

HYPERSURFACES OF CONSTANT HIGHER ORDER MEAN CURVATURE IN WARPED PRODUCTS

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ABSTRACT. In this paper we characterize compact and complete hypersurfaces with some constant higher order mean curvature into warped product spaces. Our approach is based on the use of a new trace operator version of the Omori-Yau maximum principle which seems to be interesting in its own.

1. INTRODUCTION

A classical result by Alexandrov [1] states that a compact hypersurface with constant mean curvature embedded in Euclidean space must be a round sphere. The original proof is based on a clever use of the maximum principle for elliptic partial differential equations. This method, now called the Alexandrov reflexion method, also works for hypersurfaces in ambient spaces having a sufficiently large number of isometric reflexions, for instance in the hyperbolic space.

To extend the above result to a larger class of Riemannian spaces it appears convenient to consider manifolds with a sufficiently large family of complete embedded constant mean curvature hypersurfaces. Such a family plays the role of the umbilical hypersurfaces in spaces of constant sectional curvature. In this setting, given an immersed hypersurface, the next step is to look for geometric assumptions that force the hypersurface to be one of the selected family. In the compact case, this was first done by Montiel [11], which considers as a natural class of ambient manifolds that of warped products $M^{n+1} = \mathbb{R} \times_{\rho} \mathbb{P}^n$, where \mathbb{P}^n is a complete n -dimensional Riemannian manifold and $\rho : \mathbb{R} \rightarrow \mathbb{R}_+$ is a smooth warping function. Then each leaf $\mathbb{P}_t = \{t\} \times \mathbb{P}^n$ (called here a *slice*) of the foliation $t \in \mathbb{R} \mapsto \mathbb{P}_t$ of M^{n+1} is a complete hypersurface with constant mean curvature. This approach was later considered in [4] where Alías and Dajczer generalized Montiel's results. Some of those generalizations hold even for complete, not necessarily compact, hypersurfaces.

The aim of the present paper is to extend the investigation to hypersurfaces with constant higher order mean curvatures, both in the compact and in the complete case. Our main analytical tools to reach the goal are provided by the Newton tensors P_k of the hypersurface, their associated second order differential operators L_k , and further various combinations of them. We underline that in the complete

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case, we tailor an appropriate version of the Omori-Yau maximum principle for very general operators to deal with the problems at hand.

The paper begins with Section 2, collecting a number of preliminary results and fixing notation. Section 3 is devoted to a proof of a generalized version of the Omori-Yau maximum principle for a wide class of trace operators and to determining some geometrical assumptions guaranteeing the validity of the principle (see for instance Corollary 4). In Section 4, as a first application of our method and inspired by the mean curvature estimates given in [3], we derive higher order mean curvature estimates for complete immersed hypersurfaces. In Sections 5 and 6 we determine sufficient conditions for hypersurfaces with constant higher order mean curvatures contained in a slab to be a slice of the ambient space, extending previous results for the case of constant mean curvature hypersurfaces given in [4]. Finally, in Section 7 we give a number of further results recovering this uniqueness property.

2. PRELIMINARIES

Let $f : \Sigma^n \rightarrow M^{n+1}$ be a connected hypersurface isometrically immersed into the Riemannian manifold M^{n+1} . We let A denote the second fundamental form of the immersion with respect to a (locally defined) normal vector field N . Its eigenvalues, $\kappa_1, \dots, \kappa_n$, are the principal curvatures of the hypersurface (in the direction of N). Their elementary symmetric functions $S_k, k = 0, \dots, n, S_0 = 1$, define the k -mean curvatures of the immersion via the formula

$$H_k = \binom{n}{k}^{-1} S_k.$$

Thus $H_1 = H$ is the mean curvature, H_n is the Gauss-Kronecker curvature, and H_2 is, when the ambient space is Einstein, a multiple of the scalar curvature modulo a constant.

The Newton tensors associated to the immersion are inductively defined by

$$P_0 = I, \quad P_k = S_k I - A P_{k-1}.$$

Note, for further use, that $\text{Tr} P_k = (n-k)S_k$ and $\text{Tr} A P_k = (k+1)S_{k+1}$. In the sequel we shall need to have the operators P_k to be globally defined on $T\Sigma$. Obviously, the sign of the second fundamental form A depends on the chosen local unit field N . However, when k is even the sign of S_k (and hence H_k) does not depend on the chosen N , which implies, by its very definition, that the operator P_k is a globally defined tensor field on $T\Sigma$. On the other hand, when k is odd in order to have P_k globally defined, we need to assume that Σ is *two sided*. Recall that a hypersurface $f : \Sigma^n \rightarrow M^{n+1}$ is called two sided if its normal bundle is trivial, i.e. there exists a globally defined unit normal vector field N . For instance, every hypersurface with never vanishing mean curvature is trivially two sided. When the hypersurface is two sided, a choice of N on Σ makes the second fundamental form A and its associated Newton tensors P_k globally defined tensor fields on $T\Sigma$.

Let ∇ stand for the Levi-Civita connection of Σ . For a given function $u \in C^2(\Sigma)$, we denote by $\text{hess } u : T\Sigma \rightarrow T\Sigma$ the symmetric operator given by $\text{hess } u(X) = \nabla_X \nabla u$ for every $X \in T\Sigma$, and by $\text{Hess } u : T\Sigma \times T\Sigma \rightarrow C^\infty(\Sigma)$ the metrically equivalent bilinear form given by

$$\text{Hess } u(X, Y) = \langle \text{hess } u(X), Y \rangle.$$

Associated to each globally defined Newton tensor $P_k : T\Sigma \rightarrow T\Sigma$, we may consider the second order differential operator $L_k : C^\infty(\Sigma) \rightarrow C^\infty(\Sigma)$ given by $L_k = \text{Tr}(P_k \circ \text{hess})$. In particular, L_0 is the Laplace-Beltrami operator Δ . Observe that

$$L_k(u) = \text{div}(P_k \nabla u) - \langle \text{div} P_k, \nabla u \rangle,$$

where $\text{div} P_k = \text{Tr} \nabla P_k$. This implies that L_k is elliptic if and only if P_k is positive definite, and in this case the maximum principle holds for L_k . See for instance Theorem 3.1 in [10].

Note that the ellipticity of the operator L_1 is guaranteed by the assumption $H_2 > 0$. Indeed, if this happens the mean curvature does not vanish on Σ , because of the basic inequality $H_1^2 \geq H_2$. Therefore, the immersion is two sided and we can choose the normal unit vector N on Σ so that $H_1 > 0$. Furthermore

$$n^2 H_1^2 = \sum_{j=1}^n \kappa_j^2 + n(n-1)H_2 > \kappa_i^2$$

for every $i = 1, \dots, n$, and then the eigenvalues of P_1 satisfy $\mu_{1,i} = nH_1 - \kappa_i > 0$ for every i (see, for instance, Lemma 3.10 in [8]). This shows ellipticity of L_1 . Regarding the ellipticity of L_j when $j \geq 2$, we will assume that there exists an elliptic point in Σ , that is, a point $p \in \Sigma$ at which the second fundamental form A is positive definite with respect to an appropriate orientation. The existence of an elliptic point implies that H_k is positive at that point, and applying Gårding inequalities [9], we have

$$(1) \quad H_1 \geq H_2^{1/2} \geq \dots \geq H_{k-1}^{1/(k-1)} \geq H_k^{1/k} > 0,$$

with equality at any stage only for an umbilical point. Therefore, in case H_k is constant, the immersion is two sided and $H_1 > 0$ for the chosen orientation. Moreover, in this case, for every $1 \leq j \leq k-1$, the operators L_j are elliptic or, equivalently, the operators P_j are positive definite (for a proof of this fact, see [7, Proposition 3.2]). Observe that the existence of an elliptic point is not guaranteed, in general, even in the compact case. For instance, it is clear that totally geodesic spheres and Clifford tori in \mathbb{S}^{n+1} are examples of compact isoparametric hypersurfaces without elliptic points. On the contrary, it is not difficult to see that every compact hypersurface in an open hemisphere has elliptic points (see for instance the proof of Theorem 11.1 in [5]).

In what follows, we consider the case when the ambient space is a warped product $M^{n+1} = I \times_\rho \mathbb{P}^n$, where $I \subseteq \mathbb{R}$ is an open interval, \mathbb{P}^n is a complete n -dimensional Riemannian manifold, and $\rho : I \rightarrow \mathbb{R}_+$ is a smooth function. The product manifold $I \times \mathbb{P}^n$ is endowed with the Riemannian metric

$$\langle \cdot, \cdot \rangle = \pi_I^*(dt^2) + \rho^2(\pi_I)\pi_{\mathbb{P}}^*(\langle \cdot, \cdot \rangle_{\mathbb{P}}).$$

Here π_I and $\pi_{\mathbb{P}}$ denote the projections onto the corresponding factor and $\langle \cdot, \cdot \rangle_{\mathbb{P}}$ is the Riemannian metric on \mathbb{P}^n . In particular, $M^{n+1} = I \times_\rho \mathbb{P}^n$ is complete if and only if $I = \mathbb{R}$. We also observe that each leaf $\mathbb{P}_t = \{t\} \times \mathbb{P}^n$ of the foliation $t \rightarrow \mathbb{P}_t$ of M^{n+1} is a complete totally umbilical hypersurface with constant k -mean curvature

$$\mathcal{H}_k(t) = \left(\frac{\rho'(t)}{\rho(t)} \right)^k, \quad 0 \leq k \leq n,$$

with respect to $-\partial/\partial t$.

Let $f : \Sigma^n \rightarrow M^{n+1} = I \times_\rho \mathbb{P}^n$ be an isometrically immersed hypersurface. We define the *height function* $h \in C^\infty(\Sigma)$ by setting $h = \pi_I \circ f$. In this context and following the terminology introduced in [3], we will say that the hypersurface is *contained in a slab* if $f(\Sigma)$ lies between two leaves $\mathbb{P}_{t_1}, \mathbb{P}_{t_2}$ with $t_1 < t_2$ of the foliation.

We observe that results similar to those of the present paper hold for spacelike hypersurfaces in a generalized Robertson-Walker spacetime. These will appear in our paper [6].

3. THE GENERALIZED OMORI-YAU MAXIMUM PRINCIPLE FOR TRACE OPERATORS

Let Σ^n be an n -dimensional Riemannian manifold. Following the terminology introduced in [14], the *Omori-Yau maximum principle* is said to hold on Σ for the Laplace operator if, for any smooth function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence of points $\{p_j\}_{j \in \mathbb{N}}$ in Σ with the properties

$$(i) \ u(p_j) > u^* - \frac{1}{j}, \quad (ii) \ \|\nabla u(p_j)\| < \frac{1}{j}, \quad \text{and} \quad (iii) \ \Delta u(p_j) < \frac{1}{j}.$$

Equivalently, for any $u \in C^2(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$ there exists a sequence $\{p_j\}_{j \in \mathbb{N}}$ in Σ satisfying

$$(i) \ u(p_j) < u_* + \frac{1}{j}, \quad (ii) \ \|\nabla u(p_j)\| < \frac{1}{j}, \quad \text{and} \quad (iii) \ \Delta u(p_j) > -\frac{1}{j}.$$

In this sense, the classical result given by Omori [12] and Yau [15] states that the Omori-Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below. More generally, as shown by Pigola, Rigoli and Setti [14, Example 1.13], a sufficiently controlled decay of the radial Ricci curvature of the form

$$\text{Ric}_\Sigma(\nabla r, \nabla r) \geq -C^2 G(r),$$

where r is the distance function on Σ to a fixed point, C a positive constant, and $G : [0, +\infty) \rightarrow \mathbb{R}$ is a smooth function satisfying

$$(i) \ G(0) > 0, \quad (ii) \ G'(t) \geq 0, \quad (iii) \ \int_0^{+\infty} 1/\sqrt{G(t)} = +\infty, \quad \text{and}$$

$$(iv) \ \limsup_{t \rightarrow +\infty} tG(\sqrt{t})/G(t) < +\infty,$$

suffices to imply the validity of the Omori-Yau maximum principle.

On the other hand, as observed again in [14], the validity of Omori-Yau maximum principle on Σ^n does not depend on curvature bounds as much as one would expect. For instance, the Omori-Yau maximum principle holds on every Riemannian manifold admitting a non-negative C^2 function γ satisfying the following requirements: (i) $\gamma(p) \rightarrow +\infty$ as $p \rightarrow \infty$; (ii) there exists $A > 0$ such that $\|\nabla \gamma\| \leq A\sqrt{\gamma}$ off a compact set; and (iii) there exists $B > 0$ such that $\Delta \gamma \leq B\sqrt{\gamma}\sqrt{G(\sqrt{\gamma})}$ off a compact set, where G is as above (see [14, Theorem 1.9]).

For the proof of our main results in this paper, we will use the following generalization of [14, Theorem 1.9] for trace type differential operators which includes the operators L_k .

Theorem 1. *Let $(\Sigma, \langle \cdot, \cdot \rangle)$ be a Riemannian manifold, and let $L = \text{Tr}(P \circ \text{hess})$ be a semi-elliptic operator, where $P : T\Sigma \rightarrow T\Sigma$ is a positive semi-definite symmetric tensor satisfying $\sup_{\Sigma} \text{Tr}P < +\infty$. Assume the existence of a non-negative C^2 function γ with the properties*

- (2) $\gamma(p) \rightarrow +\infty$ as $p \rightarrow \infty$,
- (3) $\exists A > 0$ such that $\|\nabla\gamma\| \leq A\gamma^{\frac{1}{2}}$ off a compact set,
- (4) $\exists B > 0$ such that $L\gamma \leq B\gamma^{\frac{1}{2}}G(\gamma^{\frac{1}{2}})^{\frac{1}{2}}$ off a compact set,

where G is a smooth function on $[0, +\infty)$ such that

- (5) (i) $G(0) > 0$; (ii) $G'(t) \geq 0$ on $[0, +\infty)$;
- (iii) $G(t)^{-\frac{1}{2}} \notin L^1(+\infty)$; (iv) $\limsup_{t \rightarrow \infty} \frac{tG(t^{\frac{1}{2}})}{G(t)} < +\infty$.

Then, given any function $u \in C^2(\Sigma)$ with $u^* = \sup_{\Sigma} u < +\infty$, there exists a sequence $\{p_j\}_{j \in \mathbb{N}} \subset \Sigma$ with the properties

- (6) (i) $u(p_j) > u^* - \frac{1}{j}$, (ii) $\|\nabla u(p_j)\| < \frac{1}{j}$, (iii) $Lu(p_j) < \frac{1}{j}$, $\forall j \in \mathbb{N}$.

Proof. Define the function

$$\varphi(t) = e^{\int_0^t G(s)^{-\frac{1}{2}} ds}.$$

Note that $\varphi(t)$ is a well defined, smooth, positive function such that $\varphi(t) \rightarrow +\infty$ as $t \rightarrow +\infty$. Moreover

$$\varphi'(t) = G(t)^{-\frac{1}{2}}\varphi(t) \quad \text{and} \quad \varphi''(t) = \left(G(t)^{-1} - 2G(t)^{-\frac{3}{2}}G'(t)\right)\varphi(t),$$

and therefore

$$(7) \quad \left(\frac{\varphi'(t)}{\varphi(t)}\right)^2 - \frac{\varphi''(t)}{\varphi(t)} = 2G(t)^{-\frac{3}{2}}G'(t) \geq 0.$$

Then, using assumption (5),(iv) we get

$$(8) \quad \frac{\varphi'(t)}{\varphi(t)} \leq c(tG(t^{\frac{1}{2}}))^{-\frac{1}{2}},$$

for some constant $c > 0$.

Fix a point $p_0 \in \Sigma$ and, for a fixed $j \in \mathbb{N}$ define

$$f_j(p) = \frac{u(p) - u(p_0) + 1}{\varphi(\gamma(p))^{\frac{1}{j}}}.$$

Then $f_j(p_0) = 1/\varphi(\gamma(p_0))^{1/j} > 0$. Moreover, since $u^* < +\infty$ and $\varphi(\gamma(p)) \rightarrow +\infty$ as $p \rightarrow \infty$, we have $\limsup_{p \rightarrow \infty} f_j(p) \leq 0$. Thus, f_j attains a positive absolute maximum at $p_j \in \Sigma$. Iterating this procedure, we produce a sequence $\{p_j\}_{j \in \mathbb{N}} \subset \Sigma$. The proof of (6),(i) and (6),(ii) is the same as in [14], so we only prove (6),(iii). Proceeding as in Theorem 1.9 of [14], up to passing to a subsequence, we have $\lim_{j \rightarrow +\infty} u(p_j) = u^*$. If $\{p_j\}$ is contained in a compact set, then $p_j \rightarrow \bar{p} \in \Sigma$ as $j \rightarrow +\infty$ and u attains its absolute maximum. Hence we have

$$u(\bar{p}) = u^*, \quad \|\nabla u(\bar{p})\| = 0, \quad \text{Hess } u(\bar{p}) \leq 0.$$

In particular, since P is positive semi-definite, it holds that $Lu(\bar{p}) \leq 0$. Hence the sequence $p_j = \bar{p}$, for each j , satisfies all the requirements. Now consider the case

when $\{p_j\}$ diverges off a compact set, so that, according to (2), $\gamma(p_j) \rightarrow +\infty$. Since f_j attains a positive maximum at p_j , we have

$$(i) (\nabla \log f_j)(p_j) = 0, \quad (ii) \text{Hess } \log f_j(p_j) \leq 0.$$

A simple computation then gives

$$\begin{aligned} \text{Hess } u(p_j)(v, v) &\leq \frac{1}{j}(u(p_j) - u(p_0) + 1) \left\{ \frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} \text{Hess } \gamma(p_j)(v, v) \right. \\ &\quad \left. + \left[\left(\frac{1}{j} - 1 \right) \left(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} \right)^2 + \frac{\varphi''(\gamma(p_j))}{\varphi(\gamma(p_j))} \right] \langle \nabla \gamma(p_j), v \rangle^2 \right\} \\ &\leq \frac{1}{j}(u(p_j) - u(p_0) + 1) \left\{ \frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} \text{Hess } \gamma(p_j)(v, v) \right. \\ &\quad \left. + \frac{1}{j} \left(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} \right)^2 \langle \nabla \gamma(p_j), v \rangle^2 \right\}, \end{aligned}$$

for every $v \in T_{p_j}\Sigma$, where we have used (7). Let $\{e_1, \dots, e_n\} \subset T_{p_j}\Sigma$ be an orthonormal basis of eigenvectors of $P(p_j)$ corresponding to the eigenvalues $\mu_i(p_j) = \langle P(p_j)e_i, e_i \rangle \geq 0$. Then, for every $1 \leq i \leq n$, we have

$$\begin{aligned} \langle P \text{hess } u(p_j)e_i, e_i \rangle &= \mu_i(p_j) \text{Hess } u(p_j)(e_i, e_i) \\ &\leq \frac{1}{j}(u(p_j) - u(p_0) + 1) \left\{ \frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} \langle P \text{hess } \gamma(p_j)e_i, e_i \rangle \right. \\ &\quad \left. + \frac{1}{j} \left(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} \right)^2 \mu_i(p_j) \langle \nabla \gamma(p_j), e_i \rangle^2 \right\}. \end{aligned}$$

Taking traces here and using the fact that

$$\langle P \nabla \gamma, \nabla \gamma \rangle = \sum_{i=1}^n \mu_i \langle \nabla \gamma, e_i \rangle^2 \leq \text{Tr} P \|\nabla \gamma\|^2,$$

we obtain that

$$\begin{aligned} Lu(p_j) &\leq \frac{1}{j}(u(p_j) - u(p_0) + 1) \left\{ \frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} L\gamma(p_j) \right. \\ &\quad \left. + \frac{1}{j} \left(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} \right)^2 \langle P \nabla \gamma(p_j), \nabla \gamma(p_j) \rangle \right\} \\ &\leq \frac{1}{j}(u(p_j) - u(p_0) + 1) \left\{ \frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} L\gamma(p_j) \right. \\ &\quad \left. + \frac{1}{j} \left(\frac{\varphi'(\gamma(p_j))}{\varphi(\gamma(p_j))} \right)^2 \text{Tr} P \|\nabla \gamma(p_j)\|^2 \right\}. \end{aligned}$$

Since (3) and (4) hold outside a compact set, they hold at p_j for j sufficiently large. Then, using (8),

$$\begin{aligned} Lu(p_j) &\leq \frac{1}{j}(u(p_j) - u(p_0) + 1) \left\{ Bc + \frac{1}{j} c^2 A^2 CG(\gamma(p_j)^{\frac{1}{2}})^{-1} \right\} \\ &\leq C \frac{u^* - u(p_0) + 1}{j} \end{aligned}$$

for some constant $C > 0$. Since the right-hand side tends to zero as $j \rightarrow +\infty$, this proves condition (iii) in (6). □

Following the terminology introduced in [14], we introduce the next

Definition 2. Let Σ be a Riemannian manifold, and let L be an operator as in Theorem 1. The Omori-Yau maximum principle is said to hold on Σ for the operator L if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup_{\Sigma} u < +\infty$, there exists a sequence $\{p_j\}_{j \in \mathbb{N}} \subset \Sigma$ with the properties

$$(i) \ u(p_j) > u^* - \frac{1}{j}, \ (ii) \ \|\nabla u(p_j)\| < \frac{1}{j}, \ (iii) \ Lu(p_j) < \frac{1}{j}$$

for every $j \in \mathbb{N}$. Equivalently, for any function $u \in C^2(\Sigma)$ with $u_* = \inf_{\Sigma} u > -\infty$, there exists a sequence $\{p_j\}_{j \in \mathbb{N}} \subset \Sigma$ with the properties

$$(i) \ u(p_j) < u_* + \frac{1}{j}, \ (ii) \ \|\nabla u(p_j)\| < \frac{1}{j}, \ (iii) \ Lu(p_j) > -\frac{1}{j}$$

for every $j \in \mathbb{N}$.

The function theoretic approach to the generalized Omori-Yau maximum principle given in Theorem 1 allows us to apply it in different situations, where the choices of the functions γ and G are suggested by the geometric setting. The next are two significant and useful examples of an intrinsic and extrinsic nature, respectively.

Let $(\Sigma, \langle \cdot, \cdot \rangle)$ be a complete, non-compact Riemannian manifold, and let $o \in \Sigma$ be a fixed reference point. Denote with $r(p)$ the distance function from o and set $\gamma(p) = r(p)^2$. Then γ satisfies assumptions (2) and (3) of Theorem 1. Furthermore, γ is smooth within the cut locus of o . Assume that the radial sectional curvature of Σ , that is, the sectional curvature of the 2-planes containing ∇r , satisfies

$$(9) \quad K_{\Sigma}^{\text{rad}} \geq -G(r),$$

where G is a smooth function on $[0, +\infty)$ even at the origin, i.e. $G^{(2k+1)}(0) = 0$ for each $k = 0, 1, \dots$, and satisfying conditions (i)–(iv) listed in (5). Then assumption (4) is satisfied.

Indeed, assuming that (9) holds, by the Hessian comparison theorem within the cut locus of o , one has

$$(10) \quad \text{Hess } r(p)(v, v) \leq \frac{\phi'(r(p))}{\phi(r(p))} (\|v\|^2 - \langle \nabla r(p), v \rangle^2)$$

for every $v \in T_p \Sigma$, where $\phi(t)$ is the (positive) solution of the initial value problem

$$\begin{cases} \phi'' - G\phi = 0, \\ \phi(0) = 0, \ \phi'(0) = 1. \end{cases}$$

Now let

$$\psi(t) = \frac{1}{\sqrt{G(0)}} \left(e^{\int_0^t \sqrt{G(s)} ds} - 1 \right).$$

Then $\psi(0) = 0, \ \psi'(0) = 1$ and

$$\psi''(t) - G(t)\psi(t) = \frac{1}{\sqrt{G(0)}} \left(G(t) + \frac{G'(t)}{2\sqrt{G(t)}} e^{\int_0^t \sqrt{G(s)} ds} \right) \geq 0.$$

Hence, by the Sturm comparison theorem

$$(11) \quad \frac{\phi'(t)}{\phi(t)} \leq \frac{\psi'(t)}{\psi(t)} = \sqrt{G(t)} \frac{e^{\int_0^t \sqrt{G(s)} ds}}{e^{\int_0^t \sqrt{G(s)} ds} - 1} \leq c\sqrt{G(t)},$$

where the last inequality holds for a constant $c > 0$ and t sufficiently large. Therefore, if r is sufficiently large

$$\text{Hess } r \leq c\sqrt{G(r)}(\langle, \rangle - dr \otimes dr).$$

Since $\text{Hess } \gamma = 2r \text{Hess } r + 2dr \otimes dr$, we obtain from here that

$$(12) \quad \text{Hess } \gamma \leq c\sqrt{\gamma G(\sqrt{\gamma})}\langle, \rangle$$

for a constant c and γ sufficiently large. Then, using the fact that P is positive semi-definite

$$L\gamma \leq c\text{Tr}P\sqrt{\gamma G(\sqrt{\gamma})}$$

for a constant c and γ sufficiently large. We have thus proved the following:

Corollary 3. *Let $(\Sigma, \langle, \rangle)$ be a complete, non-compact Riemannian manifold whose radial sectional curvature satisfies condition (9). Then the Omori-Yau maximum principle holds on Σ for any semi-elliptic operator $L = \text{Tr}(P \circ \text{hess})$ with $\sup_{\Sigma} \text{Tr}P < +\infty$.*

On the other hand, the following example, which is of an extrinsic nature, will be useful in the sequel for the case of *properly immersed* hypersurfaces. Consider \mathbb{P}^n a complete, non-compact, Riemannian manifold, let $o \in \mathbb{P}^n$ be a reference point, and denote by \hat{r} the distance function from o . We will assume that the radial sectional curvature of \mathbb{P}^n satisfies the condition

$$(13) \quad K_{\mathbb{P}}^{\text{rad}} \geq -G(\hat{r}),$$

where G is a smooth function on $[0, +\infty)$ even at the origin, i.e. $G^{(2k+1)}(0) = 0$ for each $k = 0, 1, \dots$, and satisfying conditions (i)–(iv) listed in (5). Let $f : \Sigma^n \rightarrow M^{n+1} = I \times_{\rho} \mathbb{P}^n$ be a hypersurface. Observe that if Σ is compact, then every immersion $f : \Sigma^n \rightarrow I \times_{\rho} \mathbb{P}^n$ is proper and contained in a slab, and the Omori-Yau maximum principle trivially holds on Σ for any semi-elliptic operator. Assume then that Σ is non-compact, and let $f : \Sigma^n \rightarrow M^{n+1} = I \times_{\rho} \mathbb{P}^n$ be a properly immersed hypersurface which is contained in a slab, that is, $f(\Sigma) \subset [t_1, t_2] \times \mathbb{P}^n$.

Let $\hat{\gamma} : \mathbb{P}^n \rightarrow \mathbb{R}$ be the function given by $\hat{\gamma}(x) = \hat{r}(x)^2$ for every $x \in \mathbb{P}^n$, and set $\gamma : \Sigma \rightarrow \mathbb{R}$ for the associated function, defined as

$$\gamma(p) = \tilde{\gamma}(f(p)) = \hat{\gamma}(x(p)) = \hat{r}(x(p))^2$$

for every $p \in \Sigma$, where $\tilde{\gamma}(t, x) = \hat{\gamma}(x)$ and $f(p) = (h(p), x(p))$. Since f is proper, if $p \rightarrow \infty$ in Σ , then $f(p) \rightarrow \infty$ in $M^{n+1} = I \times_{\rho} \mathbb{P}^n$, but being f contained in a slab, this means that $x(p) \rightarrow \infty$ in \mathbb{P}^n . It follows that $\gamma(p) = \hat{r}(x(p))^2 \rightarrow +\infty$ as $p \rightarrow \infty$ in Σ , and γ satisfies condition (2) in Theorem 1.

Let us denote by $\tilde{\nabla}$, $\hat{\nabla}$, and ∇ the Levi-Civita connection (and the gradient operators) in M^{n+1} , \mathbb{P}^n , and Σ^n , respectively. Since $\gamma = \tilde{\gamma} \circ f$, along the immersion f we have

$$(14) \quad \tilde{\nabla}\tilde{\gamma} = \nabla\gamma + \langle \tilde{\nabla}\tilde{\gamma}, N \rangle N,$$

where N is a (local) smooth unit normal field along f . On the other hand, from $\tilde{\gamma}(t, x) = \hat{\gamma}(x)$ we have

$$\langle \tilde{\nabla}\tilde{\gamma}, T \rangle = 0,$$

where, as usual, T stands for the lift of ∂_t to the product $I \times \mathbb{P}^n$, and

$$\langle \tilde{\nabla}\tilde{\gamma}, V \rangle = \langle \hat{\nabla}\hat{\gamma}, V \rangle_{\mathbb{P}}$$

for every V , where V denotes the lift of a vector field $V \in TP$ to $I \times \mathbb{P}^n$. Since

$$\langle \tilde{\nabla} \tilde{\gamma}, V \rangle = \rho^2 \langle \tilde{\nabla} \tilde{\gamma}, V \rangle_{\mathbb{P}},$$

we conclude from here that

$$(15) \quad \tilde{\nabla} \tilde{\gamma} = \frac{1}{\rho^2} \hat{\nabla} \hat{\gamma} = \frac{2\hat{r}}{\rho^2} \hat{\nabla} \hat{r}.$$

Therefore, since $\|\hat{\nabla} \hat{r}\| = \rho \|\tilde{\nabla} \hat{r}\|_{\mathbb{P}} = \rho$ and $\rho(h) \geq \min_{[t_1, t_2]} \rho(t) > 0$, along the immersion we have

$$(16) \quad \|\nabla \gamma\| \leq \|\tilde{\nabla} \tilde{\gamma}\| = \frac{2\sqrt{\gamma}}{\rho(h)} \leq c\sqrt{\gamma}$$

for a positive constant c . Thus, γ also satisfies condition (3) in Theorem 1. In particular, Σ is complete (see [14] pag 10).

Next, we will see that, under appropriate extrinsic restrictions, condition (4) in Theorem 1 is also satisfied. From (14) it follows that

$$\text{Hess } \gamma(X, X) = \text{Hess } \tilde{\gamma}(X, X) + \langle \tilde{\nabla} \tilde{\gamma}, N \rangle \langle AX, X \rangle$$

for every tangent vector field $X \in T\Sigma$. From (15)

$$(17) \quad \tilde{\nabla}_T \tilde{\nabla} \tilde{\gamma} = -\frac{\rho'}{\rho^3} \hat{\nabla} \hat{\gamma} = -\mathcal{H} \tilde{\nabla} \tilde{\gamma},$$

where $\mathcal{H}(t) = \rho'(t)/\rho(t)$. In particular, $\text{Hess } \tilde{\gamma}(T, T) = 0$. Then, writing $X = X^* + \langle X, T \rangle T$, where $X^* = \pi_{\mathbb{P}*} X$, we have

$$\text{Hess } \tilde{\gamma}(X, X) = \text{Hess } \tilde{\gamma}(X^*, X^*) + 2\langle X, T \rangle \text{Hess } \tilde{\gamma}(X^*, T).$$

From (17) we have that

$$\text{Hess } \tilde{\gamma}(X^*, T) = -\mathcal{H}(h) \langle \tilde{\nabla} \tilde{\gamma}, X \rangle = -\mathcal{H}(h) \langle \nabla \gamma, X \rangle.$$

On the other hand, using

$$\tilde{\nabla}_{X^*} \tilde{\nabla} \tilde{\gamma} = \frac{1}{\rho^2} \hat{\nabla}_{X^*} \hat{\nabla} \hat{\gamma} - \frac{\rho'}{\rho^3} \langle \hat{\nabla} \hat{\gamma}, X^* \rangle T,$$

we also have

$$\text{Hess } \tilde{\gamma}(X^*, X^*) = \frac{1}{\rho^2} \langle \hat{\nabla}_{X^*} \hat{\nabla} \hat{\gamma}, X^* \rangle = \langle \hat{\nabla}_{X^*} \hat{\nabla} \hat{\gamma}, X^* \rangle_{\mathbb{P}} = \text{Hess } \hat{\gamma}(X^*, X^*).$$

Summing up,

$$(18) \quad \begin{aligned} \text{Hess } \gamma(X, X) &= \text{Hess } \hat{\gamma}(X^*, X^*) - 2\mathcal{H}(h) \langle \nabla \gamma, X \rangle \langle T, X \rangle \\ &\quad + \langle \tilde{\nabla} \tilde{\gamma}, N \rangle \langle AX, X \rangle \end{aligned}$$

for every tangent vector field $X \in T\Sigma$.

Observe that, using (16),

$$|\mathcal{H}(h) \langle \nabla \gamma, X \rangle \langle T, X \rangle| \leq |\mathcal{H}(h)| \|\nabla \gamma\| \|X\|^2 \leq c\sqrt{\gamma} \|X\|^2$$

for a constant $c > 0$, since $|\mathcal{H}(h)| \leq \max_{[t_1, t_2]} |\mathcal{H}(t)|$. On the other hand, reasoning as we did before in deriving (12), from condition (13) and using the Hessian comparison theorem for \mathbb{P}^n it follows that, if γ is sufficiently large, then

$$\text{Hess } \hat{\gamma}(X^*, X^*) \leq c\sqrt{\gamma G(\sqrt{\gamma})} \|X\|^2$$

for a certain positive constant c , where we are using the fact that

$$\|X^*\|_{\mathbb{P}} \leq \frac{1}{\inf_{\Sigma} \rho(h)} \|X\| \leq \frac{1}{\min_{[t_1, t_2]} \rho(t)} \|X\|.$$

Therefore, since $\lim_{t \rightarrow +\infty} G(t) = +\infty$, from (18) we conclude

$$(19) \quad \text{Hess } \gamma(X, X) \leq c\sqrt{\gamma G(\sqrt{\gamma})} \|X\|^2 + \langle \tilde{\nabla} \tilde{\gamma}, N \rangle \langle AX, X \rangle$$

for every tangent vector field $X \in T\Sigma$, outside a compact subset of Σ .

Assume now that $\sup_{\Sigma} |H| < +\infty$. Tracing (19) we obtain

$$\Delta \gamma \leq nc\sqrt{\gamma G(\sqrt{\gamma})} + nH \langle \tilde{\nabla} \tilde{\gamma}, N \rangle$$

outside a compact set. Furthermore, by (16)

$$|H \langle \tilde{\nabla} \tilde{\gamma}, N \rangle| \leq \sup_{\Sigma} |H| \|\tilde{\nabla} \tilde{\gamma}\| \leq c\sqrt{\gamma} \leq c\sqrt{\gamma G(\sqrt{\gamma})}$$

for some constant $c > 0$. Thus, we conclude that, outside a compact subset of Σ ,

$$\Delta \gamma \leq c\sqrt{\gamma G(\sqrt{\gamma})}$$

for some constant $c > 0$, which means that condition (4) in Theorem 1 is fulfilled for the Laplacian operator. Therefore, the Omori-Yau maximum principle holds on Σ for the Laplacian.

On the other hand, if we assume instead that $\sup_{\Sigma} \|A\|^2 < +\infty$ then, using again (16), we have

$$|\langle \tilde{\nabla} \tilde{\gamma}, N \rangle \langle AX, X \rangle| \leq \|\tilde{\nabla} \tilde{\gamma}\| \|A\| \|X\|^2 \leq c\sqrt{\gamma G(\sqrt{\gamma})} \|X\|^2$$

for a positive constant c , if γ is sufficiently large. From (19) we therefore obtain

$$(20) \quad \text{Hess } \gamma(X, X) \leq c\sqrt{\gamma G(\sqrt{\gamma})} \|X\|^2,$$

for every tangent vector field $X \in T\Sigma$, outside a compact subset of Σ . Thus, if P is a positive semi-definite operator with $\sup_{\Sigma} \text{Tr} P < +\infty$, we conclude from here that

$$L\gamma \leq nc \sup_{\Sigma} \text{Tr} P \sqrt{\gamma G(\sqrt{\gamma})}$$

if γ is sufficiently large, which means that condition (4) in Theorem 1 is fulfilled for the operator L . Therefore the Omori-Yau maximum principle holds on Σ for L . We summarize the above discussion in the following:

Corollary 4. *Let \mathbb{P}^n be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies the condition (13). Let $f : \Sigma^n \rightarrow M^{n+1} = I \times_{\rho} \mathbb{P}^n$ be a properly immersed hypersurface contained in a slab.*

- (1) *If $\sup_{\Sigma} |H| < +\infty$, then Σ is complete and the Omori-Yau maximum principle holds on Σ for the Laplacian.*
- (2) *If $\sup_{\Sigma} \|A\| < +\infty$, then Σ is complete and the Omori-Yau maximum principle holds on Σ for any semi-elliptic operator $L = \text{Tr}(P \circ \text{hess})$ with $\sup_{\Sigma} \text{Tr} P < +\infty$.*

Remark 5. From the equality

$$\|A\|^2 = n^2 H_1^2 - n(n-1)H_2$$

it follows that under the assumption $\inf_{\Sigma} H_2 > -\infty$ the condition $\sup_{\Sigma} \|A\|^2 < +\infty$ is equivalent to $\sup_{\Sigma} |H_1| < +\infty$.

4. CURVATURE ESTIMATES FOR HYPERSURFACES IN WARPED PRODUCTS

In this section we will derive some estimates for the k -mean curvatures of a hypersurface in a slab of a warped product space. Towards this aim we need the next computational

Proposition 6. *Let $f : \Sigma^n \rightarrow M^{n+1} = I \times_{\rho} \mathbb{P}^n$ be an isometric immersion into a warped product space. Let h be the height function and define*

$$\sigma(t) = \int_{t_0}^t \rho(u)du.$$

Then

$$(21) \quad L_k h = \mathcal{H}(h)(c_k H_k - \langle P_k \nabla h, \nabla h \rangle) + c_k \Theta H_{k+1},$$

and

$$(22) \quad L_k \sigma(h) = c_k \rho(h)(\mathcal{H}(h)H_k + \Theta H_{k+1}),$$

where $c_k = (n-k) \binom{n}{k} = (k+1) \binom{n}{k+1}$, $\mathcal{H}(t) = \rho'(t)/\rho(t)$, and $\Theta = \langle N, T \rangle$ is the angle function.

Proof. The gradient of $\pi_I \in C^\infty(M)$ is $\bar{\nabla} \pi_I = T$, hence,

$$(23) \quad \nabla h = (\bar{\nabla} \pi_I)^\top = T - \Theta N.$$

Recall that the Levi-Civita connection of a warped product satisfies

$$\bar{\nabla}_X T = \mathcal{H}(X - \langle X, T \rangle T), \quad \text{for any } X \in TM.$$

Thus

$$\bar{\nabla}_X \nabla h = \mathcal{H}(h)(X - \langle X, T \rangle T) - X(\Theta)N + \Theta AX,$$

for any $X \in T\Sigma$. Then

$$(24) \quad \text{hess } h(X) = \nabla_X \nabla h = \mathcal{H}(h)(X - \langle X, \nabla h \rangle \nabla h) + \Theta AX,$$

where ∇ denotes the Levi-Civita connection on Σ^n . Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame on Σ . Then, using the expressions of the traces of P_k and $P_k A$

$$\begin{aligned} L_k h &= \text{Tr}(P_k \circ \text{hess } h) = \sum_i \langle P_k \text{hess } h(e_i), e_i \rangle \\ &= \mathcal{H}(h)(\text{Tr} P_k - \langle P_k \nabla h, \nabla h \rangle) + \Theta \text{Tr}(P_k A) \\ &= \mathcal{H}(h)(c_k H_k - \langle P_k \nabla h, \nabla h \rangle) + c_k \Theta H_{k+1}. \end{aligned}$$

On the other hand, since $\nabla \sigma(h) = \rho(h) \nabla h$, we have

$$\text{hess } \sigma(h)(X) = \rho'(h) \langle \nabla h, X \rangle \nabla h + \rho(h) \text{hess } h(X).$$

Therefore

$$\begin{aligned} L_k\sigma(h) &= \text{Tr}(P_k \circ \text{hess } \sigma(h)) = \sum_i \langle P_k \text{hess } \sigma(h)(e_i), e_i \rangle \\ &= \rho'(h) \langle P_k \nabla h, \nabla h \rangle + \rho(h) \mathcal{H}(h) (\text{Tr} P_k - \langle P_k \nabla h, \nabla h \rangle) \\ &\quad + \rho(h) \Theta \text{Tr}(P_k A) \\ &= c_k \rho(h) (\mathcal{H}(h) H_k + \Theta H_{k+1}). \end{aligned}$$

□

As a first application of the computations above, we derive the following:

Theorem 7. *Let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be an immersed hypersurface. If the Omori-Yau maximum principle holds on Σ for the Laplacian and $h^* = \sup_\Sigma h < +\infty$, then*

$$\sup_\Sigma |H| \geq \inf_\Sigma \mathcal{H}(h).$$

In particular, and as an application of Corollary 4, we deduce the following result, which generalizes Theorem 2 in [3].

Corollary 8. *Let \mathbb{P}^n be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition (13). If $f : \Sigma^n \rightarrow M^{n+1} = I \times_\rho \mathbb{P}^n$ is a properly immersed hypersurface contained in a slab, then*

$$(25) \quad \sup_\Sigma |H| \geq \inf_\Sigma \mathcal{H}(h).$$

In other words, there is no properly immersed hypersurface contained in a slab $[t_1, t_2] \times \mathbb{P}^n$ with

$$\sup_\Sigma |H| < \inf_{[t_1, t_2]} \mathcal{H}(t).$$

For the proof of Corollary 8, observe that if $\sup_\Sigma |H| = +\infty$, then the inequality (25) trivially holds. On the other hand, if $\sup_\Sigma |H| < +\infty$, then by Corollary 4 we know that the Omori-Yau maximum principle holds on Σ and the result follows from Theorem 7.

Proof of Theorem 7. Since h is bounded from above, we may find a sequence $\{q_j\} \subset \Sigma^n$ such that

$$\begin{aligned} \lim_{j \rightarrow +\infty} h(q_j) &= h^* := \sup h, \\ \|\nabla h(q_j)\|^2 &= 1 - \Theta^2(q_j) < \left(\frac{1}{j}\right)^2, \\ \Delta h(q_j) &= \mathcal{H}(h(q_j))(n - \|\nabla h(q_j)\|^2) + nH(q_j)\Theta(q_j) < \frac{1}{j}. \end{aligned}$$

Then

$$\frac{1}{j} > \Delta h(q_j) \geq \mathcal{H}(h(q_j))(n - \|\nabla h(q_j)\|^2) - n \sup_\Sigma |H|.$$

Making $j \rightarrow +\infty$, we get

$$0 \geq \mathcal{H}(h^*) - \sup_\Sigma |H|,$$

so that

$$\sup_\Sigma |H| \geq \mathcal{H}(h^*) \geq \inf_\Sigma \mathcal{H}(h).$$

□

Corollary 9. *Let \mathbb{P}^n be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition (13). If $f : \Sigma^n \rightarrow M^{n+1} = I \times_{e^t} \mathbb{P}^n$ is a parabolic, properly immersed hypersurface with constant mean curvature $|H| \leq 1$ contained in a slab, then $f(\Sigma)$ is slice.*

For the proof of this corollary observe that from Corollary 8 it must be $|H| = 1$. Choose the orientation so that $H = 1$. In this case $\sigma(h) = e^h$ and by (22)

$$\Delta e^h = ne^h(1 + \Theta) \geq 0.$$

Therefore, since $e^h \leq e^{h^*}$ it follows that e^h is a subharmonic function on Σ which is bounded from above. The conclusion now follows from parabolicity.

For the next results, we normalize the operators P_k to the following

$$\hat{P}_k = \frac{1}{H_k} P_k$$

where, of course, we are assuming $H_k > 0$. We will denote by \hat{L}_k the corresponding differential operator, that is

$$\hat{L}_k = \text{Tr}(\hat{P}_k \circ \text{hess}).$$

Observe that

$$\text{Tr}(\hat{P}_k) = c_k$$

so that the operators \hat{P}_k have trace always bounded from above. With this preparation we state the next

Theorem 10. *Let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be an immersed hypersurface with $H_2 > 0$. If the Omori-Yau maximum principle holds on Σ for \hat{L}_1 and $h^* = \sup_\Sigma h < +\infty$, then*

$$\sup_\Sigma H_2^{1/2} \geq \inf_\Sigma \mathcal{H}(h).$$

Proof. We may assume without loss of generality that $\sup_\Sigma H_2 < +\infty$ and $\inf_\Sigma \mathcal{H}(h) \geq 0$. Otherwise the desired conclusion trivially holds. Using the basic inequality $H_1 \geq \sqrt{H_2}$ we know that $H_1 > 0$ and \hat{L}_1 is a well defined elliptic operator. Since h is bounded from above and $\sup_\Sigma \sigma(h) = \sigma(h^*)$, we may find a sequence $\{q_j\} \subset \Sigma^n$ such that

$$\begin{aligned} \lim_{j \rightarrow +\infty} (\sigma \circ h)(q_j) &= \sigma(h^*) := \sup(\sigma \circ h), \\ \|\nabla(\sigma \circ h)(q_j)\|^2 &= \rho(h(q_j))^2(1 - \Theta^2(q_j)) < \left(\frac{1}{j}\right)^2, \\ \hat{L}_1(\sigma \circ h)(q_j) &< \frac{1}{j}. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{1}{j} > \hat{L}_1(\sigma \circ h)(q_j) &= n(n-1)\rho(h(q_j)) \left(\mathcal{H}(h(q_j)) + \Theta(q_j) \frac{H_2}{H_1}(q_j) \right) \\ &\geq n(n-1)\rho(h(q_j)) \left(\mathcal{H}(h(q_j)) - \frac{H_2}{H_1}(q_j) \right) \\ &\geq n(n-1)\rho(h(q_j)) \left(\mathcal{H}(h(q_j)) - \sqrt{H_2}(q_j) \right). \end{aligned}$$

Observe that

$$\lim_{j \rightarrow +\infty} (\sigma \circ h)(q_j) = \sigma(h^*) := \sup(\sigma \circ h)$$

implies $\lim_{j \rightarrow +\infty} h(q_j) = h^*$, because $\sigma(t)$ is strictly increasing. Making $j \rightarrow +\infty$, and, if necessary, up to passing to a subsequence we get

$$0 \geq \mathcal{H}(h^*) - \sup_{\Sigma} \sqrt{H_2}.$$

So

$$\sup_{\Sigma} H_2^{1/2} \geq \mathcal{H}(h^*) \geq \inf_{\Sigma} \mathcal{H}(h).$$

□

As a consequence of the previous theorem and of Corollary 4 (see also Remark 5) we have

Corollary 11. *Let \mathbb{P}^n be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition (13). If $f : \Sigma^n \rightarrow M^{n+1} = I \times_{\rho} \mathbb{P}^n$ is a properly immersed hypersurface with $H_2 > 0$, $\sup_{\Sigma} |H_1| < +\infty$ and contained in a slab, then*

$$\sup_{\Sigma} H_2^{1/2} \geq \inf_{\Sigma} \mathcal{H}(h).$$

In other words, there is no properly immersed hypersurface with $H_2 > 0$ and $\sup_{\Sigma} |H_1| < +\infty$ contained in a slab $[t_1, t_2] \times \mathbb{P}^n$ with

$$\sup_{\Sigma} H_2^{1/2} < \inf_{[t_1, t_2]} \mathcal{H}(t).$$

In the next theorem, the existence of an elliptic point enable us to guarantee that H_{k-1} is strictly positive and to guarantee ellipticity of the operator \hat{L}_{k-1} . Reasoning as in the previous results gives the next

Theorem 12. *Let $f : \Sigma^n \rightarrow I \times_{\rho} \mathbb{P}^n$ be an immersed hypersurface having an elliptic point, with $H_k > 0$. If the Omori-Yau maximum principle holds on Σ for \hat{L}_{k-1} , with $3 \leq k \leq n$, and $h^* < +\infty$ then*

$$\sup_{\Sigma} H_k^{1/k} \geq \inf_{\Sigma} \mathcal{H}(h).$$

Corollary 13. *Let \mathbb{P}^n be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition (13). Assume that $f : \Sigma^n \rightarrow M^{n+1} = I \times_{\rho} \mathbb{P}^n$ is a properly immersed hypersurface having an elliptic point, with $H_k > 0$ and $\sup_{\Sigma} |H_1| < +\infty$. If $f(\Sigma)$ is contained in a slab, then*

$$\sup_{\Sigma} H_k^{1/k} \geq \inf_{\Sigma} \mathcal{H}(h)$$

for every $3 \leq k \leq n$. In other words, there is no properly immersed hypersurface having an elliptic point, with $H_k > 0$, $\sup_{\Sigma} |H_1| < +\infty$ and contained in a slab $[t_1, t_2] \times \mathbb{P}^n$ with

$$\sup_{\Sigma} H_k^{1/k} < \inf_{[t_1, t_2]} \mathcal{H}(t).$$

5. HYPERSURFACES WITH CONSTANT 2-MEAN CURVATURE

In this section we will derive some applications for hypersurfaces with positive constant 2-mean curvature H_2 . Before stating the main results, let us introduce an auxiliary lemma that will be useful in the sequel.

Lemma 14. *Let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be a hypersurface with non-vanishing mean curvature which is contained in a slab. Assume that $\mathcal{H}' \geq 0$ and that the angle function Θ does not change sign. Choose on Σ the orientation so that $H_1 > 0$. Suppose the Omori-Yau maximum principle for the Laplacian holds on Σ . We have that*

- (i) if $\Theta \leq 0$, then $\mathcal{H}(h) \geq 0$,
- (ii) if $\Theta \geq 0$, then $\mathcal{H}(h) \leq 0$.

Proof. Since h is bounded from below and the Omori-Yau maximum principle for the Laplacian operator holds on Σ , we can find a sequence $\{p_j\} \subset \Sigma^n$ such that

$$\begin{aligned} \lim_{j \rightarrow +\infty} h(p_j) &= h_* := \inf h, \\ \|\nabla h(p_j)\|^2 &= 1 - \Theta^2(p_j) < \left(\frac{1}{j}\right)^2, \\ \Delta h(p_j) &= \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2) + nH_1(p_j)\Theta(p_j) > -\frac{1}{j}. \end{aligned}$$

Then

$$(26) \quad -nH_1(p_j)\Theta(p_j) < \frac{1}{j} + \mathcal{H}(h(p_j))(n - \|\nabla h(p_j)\|^2).$$

Similarly, since h is bounded from above, we can also find a second sequence $\{q_j\} \subset \Sigma^n$ such that

$$\begin{aligned} \lim_{j \rightarrow +\infty} h(q_j) &= h^* := \sup h, \\ \|\nabla h(q_j)\|^2 &= 1 - \Theta^2(q_j) < \left(\frac{1}{j}\right)^2, \\ \Delta h(q_j) &= \mathcal{H}(h(q_j))(n - \|\nabla h(q_j)\|^2) + nH_1(q_j)\Theta(q_j) < \frac{1}{j}. \end{aligned}$$

Then

$$(27) \quad -nH_1(q_j)\Theta(q_j) > -\frac{1}{j} + \mathcal{H}(h(q_j))(n - \|\nabla h(q_j)\|^2).$$

Assume first that $\Theta \leq 0$. Since $\lim_{j \rightarrow +\infty} -\Theta(p_j) = -\text{sgn } \Theta = 1 > 0$, we have $-\Theta(p_j) > 0$ for sufficiently large j . Since $H_1(p_j) > 0$, using (26) it follows from (26) that

$$0 \leq \liminf_{j \rightarrow +\infty} \left(-H_1(p_j)\Theta(p_j) \right) \leq \mathcal{H}(h_*).$$

Therefore $\mathcal{H}(h_*) \geq 0$ and, by $\mathcal{H}' \geq 0$, we conclude that

$$\mathcal{H}(h) \geq \mathcal{H}(h_*) \geq 0.$$

Assume now that $\Theta \geq 0$; then $\lim_{j \rightarrow +\infty} \Theta(q_j) = \text{sgn } \Theta = 1 > 0$, so that $\Theta(q_j) > 0$ for sufficiently large j . Therefore, since $H_1(q_j) > 0$, from (27) we deduce

$$0 \leq \liminf_{j \rightarrow +\infty} \left(H_1(q_j)\Theta(q_j) \right) \leq -\mathcal{H}(h^*).$$

Therefore $\mathcal{H}(h^*) \leq 0$ and, by $\mathcal{H}' \geq 0$, we conclude that

$$\mathcal{H}(h) \leq \mathcal{H}(h^*) \leq 0.$$

This concludes the proof. □

In the rest of this section we will work basically with the operator L_1 . We will assume that H_2 is a positive constant. Recall that this implies, in this case, that the immersion is two sided. We can choose the normal unit vector N on Σ such that $H_1 > 0$ and the operator L_1 is elliptic (see the discussion in the Preliminaries).

Let $\sigma(t) = \int_{t_0}^t \rho(s) ds$. By Proposition 6 we know that

$$\begin{aligned} \Delta\sigma(h) &= n\rho(h)(\mathcal{H}(h) + \Theta H_1), \\ (28) \quad L_1\sigma(h) &= n(n-1)\rho(h)(\mathcal{H}(h)H_1 + \Theta H_2). \end{aligned}$$

Therefore,

$$(29) \quad \mathcal{L}_1\sigma(h) = n(n-1)\rho(h)(\mathcal{H}(h)^2 - \Theta^2 H_2),$$

where \mathcal{L}_1 is the operator given by

$$\mathcal{L}_1 = (n-1)\mathcal{H}(h)\Delta - \Theta L_1 = \text{Tr}(\mathcal{P}_1 \circ \text{hess}),$$

with

$$\mathcal{P}_1 = (n-1)\mathcal{H}(h)I - \Theta P_1.$$

Let us now state the first main result of this section, which extends Theorem 2.4 in [4] to the case of constant 2-mean curvature H_2 .

Theorem 15. *Let $f : \Sigma^n \rightarrow I \times_{\rho} \mathbb{P}^n$ be a compact hypersurface of constant positive 2-mean curvature H_2 . If $\mathcal{H}'(t) \geq 0$ and the angle function Θ does not change sign, then \mathbb{P}^n is necessarily compact and $f(\Sigma^n)$ is a slice.*

Proof. As indicated above, we choose the orientation of Σ so that $H_1 > 0$. Since Σ^n is compact, we may apply Lemma 14. Let us consider first the case where $\Theta \leq 0$, for which $\mathcal{H}(h) \geq 0$. Thus, the operator \mathcal{P}_1 is positive semi-definite or, equivalently, \mathcal{L}_1 is semi-elliptic.

Since Σ is compact, there exist points $p_{\max} \in \Sigma$ and $p_{\min} \in \Sigma$ such that

$$h(p_{\max}) = h^* = \max_{\Sigma} h \quad \text{and} \quad h(p_{\min}) = h_* = \min_{\Sigma} h.$$

Therefore, $\|\nabla h(p_{\max})\| = \|\nabla h(p_{\min})\| = 0$, which yields

$$\Theta(p_{\max}) = \Theta(p_{\min}) = -1$$

because of (23). Observe that

$$(\sigma \circ h)^* = \max_{\Sigma}(\sigma \circ h) = \sigma(h^*) = \sigma(h(p_{\max}))$$

and

$$(\sigma \circ h)_* = \min_{\Sigma}(\sigma \circ h) = \sigma(h_*) = \sigma(h(p_{\min})),$$

because $\sigma(t)$ is strictly increasing. In particular,

$$\text{Hess } \sigma(h)(p_{\max}) \leq 0 \quad \text{and} \quad \text{Hess } \sigma(h)(p_{\min}) \geq 0.$$

Taking into account that \mathcal{P}_1 is positive semi-definite, yields

$$\mathcal{L}_1\sigma(h)(p_{\max}) = n(n-1)\rho(h^*)(\mathcal{H}(h^*)^2 - H_2) \leq 0$$

and

$$\mathcal{L}_1\sigma(h)(p_{\min}) = n(n-1)\rho(h_*)(\mathcal{H}(h_*)^2 - H_2) \geq 0.$$

Then, since $\mathcal{H}(h) \geq 0$ on Σ , we obtain

$$\mathcal{H}(h_*) \geq H_2^{1/2} \geq \mathcal{H}(h^*).$$

On the other hand, by $\mathcal{H}' \geq 0$ we also have $\mathcal{H}(h_*) \leq \mathcal{H}(h^*)$. Thus the validity of the equality $\mathcal{H}(h_*) = \mathcal{H}(h^*)$ and $\mathcal{H}(h) = H_2^{1/2}$ is constant on Σ . By (28), using the basic inequality $H_1 \geq H_2^{1/2}$ and the fact that $\Theta \geq -1$, we obtain

$$\begin{aligned} L_1\sigma(h) &= n(n-1)\rho(h)H_2^{1/2}(H_1 + \Theta H_2^{1/2}) \\ &\geq n(n-1)\rho(h)H_2^{1/2}(H_1 - H_2^{1/2}) \geq 0. \end{aligned}$$

That is, $L_1\sigma(h) \geq 0$ on the compact manifold Σ . Thus, by the maximum principle applied to the elliptic operator L_1 , we conclude that $\sigma(h)$, and hence h , is constant.

Finally, in the case where $\Theta \geq 0$, we know from Lemma 14 that $\mathcal{H}(h) \leq 0$ on Σ , so that the operator $-\mathcal{L}_1$ is semi-elliptic. The proof then follows as in the case $\Theta \leq 0$, working with $-\mathcal{L}_1$ instead of \mathcal{L}_1 . □

In our next result, we consider the case of complete (and non-compact) hypersurfaces, extending Theorem 2.9 in [4] to the case of constant 2-mean curvature hypersurfaces.

Theorem 16. *Let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be a complete hypersurface of constant positive 2-mean curvature H_2 such that*

$$(30) \quad K_\Sigma^{\text{rad}} \geq -G(r).$$

Here G is a smooth function on $[0, +\infty)$ which is even at the origin and satisfies conditions (i)–(iv) listed in Theorem 1. Assume that $\sup_\Sigma |H_1| < +\infty$ and that Σ is contained in a slab, that is,

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n,$$

where $t_1, t_2 \in I$ are finite. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.

Proof. Choose the orientation of Σ so that $H_1 > 0$. By Corollary 3 we know that the Omori-Yau maximum principle holds on Σ for the Laplacian operator, so that we may apply Lemma 14.

In the case where $\Theta \leq 0$, by Lemma 14 we have $\mathcal{H}(h) \geq 0$, and therefore the operator \mathcal{P}_1 is positive semi-definite. In other words, the differential operator \mathcal{L}_1 is semi-elliptic. Furthermore

$$\text{Tr}\mathcal{P}_1 = n(n-1)\mathcal{H}(h) - n(n-1)H_1\Theta \leq n(n-1)(\mathcal{H}(h^*) + H_1^*),$$

where $h^* = \sup_\Sigma h < +\infty$ and $H_1^* = \sup_\Sigma H_1 < +\infty$. Hence by Corollary 3 we know that the Omori-Yau maximum principle holds on Σ for the operator \mathcal{L}_1 .

Since $\sup_\Sigma \sigma(h) = \sigma(h^*) < +\infty$, there exists a sequence $\{p_j\}_{j \in \mathbb{N}} \subset \Sigma$ such that

- (i) $\lim_{j \rightarrow +\infty} \sigma(h(p_j)) = \sup_\Sigma \sigma(h) = \sigma(h^*),$
- (ii) $\|\nabla(\sigma \circ h)(p_j)\| = \rho(h(p_j))\|\nabla h(p_j)\| < \frac{1}{j},$
- (iii) $\mathcal{L}_1(\sigma \circ h)(p_j) < \frac{1}{j}.$

Observe that condition (i) implies that $\lim_{j \rightarrow +\infty} h(p_j) = h^*$, because $\sigma(t)$ is strictly increasing. Thus by condition (ii) we also have $\lim_{j \rightarrow +\infty} \|\nabla h(p_j)\| = 0$. Therefore

$$\mathcal{L}_1\sigma(h)(p_j) = n(n-1)\rho(h(p_j))(\mathcal{H}(h(p_j))^2 - \Theta^2(p_j)H_2) < \frac{1}{j},$$

and taking the limit for $j \rightarrow +\infty$ and observing that $\Theta^2(p_j) = 1 - \|\nabla h(p_j)\|^2 \rightarrow 1$ as $j \rightarrow +\infty$, we find

$$\mathcal{H}(h^*)^2 - H_2 \leq 0.$$

On the other hand, since h is also bounded from below, $\inf_\Sigma \sigma(h) = \sigma(h_*) > -\infty$, where $h_* = \inf_\Sigma h > -\infty$. Thus, we can find a sequence $\{q_j\}_{j \in \mathbb{N}} \subset \Sigma$ such that

- (i) $\lim_{j \rightarrow +\infty} \sigma(h(q_j)) = \inf_\Sigma \sigma(h) = \sigma(h_*)$,
- (ii) $\|\nabla(\sigma \circ h)(q_j)\| = \rho(h(q_j))\|\nabla h(q_j)\| < \frac{1}{j}$,
- (iii) $\mathcal{L}_1(\sigma \circ h)(q_j) > -\frac{1}{j}$.

Hence, proceeding as above and using that

$$\mathcal{L}_1\sigma(h)(q_j) = n(n-1)\rho(h(q_j))(\mathcal{H}(h(q_j))^2 - \Theta^2(q_j)H_2) > -\frac{1}{j},$$

we find

$$\mathcal{H}(h_*)^2 - H_2 \geq 0.$$

Thus $\mathcal{H}(h_*)^2 \geq \mathcal{H}(h^*)^2$ and, taking into account that $\mathcal{H}(h_*), \mathcal{H}(h^*) \geq 0$, this gives $\mathcal{H}(h_*) \geq \mathcal{H}(h^*)$. Therefore, since $\mathcal{H}(t)$ is an increasing function, we conclude that $h^* = h_*$.

Finally, let us consider the case where $\Theta \geq 0$. By Lemma 14 we find that $\mathcal{H}(h) \leq 0$ and then the operator $-\mathcal{L}_1$ is semi-elliptic. Moreover

$$\text{Tr}(-\mathcal{P}_1) = -n(n-1)\mathcal{H}(h) + n(n-1)H_1\Theta \leq n(n-1)(-\mathcal{H}(h_*) + H_1^*).$$

Hence the trace of $-\mathcal{P}_1$ is bounded from above and by Corollary 3 the Omori-Yau maximum principle holds for the operator $-\mathcal{L}_1$. Proceeding as above, we arrive at the two inequalities

$$H_2 - \mathcal{H}(h_*)^2 \geq 0 \quad \text{and} \quad H_2 - \mathcal{H}(h^*)^2 \leq 0.$$

Thus $\mathcal{H}(h_*)^2 \leq \mathcal{H}(h^*)^2$. Since $\mathcal{H}(h_*), \mathcal{H}(h^*) \leq 0$, this implies $\mathcal{H}(h_*) \geq \mathcal{H}(h^*)$. But $\mathcal{H}(t)$ being increasing, this gives $h_* = h^*$ concluding the proof. \square

In particular, Theorem 16 remains true if we replace condition (30) by the stronger condition of Σ^n having radial sectional curvature bounded from below by a constant. This happens, for instance, when the sectional curvature of \mathbb{P}^n is itself bounded from below. This observation yields the next

Corollary 17. *Let \mathbb{P}^n be a complete Riemannian manifold with sectional curvature bounded from below, and let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be a complete hypersurface of constant positive 2-mean curvature H_2 . Assume that $\sup_\Sigma |H_1| < +\infty$ and that Σ is contained in a slab, that is,*

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n,$$

where $t_1, t_2 \in I$ are finite. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.

As already observed, for the proof of Corollary 17, it suffices to show that K_Σ^{rad} is bounded from below by a constant, and the result then follows from Theorem 16. Actually, we can prove the following stronger result, which will be useful in the sequel.

Lemma 18. *Let \mathbb{P}^n be a Riemannian manifold with sectional curvature bounded from below, and let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be an immersed hypersurface. Assume that $\sup_\Sigma \|A\|^2 < +\infty$ and that Σ is contained in a slab. Then the sectional curvature of Σ is bounded from below by a constant.*

Given the validity of Lemma 18 and taking into account the equality

$$\|A\|^2 = \text{Tr}(A^2) = n^2 H_1^2 - n(n-1)H_2,$$

it follows that, under the assumptions of Corollary 17,

$$\sup_\Sigma \|A\|^2 \leq n^2(\sup_\Sigma H_1)^2 - n(n-1)H_2 < +\infty.$$

Thus K_Σ^{rad} is bounded from below by a constant.

Proof of Lemma 18. Recall that the Gauss equation for a hypersurface $f : \Sigma^n \rightarrow M^{n+1}$ is given by

$$\langle R(X, Y)Z, V \rangle = \langle \bar{R}(X, Y)Z, V \rangle - \langle AY, V \rangle \langle AX, Z \rangle + \langle AX, V \rangle \langle AY, Z \rangle,$$

for $X, Y, Z, V \in T\Sigma$, where R and \bar{R} are the curvature tensors of Σ^n and M^{n+1} , respectively. Then, if $\{X, Y\}$ is an orthonormal basis for an arbitrary 2-plane tangent to Σ , we have

$$\begin{aligned} K_\Sigma(X, Y) &= \bar{K}(X, Y) + \langle AX, X \rangle \langle AY, Y \rangle - \langle AX, Y \rangle^2 \\ (31) \quad &\geq \bar{K}(X, Y) - \|AX\| \|AY\| - \|AX\|^2 \\ &\geq \bar{K}(X, Y) - 2\|A\|^2, \end{aligned}$$

where the last inequality follows from the fact that

$$\|AX\|^2 \leq \text{Tr}(A^2)\|X\|^2 = \|A\|^2$$

for every unit vector X tangent to Σ . Since we are assuming that $\sup_\Sigma \|A\|^2 < +\infty$, it suffices to have $\bar{K}(X, Y)$ bounded from below.

The curvature tensor of M^{n+1} expressed in terms of the curvature tensor of \mathbb{P}^n is

$$\begin{aligned} \bar{R}(U, V)W &= R_{\mathbb{P}}(U^*, V^*)W^* - \mathcal{H}^2(\pi_I)(\langle V, W \rangle U - \langle U, W \rangle V) \\ &\quad + \mathcal{H}'(\pi_I)\langle W, T \rangle(\langle U, T \rangle V - \langle V, T \rangle U) \\ &\quad - \mathcal{H}'(\pi_I)(\langle V, W \rangle \langle U, T \rangle - \langle U, W \rangle \langle V, T \rangle)T, \end{aligned}$$

for every $U, V, W \in TM$, where $T = \partial_t$ and we are using the notation U^* to denote $\pi_{\mathbb{P}*}U$ for an arbitrary $U \in TM$. Then, for the orthonormal basis $\{X, Y\}$ we find that

$$\begin{aligned} \bar{K}(X, Y) &= \frac{1}{\rho^2(h)} K_{\mathbb{P}}(X^*, Y^*) \|X^* \wedge Y^*\|^2 \\ (32) \quad &\quad - \mathcal{H}^2(h) - \mathcal{H}'(h)(\langle X, \nabla h \rangle^2 + \langle Y, \nabla h \rangle^2) \\ &\geq \frac{1}{\rho^2(h)} K_{\mathbb{P}}(X^*, Y^*) \|X^* \wedge Y^*\|^2 - \mathcal{H}^2(h) - |\mathcal{H}'(h)|, \end{aligned}$$

since

$$\langle X, \nabla h \rangle^2 + \langle Y, \nabla h \rangle^2 \leq \|\nabla h\|^2 \leq 1.$$

On the other hand,

$$\begin{aligned}\|X^* \wedge Y^*\|^2 &= \|X^*\|^2 \|Y^*\|^2 - \langle X^*, Y^* \rangle^2 \\ &= 1 - \langle X, T \rangle^2 - \langle Y, T \rangle^2 \leq 1.\end{aligned}$$

Therefore, if $K_{\mathbb{P}} \geq c$ for some constant c , we deduce

$$(33) \quad \frac{1}{\rho^2(h)} K_{\mathbb{P}}(X^*, Y^*) \|X^* \wedge Y^*\|^2 \geq -\frac{|c|}{\rho^2(h)}.$$

Finally, since h is a bounded function, we conclude from (31), (32), and (33) that the sectional curvature $K(X, Y)$ is bounded from below by an absolute constant. \square

We observe that condition (30) has been used in the proof of Theorem 16 only to guarantee that the Omori-Yau maximum principle holds on Σ for the Laplacian and for the semi-elliptic operator \mathcal{L}_1 (or $-\mathcal{L}_1$). Therefore, the theorem remains true under any other hypothesis guaranteeing this latter fact. Thus, and as a consequence of Corollary 4, we can also state the following:

Theorem 19. *Let \mathbb{P}^n be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition (13). Let $f : \Sigma^n \rightarrow I \times_{\rho} \mathbb{P}^n$ be a properly immersed hypersurface of constant positive 2-mean curvature H_2 . Assume that $\sup_{\Sigma} |H_1| < +\infty$ and that Σ is contained in a slab. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

As pointed out before, for the validity of Theorem 19 it suffices to show that the Omori-Yau maximum principle holds on Σ for the Laplacian and for the semi-elliptic operator \mathcal{L}_1 (or $-\mathcal{L}_1$). But this follows directly from Corollary 4, since $\|A\|^2 = \text{Tr}(A^2) = n^2 H_1^2 - n(n-1)H_2$ and

$$\sup_{\Sigma} \|A\|^2 \leq n^2 (\sup_{\Sigma} H_1)^2 - n(n-1)H_2 < +\infty.$$

See Remark 5 following Corollary 4.

6. HYPERSURFACES WITH CONSTANT HIGHER ORDER MEAN CURVATURE

In this section we will extend our previous results to the case of hypersurfaces with non-zero constant k -mean curvature H_k , when $k \geq 3$. To this end, we will work with the operator L_{k-1} , and we will assume that there exists an elliptic point in Σ . Note that the existence of an elliptic point is always guaranteed when Σ is compact and $\rho' \neq 0$ on Σ (see the proof of Theorem 24 below and Lemma 5.3 in [2] in a Lorentzian ambient space). Recall from the discussion in the Preliminaries that the existence of an elliptic point implies that H_k is positive, the immersion is two sided and $H_1 > 0$ for the chosen orientation. Moreover, it implies also that for every $1 \leq j \leq k-1$, the operators L_j are elliptic or, equivalently, the operators P_j are positive definite.

In order to extend our previous results to the case of higher order mean curvatures, we introduce a family of operators, extending \mathcal{L}_1 . For $2 \leq k \leq n$, we define the operator

$$\mathcal{L}_{k-1} = \text{Tr} \left(\left[\sum_{j=0}^{k-1} (-1)^j \frac{c_{k-1}}{c_j} \mathcal{H}(h)^{k-1-j} \Theta^j P_j \right] \circ \text{hess} \right) = \text{Tr}(\mathcal{P}_{k-1} \circ \text{hess}),$$

where

$$(34) \quad \mathcal{P}_{k-1} = \sum_{j=0}^{k-1} (-1)^j \frac{c_{k-1}}{c_j} \mathcal{H}(h)^{k-1-j} \Theta^j P_j.$$

We claim that

$$\mathcal{L}_{k-1}\sigma(h) = c_{k-1}\rho(h)(\mathcal{H}(h)^k + (-1)^{k-1}\Theta^k H_k),$$

and we prove the claim by induction. We have already seen in (29) that the claim is true for $k = 2$. For $k \geq 3$, we observe that

$$\mathcal{P}_{k-1} = \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \mathcal{P}_{k-2} + (-1)^{k-1} \Theta^{k-1} P_{k-1}$$

and then

$$\mathcal{L}_{k-1} = \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \mathcal{L}_{k-2} + (-1)^{k-1} \Theta^{k-1} L_{k-1},$$

Therefore, if $k \geq 3$ and we assume that the claim is true for \mathcal{L}_{k-2} , then using (22) we conclude that

$$\begin{aligned} \mathcal{L}_{k-1}\sigma(h) &= \frac{c_{k-1}}{c_{k-2}} \mathcal{H}(h) \mathcal{L}_{k-2}\sigma(h) + (-1)^{k-1} \Theta^{k-1} L_{k-1}\sigma(h) \\ &= c_{k-1}\rho(h)(\mathcal{H}(h)^k + (-1)^{k-2} \mathcal{H}(h) \Theta^{k-1} H_{k-1} \\ &\quad + (-1)^{k-1} \mathcal{H}(h) \Theta^{k-1} H_{k-1} + (-1)^{k-1} \Theta^k H_k) \\ &= c_{k-1}\rho(h)(\mathcal{H}(h)^k + (-1)^{k-1} \Theta^k H_k). \end{aligned}$$

We are now ready to give the following extension of Theorem 15.

Theorem 20. *Let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be a compact hypersurface with constant k -mean curvature H_k , with $3 \leq k \leq n$. Assume that there exists an elliptic point in Σ . If $\mathcal{H}'(t) \geq 0$ and the angle function Θ does not change sign, then \mathbb{P}^n is necessarily compact and $f(\Sigma^n)$ is a slice.*

Proof. Choose the orientation of Σ so that $H_1 > 0$. Since Σ^n is compact, we may apply Lemma 14. Let us consider first the case when $\Theta \leq 0$, so that $\mathcal{H}(h) \geq 0$. Thus, by (34) the operator \mathcal{P}_{k-1} is positive semi-definite or, equivalently, \mathcal{L}_{k-1} is semi-elliptic. Reasoning as in the proof of Theorem 15, yields

$$\mathcal{L}_{k-1}\sigma(h)(p_{\max}) = c_{k-1}\rho(h^*)(\mathcal{H}(h^*)^k - H_k) \leq 0$$

and

$$\mathcal{L}_{k-1}\sigma(h)(p_{\min}) = c_{k-1}\rho(h_*)(\mathcal{H}(h_*)^k - H_k) \geq 0,$$

with $p_{\max} \in \Sigma$ and $p_{\min} \in \Sigma$ such that $h(p_{\max}) = h^* = \max_\Sigma h$ and $h(p_{\min}) = h_* = \min_\Sigma h$.

Then, since $\mathcal{H}(h) \geq 0$ on Σ , we obtain

$$\mathcal{H}(h_*) \geq H_k^{1/k} \geq \mathcal{H}(h^*).$$

On the other hand, by $\mathcal{H}' \geq 0$ we also have $\mathcal{H}(h_*) \leq \mathcal{H}(h^*)$. Thus, we have the equality $\mathcal{H}(h_*) = \mathcal{H}(h^*)$ and $\mathcal{H}(h) = H_k^{1/k}$ is constant on Σ . Therefore, by (22) and using the Gårding inequality $H_{k-1} \geq H_k^{(k-1)/k}$ (see (1)) and the fact that $\Theta \geq -1$, we obtain

$$\begin{aligned} L_{k-1}\sigma(h) &= c_{k-1}\rho(h)H_k^{1/k}(H_{k-1} + \Theta H_k^{(k-1)/k}) \\ &\geq c_{k-1}\rho(h)H_k^{1/k}(H_{k-1} - H_k^{(k-1)/k}) \geq 0. \end{aligned}$$

That is, $L_{k-1}\sigma(h) \geq 0$ on the compact manifold Σ . Therefore, by the maximum principle applied to the elliptic operator L_{k-1} , we conclude that $\sigma(h)$, and hence h , is constant.

Finally, in the case where $\Theta \geq 0$, we know from Lemma 14 that $\mathcal{H}(h) \leq 0$ on Σ , so that the operator $(-1)^{k-1}\mathcal{L}_{k-1}$ is semi-elliptic. The proof then follows as in the case $\Theta \leq 0$, working with $(-1)^{k-1}\mathcal{L}_{k-1}$ instead. \square

For the case of complete (and non-compact) hypersurfaces, we can state the following extension of Theorem 16.

Theorem 21. *Let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be a complete hypersurface with constant k -mean curvature H_k , $3 \leq k \leq n$, which satisfies condition (30). Assume that there exists an elliptic point in Σ , $\sup_\Sigma |H_1| < +\infty$ and Σ is contained in a slab. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

Proof. By Corollary 3, we know that the Omori-Yau maximum principle holds on Σ for the Laplacian operator, so that we may apply Lemma 14. Thus, in the case where $\Theta \leq 0$, we have $\mathcal{H}(h) \geq 0$ and therefore

$$(-1)^j \mathcal{H}(h)^{k-1-j} \Theta^j \geq 0$$

for every $j = 0, \dots, k-1$. Since the operators $P_0 = I, P_1, \dots, P_{k-1}$ are all positive definite, it follows from here that the operator \mathcal{P}_{k-1} is positive semi-definite or, in other words, that the differential operator \mathcal{L}_{k-1} is semi-elliptic. Furthermore, since $0 \leq -\Theta \leq 1$,

$$\text{Tr}(\mathcal{P}_{k-1}) = c_{k-1} \sum_{j=0}^{k-1} (-1)^j \mathcal{H}(h)^{k-1-j} \Theta^j H_j \leq c_{k-1} \sum_{j=0}^{k-1} \mathcal{H}(h^*)^{k-1-j} H_j^*,$$

where $h^* = \sup_\Sigma h < +\infty$ and $H_j^* = \sup_\Sigma H_j \leq (\sup_\Sigma H_1)^j < +\infty$ because of (1). Hence by Corollary 3, the Omori-Yau maximum principle holds for the operator \mathcal{L}_{k-1} and, proceeding as in the proof of Theorem 16, we may find two sequences $\{p_j\}_{j \in \mathbb{N}} \subset \Sigma$ and $\{q_j\}_{j \in \mathbb{N}} \subset \Sigma$ satisfying

$$\begin{aligned} \lim_{j \rightarrow +\infty} h(p_j) &= h^*, & \text{and} & & \lim_{j \rightarrow +\infty} h(q_j) &= h_*, \\ \lim_{j \rightarrow +\infty} \Theta(p_j) &= \lim_{j \rightarrow +\infty} \Theta(q_j) &= & & -1, \end{aligned}$$

$$\mathcal{L}_{k-1}\sigma(h)(p_j) = c_{k-1}\rho(h(p_j))(\mathcal{H}(h(p_j)))^k + (-1)^{k-1}\Theta^k(p_j)H_k < \frac{1}{j},$$

and

$$\mathcal{L}_{k-1}\sigma(h)(q_j) = c_{k-1}\rho(h(q_j))(\mathcal{H}(h(q_j)))^k + (-1)^{k-1}\Theta^k(q_j)H_k < \frac{1}{j}.$$

Making $j \rightarrow +\infty$ in the inequalities above, we obtain that

$$\mathcal{H}(h^*)^k \leq H_k \leq \mathcal{H}(h_*)^k,$$

which implies that $h_* = h^*$, as in the proof of Theorem 16.

Finally, in the case where $\Theta \geq 0$ we proceed again as in the proof of Theorem 16, working now with the operator $(-1)^{k-1}\mathcal{L}_{k-1}$, which in this case is semi-elliptic and with $\text{Tr}((-1)^{k-1}\mathcal{P}_{k-1})$ bounded from above. \square

As in the previous section for Theorem 19, Theorem 21 remains true if we replace condition (30) by the stronger condition of Σ^n having radial sectional curvature bounded from below by a constant. By applying Lemma 18, we see that this happens when the sectional curvature of \mathbb{P}^n is itself bounded from below.

Corollary 22. *Let \mathbb{P}^n be a complete Riemannian manifold with sectional curvature bounded from below, and let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be a complete hypersurface with constant k -mean curvature H_k , $3 \leq k \leq n$. Assume that there exists an elliptic point in Σ , $\sup_\Sigma |H_1| < +\infty$ and Σ is contained in a slab. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

Indeed, by (1) we know that $H_2 > 0$, so that $\sup_\Sigma \|A\|^2 \leq n^2(\sup_\Sigma H_1)^2 < +\infty$ and we may apply Lemma 14 to conclude that the radial sectional curvature of Σ is bounded from below. The result then follows from Theorem 21.

Finally, similar to what happened in the previous section, condition (30) has been used in the proof of Theorem 21 only to guarantee that the Omori-Yau maximum principle holds on Σ for the Laplacian and for the semi-elliptic operator \mathcal{L}_{k-1} (or $-\mathcal{L}_{k-1}$). Therefore, the theorem remains true under any other hypothesis guaranteeing that property. Then, and as a consequence of Corollary 4, we can also state the following:

Theorem 23. *Let \mathbb{P}^n be a complete, non-compact, Riemannian manifold whose radial sectional curvature satisfies condition (13). Let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be a properly immersed hypersurface of constant k -mean curvature, $3 \leq k \leq n$. Assume that there exists an elliptic point in Σ , $\sup_\Sigma |H_1| < +\infty$ and that Σ is contained in a slab. If $\mathcal{H}'(t) > 0$ almost everywhere and the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.*

7. FURTHER RESULTS FOR HYPERSURFACES
WITH CONSTANT HIGHER ORDER MEAN CURVATURES

In this last section we introduce some further results for the case of constant higher order mean curvatures. As a first result in this direction we shall prove the next

Theorem 24. *Let $f : \Sigma^n \rightarrow M^{n+1} = I \times_\rho \mathbb{P}^n$ be a compact hypersurface of constant k -mean curvature, $2 \leq k \leq n$, and suppose that \mathcal{H} does not vanish. Assume that*

$$(35) \quad K_{\mathbb{P}^n} \geq \sup_I \{\rho'^2 - \rho''\rho\},$$

$K_{\mathbb{P}^n}$ being the sectional curvature of \mathbb{P}^n , and that the angle function Θ does not change sign. Then either $f(\Sigma^n)$ is a slice over a compact \mathbb{P}^n or M^{n+1} has constant sectional curvature and Σ^n is a geodesic hypersphere. The latter case cannot occur if the inequality (35) is strict.

First we proceed with the proof of two important lemmas that will be essential in the proof of Theorem 24. For Lemma 25 see also Lemma 3.1 in [5].

Lemma 25. *Let $\Sigma^n \rightarrow \overline{M}^{n+1}$ be an isometric immersion. Let E_1, \dots, E_n be a local orthonormal frame on Σ , and let N be local unit normal. Then*

$$(36) \quad \sum_{i=1}^n \langle (\nabla_{E_i} P_k) X, E_i \rangle = \sum_{j=0}^{k-1} \sum_{i=1}^n (-1)^{k-1-j} \langle \overline{R}(E_i, A^{k-1-j} X) N, P_j E_i \rangle,$$

for every vector field $X \in T\Sigma$.

Proof. We will prove equation (36) by induction on k , $1 \leq k \leq n - 1$. It is not difficult to prove that this is true for $k = 1$ using the Codazzi equation and the definition of P_1 . Assume that the equation holds for $k - 1$. Then, again using the Codazzi equation, we get

$$\begin{aligned} \sum_{i=1}^n \langle (\nabla_{E_i} P_k) X, E_i \rangle &= - \sum_{i=1}^n \langle (\nabla_{E_i} P_{k-1}) A X, E_i \rangle + \sum_{i=1}^n \langle \overline{R}(E_i, X) N, P_{k-1} E_i \rangle \\ &= - \sum_{j=0}^{k-2} \sum_{i=1}^n (-1)^{k-2-j} \langle \overline{R}(E_i, A^{k-1-j} X) N, P_j E_i \rangle \\ &\quad + \sum_{i=1}^n \langle \overline{R}(E_i, X) N, P_{k-1} E_i \rangle \\ &= \sum_{j=0}^{k-1} \sum_{i=1}^n (-1)^{k-1-j} \langle \overline{R}(E_i, A^{k-1-j} X) N, P_j E_i \rangle. \end{aligned}$$

For further details see [5], paying attention to the different conventions for the sign of \overline{R} . □

Corollary 26. *Let $\Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be an immersed hypersurface, and assume that \mathbb{P}^n has constant sectional curvature κ . Then*

$$(37) \quad \operatorname{div} P_k = -(n - k) \Theta \left(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h) \right) P_{k-1} \nabla h.$$

Proof. Let E_1, \dots, E_n be a local orthonormal frame on Σ^n , and observe that

$$\langle \operatorname{div} P_k, X \rangle = \sum_{i=1}^n \langle (\nabla_{E_i} P_k) X, E_i \rangle$$

for every vector field $X \in T\Sigma$. Fix j , $j = 0, \dots, k - 1$. Then

$$\begin{aligned} \sum_{i=1}^n \langle \overline{R}(E_i, A^{k-1-j} X) N, P_j E_i \rangle &= \sum_{i=1}^n \langle R_{\mathbb{P}}(\pi_{\mathbb{P}*} E_i, \pi_{\mathbb{P}*} A^{k-1-j} X) \pi_{\mathbb{P}*} N, P_j E_i \rangle \\ &\quad + \Theta \mathcal{H}'(h) \langle P_j \nabla h, A^{k-1-j} X \rangle \\ &\quad - c_j H_j \langle \nabla h, A^{k-1-j} X \rangle. \end{aligned}$$

Since \mathbb{P}^n has constant sectional curvature κ it follows that

$$R_{\mathbb{P}}(Y, Z)W = \kappa(\langle Z, W \rangle_{\mathbb{P}} Y - \langle Y, W \rangle_{\mathbb{P}} Z).$$

Hence a direct calculation shows that

$$\begin{aligned} \sum_{i=1}^n \langle R_{\mathbb{P}}(\pi_{\mathbb{P}*} E_i, \pi_{\mathbb{P}*} A^{k-1-j} X) \pi_{\mathbb{P}*} N, P_j E_i \rangle &= \frac{\kappa}{\rho^2(h)} \Theta \langle P_j \nabla h, A^{k-1-j} X \rangle \\ &\quad - c_j H_j \langle \nabla h, A^{k-1-j} X \rangle. \end{aligned}$$

We claim that

$$B_k := \sum_{j=0}^{k-1} (-1)^{k-j-1} (P_j A^{k-1-j} - c_j H_j A^{k-1-j}) = -(n-k)P_{k-1}.$$

In this case the conclusion of Corollary 26 is immediate. We prove the claim by induction on k , $k = 0, \dots, n - 1$. The case $k = 0$ is trivial. Assume that we proved the equation for $k - 1$. Then

$$\begin{aligned} B_k &= P_{k-1} - c_{k-1}H_{k-1}I - B_{k-1} \circ A \\ &= P_{k-1} - c_{k-1}H_{k-1}I + (n-k+1)P_{k-2}A \\ &= -(n-k)P_{k-1}. \end{aligned}$$

□

Lemma 27. *Let Σ^n be a hypersurface immersed into a warped product space $I \times_\rho \mathbb{P}^n$, with angle function Θ and height function h . Let $\hat{\Theta} = \rho\Theta$. Then, for every $k = 0, \dots, n - 1$, we have*

$$\begin{aligned} L_k \hat{\Theta} &= - \binom{n}{k+1} \rho(h) \langle \nabla h, \nabla H_{k+1} \rangle - \rho'(h) c_k H_{k+1} \\ &\quad - \hat{\Theta} \mathcal{H}'(h) (\|\nabla h\|^2 c_k H_k - \langle P_k \nabla h, \nabla h \rangle) - \frac{\hat{\Theta}}{\rho(h)^2} \beta_k \\ &\quad - \hat{\Theta} \binom{n}{k+1} (nH_1 H_{k+1} - (n-k-1)H_{k+2}), \end{aligned}$$

where

$$\beta_k = \sum_{i=1}^n \mu_{k,i} K_{\mathbb{P}}(\pi_{\mathbb{P}*} E_i, \pi_{\mathbb{P}*} N) \|\pi_{\mathbb{P}*} E_i \wedge \pi_{\mathbb{P}*} N\|^2.$$

Here the $\mu_{k,i}$'s stand for the eigenvalues of P_k and $\{E_1, \dots, E_n\}$ is a local orthonormal frame on Σ diagonalizing A .

Proof. Since $\rho(t)T$ is a conformal vector field

$$\nabla \hat{\Theta} = -\rho(h)A\nabla h.$$

Therefore, using equation (24) we find

$$\nabla_X \nabla \hat{\Theta} = -\rho(h)(\nabla_X A)\nabla h - \rho'(h)AX - \hat{\Theta}A^2X.$$

Hence

$$\begin{aligned} L_k \hat{\Theta} &= -\rho(h) \sum_{i=1}^n \langle P_k(\nabla_{E_i} A)\nabla h, E_i \rangle \\ &\quad - \rho'(h) c_k H_{k+1} - \binom{n}{k+1} \hat{\Theta} (H_1 H_{k+1} - (n-k-1)H_{k+2}). \end{aligned}$$

Using the expression of the covariant derivative of a tensor field, we get

$$\begin{aligned} -P_k(\nabla_{E_i} A)\nabla h &= (\nabla_{E_i} P_k)A\nabla h - (\nabla_{E_i} P_k A)\nabla h \\ &= (\nabla_{E_i} P_k)A\nabla h + (\nabla_{E_i} P_{k+1})\nabla h - E_i(S_{k+1})\nabla h. \end{aligned}$$

By equation (36) it follows that

$$\begin{aligned}
 -\sum_{i=1}^n \langle P_k(\nabla_{E_i} A)\nabla h, E_i \rangle &= \sum_{i=1}^n \langle (\nabla_{E_i} P_k)A\nabla h, E_i \rangle \\
 &\quad + \sum_{i=1}^n \langle (\nabla_{E_i} P_{k+1})\nabla h, E_i \rangle - \nabla h(S_{k+1}) \\
 &= \sum_{i=1}^n \langle \bar{R}(E_i, \nabla h)N, P_k E_i \rangle - \nabla h(S_{k+1}).
 \end{aligned}$$

Since $\nabla h = T - \Theta N$, we can write

$$\bar{R}(E_i, \nabla h)N = \bar{R}(E_i, T)N - \Theta \bar{R}(E_i, N)N.$$

Using the Gauss equation and observing that $\pi_{\mathbb{P}*}T = 0$, we get

$$\bar{R}(E_i, T)N = -(\mathcal{H}(h)^2 + \mathcal{H}'(h))\Theta E_i = -\frac{\rho''(h)}{\rho(h)}\Theta E_i.$$

So

$$\sum_{i=1}^n \langle \bar{R}(E_i, T)N, P_k E_i \rangle = -\frac{\rho''(h)}{\rho(h)}\Theta c_k H_k.$$

Again by the Gauss equation

$$\begin{aligned}
 \bar{R}(E_i, N)N &= R_{\mathbb{P}}(\pi_{\mathbb{P}*}E_i, \pi_{\mathbb{P}*}N)\pi_{\mathbb{P}*}N - \mathcal{H}(h)^2 E_i \\
 &\quad + \mathcal{H}'(h)\Theta(\langle E_i, \nabla h \rangle N - \Theta E_i) - \mathcal{H}'(h)\langle E_i, \nabla h \rangle T.
 \end{aligned}$$

Assume that the orthonormal basis $\{E_i\}_1^n$ diagonalizes A and hence P_k , that is $P_k E_i = \mu_{k,i} E_i$. Then

$$\begin{aligned}
 \sum_{i=1}^n \langle \bar{R}(E_i, N)N, P_k E_i \rangle &= \frac{1}{\rho(h)^2} \sum_{i=1}^n \mu_{k,i} K_{\mathbb{P}}(\pi_{\mathbb{P}*}E_i, \pi_{\mathbb{P}*}N) \|\pi_{\mathbb{P}*}E_i \wedge \pi_{\mathbb{P}*}N\|^2 \\
 &\quad - \frac{\rho''(h)}{\rho(h)} c_k H_k + \mathcal{H}'(h)(\|\nabla h\|^2 c_k H_k - \langle P_k \nabla h, \nabla h \rangle).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 \sum_{i=1}^n \langle \bar{R}(E_i, \nabla h)N, P_k E_i \rangle &= \sum_{i=1}^n \langle \bar{R}(E_i, T)N, P_k E_i \rangle - \Theta \sum_{i=1}^n \langle \bar{R}(E_i, N)N, P_k E_i \rangle \\
 &= -\frac{\Theta}{\rho(h)^2} \sum_{i=1}^n \mu_{k,i} K_{\mathbb{P}}(\pi_{\mathbb{P}*}E_i, \pi_{\mathbb{P}*}N) \|\pi_{\mathbb{P}*}E_i \wedge \pi_{\mathbb{P}*}N\|^2 \\
 &\quad - \Theta \mathcal{H}'(h)(\|\nabla h\|^2 c_k H_k - \langle P_k \nabla h, \nabla h \rangle),
 \end{aligned}$$

and this concludes the proof of the lemma. □

Corollary 28. *Let Σ^n be a hypersurface immersed into a warped product space $I \times_{\rho} \mathbb{P}^n$ with angle function Θ and height function h . Assume that \mathbb{P}^n has constant*

sectional curvature κ , and let $\hat{\Theta} = \rho(h)\Theta$. Then, for every $k = 0, \dots, n - 1$ we have

$$\begin{aligned} L_k \hat{\Theta} &= - \binom{n}{k+1} \rho(h) \langle \nabla h, \nabla H_{k+1} \rangle - \rho'(h) c_k H_{k+1} \\ &\quad - \hat{\Theta} \left(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h) \right) (\|\nabla h\|^2 c_k H_k - \langle P_k \nabla h, \nabla h \rangle) \\ &\quad - \hat{\Theta} \binom{n}{k+1} (n H_1 H_{k+1} - (n - k - 1) H_{k+2}). \end{aligned}$$

We are now ready to give the

Proof of Theorem 24. We may assume without loss of generality that $\mathcal{H}(h) > 0$ on Σ . Since Σ^n is compact, there exists a point $p_0 \in \Sigma$ where the height function attains its maximum. Then $\nabla h(p_0) = 0$, $\Theta(p_0) = \pm 1$ and by (24)

$$\text{Hess } h(p_0)(v, v) = \mathcal{H}(h^*) \langle v, v \rangle + \Theta(p_0) \langle Av, v \rangle(p_0) \leq 0.$$

If $\Theta(p_0) = -1$, then

$$\langle Av, v \rangle(p_0) \geq \mathcal{H}(h^*) \langle v, v \rangle > 0,$$

for any $v \neq 0$. Thus p_0 is an elliptic point, H_k is a positive constant and by the Gårding inequalities

$$H_1 \geq H_2^{\frac{1}{2}} \geq \dots \geq H_k^{\frac{1}{k}} > 0$$

with equality only at umbilical points. In particular, Σ is two sided and then $\Theta \leq 0$. If $\Theta(p_0) = 1$, changing the orientation we have the same conclusion.

Consider the function

$$\phi = \sigma(h) H_k^{\frac{1}{k}} + \rho(h)\Theta.$$

Let us prove that $L_{k-1}\phi \geq 0$. Since H_k is constant, we have

$$\begin{aligned} L_{k-1}\phi &= H_k^{\frac{1}{k}} L_{k-1}\sigma(h) + L_{k-1}\hat{\Theta} \\ &= c_{k-1} H_k^{\frac{1}{k}} (\rho'(h) H_{k-1} + \hat{\Theta} H_k) - c_{k-1} H_{k-1} \hat{\Theta} \mathcal{H}'(h) \|\nabla h\|^2 \\ &\quad + \hat{\Theta} \mathcal{H}'(h) \langle P_{k-1} \nabla h, \nabla h \rangle - \hat{\Theta} \binom{n}{k} (n H_1 H_k - (n - k) H_{k+1}) \\ &\quad - \rho'(h) c_{k-1} H_k - \frac{\hat{\Theta}}{\rho(h)^2} \sum_{i=1}^n \mu_{k-1,i} K_{\mathbb{P}}(\pi_{\mathbb{P}*} E_i, \pi_{\mathbb{P}*} N) \|\pi_{\mathbb{P}*} E_i \wedge \pi_{\mathbb{P}*} N\|^2 \\ &= A + B + C, \end{aligned}$$

where

$$\begin{aligned} A &= -\hat{\Theta} \binom{n}{k} (n H_1 H_k - (n - k) H_{k+1} - k H_k^{\frac{k+1}{k}}), \\ B &= c_{k-1} \rho'(h) (H_{k-1} H_k^{\frac{1}{k}} - H_k) \end{aligned}$$

and

$$\begin{aligned} C &= -\hat{\Theta} \mathcal{H}'(h) (\|\nabla h\|^2 c_{k-1} H_{k-1} - \langle P_{k-1} \nabla h, \nabla h \rangle) \\ &\quad - \frac{\hat{\Theta}}{\rho(h)^2} \sum_{i=1}^n \mu_{k-1,i} K_{\mathbb{P}}(\pi_{\mathbb{P}*} E_i, \pi_{\mathbb{P}*} N) \|\pi_{\mathbb{P}*} E_i \wedge \pi_{\mathbb{P}*} N\|^2. \end{aligned}$$

Then by the Gårding inequalities,

$$H_{k-1} H_k^{\frac{1}{k}} - H_k = H_k^{\frac{1}{k}} (H_{k-1} - H_k^{\frac{k-1}{k}}) \geq 0.$$

Moreover,

$$nH_1H_k - kH_k^{\frac{k+1}{k}} \geq nH_k^{\frac{k+1}{k}} - kH_k^{\frac{k+1}{k}} = (n - k)H_k^{\frac{k+1}{k}},$$

hence

$$nH_1H_k - kH_k^{\frac{k+1}{k}} - (n - k)H_{k+1} \geq (n - k)(H_k^{\frac{k+1}{k}} - H_{k+1}) \geq 0.$$

Finally, let $\alpha := \sup_I\{\rho'^2 - \rho''\rho\}$. Since

$$\|\pi_{\mathbb{P}^*}E_i \wedge \pi_{\mathbb{P}^*}N\|^2 = \|\nabla h\|^2 - \langle E_i, \nabla h \rangle^2,$$

taking into account that the $\mu_{k-1,i}$'s are positive, we have

$$\begin{aligned} & \sum_{i=1}^n \mu_{k-1,i} K_{\mathbb{P}}(\pi_{\mathbb{P}^*}E_i, \pi_{\mathbb{P}^*}N) \|\pi_{\mathbb{P}^*}E_i \wedge \pi_{\mathbb{P}^*}N\|^2 \\ & \geq \alpha \sum_{i=1}^n \mu_{k-1,i} \|\pi_{\mathbb{P}^*}E_i \wedge \pi_{\mathbb{P}^*}N\|^2 \\ & = \alpha(c_{k-1}H_{k-1} \|\nabla h\|^2 - \langle P_{k-1} \nabla h, \nabla h \rangle). \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{1}{\rho(h)^2} \sum_{i=1}^n \mu_{k-1,i} K_{\mathbb{P}}(\pi_{\mathbb{P}^*}E_i, \pi_{\mathbb{P}^*}N) \|\pi_{\mathbb{P}^*}E_i \wedge \pi_{\mathbb{P}^*}N\|^2 \\ & \quad + \mathcal{H}'(h)(\|\nabla h\|^2 c_{k-1}H_{k-1} - \langle P_{k-1} \nabla h, \nabla h \rangle) \\ & \geq \left(\frac{\alpha}{\rho(h)^2} + \mathcal{H}'(h) \right) (\|\nabla h\|^2 c_{k-1}H_{k-1} - \langle P_{k-1} \nabla h, \nabla h \rangle) \geq 0, \end{aligned}$$

where the last inequality follows from $\alpha = \sup_I\{-\rho^2\mathcal{H}'\}$ and from the fact that P_{k-1} is a positive definite operator. Thus, $L_{k-1}\phi \geq 0$. Since L_{k-1} is an elliptic operator and Σ is compact, we conclude by the maximum principle that ϕ must be constant. Hence $L_{k-1}\phi = 0$ and the three terms A , B , and C in $L_{k-1}\phi$ vanish on Σ .

In particular $B = 0$ implies that Σ is a totally umbilical hypersurface. Moreover, since H_k is a positive constant and Σ is totally umbilical, all the higher order mean curvatures are constant. In particular, H_1 is constant and the conclusion follows by Theorem 3.4 [4]. □

On the other hand, using Theorem 3.1 in [4] we can give the following version of Theorem 3.4 in [4] for the complete case.

Proposition 29. *Let $M^{n+1} = I \times_{\rho} \mathbb{P}^n$ be a warped product space, and assume that the Ricci curvature of \mathbb{P}^n satisfies*

$$(38) \quad \text{Ric}_{\mathbb{P}} > \sup_I\{\rho'^2 - \rho''\rho\}.$$

Let $f : \Sigma^n \rightarrow I \times_{\rho} \mathbb{P}^n$ be a complete, parabolic, two sided hypersurface with constant mean curvature. Suppose that

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n,$$

where t_1, t_2 are finite. If the angle function Θ does not change sign, then $f(\Sigma^n)$ is a slice.

For the proof, observe that since the function $\phi = H\sigma(h) + \hat{\Theta}$ is subharmonic and Σ is parabolic, then it must be constant. In particular, $\Delta\phi = 0$ and by equation (3.8) in [4] we conclude that h has to be constant, because of the strict inequality in (38).

In what follows, we extend this result to higher order mean curvatures. Towards this aim, we let \mathfrak{L}_k be the operator

$$\mathfrak{L}_k f = \operatorname{div}(P_k \nabla f),$$

where $f \in C^\infty(\Sigma)$. Notice that

$$\mathfrak{L}_k f = \langle \operatorname{div} P_k, \nabla f \rangle + L_k f.$$

We introduce the following

Definition 30. We will say that the manifold $\Sigma^n \hookrightarrow I \times_\rho \mathbb{P}^n$ is \mathfrak{L}_k -parabolic if the only bounded above C^1 solutions of the inequality

$$\mathfrak{L}_k f \geq 0$$

are constant.

The following theorem is a special case of Theorem 2.6 in [13].

Theorem 31. Let $\Sigma^n \hookrightarrow I \times_\rho \mathbb{P}^n$ be a complete manifold. Fix an origin $o \in \Sigma$. If

$$(39) \quad \left(\sup_{\partial B_t} H_{k-1} \operatorname{vol}(\partial B_t) \right)^{-1} \notin L^1(+\infty),$$

where ∂B_t is the geodesic sphere of radius t centered at o , then Σ^n is \mathfrak{L}_{k-1} -parabolic.

We are ready to state our last result.

Theorem 32. Let $M^{n+1} = I \times_\rho \mathbb{P}^n$ be a warped product space, and assume that \mathbb{P}^n has constant sectional curvature κ satisfying

$$(40) \quad \kappa > \sup_I \{\rho'^2 - \rho''\rho\}.$$

Let $f : \Sigma^n \rightarrow I \times_\rho \mathbb{P}^n$ be a complete hypersurface with $\sup_\Sigma |H_1| < +\infty$ and satisfying condition (39). Suppose that f has constant k -mean curvature, $2 \leq k \leq n$, and

$$f(\Sigma^n) \subset [t_1, t_2] \times \mathbb{P}^n,$$

where t_1, t_2 are finite. Assume that either $k = 2$ and H_2 is positive or $k \geq 3$ and there exists an elliptic point $p \in \Sigma^n$. If $\mathcal{H}(h)$ and the angle function Θ do not change sign, then $f(\Sigma^n)$ is a slice.

Remark 33. Comparing with Theorem 24, we have relaxed the condition on \mathcal{H} but we are requiring, as it will be clear from the proof, the existence of an elliptic point. That, on a compact manifold was guaranteed by the assumption $\mathcal{H} > 0$. Moreover, we observe that the angle function is indeed well defined because Σ is two sided. For $k = 2$, this follows from the positivity of H_2 since $H_1^2 \geq H_2 > 0$. In the remaining cases this property follows from the Gårding inequalities, as in the compact case. In any case we choose the orientation so that $H_1 > 0$.

Proof. It follows from the hypotheses that $\sup_\Sigma \|A\| < +\infty$ and therefore by Lemma 18 that the sectional curvature of Σ is bounded from below. We then deduce the validity of the Omori-Yau maximum principle for the Laplacian. Assume $\mathcal{H}(h) \geq 0$.

Applying the Omori-Yau maximum principle to the Laplace operator and using equation (21) we find that

$$-\operatorname{sgn} \Theta \liminf_{j \rightarrow +\infty} H_1(q_j) \geq \mathcal{H}(h^*) \geq 0.$$

Therefore for the chosen orientation, $\operatorname{sgn} \Theta = -1$ and $\Theta \leq 0$ on Σ . Consider the operator

$$\mathfrak{L}_{k-1}f = \operatorname{div}(P_{k-1}\nabla f)$$

and the function

$$\phi = H_k^{\frac{1}{k}}\sigma(h) + \hat{\Theta},$$

where $\hat{\Theta} = \rho(h)\Theta$. Since \mathbb{P}^n has constant sectional curvature κ , it follows by equation (37) that

$$\begin{aligned} \mathfrak{L}_{k-1}\phi &= -(n-k+1)\Theta\left(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\right)\langle P_{k-2}\nabla h, \nabla\phi \rangle + L_{k-1}\phi \\ &= -(n-k+1)\hat{\Theta}\left(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\right)\langle P_{k-2}\nabla h, \nabla h \rangle H_k^{\frac{1}{k}} \\ &\quad + (n-k+1)\hat{\Theta}\left(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\right)\langle P_{k-2}A\nabla h, \nabla h \rangle \\ &\quad + H_k^{\frac{1}{k}}L_{k-1}\sigma(h) + L_{k-1}\hat{\Theta}. \end{aligned}$$

Using equation (22) and Corollary 28, we find

$$\begin{aligned} \mathfrak{L}_{k-1}\phi &= c_{k-1}\rho'(h)H_k^{\frac{1}{k}}(H_{k-1} - H_k^{\frac{k-1}{k}}) \\ &\quad - \binom{n}{k}\hat{\Theta}\left(nH_1H_k - (n-k)H_{k+1} - kH_k^{\frac{k+1}{k}}\right) \\ (41) \quad &\quad - (n-k)\hat{\Theta}\left(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\right)\langle P_{k-1}\nabla h, \nabla h \rangle \\ &\quad - (n-k+1)\hat{\Theta}H_k^{\frac{1}{k}}\left(\frac{\kappa}{\rho^2(h)} + \mathcal{H}'(h)\right)\langle P_{k-2}\nabla h, \nabla h \rangle. \end{aligned}$$

Using Gårding inequalities as in Theorem 24, it is easy to prove that the first and the second terms are nonnegative. By the fact that each P_j is an elliptic operator, $j = 0, \dots, k-1$, and by equation (35), it follows that also all the remaining terms in the previous equation are nonnegative. Thus $\mathfrak{L}_{k-1}\phi \geq 0$. Since, by assumption (39) Σ^n is \mathfrak{L}_{k-1} -parabolic, we conclude that ϕ has to be constant. In particular, $\mathfrak{L}_{k-1}\phi = 0$ and the four terms on the right-hand side of equation (41) vanish. Let us prove that $\mathcal{U} = \{p \in \Sigma^n : \Theta(p) = 0\}$ has empty interior. Indeed, assume the contrary, and let $\mathcal{V} \neq \emptyset$ be an open subset of \mathcal{U} . On \mathcal{V} the function $\phi = \sigma(h)H_k^{1/k}$ is constant. Hence, since $H_k \neq 0$, then $\sigma(h)$ and, equivalently h , is constant on \mathcal{V} , which is not possible since $\|\nabla h\|^2 = 1 - \Theta^2 = 1$ on \mathcal{V} . Therefore, since the third term on the right-hand of (41) vanishes identically, we have

$$\langle P_{k-1}\nabla h, \nabla h \rangle = 0.$$

Since P_{k-1} is positive definite, this means that h has to be constant. □

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