = \frac{1}{2}\frac{\sqrt{E_n/2}}{\mathcal{N}_n}\left[\frac{1}{8}W_1(n)^2 - \frac{1}{4}W_2(n)^2 - \frac{1}{4}W_3(n)^2 - \frac{1}{2}W_4(n)^2 + \frac{1}{8} + o_{\mathbb{P}}(1)\right].$$

[25, Section 2.1 and Section 4] shows that, in the case of a non-static curve, the second chaotic projection dominates the chaos expansion of $\mathcal{Z}_n(C)$, and it has the form

$$\mathcal{Z}_n(C)[2] = \mathcal{Z}_n^a(C)[2] + \mathcal{Z}_n^b(C)[2],$$

where $\mathrm{Var}(\mathcal{Z}_n^b(C)[2]) = o(\mathrm{Var}(\mathcal{Z}_n^a(C)[2]))$, and

$$\mathcal{Z}_n^a(C)[2] = \frac{\sqrt{2\pi^2 n}}{2\pi}\frac{1}{\mathcal{N}_n/2}L\sum_{\lambda\in\Lambda_n^+}(|a_\lambda|^2-1)\big(2I_{\lambda,\lambda'}(2,0)-1\big),$$

where we have introduced the notation

$$I_{\lambda,\lambda'}(k,k') := \frac{1}{L}\int_0^L\left\langle\frac{\lambda}{|\lambda|},\dot\gamma(t)\right\rangle^k\left\langle\frac{\lambda'}{|\lambda'|},\dot\gamma(t)\right\rangle^{k'}dt.$$

Now if the curve is static [25, Lemma 6.5], the leading term in the chaotic expansion of $\mathcal{Z}_n(C')$ is no longer the projection onto the second chaos, but the projection onto the fourth chaos:

$$\mathcal{Z}_n(C')[4] = \mathcal{Z}_n^a(C')[4] + \mathcal{Z}_n^b(C')[4]$$

where $\mathrm{Var}(\mathcal{Z}_n^b(C')[4]) = o(\mathrm{Var}(\mathcal{Z}_n^a(C')[4]))$, and

$$\mathcal{Z}_n^a(C')[4] = \frac{\sqrt{2n}}{4\mathcal{N}_n}L\Big[\frac{1}{\mathcal{N}_n/2}\sum_{\lambda,\lambda'\in\Lambda_n^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)(-4I_{\lambda,\lambda'}(2,2)-1+4I_{\lambda,\lambda'}(2,0)) + \frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(4I_{\lambda,\lambda'}(4,0)-1)\Big]$$

but, for any static curve we have that $I_{\lambda,\lambda'}(2,0) = 1/2$ via Lemma 3.5, so

$$\mathcal{Z}_n^a(C')[4] = \frac{\sqrt{2n}}{4\mathcal{N}_n}L\Big[\frac{1}{\mathcal{N}_n/2}\sum_{\lambda,\lambda'\in\Lambda_n^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)(1-4I_{\lambda,\lambda'}(2,2)) + \frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(4I_{\lambda,\lambda'}(4,0)-1)\Big].$$

And in the particular case of a doubly static curve we have (using (A.36) – see the proof of Lemma 3.10):

$$I_{\lambda,\lambda'}(4,0) = \frac{3}{8}, \qquad I_{\lambda,\lambda'}(2,2) = \frac{1}{8}\left(1+2\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2\right),$$

so

$$\mathcal{Z}_n^a(C'')[4] = \frac{\sqrt{2n}}{4\mathcal{N}_n}L\Big[\frac{1}{\mathcal{N}_n/2}\sum_{\lambda,\lambda'\in\Lambda_n^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)\Big(\frac{1}{2}-\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2\Big)+\frac{1}{2}\Big].$$

3. Main results

3.1. The correlation structure in the 2-dimensional case

Let us start investigating the correlation between boundary lengths; recall that S is defined in (2.2) as the collection of all numbers expressible as a sum of two squares.

Theorem 3.1. *Let $u_1, u_2 \in \mathbb{R}$, for $n \subset \{S\}$ sequence of energies such that $\mathcal{N}_n \to +\infty$,*

$$\mathrm{Corr}(\mathcal{L}_n^{u_1}, \mathcal{L}_n^{u_2}) \to \begin{cases} 1, & u_1, u_2 \neq 0 \text{ or } u_1 = u_2 = 0, \\ 0, & \text{otherwise.}\end{cases}$$

Theorem 3.1 follows immediately from [27, Corollary 2.7] and [11, Section 1.4]. The correlation structure between boundary length and the intersection number with a fixed smooth reference curve of nowhere zero curvature is given below.

Theorem 3.2. *Let $u \in \mathbb{R}$, and $C \subset \mathbb{T}^2$ be a smooth curve of total length L with nowhere zero curvature and for which $\{4\mathcal{B}_C(\mu_n) - L^2\}_n$ is bounded away from zero. Then for $n \subset \{S\}$ such that $\mathcal{N}_n \to +\infty$,*

$$\mathrm{Corr}(\mathcal{L}_n^u, \mathcal{Z}_n(C)) \to 0. \tag{3.14}$$

The proof of Theorem 3.2 is in Section 4.

Remark 3.3 (*[25]*)**.** observe that the assumptions in Theorem 3.2 are satisfied for the full sequence $n \subset \{S\}$ of energy levels (still assuming $\mathcal{N}_n \to +\infty$), if both the imaginary and real parts of $I(\gamma)$ do not vanish, where

$$I(\gamma) := \int_0^L e^{2i\phi(t)} dt,$$

and $\phi(t) \in [0, 2\pi]$ is the argument of $\dot\gamma(t)$ i.e. $\dot\gamma(t) = e^{i\phi(t)}$. A curve is static if and only if $I(\gamma) = 0$ [18, Corollary 7.2]. If $I(\gamma) \neq 0$, $\Re I(\gamma) = 0$ and the lattice points converge to the Cilleruelo measure (resp. $\Im I(\gamma) = 0$ and the lattice points converge to the tilted Cilleruelo measure), the leading coefficient $\{4\mathcal{B}_C(\mu_n) - L^2\}_n$ vanishes asymptotically (at least under the δ-separatedness assumption); in these last two cases the rate of convergence of $\{4\mathcal{B}_C(\mu_n) - L^2\}_n$ decides which is the leading term in the chaotic expansion. These are the two only cases not covered by Theorem 3.2.

To deal with the static case, we need to define the following

$$\mathcal{I}_4' = \mathcal{I}_4'(C') := \frac{1}{L}\int_0^L (\dot\gamma_1(t)^4 + \dot\gamma_2(t)^4 - 6\dot\gamma_1(t)^2\dot\gamma_2(t)^2) dt. \tag{3.15}$$

Theorem 3.4. *Let $C' \subset \mathbb{T}^2$ be a static curve. For δ-separated sequences of energy levels $n \subset \{S\}$ such that $\mathcal{N}_n \to +\infty$,*

$$\mathrm{Corr}(\mathcal{L}_n^u, \mathcal{Z}_n(C')) \to 0, \quad \text{if } u \neq 0. \tag{3.16}$$

If moreover $\widehat{\mu}_n(4) \to \eta$, we have that

$$\mathrm{Corr}(\mathcal{L}_n, \mathcal{Z}_n(C')) \to f_{C'}(\eta), \tag{3.17}$$

where

$$f_{C'}(\eta) := \frac{1 + 2\eta\mathcal{I}_4' + \eta^2}{\sqrt{2}\sqrt{1+\eta^2}\sqrt{2(1-\eta^2)(2\mathcal{I}_4 - 1) + (\eta\mathcal{I}_4' + 1)^2}}. \tag{3.18}$$

3.1.1. Doubly static curves

There are special static curves for which $f_{C'}(\eta) = 1$. In order to investigate this case, we need to deeply understand the geometry of static curves.

Lemma 3.5. *A curve is static if and only if*

$$\int_0^L \dot\gamma_1(t)^2 dt = \int_0^L \dot\gamma_2(t)^2 dt = L/2 \quad \text{and} \quad \int_0^L \dot\gamma_1(t)\dot\gamma_2(t) dt = 0.$$

Remark 3.6. If C is static, then also $\int_0^L \dot\gamma_1(t)^4 dt = \int_0^L \dot\gamma_2(t)^4 dt$.

Let us now define

$$A = A_C := \frac{1}{L}\int_0^L \dot\gamma_1(t)^2\dot\gamma_2(t)^2 dt, \qquad B = B_C := \frac{1}{L}\int_0^L \dot\gamma_1(t)^3\dot\gamma_2(t) dt,$$

$$\mathcal{I}_4 = \mathcal{I}_{4,C} := \frac{1}{L^2}\int_0^L\int_0^L \langle\dot\gamma(t), \dot\gamma(u)\rangle^4 dt du.$$

Hence

$$\mathcal{I}_4' = 1 - 8A.$$

The following result may look technical, but it is instrumental to introduce the notion of a doubly static curve.

Lemma 3.7. *If C is a static curve, then we have the relation,*

$$\mathcal{I}_4 = \frac{1}{2} + 8A^2 - 2A + 8B^2, \tag{3.19}$$

and the inequalities

$$0 < A < \frac{1}{4}, \qquad B^2 < \frac{A(1-4A)}{4}, \qquad \frac{3}{8} \leq \mathcal{I}_4 < \frac{1}{2}, \qquad -1 < \mathcal{I}_4' < 1.$$

The proof of Lemma 3.7 is postponed to Appendix. We can now introduce the notion of a doubly static curve.

Definition 3.8. We say the curve C doubly static if $\mathcal{I}_4 = 3/8$.

Example 3.9. Circles and semicircles are doubly static.

Our characterization of doubly static curves is given in the following lemmas, also proved in Appendix.

Lemma 3.10. *One has $\mathcal{I}_4 = 3/8$ if and only if C is static, $A = 1/8$, and $B = 0$; this implies also $\mathcal{I}_4' = 0$.*

Corollary 3.11. *Let $C'' \subset \mathbb{T}^2$ be a doubly static curve. For δ-separated sequences of energy levels $n \subset \{S\}$ such that $\mathcal{N}_n \to +\infty$ and $\widehat{\mu}_n(4) \to \eta$,*

$$\mathrm{Corr}(\mathcal{L}_n, \mathcal{Z}_n(C'')) \to 1. \tag{3.20}$$

Lemma 3.12 (*cf.* *[25, Appendix G]*). *Let $C \subset \mathbb{T}^2$ be a smooth closed curve with nowhere 0 curvature, invariant with respect to rotations by $2\pi/k$, for integer $k = 3$ or $k \geq 5$. Then C is doubly static.*

3.1.2. Discussion

Our first main results on the asymptotic correlation structure among the functionals that we introduced in Section 2.2 can be conveniently summarized in the following (symmetric) correlation matrix. As mentioned above, C' (resp. C'') denotes a static (resp. doubly static) curve.

Asymptotic correlation structure, $d = 2$.

	$\mathcal{L}_n^0$	$\mathcal{L}_n^{u_1}$	$\mathcal{L}_n^{u_2}$	$\mathcal{Z}_n(C)$	$\mathcal{Z}_n(C')$	$\mathcal{Z}_n(C'')$
$\mathcal{L}_n^0$	1					
$\mathcal{L}_n^{u_1}$	0	1				
$\mathcal{L}_n^{u_2}$	0	1	1			
$\mathcal{Z}_n(C)$	0	0	0	1		
$\mathcal{Z}_n(C')$	$f_{C'}(\eta)$	0	0	0	1	
$\mathcal{Z}_n(C'')$	1	0	0	0	$f_{C'}(\eta)$	1

Remark 3.13. Level curves have asymptotically full correlation at different non-zero thresholds u_1, u_2; this is the phenomenon noted by [15] for random spherical harmonics, using the expansion of the 2-point correlation function, and then related to the domination of the second chaos by [3,6,11,14,16,27] and others. On the other hand, similarly to what was noted earlier in [16] for eigenfunctions on the sphere, Theorem 3.1 shows that the nodal length and the boundary lengths of excursion sets at non-zero levels are asymptotically fully uncorrelated in the high-energy regime. This can be interpreted as a spurious effect: boundary lengths at non-zero levels are dominated by the second-order chaos, which is proportional to the random norm of the eigenfunctions, and the latter of course has no impact on the nodal length (which is invariant to normalizations).

Remark 3.14. It is important to note that the correlation between boundary length and the number of nodal intersections with static or non-static curves is always zero in the asymptotic limit, excluding the nodal case (i.e., $u = 0$). However, the mechanism here is different than what we observed in the previous remark: indeed, the second-order chaos component in the intersection of the nodal length with a non-static curve does *not* vanish, although it is still uncorrelated with the random norm of the eigenfunctions, which dominates the behaviour of the boundary length. See the proof of Theorem 3.2 for more details. On the other hand for intersections with static curves the second order chaos is of lower order, (see [25], Section 2.1) and therefore the asymptotic correlation with the boundary lengths is obviously zero. In some sense, intersections with static curves have some form of invariance with respect to normalization factors for the Arithmetic Random Waves, and this makes their behaviour somewhat analogous to nodal lines, see our following discussion on partial autocorrelation results.

Remark 3.15. In the special case where $\eta = 0$ the limiting spectral measure is Lebesgue, i.e. lattice points are equidistributed in the limit. In these circumstances, we have

$$\mathrm{Corr}(\mathcal{L}_n^0, \mathcal{Z}_n(C')) \to \frac{1}{\sqrt{2}} \frac{1}{\sqrt{4\mathcal{I}_4 - 1}};$$

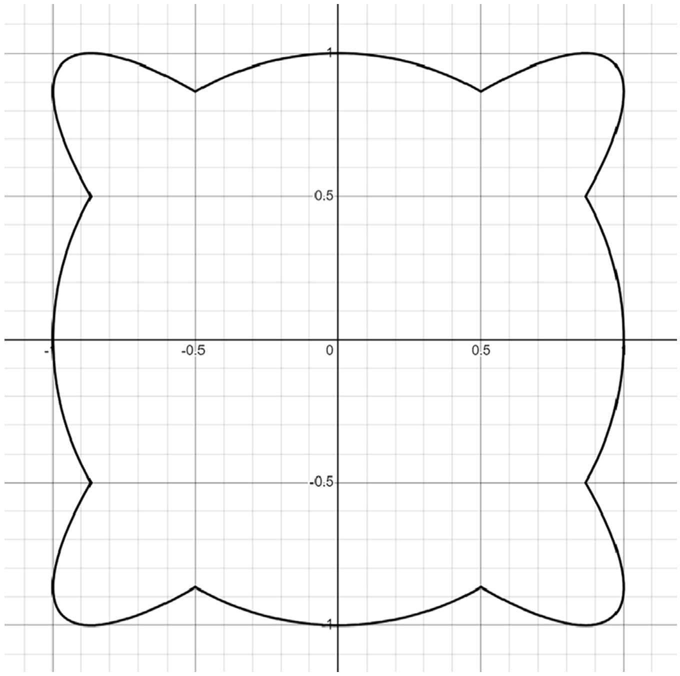

Fig. 1. A static but not doubly static curve C'.

this result can be compared to Theorem 3.28 below for the three-dimensional case. At the other extreme we have $\eta = \pm 1$, where the limiting measure is Cilleruelo or tilted Cilleruelo, namely the spectral measure exhibits the maximal concentration. In these cases $\mathrm{Corr}(\mathcal{L}_n^0, \mathcal{Z}_n(C')) \to 1$ for any static curve C'.

Remark 3.16. We have that $\mathrm{Corr}(\mathcal{L}_n^0, \mathcal{Z}_n(C')) > 0$ for any η and any static curve C'.

Remark 3.17. In the doubly static case, the dominant terms in $\mathcal{L}^0[4]$ and $\mathcal{Z}(C'')[4]$ coincide, up to a factor depending on the energy and L, so that we get full correlation for any $\eta \in [-1, 1]$. More explicitly, $\mathcal{L}_n^0$ and $\mathcal{Z}_n(C'')$ are asymptotically the same random variable up to a constant. Once again, we recall that circles and semicircles are doubly static.

Remark 3.18. Let C' be a fixed toral curve, static, but not doubly static. Then (as noted above) we have $f_{C'}(\eta) = 1$ for $\eta = \pm 1$, and $f_{C'}(\eta) = 0$ has exactly one solution for $\eta \in (-1, 1)$. For instance, we could take C' to be defined in the first quadrant as the union of the unit circle in the ranges $[0, \pi/6]$ and $[\pi/3, \pi/2]$, and the ellipse arc

$$(\cos(t),\ \sin(t + \pi/3)), \quad t \in [-\pi/6, \pi/3],$$

and symmetrically in the other quadrants, see Fig. 1.

Remark 3.19. Above we consider the case $\eta = \pm 1$ under the assumption of well separated sequences of eigenvalues. We stress that it is possible that $\eta = \pm 1$ may not be attainable under the well separated assumption.

3.2. The partial correlation structure in the 2-dimensional case

To get deeper insights into the correlation structure for the geometry of level sets of Arithmetic Random Waves, it is of greater interest to get rid of the effect of the random L^2 norm of the eigenfunctions. More precisely, it is of interest to investigate the so-called partial correlation structure, where the effect of the fluctuations in the eigenfunctions norm is removed.

Let X, Y, Z be square integrable random variables; we define the partial correlation coefficient between X and Y conditional on Z as

$$\mathrm{Corr}_Z(X, Y) := \mathrm{Corr}(X^*, Y^*),$$

where X^* and Y^* are the residuals after projecting X, Y onto the explanatory variable Z.

In analogous circumstances, it was shown in [16] that for random spherical harmonics perfect autocorrelation holds in the high energy limit (see also [28] for a similar result on critical points). For Arithmetic Random Waves, the correlation structure is much more subtle, as detailed below.

Proposition 3.20. *Let $u_1, u_2 \in \mathbb{R}$, then for subsequences of energy levels $\{n\} \subset S$ such that $\widehat{\mu_n}(4) \to \eta \in [-1,1]$ as $\mathcal{N}_n \to +\infty$ we have*

$$Cov\,(\mathcal{L}_n^{u_1}[4], \mathcal{L}_n^{u_2}[4]) \sim \phi(u_1)\phi(u_2)\frac{\pi}{4}\frac{E_n}{\mathcal{N}_n^2}\left(2a(u_1)a(u_2) - \frac{1}{4}(a(u_1)+a(u_2)) + \frac{3+\eta^2}{8^2}\right).$$

Note that for $u_1 = u_2 = 0$, since $a(0) = \frac{1}{8}$, we retrieve

$$\mathrm{Var}(\mathcal{L}_n[4]) \sim \frac{4\pi^2 n}{\mathcal{N}_n^2}\frac{1+\eta^2}{512},$$

as expected.

Theorem 3.21. *Let $u \in \mathbb{R}$, for a δ-separated subsequences of energies $\{n\} \subset S$ such that $\mathcal{N}_n \to \infty$ and $\hat{\mu}_n(4) \to \eta \in [-1,1]$, we have*

$$Cov\,(\mathcal{L}_n^{u}[4], \mathcal{Z}_n(C')[4]) \sim \phi(u)\sqrt{\frac{\pi}{2}}\frac{\sqrt{E_n/2}}{\mathcal{N}_n}\frac{\sqrt{2n}}{4\mathcal{N}_n}L\frac{1}{16}\left[1+2\eta\mathcal{I}_4'+\eta^2\right].$$

3.2.1. Discussion

For 2-dimensional Arithmetic Random Waves, the following asymptotic partial correlation structure holds

Partial Correlation structure, dimension $d = 2$.

	$\mathcal{L}_n^0$	$\mathcal{L}_n^{u_1}$	$\mathcal{L}_n^{u_2}$	$\mathcal{Z}_n(C)$	$\mathcal{Z}_n(C')$	$\mathcal{Z}_n(C'')$
$\mathcal{L}_n^0$	1					
$\mathcal{L}_n^{u_1}$	$M(0,u_1;\eta)$	1				
$\mathcal{L}_n^{u_2}$	$M(0,u_2;\eta)$	$M(u_1,u_2;\eta)$	1			
$\mathcal{Z}_n(C)$	0	0	0	1		
$\mathcal{Z}_n(C')$	$f_{C'}(\eta)$	$f_{C'}(u_1;\eta)$	$f_{C'}(u_2;\eta)$	0	1	
$\mathcal{Z}_n(C'')$	1	$M(0,u_1;\eta)$	$M(0,u_2;\eta)$	0	$f_{C'}(\eta)$	1

where for $u \in \mathbb{R}$

$$f_{C'}(u;\eta) := \frac{\frac{\sqrt{2}}{16}\left[1+2\eta\mathcal{I}_4'+\eta^2\right]}{\sqrt{(2a(u)^2 - \frac{1}{2}a(u) + \frac{3+\eta^2}{8^2})}\cdot\sqrt{2(1-\eta^2)(2\mathcal{I}_4 - 1) + (\eta\mathcal{I}_4'+1)^2}},$$

(note that $f_{C'}(\eta) = f_{C'}(0;\eta)$ as in (4.26) since $a(0) = \frac{1}{8}$). Moreover,

$$M(u_1,u_2;\eta) = \frac{\left\{2a(u_1)a(u_2) - \frac{1}{4}(a(u_1)+a(u_2)) + \frac{3+\eta^2}{8^2}\right\}}{\sqrt{\left\{2a^2(u_1) - \frac{1}{2}a(u_1) + \frac{3+\eta^2}{8^2}\right\}\left\{2a^2(u_2) - \frac{1}{2}a(u_2) + \frac{3+\eta^2}{8^2}\right\}}},$$

for

$$a(u) = \frac{H_4(u)}{4} + \frac{H_2(u)}{2} - \frac{1}{8}\,.$$

Note that

$$M(0,u;\eta) = f_{C''}(u;\eta),$$

where C'' is a doubly static curve.

Remark 3.22. The rationale behind the previous correlation table can be explained as follows. Considering partial correlation, the second-order chaos term disappears from the boundary lengths measure at non-zero thresholds. As a consequence their correlation with the intersections for non static curves becomes zero, because the latter is dominated by the second-order chaos. On the other hand, for static or doubly static curves the second-order chaos is lower order, hence the partial correlation becomes basically the correlation between the fourth-chaos components of intersections and boundary lengths.

An important consequence of the previous results is the existence of *resonant pairs*, that is, sets of threshold levels with asymptotically full correlation between boundary length and/or nodal intersections; this is illustrated in the following corollary.

Corollary 3.23. *We have that*

$$\lim_{n\to\infty} \mathrm{Corr}(\mathcal{L}_n^{u_1}, \mathcal{L}_n^{u_2}) = 1,$$

if and only if

$$u_2^4 - 4u_2^2 = u_1^4 - 4u_1^2. \tag{3.21}$$

Example 3.24. For $u_1 = 0$ we obtain that the nodal lines (and the interesections with doubly static curves) have asymptotically perfect correlations with the levels $u_2 = \pm 2$. For $u_1 = 1$ we obtain $u_2^4 - 4u_2^2 + 3 = 0$, with solutions $u_2 = \sqrt{3}, -\sqrt{3}, -1, 1$, so that resonant pairs are given by $(1,3), (1,-3)$ and $(1,-1)$.

We call (3.21) the *Full Correlation Curve* for boundary lengths of Arithmetic Random Waves. We believe that analogous algebraic curves characterize full correlation pairs for other geometric functionals, such as Lipschitz-Killing curvatures and critical values. We leave the investigation of this for future research.

3.3. Some results in the 3-dimensional case

It is of obvious interest to investigate the existence of a similar correlation structure for higher-dimensional arithmetic random waves (as studied for instance in [20,21,29]). For brevity's sake, we do not explore fully this possibility here; we focus just on a special case, that is in 3 dimensions the correlation between the nodal area ($\mathcal{A}_n = \mathcal{A}_n^0$) and the length of intersections of the nodal area with static surfaces or doubly-static surfaces ($\mathcal{M}_n(\Sigma')$, $\mathcal{M}_n(\Sigma'')$). Let us first define

$$\mathcal{A} = \mathcal{A}_n := \mathcal{H}^2(\{x \in \mathbb{T}^3 : T_n(x) = 0\}),$$

and, for $\Sigma \subset \mathbb{T}^3$ a fixed compact regular toral surface, of finite area $A := |\Sigma|$,

$$\mathcal{M} = \mathcal{M}_n(\Sigma) := \mathcal{H}^1(\{x \in \mathbb{T}^3 : T_n(x) = 0, x \in \Sigma\}).$$

From [29, Theorem 1.2], as $n \to +\infty$, $n \not\equiv 0, 4, 7 (\mathrm{mod}\ 8)$,

$$\mathrm{Var}(\mathcal{A}_n) \sim \frac{32}{375}\frac{n}{\mathcal{N}_n^2}.$$

Assume that Σ admits a smooth normal vector locally, and call $n(\sigma)$ the unit normal vector to Σ at the point σ. For $k \geq 0$ even, call

$$\mathcal{I}_k = \mathcal{I}_{k,\Sigma} := \frac{1}{A^2}\iint_{\Sigma^2} \langle n(\sigma), n(\sigma')\rangle^k \, d\sigma d\sigma'. \tag{3.22}$$

Definition 3.25. We call Σ of nowhere 0 Gauss-Kronecker curvature static if $\mathcal{I}_2 = 1/3$, and doubly static if $\mathcal{I}_4 = 1/5$.

Remark 3.26. Also for surfaces doubly static implies static (see Lemma 6.4 below). Simple examples of doubly static surfaces are the sphere and hemisphere.

From [30], we have as $n \to +\infty$, $n \not\equiv 0, 4, 7 (\mathrm{mod}\ 8)$, along a well separated[2] sequence of eigenvalues

$$\mathrm{Var}(\mathcal{M}_n) \sim \frac{\pi^2}{9600}\frac{n}{N^2}(81\mathcal{I}_4 + 35A^2). \tag{3.23}$$

Remark 3.27. Any surface Σ of finite area and nowhere 0 Gauss-Kronecker curvature, invariant with respect to any permutation and sign change of the coordinates is static. To see this, note that under this condition Σ verifies the criterion for staticity given by Lemma 6.2 below. For instance, Σ may be given piecewise by symmetric trivariate polynomials where all variables appear to even powers (as long as the assumption on the curvature is met everywhere).

Our main result is the following.

Theorem 3.28. *Let $\mathcal{A}$ be the nodal area and $\mathcal{M}$ the nodal intersection length. For static surfaces Σ of area A, we have as $n \to \infty$, $n \not\equiv 0, 4, 7 \ (\mathrm{mod}\ 8)$ along a well separated sequence of eigenvalues*

$$Cov\ (\mathcal{A}_n, \mathcal{M}_n) \sim \frac{n}{\mathcal{N}^2}\cdot\frac{8\pi A}{375}, \tag{3.24}$$

so that

$$\mathrm{Corr}(\mathcal{A}_n, \mathcal{M}_n) \longrightarrow \frac{16}{\sqrt{405\cdot\mathcal{I}_4 + 175}}.$$

[2] See Definition 1.6 in [30].

Remark 3.29. Static surfaces verify $1/5 \leq \mathcal{I}_4 \leq 1/3$ (Lemma 6.4). The above limit is 1 for 'doubly static' surfaces i.e. those satisfying $\mathcal{I}_4 = 1/5$, for instance sphere and hemisphere.

4. The 2-dimensional correlation structure

Let us start investigating the correlation between the boundary length at level u and the number of nodal intersections with respect to a non-static curve.

Proof of Theorem 3.2. For $u = 0$ the result follows immediately from the orthogonality of the projections in the chaos expansion. In fact in [25, Section 2.1] it is shown that, in the case of a non-static curve, the second chaotic projection dominates the chaos expansion of $\mathcal{Z}_n(C)$, while it is known, see [11, Section 1.4], that $\mathcal{L}_n^0$ is dominated by the fourth chaotic projection. For $u \neq 0$ the boundary length is dominated by the second order chaos [27, Theorem 2.4], so we have

$$\begin{aligned}
&\text{Cov}\,(\mathcal{L}_n^u, \mathcal{Z}_n(C)) \\
&= \sqrt{2\pi^2 n}\sqrt{\frac{\pi}{8}}\phi(u)u^2\frac{\sqrt{n}}{\sqrt{2}}L\frac{1}{\mathcal{N}_n^2/4}\sum_{\lambda,\lambda'\in\Lambda_n^+}\mathbb{E}[(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)](2I_{\lambda',\lambda'}(2,0)-1)+o\left(\frac{n}{\mathcal{N}_n}\right) \\
&= \sqrt{2\pi^2 n}\sqrt{\frac{\pi}{8}}\phi(u)u^2\frac{\sqrt{n}}{\sqrt{2}}L\frac{1}{\mathcal{N}_n^2/4}\sum_{\lambda\in\Lambda_n^+}(\nvDash I_{\lambda,\lambda}(2,0)-1)+o\left(\frac{n}{\mathcal{N}_n}\right)
\end{aligned}$$

where the last step follows by observing that $2|a_\lambda|^2$ has a chi-squared distribution with 2 degrees of freedom, and by recalling that a_λ and a'_λ are independent for $\lambda \neq \lambda'$. The statement immediately follows by observing that

$$\sum_{\lambda\in\Lambda_n^+}(-1+2I_{\lambda,\lambda}(2,0)) = \frac{1}{2}\sum_{\lambda\in\Lambda_n}(-1+2I_{\lambda,\lambda}(2,0)) = 0,$$

since, as shown in Lemma (A.37) Eq. (A.37), we have

$$\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} I_{\lambda,\lambda}(2,0) = \frac{1}{2}. \quad \square$$

The main result to prove Theorem 3.4, i.e. the correlation structure among nodal length and nodal intersection numbers, is the following proposition.

Proposition 4.1. *Let $\mathcal{L}_n$ be the nodal length, and $\mathcal{Z}_n(C')$ the nodal intersections number with a static curve. For a δ-separated subsequence of energies $\{n\} \subset S$ such that $\mathcal{N}_n \to \infty$ and $\hat{\mu}_n(4) \to \eta \in [-1, 1]$, we have*

$$\text{Cov}\,(\mathcal{L}_n, \mathcal{Z}_n(C')) \sim \frac{\sqrt{\pi^2 n}}{\mathcal{N}_n}\frac{\sqrt{n}}{4\mathcal{N}_n}L\frac{1}{16}\left[1+2\eta\mathcal{I}_4'+\eta^2\right], \tag{4.25}$$

and

$$\text{Corr}(\mathcal{L}_n, \mathcal{Z}_n(C')) \sim \frac{1+2\eta\mathcal{I}_4'+\eta^2}{\sqrt{2}\sqrt{1+\eta^2}\sqrt{2(1-\eta^2)(2\mathcal{I}_4-1)+(\eta\mathcal{I}_4'+1)^2}}. \tag{4.26}$$

Proof of Proposition 4.1. The covariance in (4.25) follows immediately from Theorem 3.21 with $u = 0$. Now we take into account the expression for $\text{Var}(\mathcal{L}_n[4])$ in [11, (2.20)], and, in our notation, [25, (3.4)] is

$$\text{Var}(\mathcal{Z}[4]) \sim \frac{n}{4\mathcal{N}^2}\cdot\frac{L^2}{4}[2(1-\eta^2)(2\mathcal{I}_4-1)+(\eta\mathcal{I}_4'+1)^2]. \quad \square$$

5. The partial correlation structure in the 2-dimensional case

Proof of Proposition 3.20. Let us write

$$\begin{aligned}
\text{Cov}\,(\mathcal{L}_n^{u_1}[4], \mathcal{L}_n^{u_2}[4]) \sim\; &\phi(u_1)\phi(u_2)\frac{\pi}{4}\frac{4\pi^2 n}{\mathcal{N}_n^2} \\
&\times\left\{a(u_1)a(u_2)\text{Var}(W_1(n)^2)\right. \\
&+[a(u_1)+a(u_2)]\text{Cov}\left(W_1(n)^2, -\frac{1}{4}W_2(n)^2-\frac{1}{4}W_3(n)^2-\frac{1}{2}W_4(n)^2\right) \\
&\left.+\text{Var}\left(-\frac{1}{4}W_2(n)^2-\frac{1}{4}W_3(n)^2-\frac{1}{2}W_4(n)^2\right)\right\}.
\end{aligned}$$

where the random variables $W_i(n)$ are defined in Eqs. (2.11), (2.12) and (2.13). From Lemma 4.3 in [11] and a simple computations of Gaussian moments we have, as $n \to +\infty$,

$$\mathrm{Var}(W_1(n)^2) \to 2,$$

and moreover

$$\mathrm{Cov}\left(W_1(n)^2, -\frac{1}{4}W_2(n)^2 - \frac{1}{4}W_3(n)^2 - \frac{1}{2}W_4(n)^2\right) \to -\frac{1}{4}.$$

Finally,

$$\begin{aligned}&\mathrm{Var}\left(-\frac{1}{4}W_2(n)^2 - \frac{1}{4}W_3(n)^2 - \frac{1}{2}W_4(n)^2\right)\\ &\to \frac{1}{16}\cdot 2\left(\frac{3+\eta}{8}\right)^2 + \frac{1}{16}\cdot 2\left(\frac{3+\eta}{8}\right)^2 + \frac{1}{4}\cdot 2\cdot\left(\frac{1-\eta}{8}\right)^2 + 2\cdot\frac{1}{16}\cdot 2\left(\frac{1-\eta}{8}\right)^2\\ &= \frac{1}{8^2}(3+\eta^2),\end{aligned}$$

thus concluding the proof. □

Let us now investigate partial correlation between the boundary length and the number of intersections with a static curve. Define

$$\begin{aligned}\mathcal{I} &:= \frac{1}{L}\int_0^L \Big[\frac{3+\eta}{8}\dot{\gamma}_1^4(t) + \frac{3+\eta}{8}\dot{\gamma}_2^4(t) + 6\frac{1-\eta}{8}\dot{\gamma}_1^2(t)\dot{\gamma}_2^2(t)\Big]dt,\\ \mathcal{J} &:= \frac{1}{L}\int_0^L \Big[\Big(\frac{3+\eta}{8}\Big)^2\dot{\gamma}_1^4(t) + \Big(\frac{1-\eta}{8}\Big)^2\dot{\gamma}_1^4(t) + 4\Big(\frac{3+\eta}{8}\Big)\Big(\frac{1-\eta}{8}\Big)\dot{\gamma}_1^2(t)\dot{\gamma}_2^2(t)\\ &\qquad + \Big(\frac{1-\eta}{8}\Big)^2\dot{\gamma}_2^4(t) + \Big(\frac{3+\eta}{8}\Big)^2\dot{\gamma}_2^4(t) + 8\Big(\frac{1-\eta}{8}\Big)^2\dot{\gamma}_1^2(t)\dot{\gamma}_2^2(t)\Big]dt.\end{aligned}$$

Proof of Theorem 3.21. Using the expressions of the 4-th order chaos, see Section 2.3, we write

$$\begin{aligned}&\mathrm{Cov}\,(\mathcal{L}_n^u[4], \mathcal{Z}_n(\mathcal{C}')[4])\\ &\sim \phi(u)\sqrt{\frac{\pi}{2}}\frac{\sqrt{E_n/2}}{\mathcal{N}_n}\frac{\sqrt{2n}}{4\mathcal{N}_n}L\,\mathbb{E}\Big\{\Big[a(u)W_1(n)^2 - \frac{1}{4}W_2(n)^2 - \frac{1}{4}W_3(n)^2 - \frac{1}{2}W_4(n)^2\Big]\\ &\qquad\times\Big[\frac{2}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)(1-4I_{\lambda,\lambda'}(2,2))\Big]\Big\}.\end{aligned}$$

The first term is

$$\begin{aligned}A &= a(u)\mathbb{E}\left[W_1(n)^2\frac{2}{\mathcal{N}_n}\sum_{\lambda,\lambda'\in\Lambda_n^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)(1-4I_{\lambda,\lambda'}(2,2))\right]\\ &= a(u)\frac{4}{\mathcal{N}_n^2}\mathbb{E}\Big[\Big(\sum_{\lambda_1,\lambda_2\in\Lambda_n^+}(|a_{\lambda_1}|^2-1)(|a_{\lambda_2}|^2-1)\Big)\Big(\sum_{\lambda_3,\lambda_4\in\Lambda_n^+}(|a_{\lambda_3}|^2-1)(|a_{\lambda_4}|^2-1)(1-4I_{\lambda_3,\lambda_4}(2,2))\Big)\Big]\\ &= a(u)\frac{4}{\mathcal{N}_n^2}\Big[\sum_{\lambda,\lambda'\in\Lambda_n^+}(1-4I_{\lambda',\lambda'}(2,2)) + 2\sum_{\lambda,\lambda'\in\Lambda_n^+}(1-4I_{\lambda,\lambda'}(2,2))\Big]\\ &= a(u)\frac{4}{\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n^+}(1-4I_{\lambda',\lambda'}(2,2))\end{aligned}$$

where in the last step we apply (A.39). The second and third term have the form

$$B_i = -\frac{1}{4}\frac{4}{n^2\mathcal{N}_n^2}\Big[\sum_{\lambda,\lambda'\in\Lambda_n^+}\lambda_i^4(1-4I_{\lambda',\lambda'}(2,2)) + 2\sum_{\lambda,\lambda'\in\Lambda_n^+}\lambda_i^2(\lambda_i')^2(1-4I_{\lambda,\lambda'}(2,2))\Big]$$

for $i = 1, 2$, and the last term is given by

$$C = -\frac{1}{2}\frac{4}{n^2\mathcal{N}_n^2}\Big[\sum_{\lambda,\lambda'\in\Lambda_n^+}\lambda_1^2\lambda_2^2(1-4I_{\lambda',\lambda'}(2,2)) + 2\sum_{\lambda,\lambda'\in\Lambda_n^+}\lambda_1\lambda_2\lambda_1'\lambda_2'(1-4I_{\lambda,\lambda'}(2,2))\Big].$$

We observe that

$$B_1 + B_2 + C = -\frac{1}{4}\frac{4}{n^2\mathcal{N}_n^2}\Big[\sum_{\lambda,\lambda'\in\Lambda_n^+}\langle\lambda,\lambda\rangle^2(1-4I_{\lambda',\lambda'}(2,2)) + 2\sum_{\lambda,\lambda'\in\Lambda_n^+}\langle\lambda,\lambda'\rangle^2(1-4I_{\lambda,\lambda'}(2,2))\Big].$$

In view of Lemma A.1 we write

$$\begin{aligned}A &= a(u)\frac{1}{\mathcal{N}_n^2}[\mathcal{N}_n^2 - 4\mathcal{N}_n^2\mathcal{I}] = a(u)[1-4\,\mathcal{I}]\\ B_1 + B_2 + C &= -\frac{1}{4}\frac{1}{n^2\mathcal{N}_n^2}[n^2\mathcal{N}_n^2 - 4n^2\mathcal{N}_n^2\mathcal{I} + 2\frac{1}{2}n^2\mathcal{N}_n^2 - 2\cdot 4n^2\mathcal{N}_n^2\mathcal{J}] = -1 + 4\mathcal{I} - 2\frac{1}{2} + 2\cdot 4\mathcal{J};\end{aligned}$$

i.e.

$$A + B_1 + B_2 + C = a(u)[1 - 4\mathcal{I}] - \frac{1}{4} + \mathcal{I} - \frac{1}{4} + 2\mathcal{J} = \left[a(u) - \frac{1}{4}\right][1 - 4\mathcal{I}] - \frac{1}{4} + 2\mathcal{J}.$$

Moreover,

$$-\left(a(u) - \frac{1}{4}\right)\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(4\mathcal{I}_{\lambda,\lambda}(4,0) - 1) = -\left(a(u) - \frac{1}{4}\right)(4\mathcal{I} - 1),$$

and

$$-\left(a(u) - \frac{1}{4}\right)\frac{1}{\mathcal{N}_n/2}\sum_{\lambda,\lambda'\in\Lambda_n^+}(|a_\lambda|^2 - 1)(|a_{\lambda'}|^2 - 1)(1 - 4\mathcal{I}_{\lambda,\lambda'}(2,2)) = -\left(a(u) - \frac{1}{4}\right)(1 - 4\mathcal{I}).$$

Finally,

$$\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}(4\mathcal{I}_{\lambda,\lambda}(4,0) - 1)\frac{1}{\mathcal{N}_n/2}\sum_{\lambda,\lambda'\in\Lambda_n^+}(|a_\lambda|^2 - 1)\left(|a_{\lambda'}|^2 - 1)(a(u) - \frac{1}{4}\left\langle\frac{\lambda}{|\lambda|}, \frac{\lambda'}{|\lambda'|}\right\rangle^2\right)$$
$$= (4\mathcal{I} - 1)\left(a(u) - \frac{1}{4}\right).$$

So we obtain that

$$\begin{aligned}\mathrm{Cov}\,(\mathcal{L}_n^u[4], \mathcal{Z}_n(C')[4]) &\sim \phi(u)\sqrt{\frac{\pi}{2}}\frac{\sqrt{E_n/2}}{\mathcal{N}_n}\frac{\sqrt{2n}}{4\mathcal{N}_n}L\left\{\left[a(u) - \frac{1}{4}\right][1 - 4\mathcal{I}] - \frac{1}{4} + 2\mathcal{J} - (a(u) - \frac{1}{4})(1 - 4\mathcal{I})\right\}\\ &= \phi(u)\sqrt{\frac{\pi}{2}}\frac{\sqrt{E_n/2}}{\mathcal{N}_n}\frac{\sqrt{2n}}{4\mathcal{N}_n}L\left\{-\frac{1}{4} + 2\mathcal{J}\right\}.\end{aligned}$$

Observing that,

$$\mathcal{J} = 5\frac{2}{8^2} + \frac{4}{8^2}\eta\mathcal{I}_4' + \frac{2}{8^2}\eta^2,$$

we arrive at

$$\mathrm{Cov}\,(\mathcal{L}_n^u[4], \mathcal{Z}(C')[4]) \sim \phi(u)\sqrt{\frac{\pi}{2}}\frac{\sqrt{E_n/2}}{\mathcal{N}_n}\frac{\sqrt{2n}}{4\mathcal{N}_n}L\frac{1}{16}\left[1 + 2\eta\mathcal{I}_4' + \eta^2\right]. \quad \square$$

Proof of Corollary 3.23. We need to study the covariance expression for the boundary lengths at thresholds u_1, u_2, that is

$$\mathrm{Cov}(\mathcal{L}_n^{u_1}, \mathcal{L}_n^{u_2}) = 2a(u_1)a(u_2) - \frac{1}{4}(a(u_1) + a(u_2)) + \frac{3 + \eta^2}{64}$$

where we have that

$$a(u) = \frac{u^4 - 6u^2 + 3}{4} + \frac{u^2 - 1}{2} - \frac{1}{8} = \frac{1}{4}u^4 - u^2 + \frac{1}{8}.$$

We have that

$$\begin{aligned}\mathrm{Cov}(\mathcal{L}_n^u, \mathcal{L}_n^0) &= 2a(u)a(0) - \frac{1}{4}(a(u) + a(0)) + \frac{3 + \eta^2}{64}\\ &= 2\left(\frac{1}{4}u^4 - u^2 + \frac{1}{8}\right)\frac{1}{8} - \frac{1}{4}\left(\frac{1}{4}u^4 - u^2 + \frac{1}{8} + \frac{1}{8}\right) + \frac{3 + \eta^2}{64}\\ &= \frac{1}{64}(2(2u^4 - 8u^2 + 1) - 2(2u^4 - 8u^2 + 2) + 3 + \eta^2)\\ &= \frac{1 + \eta^2}{64}.\end{aligned}$$

On the other hand

$$\begin{aligned}\mathrm{Var}(\mathcal{L}_n^u) &= \mathrm{Cov}(\mathcal{L}_n^u, \mathcal{L}_n^u)\\ &= 2\left(\frac{1}{4}u^4 - u^2 + \frac{1}{8}\right)^2 - \frac{1}{2}\left(\frac{1}{4}u^4 - u^2 + \frac{1}{8}\right) + \frac{3 + \eta^2}{64}\\ &= \frac{1}{8}u^8 - u^6 + 2u^4 + \frac{1 + \eta^2}{64} = \frac{u^4}{8}(u^2 - 4)^2 + \frac{1 + \eta^2}{64},\end{aligned}$$

so that the squared correlation is given by

$$\frac{\{\mathrm{Cov}(\mathcal{L}_n^u, 0)\}^2}{\mathrm{Var}(\mathcal{L}_n^0)\mathrm{Var}(\mathcal{L}_n^u)} = \frac{1 + \eta^2}{8u^4(u^2 - 4)^2 + 1 + \eta^2},$$

and there are resonance points for the nodal length at $u = \pm 2$, because at those points obviously $\frac{u^4}{8}(u^2 - 4)^2 = 0$. More generally, considering any two threshold levels u_1, u_2 we obtain that

$$\mathrm{Cov}(\mathcal{L}_n^{u_1}, \mathcal{L}_n^{u_2}) = 2a(u_1)a(u_2) - \frac{1}{4}(a(u_1) + a(u_2)) + \frac{3 + \eta^2}{64}$$

$$= 2\left(\frac{1}{4}u_1^4 - u_1^2 + \frac{1}{8}\right)\left(\frac{1}{4}u_2^4 - u_2^2 + \frac{1}{8}\right) - \frac{1}{4}\left(\frac{1}{4}u_1^4 - u_1^2 + \frac{1}{8} + \frac{1}{4}u_2^4 - u_2^2 + \frac{1}{8}\right) + \frac{3}{64}$$
$$= \frac{1}{8}u_1^4u_2^4 - \frac{1}{2}u_1^4u_2^2 - \frac{1}{2}u_1^2u_2^4 + 2u_1^2u_2^2 + \frac{1+\eta^2}{64},$$

so that the correlation is one if and only if

$$1 = \frac{(\frac{1}{8}u_1^4u_2^4 - \frac{1}{2}u_1^4u_2^2 - \frac{1}{2}u_1^2u_2^4 + 2u_1^2u_2^2 + \frac{1+\eta^2}{64})^2}{(\frac{1}{8}u_1^8 - u_1^6 + 2u_1^4 + \frac{1+\eta^2}{64})(\frac{1}{8}u_2^8 - u_2^6 + 2u_2^4 + \frac{1+\eta^2}{64})},$$

and hence,

$$\left(\frac{1}{8}u_1^4u_2^4 - \frac{1}{2}u_1^4u_2^2 - \frac{1}{2}u_1^2u_2^4 + 2u_1^2u_2^2 + \frac{1+\eta^2}{64}\right)^2$$
$$- \left(\frac{1}{8}u_1^8 - u_1^6 + 2u_1^4 + \frac{1+\eta^2}{64}\right)\left(\frac{1}{8}u_2^8 - u_2^6 + 2u_2^4 + \frac{1+\eta^2}{64}\right)$$
$$= -\frac{1+\eta^2}{512}\left(-u_1^4 + 4u_1^2 + u_2^4 - 4u_2^2\right)^2 = 0. \quad \square$$

6. The 3-dimensional correlation between nodal area and nodal intersections

Proof of Theorem 3.28. We may write [30, (8.99) and Lemma 8.1] as

$$\mathcal{M}[4] \sim \sqrt{\frac{4\pi^2 m}{3}}\,\frac{3\cdot 2\cdot A}{16\cdot 8\cdot \mathcal{N}}\left[\frac{32}{15} + \frac{1}{\mathcal{N}/2}\sum_{\lambda,\lambda'\in\Lambda_{n_j}^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)\left(3 - 9I(2,2) + 14I(2,0)\right.\right.$$
$$\left.\left. -6\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2 + 12\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle I(1,1)\right)\right],$$

with

$$I(k,k') = I_{\lambda,\lambda'}(k,k') := \frac{1}{A}\int_\Sigma \left\langle\frac{\lambda}{|\lambda|}, n(\sigma)\right\rangle^k \left\langle\frac{\lambda'}{|\lambda'|}, n(\sigma)\right\rangle^{k'} d\sigma.$$

For static surfaces we have (Lemma 6.2)

$$I(2,0) = \frac{1}{3}, \qquad I(1,1) = \frac{1}{3}\langle\lambda,\lambda'\rangle$$

hence

$$\mathcal{M}[4] \sim \sqrt{\frac{4\pi^2 m}{3}}\,\frac{3\cdot 2}{16\cdot 8\cdot \mathcal{N}}\,\frac{A}{15}\left[32 + \frac{5}{\mathcal{N}/2}\sum_{\lambda,\lambda'\in\Lambda_{n_j}^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)\left(5 - 27I(2,2)\right.\right.$$
$$\left.\left. -6\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2\right)\right]. \tag{6.27}$$

Starting with the case of doubly static Σ, then

$$I(2,2) = \frac{1}{15}(1 + 2\langle\lambda,\lambda'\rangle^2)$$

so that

$$\mathcal{M}[4] \sim \sqrt{\frac{4\pi^2 m}{3}}\,\frac{3\cdot 2\cdot A}{16\cdot 8\cdot \mathcal{N}}\cdot\frac{16}{15}\left[2 + \frac{1}{\mathcal{N}/2}\sum_{\lambda,\lambda'\in\Lambda_{n_j}^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)\left(1 - 3\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2\right)\right].$$

Comparing with [20]

$$\mathcal{A}[4] \sim \frac{\sqrt{m}}{5\sqrt{3}\mathcal{N}}\cdot 2\left[2 + \frac{1}{\mathcal{N}/2}\sum_{\lambda,\lambda'\in\Lambda_{n_j}^+}(|a_\lambda|^2-1)(|a_{\lambda'}|^2-1)\left(1 - 3\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2\right)\right], \tag{6.28}$$

we see that in this case the two expressions are the same up to a multiplicative factor depending on the area A, and in particular $\mathrm{Corr}(\mathcal{A},\mathcal{M}) \to 1$.

In the general case, from (6.27) and (6.28) we compute Cov $(\mathcal{A}[4], \mathcal{M}[4])$, where many terms cancel out, leaving

$$\mathrm{Cov}\,(\mathcal{A}[4],\mathcal{M}[4]) \sim \frac{m}{\mathcal{N}^2}\frac{1}{5\sqrt{3}}\frac{2\pi}{\sqrt{3}}\frac{3\cdot 2}{16\cdot 8}\frac{A}{15}\cdot\frac{10}{\mathcal{N}^2}\sum_{\lambda,\lambda'\in\Lambda}\left(5-21\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2+18\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^4\right.$$
$$\left.-27I(2,2)+81I(2,2)\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2\right). \tag{6.29}$$

We exchange the order of sums and integral, and apply Lemma 6.1 to simplify (6.29) to (3.24).

Lastly, we take into account the expressions for $\mathrm{Var}(\mathcal{A})$ [29, Theorem 1.2] and $\mathrm{Var}(\mathcal{M})$ [30, (1.19)] to conclude the proof. □

6.1. Static surfaces

Lemma 6.1. *For any static surface Σ, we have as $m\to\infty$, $m \not\equiv 0,4,7 \pmod 8$*

$$\frac{1}{\mathcal{N}^2}\sum_{\lambda,\lambda'\in\Lambda}\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^4 \to \frac{1}{5},$$
$$\frac{1}{\mathcal{N}^2}\sum_{\lambda,\lambda'\in\Lambda} I(2,2) \to \frac{1}{9},$$
$$\frac{1}{\mathcal{N}^2}\sum_{\lambda,\lambda'\in\Lambda} I(2,2)\left\langle\frac{\lambda}{|\lambda|},\frac{\lambda'}{|\lambda'|}\right\rangle^2 \to \frac{11}{15^2}.$$

Proof. In each of the three expressions, we expand the summands and apply [29, Lemma 2.5]. Then we simply recall that the normal n is of norm one to complete the calculations in the second and third expressions. □

Lemma 6.2. *A surface Σ is static if and only if for every i,j one has*

$$\int_\Sigma n_i n_j \, d\sigma = \frac{A}{3}\delta_{ij}.$$

Proof. We write

$$\mathcal{I} := \iint_{\Sigma^2}\langle n(\sigma),n(\sigma')\rangle^2\,d\sigma d\sigma' = \sum_{i,j}\left(\int_\Sigma n_i(\sigma)n_j(\sigma)d\sigma\right)^2 \geq \sum_i\left(\int_\Sigma n_i(\sigma)^2 d\sigma\right)^2. \tag{6.30}$$

The sum of the three integrals

$$\sum_i\int_\Sigma n_i(\sigma)^2 = A$$

is fixed, so the sum of their squares is smallest when they are all equal:

$$\mathcal{I} \geq \sum_i\left(\frac{A}{3}\right)^2 = \frac{A^2}{3}.$$

All summands in (6.30) are non-negative: then $\mathcal{I}$ is minimized, i.e. Σ is static, if and only if for every i,j one has

$$\int_\Sigma n_i n_j \, d\sigma = \frac{A}{3}\delta_{ij}. \quad \square$$

Remark 6.3. Surfaces in Remark 3.27 that satisfy the further condition $\int n_1^4 d\sigma = |\Sigma|/5$ are doubly static, due to Lemma 6.4.

Lemma 6.4. *One has $\mathcal{I}_4 = 1/5$ if and only if Σ is static and*

$$\int n_i n_j n_\ell n_k \, d\sigma = \begin{cases} A/5 & \text{if } i=j=\ell=k,\\ A/15 & \text{if } i,j,\ell,k \text{ are pairwise equal},\\ 0 & \text{otherwise.}\end{cases} \tag{6.31}$$

Generic surfaces Σ satisfy $1/5 \leq \mathcal{I}_4 \leq 1$ (and the maximum is attained by surfaces contained in a plane). If Σ is static, then

$$\frac{1}{5} \leq \mathcal{I}_4 \leq \frac{1}{3}.$$

Proof. The upper bounds are due to $\mathcal{I}_4 \le \mathcal{I}$. For generic Σ,

$$\begin{aligned} A^2\mathcal{I}_4 &:= \iint_{\Sigma^2} \langle n(\sigma), n(\sigma')\rangle^4 \, d\sigma d\sigma' = \sum_{i,j,k,l} \left(\int_\Sigma n_i n_j n_k n_l d\sigma \right)^2 \\ &\ge 3 \left[\sum_i \frac{1}{3} \left(\int_\Sigma n_i^4 d\sigma \right)^2 + \sum_{i<j} 2 \left(\int_\Sigma n_i^2 n_j^2 d\sigma \right)^2 \right] \\ &= 3 \left[\left(\frac{a_{ii}^2}{9} + \frac{a_{ii}^2}{9} + \frac{a_{ii}^2}{9} + \frac{a_{jj}^2}{9} + \frac{a_{jj}^2}{9} + \frac{a_{jj}^2}{9} + a_{ij}^2 + a_{ij}^2 + a_{ik}^2 + a_{jk}^2 \right) \right. \\ &\quad \left. + \left(\frac{a_{kk}^2}{9} + \frac{a_{kk}^2}{9} + \frac{a_{kk}^2}{9} + a_{ik}^2 + a_{jk}^2 \right) \right] \end{aligned} \tag{6.32}$$

with the notation $a_{ij} := \int_\Sigma n_i^2 n_j^2 d\sigma$, where $\{i,j,k\}$ is any permutation of $\{1,2,3\}$. On the RHS of (6.32) there are two brackets: the former is the sum of squares of ten terms, the latter of another five. The latter five sum up to

$$\frac{a_{kk}}{3} + \frac{a_{kk}}{3} + \frac{a_{kk}}{3} + a_{ik} + a_{jk} = \int_\Sigma n_k^2 \, d\sigma.$$

Since the sum of all fifteen is simply A, the former ten have a total of $A - \int_\Sigma n_k^2 \, d\sigma$. With the same idea as in Lemma 6.2,

$$\mathcal{I}_4 \ge \frac{3}{10} \left[3X^2 - 2XA + A^2 \right]$$

where

$$X = X_\Sigma := \max_{k=1,2,3} \left(\int_\Sigma n_k^2 \, d\sigma \right).$$

Now $\sum_{k=1,2,3} \int_\Sigma n_k^2 \, d\sigma = A$ and the three summands are non-negative, so that $X \ge A/3$. Moreover, if A is fixed, then $3x^2 - 2xA + A^2$ is an increasing function of x for $x > A/3$. It follows that $\mathcal{I}_4 \ge 1/5$. In addition, if this minimum is achieved, then necessarily $\int_\Sigma n_k^2 \, d\sigma = A/3$ for $k = 1,2,3$, and (6.31) must hold. It then also follows that

$$\int_\Sigma n_i n_j \, d\sigma = \int_\Sigma n_i^3 n_j \, d\sigma + \int_\Sigma n_i n_j^3 \, d\sigma + \int_\Sigma n_i n_j n_k^2 \, d\sigma = 0, \qquad i \ne j. \qquad \square$$

Lemma 6.5. *One has $\mathcal{I}_k = A^2/(k+1)$ if and only if Σ is static and*

$$\int n_1^x n_2^y n_3^z d\sigma = \begin{cases} A \frac{(x-1)!!(y-1)!!(z-1)!!}{(k+1)!!} & \text{if } x,y,z \text{ are all even,} \\ 0 & \text{otherwise,} \end{cases} \tag{6.33}$$

for all $x,y,z \ge 0$ satisfying $x+y+z=k$. Generic surfaces Σ satisfy $A^2/(k+1) \le \mathcal{I}_k \le A^2$ (and the maximum is attained by surfaces contained in a plane). If Σ is static, then

$$\frac{A^2}{k+1} \le \mathcal{I}_k \le \frac{A^2}{3}.$$

Proof. The upper bounds are due to $\mathcal{I}_k \le \mathcal{I}$. For generic Σ,

$$\begin{aligned} \mathcal{I}_k &:= \iint_{\Sigma^2} \langle n(\sigma), n(\sigma')\rangle^k \, d\sigma d\sigma' = \sum_{x+y+z=k} c_{x,y,z} \left(\int_\Sigma n_1^x n_2^y n_3^z d\sigma \right)^2 \\ &\ge \sum_{\substack{x+y+z=k \\ x,y,z \text{ even}}} c_{x,y,z} \left(\int_\Sigma n_1^x n_2^y n_3^z d\sigma \right)^2 \\ &= (k-1)!! \left[\sum_{\substack{x+y=k \\ x,y \\ \text{even}}} \frac{c_{x,y,0}}{(k-1)!!} \left(\int_\Sigma n_1^x n_2^y d\sigma \right)^2 + \sum_{\substack{x+y+z=k \\ x,y,z \text{ even} \\ z \ge 2}} \frac{c_{x,y,z}}{(k-1)!!} \left(\int_\Sigma n_1^x n_2^y n_3^z d\sigma \right)^2 \right], \end{aligned}$$

with

$$c_{x,y,z} := \binom{k}{x\ y\ z}$$

the trinomial coefficient. In the RHS, we replace each summand with

$$c'_{x,y,z} := \binom{k}{x\ y\ z} \frac{(x-1)!!^2 (y-1)!!^2 (z-1)!!^2}{(k-1)!!}$$

copies of

$$\left(\int_{\Sigma}\frac{n_1^x n_2^y n_3^z}{(x-1)!!(y-1)!!(z-1)!!}\,d\sigma\right)^2.$$

The sum of all integrals is A, and there are a total of

$$\sum_{\substack{x+y+z=k\\ x,y,z\ \text{even}}} c'_{x,y,z}=(k+1)!!$$

integrals. The second sum contains a third of the terms, and they sum up to $\int_\Sigma n_3^2\,d\sigma$. With the same idea as in Lemma 6.2,

$$\mathcal{I}_k\geq(k-1)!!\left[\frac{(A-X)^2}{2(k+1)!!/3}+\frac{X^2}{(k+1)!!/3}\right]=\frac{3}{2(k+1)}\left[3X^2-2XA+A^2\right]$$

where

$$X=X_\Sigma:=\max_{k=1,2,3}\left(\int_\Sigma n_k^2\,d\sigma\right).$$

Now $\sum_{k=1,2,3}\int_\Sigma n_k^2\,d\sigma=A$ and the three summands are non-negative, so that $X\geq A/3$. Moreover, if A is fixed, then $3x^2-2xA+A^2$ is an increasing function of x for $x>A/3$. It follows that $\mathcal{I}_k\geq A^2/(k+1)$.

In addition, if this minimum is achieved, then necessarily $\int_\Sigma n_k^2\,d\sigma=A/3$ for $k=1,2,3$, and (6.33) must hold. It then also follows that

$$\int_\Sigma n_in_j\,d\sigma=\int_\Sigma n_in_j(n_1^2+n_2^2+n_3^2)^{(k-2)}\,d\sigma=0,\qquad i\neq j.\quad\square$$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

We are grateful to GNAMPA-INdAM Project 2022 *Proprietà e teoremi limite per funzionali di campi Gaussiani*, to MUR Department of Excellence Project 2023–2027 *MatModTov*, to MUR Prin 2023–2025 *Grafia*, to Swiss National Science Foundation project 200021_184927, held by Prof. Maryna Viazovska, and to Programme for Young Researchers 'Rita Levi Montalcini' *Discrete and Probabilistic Methods in Mathematics with Applications*, for financial support.

Appendix. Proofs of auxiliary lemmas

Proof of Lemma 3.5. Let C be a curve parameterized by arc-lenght, and ν be a probability measure on $\mathbb{S}^1$. We can assume that ν is invariant w.r.t. rotations by $\pi/2$, indeed the curve is static if and only if $4\mathcal{B}_C(\nu)-L^2=0$ holds for ν the uniform measure. Via some manipulations, we rewrite

$$\mathcal{B}_C(\nu)=\frac{L^2}{8}(1+2\mathcal{I}_2)+\frac{L^2}{8}\hat{\nu}(4)(1-2\mathcal{I}_2^\perp);\tag{A.34}$$

with $\hat{\nu}(k)=\int_{\mathbb{S}^1}z^{-k}d\nu(z)$, for any $k\in\mathbb{Z}$, Fourier transform of ν; note that $\hat{\nu}(4)$ is real if ν is invariant under the transformations $z\to\bar{z}$ and $z\to i\cdot z$; and

$$\mathcal{I}_2=\mathcal{I}_{2,C}=\frac{1}{L^2}\int_0^L\int_0^L\langle\dot\gamma(t),\dot\gamma(u)\rangle^2dtdu,$$

$$\mathcal{I}_2^\perp=\mathcal{I}_{2,C}^\perp=\frac{1}{L^2}\int_0^L\int_0^L\langle\dot\gamma(t),(\dot\gamma_2(u),\dot\gamma_1(u))\rangle^2dtdu.$$

Then C is static if and only if $\mathcal{I}_2=\mathcal{I}_2^\perp=1/2$: indeed if $\mathcal{I}_2=\mathcal{I}_2^\perp=1/2$ then from (A.34) it holds that $4\mathcal{B}_C(\nu)-L^2=0$ in particular for ν the uniform measure. On the other hand, we may rearrange

$$\mathcal{I}_2=\sum_{i,j=1,2}\left(\frac{1}{L}\int_0^L\dot\gamma_i(t)\dot\gamma_j(t)dt\right)^2\geq\sum_{i=1,2}\left(\frac{1}{L}\int_0^L\dot\gamma_i(t)^2dt\right)^2.$$

Since $\int_0^L\dot\gamma_1(t)^2dt+\int_0^L\dot\gamma_2(t)^2dt=L$, then $\mathcal{I}_2\geq1/2$ with equality if and only if $\int_0^L\dot\gamma_1(t)^2dt=\int_0^L\dot\gamma_2(t)^2dt=L/2$ and $\int_0^L\dot\gamma_1(t)\dot\gamma_2(t)dt=0$. These conditions also ensure that $\mathcal{I}_2^\perp=1/2$, and the proof of this lemma is complete. $\square$

Proof of Lemma 3.10. Similarly to the proof of Lemma 3.5, we have

$$\mathcal{I}_4 = \sum_{i=0}^{4} \binom{4}{i} \left[\frac{1}{L}\int_0^L \dot\gamma_1(t)^i \dot\gamma_2(t)^{4-i} dt\right]^2 \geq \sum_{i=1,2} \left(\frac{1}{L}\int_0^L \dot\gamma_i(t)^4 dt\right)^2 + 6\left(\frac{1}{L}\int_0^L \dot\gamma_1(t)^2\dot\gamma_2(t)^2 dt\right)^2. \tag{A.35}$$

Clearly $\int_0^L \dot\gamma_1(t)^4 dt + \int_0^L \dot\gamma_2(t)^4 dt + 2\int_0^L \dot\gamma_1(t)^2\dot\gamma_2(t)^2 dt = L$, hence $\mathcal{I}_4 \geq 3/8$ with equality iff

$$\frac{1}{L}\int_0^L \dot\gamma_i(t)\dot\gamma_j(t)\dot\gamma_k(t)\dot\gamma_l(t)dt = \begin{cases} 3/8 & \text{if } i=j=k=l, \\ 1/8 & \text{if } i=j\neq k=l, \\ 0 & \text{otherwise.}\end{cases} \tag{A.36}$$

If (A.36) holds true, this clearly means that $A = 1/8$, and $B = 0$, and moreover C is static due to Lemma 3.5 (e.g. $\int_0^L \dot\gamma_1(t)^2 dt = \int_0^L \dot\gamma_1(t)^4 dt + \int_0^L \dot\gamma_1(t)^2\dot\gamma_2(t)^2 dt = 3L/8 + L/8 = L/2$).

Vice versa, assume that C is static, $A = 1/8$, and $B = 0$. By staticity we have $\int_0^L \dot\gamma_1(t)^4 dt = \int_0^L \dot\gamma_2(t)^4 dt$. Using $A = 1/8$ and the fact that C is unit speed we get also the first case of (A.36). As for the third one, it suffices to point out that $\int_0^L \dot\gamma_1(t)\dot\gamma_2(t)^3 dt = -B = 0$. □

Proof of Lemma 3.12. To construct a family of doubly static curves, we adapt [25, Appendix G]. The condition is $\mathcal{I}_{4,C} = 3/8$. Bearing in mind [25, (G.4)], we impose

$$\sum_{j=0}^{k-1} \cos\left(\dot\gamma(t) - \phi(u) + j\cdot\frac{2\pi}{k}\right)^4 = \frac{3k}{8}$$

where $\dot\gamma(u) = \exp(i\phi(u))$. Due to the identity

$$\cos(x)^4 = \frac{3}{8} + \frac{\cos(2x)}{2} + \frac{\cos(4x)}{8},$$

it suffices to impose

$$\frac{k}{\gcd(2,k)} \geq 2 \qquad \text{and} \qquad \frac{k}{\gcd(4,k)} \geq 2,$$

i.e. $k = 3$ or $k \geq 5$. For $k = 4$ the curve is static, but not necessarily doubly static (for C to be static we only need $k/\gcd(2,k) \geq 2$ – cf. Fig. 1). □

Proof of Lemma 3.7. To prove (3.19), we rewrite $\mathcal{I}_4$ as

$$\mathcal{I}_4 = \sum_{i=0}^{4} \binom{4}{i} \left[\frac{1}{L}\int_0^L \dot\gamma_1(t)^i \dot\gamma_2(t)^{4-i} dt\right]^2.$$

The terms for $i = 1,2$ are simply $4B^2$ and $6A^2$. Since C is static, in light of Lemma 3.5 the term for $i = 3$ is $4(-B)^2$, and the terms for $i = 0,4$ are each equal to $(1/2 - A)^2$. Rearranging proves (3.19).

For static curves $(\int \dot\gamma_i^4 dt/L) + A = 1/2$, with $i = 1,2$. Moreover, by Cauchy–Schwarz, A is always the smaller of the two summands , hence $0 < A < 1/4$ (extrema excluded else C would be a straight line segment). Rearranging, $-1 < \mathcal{I}_4' < 1$.

The inequality $3/8 \leq \mathcal{I}_4$ is shown in Lemma 3.10. For the upper bound, by Cauchy–Schwarz $\mathcal{I}_4 < \mathcal{I}_2$ (as defined in Lemma 3.5), and $\mathcal{I}_2 = 1/2$ due to staticity (extremum excluded again else C would be a straight line segment). Lastly, we combine $\mathcal{I}_4 < 1/2$ and (3.19) to find $B^2 < A(1-4A)/4$. □

To state the technical results in Lemma A.1, we introduce the following notation

$$\mathcal{I} := \frac{1}{L}\int_0^L \left[\frac{3+\eta}{8}\dot\gamma_1^4(t) + \frac{3+\eta}{8}\dot\gamma_2^4(t) + 6\frac{1-\eta}{8}\dot\gamma_1^2(t)\dot\gamma_2^2(t)\right]dt,$$

$$\begin{aligned}\mathcal{J} := \frac{1}{L}\int_0^L \Bigg[&\left(\frac{3+\eta}{8}\right)^2\dot\gamma_1^4(t) + \left(\frac{1-\eta}{8}\right)^2\dot\gamma_1^4(t) + 4\left(\frac{3+\eta}{8}\right)\left(\frac{1-\eta}{8}\right)\dot\gamma_1^2(t)\dot\gamma_2^2(t) \\ &+ \left(\frac{1-\eta}{8}\right)^2\dot\gamma_2^4(t) + \left(\frac{3+\eta}{8}\right)^2\dot\gamma_2^4(t) + 8\left(\frac{1-\eta}{8}\right)^2\dot\gamma_1^2(t)\dot\gamma_2^2(t)\Bigg]dt\end{aligned}$$

and we observe that

$$\mathcal{I} = \frac{3}{8} + \frac{\eta}{8}\mathcal{I}_4', \qquad \mathcal{J} = 5\frac{2}{8^2} + \frac{4}{8^2}\eta\mathcal{I}_4' + \frac{2}{8^2}\eta^2.$$

Lemma A.1. *We have*

$$\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} I_{\lambda,\lambda}(2,0) = \frac{1}{2}, \tag{A.37}$$

$$\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} I_{\lambda,\lambda}(2,2) = \mathcal{I}, \tag{A.38}$$

$$\frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n} I_{\lambda,\lambda'}(2,2)=\frac{1}{4}. \tag{A.39}$$

$$\frac{1}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n}\langle\lambda,\lambda'\rangle^2=\frac{1}{2}, \tag{A.40}$$

$$\frac{1}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n}\langle\lambda,\lambda'\rangle^2 I_{\lambda,\lambda'}(2,2)=\mathcal{J}. \tag{A.41}$$

Proof. Eq. (A.37) follows immediately by observing that

$$\begin{aligned}\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} I_{\lambda,\lambda}(2,0)&=\frac{1}{n\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\frac{1}{L}\int_0^L[\lambda_1\dot\gamma_1(t)+\lambda_2\dot\gamma_2(t)]^2dt\\&=\frac{1}{n\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\frac{1}{L}\int_0^L[\lambda_1^2\dot\gamma_1^2(t)+\lambda_2^2\dot\gamma_2^2(t)+2\lambda_1\lambda_2\dot\gamma_1(t)\dot\gamma_2(t)]dt\\&=\frac{1}{L}\int_0^L\left[\frac{1}{2}\dot\gamma_1^2(t)+\frac{1}{2}\dot\gamma_2^2(t)\right]dt\\&=\frac{1}{2}\frac{1}{L}\int_0^L\langle\dot\gamma(t),\dot\gamma(t)\rangle\,dt=\frac{1}{2},\end{aligned}$$

where we used the relations

$$\frac{1}{n\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_i^2=\frac{1}{2},\quad i=1,2,\qquad\text{and}\qquad\frac{1}{n\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_1\lambda_2=0. \tag{A.42}$$

The proof of (A.38) is similar, in fact

$$\begin{aligned}\frac{1}{\mathcal{N}_n}\sum_{\lambda\in\Lambda_n} I_{\lambda,\lambda}(2,2)&=\frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\frac{1}{L}\int_0^L[\lambda_1\dot\gamma_1(t)+\lambda_2\dot\gamma_2(t)]^4dt\\&=\frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\frac{1}{L}\int_0^L[\lambda_1^4\dot\gamma_1^4(t)+\lambda_2^4\dot\gamma_2^4(t)+6\lambda_1^2\lambda_2^2\dot\gamma_1^2(t)\dot\gamma_2^2(t)]dt\\&=\mathcal{I},\end{aligned}$$

since

$$\frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_i^4=\frac{3+\eta}{8},\quad i=1,2,\qquad\text{and}\qquad\frac{1}{n^2\mathcal{N}_n}\sum_{\lambda\in\Lambda_n}\lambda_1^2\lambda_2^2=\frac{1-\eta}{8}.$$

To prove (A.39), we observe that

$$\begin{aligned}\frac{1}{\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n} I_{\lambda,\lambda'}(2,2)&=\frac{1}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n}\frac{1}{L}\int_0^L[\lambda_1\dot\gamma_1(t)+\lambda_2\dot\gamma_2(t)]^2[\lambda_1'\dot\gamma_1(t)+\lambda_2'\dot\gamma_2(t)]^2dt\\&=\frac{1}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n}\frac{1}{L}\int_0^L[\lambda_1^2\dot\gamma_1^2(t)+\lambda_2^2\dot\gamma_2^2(t)+2\lambda_1\lambda_2\dot\gamma_1(t)\dot\gamma_2(t)]\\&\qquad\times[(\lambda_1')^2\dot\gamma_1^2(t)+(\lambda_2')^2\dot\gamma_2^2(t)+2\lambda_1'\lambda_2'\dot\gamma_1(t)\dot\gamma_2(t)]dt\\&=\frac{1}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n}\frac{1}{L}\int_0^L[\lambda_1^2(\lambda_1')^2\dot\gamma_1^4(t)+\lambda_1^2(\lambda_2')^2\dot\gamma_1^2(t)\dot\gamma_2^2(t)\\&\qquad+\lambda_2^2(\lambda_1')^2\dot\gamma_1^2(t)\dot\gamma_2^2(t)+\lambda_2^2(\lambda_2')^2\dot\gamma_2^4(t)]dt\\&=\frac{1}{L}\int_0^L\left[\frac{1}{4}\dot\gamma_1^4(t)+\frac{1}{2}\dot\gamma_1^2(t)\dot\gamma_2^2(t)+\frac{1}{4}\dot\gamma_2^4(t)\right]dt\\&=\frac{1}{4}\frac{1}{L}\int_0^L\langle\dot\gamma(t),\dot\gamma(t)\rangle^2dt=\frac{1}{4}.\end{aligned}$$

Eq. (A.40) is an immediate consequence of (A.42). Eq. (A.41) follows from

$$\begin{aligned}&\frac{1}{n^2\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n}\langle\lambda,\lambda'\rangle^2 I_{\lambda,\lambda'}(2,2)\\&=\frac{1}{n^4\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n}(\lambda_1\lambda_1'+\lambda_2\lambda_2')^2\frac{1}{L}\int_0^L[\lambda_1\dot\gamma_1(t)+\lambda_2\dot\gamma_2(t)]^2[\lambda_1'\dot\gamma_1(t)+\lambda_2'\dot\gamma_2(t)]^2dt\\&=\frac{1}{n^4\mathcal{N}_n^2}\sum_{\lambda,\lambda'\in\Lambda_n}\{\lambda_1^2(\lambda_1')^2+\lambda_2^2(\lambda_2')^2+2\lambda_1\lambda_2\lambda_1'\lambda_2'\}\\&\times\frac{1}{L}\int_0^L[\lambda_1^2\dot\gamma_1^2(t)+\lambda_2^2\dot\gamma_2^2(t)+2\lambda_1\lambda_2\dot\gamma_1(t)\dot\gamma_2(t)]\,[(\lambda_1')^2\dot\gamma_1^2(t)+(\lambda_2')^2\dot\gamma_2^2(t)+2\lambda_1'\lambda_2'\dot\gamma_1(t)\dot\gamma_2(t)]dt\end{aligned}$$

$$= \frac{1}{n^4 \mathcal{N}_n^2} \sum_{\lambda,\lambda' \in \Lambda_n} \frac{1}{L} \int_0^L [\lambda_1^4 (\lambda_1')^4 \dot{\gamma}_1^4(t) + \lambda_1^2 (\lambda_1')^2 \lambda_2^2 (\lambda_2')^2 \dot{\gamma}_1^4(t) + \lambda_1^4 (\lambda_1')^2 (\lambda_2')^2 \dot{\gamma}_1^2(t) \dot{\gamma}_2^2(t)$$

$$+ \lambda_1^2 \lambda_2^2 (\lambda_2')^4 \dot{\gamma}_1^2(t) \dot{\gamma}_2^2(t) + \lambda_1^2 \lambda_2^2 (\lambda_1')^4 \dot{\gamma}_1^2(t) \dot{\gamma}_2^2(t) + (\lambda_2')^2 \lambda_2^4 (\lambda_1')^2 \dot{\gamma}_1^2(t) \dot{\gamma}_2^2(t)$$

$$+ \lambda_1^2 (\lambda_1')^2 \lambda_2^2 (\lambda_2')^2 \dot{\gamma}_2^4(t) + \lambda_2^4 (\lambda_2')^4 \dot{\gamma}_2^4(t) + 8 \lambda_1^2 \lambda_2^2 (\lambda_1')^2 (\lambda_2')^2 \dot{\gamma}_1^2(t) \dot{\gamma}_2^2(t)] dt$$

$$= \mathcal{J}. \quad \square$$

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