BOUNDED COHOMOLOGY CLASSES OF EXACT FORMS

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Abstract. On negatively curved compact manifolds, it is possible to associate to every closed form a bounded cocycle – hence a bounded cohomology class – via integration over straight simplices. The kernel of this map is contained in the space of exact forms. We show that in degree 2 this kernel is trivial, in contrast with higher degree. In other words, exact non-zero 2-forms have non-trivial bounded cohomology classes.

This result is the higher dimensional version of a classical theorem by Barge and Ghys for surfaces [BG88]. As a consequence, one gets that the second bounded cohomology of negatively curved manifolds contains an infinite dimensional space, whose classes are explicitly described by integration of forms. This also showcases that some recent results by Marasco [Mar22a, Mar22b] can be applied in higher dimension to obtain new non-trivial results on the vanishing of certain cup products and Massey products. Some other applications are discussed.

1. Introduction

Let $\Omega^2(M)$ denote the space of differential 2-forms on M and let $C\Omega^2(M)$ and $E\Omega^2(M)$ be the subspaces of closed and exact forms, respectively.

Barge and Ghys in the late 80s showed that the second bounded cohomology group of a negatively curved closed surface contains an infinite dimensional subspace given by the space of differential 2-forms:

Theorem ([BG88])**.** *Let Σ be an oriented closed connected negatively curved surface. Then, there exists an embedding*

$$\Psi\colon \Omega^2(\Sigma) \to H_b^2(\Sigma; \mathbb{R}),$$

where $H_b^2(\Sigma;\mathbb{R})$ denotes the second bounded cohomology group of Σ.

This classical theorem has attracted new attention after the recent results by Marasco on the vanishing of certain cup products and Massey products:

Theorem ([Mar22a, Mar22b])**.** *Let M be an oriented closed connected negatively curved manifold with (possibly empty) convex boundary. Let $\omega \in E\Omega^2(M)$ be an exact form and $\alpha \in H_b^k(M;\mathbb{R})$, then the cup product*

$$\Psi(\omega) \cup \alpha = 0 \in H_b^{k+2}(M;\mathbb{R}),$$

where Ψ is the Barge–Ghys straightening morphism (Section 2.2). Moreover, if $\omega \in E\Omega^2(M), \alpha_1 \in H_b^{k_1}(M;\mathbb{R})$ and $\alpha_2 \in H_b^{k_2}(M;\mathbb{R})$ with $k_1, k_2 \geq 1$, then also the triple Massey product $\langle \alpha_1, \Psi(\omega), \alpha_2 \rangle \in H_b^{k_1+k_2+1}(M;\mathbb{R})$ vanishes.

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In this short note we observe that the proof of Barge–Ghys' Theorem extends to higher dimensional manifolds, getting:

Theorem 1. *Let M be an oriented closed connected negatively curved manifold. Then, the Barge–Ghys straightening morphism*

$$\Psi\colon C\Omega^2(M) \to H^2_b(M;\mathbb{R})$$

is injective. In particular, both $\Psi(E\Omega^2(M))$ and $\Psi(C\Omega^2(M))$ are infinite dimensional subspaces of $H^2_b(M;\mathbb{R})$.

This result showcases that Marasco's result provides non-trivial information for closed manifolds of any dimension. For instance we have:

Example 1.1. Let $n \geq 2$ and M be an oriented closed connected negatively curved n-manifold. Since M is negatively curved, we can pick a non-trivial element $\alpha \in H^n_b(M;\mathbb{R})$ (e.g. the volume form [Gro82, IY82]). By Theorem 1, for every non-trivial $\omega \in E\Omega^2(M)$ the class $\Psi(\omega) \in H^2_b(M;\mathbb{R})$ is non-zero. Hence we have

$$\Psi(\omega) \cup \alpha = 0 \in H^{n+2}_b(M;\mathbb{R}),$$

where both classes $\Psi(\omega)$ and α are non-trivial. This can be interpreted as the first non-trivial vanishing result of the cup products in terms of bounded *geometric* classes of arbitrary dimension.

As an example of other kind of consequences of Theorem 1, we have:

Theorem 2. *Let M be a manifold and let N be an oriented closed connected negatively curved manifold. Suppose that there exists a continuous map $f\colon M \to N$ that induces a surjective homomorphism at the level of fundamental groups. Then, we have the following embedding:*

$$C\Omega^2(N) \hookrightarrow H^2_b(M;\mathbb{R}).$$

Other corollaries of Theorem 1 are discussed in Section 3 (see for instance Example 3.7). Moreover, in Section 4 we prove that such kind of results can be applied successfully also to the case of totally geodesic boundary (see also Proposition 4.1):

Theorem 3. *Let $n \geq 3$ and let M be an oriented compact connected negatively curved n-manifold with convex boundary and a totally geodesic boundary component. Then, $\Psi(C\Omega^2(M))$ and $\Psi(E\Omega^2(M))$ are infinite dimensional.*

On the other hand, we notice that the Barge–Ghys straightening morphism can be trivial for general negatively curved manifold with convex non-empty boundary (see Section 5).

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2. Basic definitions and notations

In this section we recall the main definitions that we need in the paper.

2.1. **Bounded cohomology.** Let X be a topological space. We denote by $(C^\bullet(X;\mathbb{R}),\delta^\bullet)$ the standard real singular cochain complex. Gromov defined the *bounded cohomology* of spaces as follows [Gro82]: given a singular cochain $\varphi \in C^k(X;\mathbb{R})$, the *$\ell^\infty$-norm* of φ is

$$\|\varphi\|_\infty \coloneqq \sup\{|\varphi(\sigma)|\,,\,\sigma \text{ is a singular } k\text{-simplex}\}.$$

We denote by $C_b^\bullet(X;\mathbb{R}) \subseteq C^\bullet(X;\mathbb{R})$ the subspace of *bounded cochains*, i.e. those cochains such that $\|\varphi\|_\infty < +\infty$. Since the standard coboundary operator sends bounded cochains to bounded cochains, we have that $(C_b^\bullet(X;\mathbb{R}),\delta^\bullet)$ is a cochain complex.

The *bounded cohomology of X* (with real coefficients) is then:

$$H_b^*(X;\mathbb{R}) \coloneqq H^*(C_b^\bullet(X;\mathbb{R}),\delta^\bullet).$$

Similarly, since bounded cohomology is a homotopy invariant [Gro82, Iva17], we can define the (real) bounded cohomology of a group Γ simply as

$$H_b^\bullet(\Gamma;\mathbb{R}) \coloneqq H_b^\bullet(B\Gamma;\mathbb{R}),$$

for any model $B\Gamma$.

2.2. **The Barge–Ghys straightening morphism.** Let M be an oriented compact connected negatively curved manifold M with (possibly empty) convex boundary. The latter condition means that there exists a fundamental domain of M in the universal cover $\widetilde{M}$ that is convex.

For every singular simplex $\sigma\colon \Delta \to M$ we can define its straightening by lifting σ to $\widetilde{\sigma} : \Delta \to \widetilde{M}$, then pulling it tight relatively to vertices, and projecting back to M. In case of hyperbolic manifolds, this coincides with the geodesic convex-hull of the vertices of $\widetilde{\sigma}$. In general the construction is slightly different, and it is done by recursively coning from a vertex to opposite faces [Fri17, Section 8.4].

This construction defines a map $\mathrm{st}\colon C_2(M;\mathbb{R}) \to C_2(M;\mathbb{R})$ which is homotopic to the identity [Gro82, Fri17, Mar22a]. We can therefore associate to every $\omega \in C\Omega^2(M)$ a 2-cocycle c_ω defined as

$$c_\omega(\sigma) = \int_{\mathrm{st}(\sigma)} \omega.$$

The following is classical:

Lemma 2.1 ([BG88, Lemma 3.1], [Mar22a, Section 2.2])**.** *The 2-cocycle c_ω is bounded.*

The proof of Lemma 2.1 goes as follows. Since M is compact, then ω is bounded on orthonormal 2-frames. In the hyperbolic case then the claim follows from the well-known bound on areas of geodesic triangles; the same being true in negative curvature setting, where the bound now depends on curvature-bounds [IY82].

Lemma 2.1 shows that there exists a well-defined cochain map

$$\psi\colon C\Omega^2(M) \to Z_b^2(M;\mathbb{R}), \qquad \omega \mapsto c_\omega$$

from the space of closed 2-forms to the space of bounded 2-cocycles of M. By passing to (bounded) cohomology on the right, we obtain the *Barge–Ghys straightening morphism*

$$\Psi\colon C\Omega^2(M) \to H_b^2(M;\mathbb{R}).$$

Remark 2.2. The result by Barge and Ghys for surfaces does not (explicitly) require that ω is closed. Indeed, in the case of closed surfaces all the differential 2-forms are closed. On the other hand, when we consider higher dimensional manifolds, we have to restrict to closed differential 2-forms.

3. Barge–Ghys Theorem for closed negatively curved manifolds

In this section we show how to prove Theorem 1 by combining the original proof by Barge–Ghys with the following result of the late 90s:

Theorem 3.1 ([CS98, Corollary 1.5])**.** *Let M be a closed negatively curved manifold, α a smooth 1-form (not a priori closed) on M, and β a closed 1-form on M. If for every closed geodesic γ in M we have*

$$\int_\gamma \alpha = \int_\gamma \beta$$

then α is closed and $[\alpha] = [\beta]$ in $H^1(M;\mathbb{R})$.

This result was only known for $n = 2$ when Barge and Ghys wrote their paper, but now one can use it to extend their argument to the case of oriented closed connected negatively curved n-manifolds. More precisely, we have the following:

Theorem 1. *Let M be an oriented closed connected negatively curved manifold. Then, the Barge–Ghys straightening morphism*

$$\Psi\colon C\Omega^2(M) \to H_b^2(M;\mathbb{R})$$

is injective. In particular, $\Psi(E\Omega^2(M))$ and $\Psi(C\Omega^2(M))$ are infinite dimensional subspaces of $H_b^2(M;\mathbb{R})$.

For the rest of the section we assume that M is an oriented closed connected negatively curved manifold. Following Barge–Ghys' proof, we first show that if $\Psi(\omega) = 0$, then ω has to be exact [BG88, Lemma 3.4]:

Lemma 3.2. *Let $\omega \in C\Omega^2(M)$ be such that $\Psi(\omega) = 0$. Then, ω is exact.*

Proof. Recall that we have the following commutative diagram [Mar22a, Section 2.2]:

$$\begin{tikzcd}
C\Omega^2(M;\mathbb{R}) \arrow[r] \arrow[d, "\Psi"] & H^2_{\mathrm{dR}}(M;\mathbb{R}) \arrow[d, "\cong"] \\
H_b^2(M;\mathbb{R}) \arrow[r] & H^2(M;\mathbb{R}),
\end{tikzcd}$$

where $H^2_{\mathrm{dR}}(M;\mathbb{R})$ denotes the de Rham cohomology of M and the lower horizontal arrow is the comparison map [Gro82]. Hence, if $\Psi(\omega) = 0$, we have that ω is mapped to the zero element in $H^2_{\mathrm{dR}}(M;\mathbb{R})$. This shows that ω has to be exact. $\square$

Remark 3.3. One standard way to prove that $H^2_b(M;\mathbb{R})$ does not vanish is to show that the kernel of the comparison map

$$\mathrm{comp}^2\colon H^2_b(M;\mathbb{R}) \to H^2(M;\mathbb{R})$$

is non-trivial. In the case of bounded cohomology of groups, the kernel of comp^2, called *second exact bounded cohomology* and denoted by $EH^2_b(M;\mathbb{R})$, is usually studied via (homogeneous) quasi-morphisms [Fri17, Proposition 2.8, Corollary 2.11]. In this setting, the construction by Barge and Ghys leads to the so-called *de Rham quasi-morphisms* [Cal09, Section 2.3.1]. From the diagram in the proof of Lemma 3.2, it follows that $\Psi(E\Omega^2(M)) \subseteq EH^2_b(M;\mathbb{R})$. A corollary of Theorem 1 is therefore that non-zero exact forms are non-trivial elements in second exact bounded cohomology group of M.

We assume now that ω is an exact 2-form such that $\Psi(\omega) = 0$. This allows us to fix the following notations: $\omega = d\alpha$ and $c_\omega = \delta\tau$, where α and τ are a differential 1-form and a bounded 1-cochain, respectively.

Lemma 3.4. *Let $\gamma_1, \dots, \gamma_s$ be closed oriented geodesics. Fix a base-point on each γ_i and a parametrization by arc-length in order to consider each γ_i as a 1-cycle. If $\sum_{i=1}^n \gamma_i = 0 \in H_1(M;\mathbb{R})$, then*

$$\sum_{i=1}^n \int_{\gamma_i} \alpha = \sum_{i=1}^n \tau(\gamma_i).$$

Proof. In the case of surfaces, this is exactly [BG88, Lemma 3.5]. Let S be a 2-chain such that $\partial c = \sum_{i=1}^n \gamma_i$. Since each γ_i is geodesic, we can straighten each simplex of S and suppose that S is made by straight simplices (recall that the map st is homotopic to the identity). Then:

$$\sum_{i=1}^n \int_{\gamma_i} \alpha = \int_{\partial S} \alpha = \int_S d\alpha = \int_S \omega = c_\omega(S) = \delta\tau(S) = \tau(\partial S) = \sum_{i=1}^n \tau(\gamma_i).$$

□

Lemma 3.5 ([BG88, Lemma 3.6])**.** *Let $\gamma_1, \dots, \gamma_s$ be closed oriented geodesics such that $\sum_{i=1}^n \gamma_i = 0 \in H_1(M;\mathbb{R})$. Then*

$$\sum_{i=1}^n \int_{\gamma_i} \alpha = 0.$$

For the convenience of the reader, we recall the proof by Barge and Ghys:

Proof. Let us denote by γ_i^N the geodesic γ_i repeated N times, considered as a single singular 1-simplex. Since the element $\sum_{i=1}^n \gamma_i^N$ is clearly null-homologous, by Lemma 3.4 we have that for every $N \in \mathbb{N}$:

$$N\left(\sum_{i=1}^n \int_{\gamma_i} \alpha\right) = \sum_{i=1}^n \int_{\gamma_i^N} \alpha = \tau\left(\sum_{i=1}^n \gamma_i^N\right).$$

Since τ is bounded, the proof is complete. □

All the previous results lead to the following important lemma (compare with Barge–Ghys [BG88, Lemma 3.7]):

Lemma 3.6. *There exists a closed 1-form β such that for every closed geodesic γ*

$$\int_\gamma \alpha = \int_\gamma \beta.$$

Proof. By Lemma 3.5, if γ_1 and γ_2 are homologous, then

$$\int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha.$$

Recall that in a negatively curved manifold each non-trivial free homotopy class of closed curves (and hence each homology class) contains a geodesic representative [Kli11, Theorem 3.8.14]. Then by Lemma 3.5 the integration of α over closed geodesics defines a map I from $H_1(M;\mathbb{R})$ to $\mathbb{R}$. This map is linear: if $\gamma_1 + \gamma_2$ is homologous to γ_3, then Lemma 3.5 shows that $I(\gamma_3) = I(\gamma_1) + I(\gamma_2)$. Hence, the map I defines an element in $H^1(M;\mathbb{R}) \cong \mathrm{Hom}(H_1(M;\mathbb{R});\mathbb{R})$. By de Rham Isomorphism Theorem there exists a closed 1-form β whose integral represent such class. This shows that

$$\int_\gamma \alpha = \int_\gamma \beta,$$

for every closed geodesic γ. ☐

Theorem 1 now immediately follows.

Proof of Theorem 1. By Lemma 3.6 and Theorem 3.1, the form α is closed, whence $\omega = d\alpha = 0$. ☐

Combining Theorem 1 with classical facts on bounded cohomology we get the following result:

Theorem 2. *Let M be a manifold and let N be an oriented closed connected negatively curved manifold. Suppose that there exists a continuous map $f\colon M \to N$ that induces a surjective homomorphism at the level of fundamental groups. Then, we have the following embedding:*

$$C\Omega^2(N) \hookrightarrow H^2_b(M;\mathbb{R}).$$

Proof. It is well known that every surjective group homomorphism induces an injective map on second bounded cohomology groups [Bou01]. Hence, we have that the surjective group homomorphism

$$\pi_1(f)\colon \pi_1(M) \to \pi_1(N)$$

induces an injective map between the second bounded cohomology groups:

$$H^2_b(\pi_1(f))\colon H^2_b(\pi_1(N);\mathbb{R}) \to H^2_b(\pi_1(M);\mathbb{R}).$$

Moreover, by Gromov's Mapping Theorem [Gro82, Iva17] (and the homotopy invariance of bounded cohomology) we have the following commutative diagram:

$$\begin{tikzcd}
H^2_b(\pi_1(N);\mathbb{R}) \arrow[r, "H^2_b(\pi_1(f))"] \arrow[d, "\cong"] & H^2_b(\pi_1(M);\mathbb{R}) \arrow[d, "\cong"] \\
H^2_b(N;\mathbb{R}) \arrow[r, "H^2_b(f)"] & H^2_b(M;\mathbb{R})
\end{tikzcd}$$

where the vertical lines are (isometric) isomorphisms. Since $H^2_b(\pi_1(f))$ is injective, the commutativity of the diagram shows that also $H^2_b(f)$ is injective. The thesis now follows from the following composition

$$C\Omega^2(N) \xrightarrow{\Psi} H^2_b(N;\mathbb{R}) \xrightarrow{H^2_b(f)} H^2_b(M;\mathbb{R}),$$

where Ψ is injective because of Thereom 1. □

Theorem 2 shows a geometric way to describe infinite-dimensional subspaces of the second bounded cohomology group of many non-positively curved manifolds:

Example 3.7. Let M be a fiber bundle over an oriented closed connected negatively curved manifold N with connected fiber, then Theorem 2 implies that

$$C\Omega^2(N) \hookrightarrow H^2_b(M;\mathbb{R}).$$

For instance, this applies when M is the product of oriented closed connected negatively curved manifolds.

4. The case of totally geodesic boundary

In this section we prove the following more general version of Theorem 3:

Proposition 4.1. *Let M be a Riemannian manifold with (possibly non-empty) convex boundary. Suppose that M contains an oriented closed connected negatively curved totally geodesic submanifold N (possibly contained in the boundary). Then, the images of $C\Omega^2(M)$ and $E\Omega^2(M)$ under the Barge–Ghys straightening morphism $\Psi\colon C\Omega^2(M) \to H^2_b(M;\mathbb{R})$ are infinite dimensional.*

Proof. Let $i\colon N \to M$ denote inclusion of N into M. We can then consider the following diagram:

$$\begin{tikzcd}
C\Omega^2(M) \arrow[r, "i^*"] \arrow[d, "\Psi_M"] & C\Omega^2(N) \arrow[d, "\Psi_N"] \\
H^2_b(M;\mathbb{R}) \arrow[r, "H^2_b(i)"] & H^2_b(N;\mathbb{R}).
\end{tikzcd}$$

By assumption, since N is an embedded closed submanifold of M and so it admits a tubular/collar neighborhood in M, the map i^* is surjective. Moreover, Theorem 1 shows that Ψ_N is also injective. Hence, in order to prove the statement it is sufficient to show that the previous diagram commutes (note that i^* sends exact forms to exact forms.) In fact, we are going to show that it commutes at the level of cochains, i.e. that the following diagram commutes:

$$\begin{tikzcd}
C\Omega^2(M) \arrow[r, "i^*"] \arrow[d, "\psi_M"] & C\Omega^2(N) \arrow[d, "\psi_N"] \\
C^2_b(M;\mathbb{R}) \arrow[r, "C^2_b(i)"] & C^2_b(N;\mathbb{R}).
\end{tikzcd}$$

First, since N is totally geodesic in M, the inclusion map i sends straight simplices to straight simplices, that is

$$i(\mathrm{st}(\sigma)) = \mathrm{st}(i(\sigma))$$

for every $\sigma\colon \Delta^2 \to N$ singular 2-simplex in N. Then, on the one hand we have:

$$\begin{aligned} C_b^2(i) \circ \psi_M(\omega) &= C_b^2(i)(c_\omega) \\ &= \Big(\sigma \mapsto c_\omega(i(\sigma)) = \int_{\mathrm{st}(i(\sigma))} \omega\Big), \end{aligned}$$

on the other hand,

$$\begin{aligned} \psi_N \circ i^*(\omega) &= \psi_N(i^*(\omega)) \\ &= \Big(\sigma \mapsto c_{i^*(\omega)}(\sigma) = \int_{\mathrm{st}(\sigma)} i^*\omega\Big) \\ &= \Big(\sigma \mapsto \int_{i(\mathrm{st}(\sigma))} \omega\Big). \end{aligned}$$

Since $i(\mathrm{st}(\sigma)) = \mathrm{st}(i(\sigma))$, we get that the two cocycles coincide. This shows that both the diagrams commute, whence we get the thesis. □

Theorem 3 is then an easy consequence of Proposition 4.1:

Proof of Theorem 3. It is sufficient to notice that when M is an oriented compact connected negatively curved manifold of dimension $\dim(M) \geq 3$ with convex boundary and totally geodesic boundary component M_0, then M_0 is an oriented closed connected negatively curved manifold. Hence, we can apply Proposition 4.1 by setting $N = M_0$. □

5. Non-injective examples

One can easily construct simple examples of negatively curved manifolds with non-empty convex boundary such that the Barge–Ghys straightening morphism Ψ is trivial (and so non-injective).

Example 5.1. Let $n \geq 2$ and let M be an oriented compact connected negatively curved n-manifold M, with non-empty convex boundary. In this case $C\Omega^2(M)$ is not trivial. Suppose moreover that M is homotopy equivalent to a space X with $H_b^2(X;\mathbb{R}) = 0$. Since the bounded cohomology is a homotopy invariant [Gro82, Iva17], then $H_b^2(M;\mathbb{R}) = 0$ and so the Barge–Ghys straightening morphism Ψ is trivial.

Since the point and S^1 have trivial second bounded cohomology [Gro82, Iva17], as examples of the above classes one can take a ball in a hyperbolic space, or a convex tubular neighbourhood of a simple closed geodesic in a hyperbolic manifold.

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