Axiomatics of the Blondel-Park Transformation

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Abstract— The doubly-fed induction generator (*DFIG*) is a key constituent of energy conversion plants. The control of a *DFIG* is a challenge, whenever the primary energy supply (e.g., the wind velocity field) is characterised by intermittency. The mathematical model and control of a *DFIG* rely on the Blondel-Park transformation, which is known to simplify the governing equations. The distinctive feature of this contribution consists of showing how the Blondel-Park transformation derives from a set of conditions to be met by a group. Such a group is shown to exist and to continuously depend on one parameter. The uniqueness of its infinitesimal generator is also shown. As an application, the well-known electric torque theorem is proved in a simple way, which relies on a "product of matrices" formula. The latter, in turn, is a by-product of the axiomatic deduction of the Blondel-Park transformation.

1. INTRODUCTION

The equations which describe the ideal doubly-fed induction generator (*DFIG*) have been studied for more than a century. Interest in the electric machine equations has obviously been motivated by the need to understand and improve the conversion from mechanical to electrical work as well as controlling the plant. Control is needed to face uncertainty in the primary energy supply (e.g., intermittency of the wind velocity field in front of a wind turbine) as well as in the electric load. In turn, the design of a control algorithm benefits from knowledge of the abstract properties of the plant model equations. Said equations, and the related changes of variables, center on two matrices, the rotor-to-stator mutual inductance matrix, $\mathbf{L}_{sr}[\cdot]$, and the Blondel [1]-Park [2] transformation matrix, $\mathbf{K}[\cdot]$. In the past, two pieces of work addressed the ideal *DFIG* from the points of view of dynamical systems [3] and, to some extent, group theory [4]. The presentation herewith differs from previous ones, because it aims at deriving the Blondel-Park transformation from first principles, i.e., basic requirements which formalise some physical properties.

2. PROPERTIES REQUIRED OF A FRAME TRANSFORMATION

Without providing too many details, electric currents, magnetic flux linkages and voltages of an induction machine form 3-vectors which are usually represented in the $\{abc\}$ stator and rotor frames. For example, the stator currents form a vector

$$\vec{j}_{\{abc\}s} = \begin{bmatrix} j_{as} \\ j_{bs} \\ j_{cs} \end{bmatrix} \in \mathbb{R}^3,$$

the components of which are functions of time $t \in \mathfrak{T}$. Similar notations and representations hold for other electric quantities. In order to remove redundancy sitting in $\{abc\}$ frames, another frame, called $\{dq0\}$, is introduced, where only two components of a vector shall matter, the direct one, d, and the quadrature component, q.

Def. 1. (The $dq0$ frame.) Let $\{dq0\}$ denote a reference frame for electric quantities of axes d and q, subject to the following specifications, $d.1$) to $d.3$).

$d.1$) Representations of any given electric variable with respect to the *direct*, a.k.a. d, axis and the stator a axis (as) shall be given by the same function evaluated at arguments which differ by the phase angle β_s. Similarly, the difference in phase angles of variables pertaining to the rotor ar axis shall be β_r. The relation of the two angles to θ_r is

$$\beta_s = \beta_r + \theta_r. \tag{1}$$

$d.2$) The *quadrature* axis, a.k.a. q, shall be orthogonal to d in the $L^2([0, 2\pi])$ sense: if $\zeta_d[\cdot]$ and $\zeta_q[\cdot]$ are the d- and q-components of a (generally complex-valued) signal which depend on an angle, η, then

$$\oint \zeta_d[\eta]^* \zeta_q[\eta] d\eta = 0, \tag{2}$$

where $\oint \equiv \int_0^{2\pi}$.
d.3) The third entry of a vector in the $\{dq0\}$ frame shall represent a zero sequence, e.g.,

$$j_a + j_b + j_c = 0, \quad \forall t \in \mathfrak{T}. \tag{3}$$

In abstract terms, the sought for relation between $\{abc\}$ and $\{dq0\}$ is formalised as follows.

PBM. 1. (The $\{abc\}$ to $\{dq0\}$ transformation problem.) Find a transformation which maps vectors (electric quantities) from the $\{abc\}s$ and, respectively, the $\{abc\}r$ frames to the $\{dq0\}$ frame, as specified by DEF. 1 and

K.1) is invertible and linear,

K.2) conserves instantaneous electric power,

K.3) has the same functional form for both stator and rotor quantities,

K.4) depends at most on one real parameter (an "electric angle") and is of class $\mathcal{C}^1$ at least with respect to that parameter.

K.5) The parameter may be different for stator as compared to rotor quantities.

K.6) Transformed flux linkages are magnetically decoupled. (Diagonalisation.)

3. THE TRANSFORMATION GROUP

There is a solution to PBM. 1 and the solution procedure is constructive.

THM. 1. (Existence and representation of the one parameter group.)

T1.1) (representation) The sought for transformations form one-parameter groups of operators

$$\mathbf{K}[\eta] = \mathbf{K}_0 \cdot e^{\eta \mathbf{F}}. \tag{4}$$

T1.2) (left multipliers) Possible $\mathbf{K}_0$'s ($\equiv \mathbf{K}[0]$) are [only two are shown for reasons of space]

$$\mathbf{K}_0^{(a)} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \quad \mathbf{K}_0^{(b)} = \sqrt{\frac{2}{3}} \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}. \tag{5}$$

T1.3) (infinitesimal generator) The matrix $\mathbf{F}$ in Eq. (4) is unique and reads

$$\mathbf{F} = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \tag{6}$$

REM. The algebra of $\mathbf{F}$ is of independent interest: the left zero divisors of $\mathbf{F}$ and recurrent formulas for its powers are also derived, which have not been provided before.

REM. Eventually, THM. 1 leads to a streamlined proof of the electric torque theorem. In fact, this is made possible by a formula for the product of the BLONDEL-PARK transformation matrices (Eq. (4)) with the θ_r derivative of the mutual inductance matrix: $\mathbf{K}[\theta] \cdot (\frac{\partial \mathbf{L}_{sr}}{\partial \theta_r})[\theta_r] \cdot \mathbf{K}[\beta]^{-1} = \frac{3}{2}\mathbf{A}_3$. No details can be worked out herewith for reasons of space.

4. CONCLUSION

The above results support the relevance of group theory in the modeling of electric machines and of *DGIG*s in particular. The axiomatic deduction, from PBM. 1 to THM. 1, of the BLONDEL-PARK transformation stated as existence of a one-parameter, continuous and differentiable group of transformations does not seem to have been addressed before and is the distinctive feature of this contribution.

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