



Some properties of the finitely additive vector integral

Gianluca Cassese¹ 

Received: 20 July 2025 / Accepted: 27 November 2025
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Abstract

We prove some results concerning the finitely additive, vector integrals of Bochner and Pettis and their representation over a countably additive probability space. An application to the non compact Choquet theorem is also provided.

Keywords Bochner integral · Choquet integral representation · Finitely additive probabilities · Integral representation · Pettis integral · Radon measure

Mathematics Subject Classification 28B05 · 28C05 · 28C15 · 46A55

1 Introduction

In this paper we are concerned with various instances of the condition

$$\int_{\Omega} h(f) dm = \int_S h(\tilde{f}) d\tilde{m}, \quad h \in \mathcal{H} \quad (1.1)$$

involving two finitely additive probabilities, m and \tilde{m} , two functions f and \tilde{f} with values in a Banach space X and an appropriate family \mathcal{H} of real valued functions on X (the case of functions $h \in \mathcal{H}$ with values in another Banach space may be treated similarly but does not seem to add much value). In the typical situation considered in this work, the pair (m, f) and the family \mathcal{H} are taken as given and the problem is finding a pair (\tilde{m}, \tilde{f}) which solves (1.1), a *representation* of (m, f) relatively to \mathcal{H} (see Definition 1).

The problem of whether \tilde{f} may represent (m, f) for some probability \tilde{m} and relatively to a sufficiently interesting family \mathcal{H} of functions is solved in Sect. 3 under fairly general conditions involving the range of f and of \tilde{f} . Our interest, however, goes beyond mere existence and aims at representations satisfying some additional,

Communicated by Timur Oikhberg.

✉ Gianluca Cassese
gianluca.cassese@unimib.it

¹ University of Milano-Bicocca, Building U7, Room 2097, via Bicocca degli Arcimboldi 8, Milano, MI, Italy

natural properties. In particular we focus on conditions hinging on the supporting set S and on the intervening measure \tilde{m} .

In Sect. 4 we consider representations supported by the set \mathbb{N} of natural numbers. We show (Theorem 2) that this kind of representation is always possible when f is measurable. If, in addition, f is integrable then, passing from \mathbb{N} to its compactification $\beta\mathbb{N}$, we deduce a countably additive representing measure.

In Sect. 5 we examine the problem of the existence of a countably additive representing measure \tilde{m} in the case in which f is just Pettis integrable. This case is at the same time more interesting and more delicate. In fact, outside of some special cases, e.g. when X is reflexive or a dual space or when f is tight, there is no natural representation (\tilde{m}, \tilde{f}) . Stone space techniques are only partly useful and, in particular, do not permit to identify \tilde{f} .

From a historical perspective, the first result concerning countably additive representations of finitely additive integrals was proved by Dubins and Savage [7] in the setting $X = S = \mathbb{R} = \Omega$ and with \mathcal{H} consisting of bounded, continuous functions on the real line. In their representation $f = \tilde{f}$. The most relevant result so far (of which I am aware) was obtained by Karandikar [18] (but see [17] as well) with the purpose of extending limit theorems to the finitely additive setting. In these works $X = \mathbb{R}^{\mathbb{N}}$ and \mathcal{H} is formed by all bounded continuous functions which only depend on finitely many coordinates. In this framework, a key observation is that measurable functions with values in \mathbb{R}^n are tight, which makes it possible to apply the theorems of Dini and of Daniell. We mention that, motivated again by limit theorems applications, this result was extended by Berti et al. [2, Theorem 2.1] to the case of functions with values in a metric space but assuming tightness explicitly. In infinite dimensional spaces, in fact, measurable functions need not be tight nor need continuous transformations of measurable functions be measurable.

In Sect. 6 we consider the case in which X possesses the Radon–Nikodým property (RNP) and f is Pettis integrable. In Theorem 5 we prove the existence of a countably additive representation for functions on X which are uniformly continuous with respect to the weak topology. In Theorem 6 we find conditions, hinging on the range of f , under which a countably additive representation obtains with $\tilde{f} = f$. In Sect. 7 we discuss the Pettis integrability property (PIP), rarely discussed in the context of finite additivity, obtaining some partial conclusions, particularly under the assumption that X is separable. Eventually, in Sect. 8 we prove a version of Choquet integral representation for non compact, non convex sets. This final result permits to appreciate advantages and disadvantages of our approach.

2 Preliminaries

Throughout the paper (Ω, \mathcal{A}, m) will be a fixed finitely additive probability space, with Ω an arbitrary non empty set, \mathcal{A} an algebra of subsets of Ω and m a finitely additive probability defined on \mathcal{A} , i.e. $m \in \mathbb{P}(\mathcal{A})$ (the symbol $\mathbb{P}_{ca}(\mathcal{A})$ denotes countably additive probabilities).¹ Also given will be a Banach space X , its dual space X^* (with

¹ We write $\mathbb{P}(S)$ for probabilities defined on the power set of a set S .

\mathbb{B} and \mathbb{B}^* the closed unit balls in X and X^* respectively) and a function f of Ω into X on which, from time to time, different measurability and integrability properties will be imposed (relatively to m).

Convergence in measure and measurability are defined differently under finite additivity and it is useful to recall these definitions explicitly: a sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ of maps from Ω to X converges in m measure to 0 (or simply m -converges to 0) whenever

$$\lim_n m^*(\|f_n\| > c) = 0, \quad c > 0. \tag{2.1}$$

In (2.1) m^* is the outer measure induced by m , i.e.

$$m^*(E) = \inf\{m(A) : A \in \mathcal{A}, E \subset A\}. \tag{2.2}$$

f is measurable, in symbols $f \in L_X^0(m)$ (or $f \in L_X^0(\mathcal{A}, m)$ if reference to \mathcal{A} is not obvious from the context), if there exists a sequence f_n of \mathcal{A} simple, X valued functions whose distance from f m -converges to 0. When m is countably additive and \mathcal{A} a σ algebra, this notion of measurability implies the classical $(\mathcal{A}(m), \mathcal{B}(X))$ definition (with $\mathcal{A}(m)$ denoting the m completion of \mathcal{A} and $\mathcal{B}(X)$ the σ algebra of Borel subsets of X) and that the two are actually equivalent if the range of f is essentially separably valued [8, III.6.10]. We speak of f as integrable, and write $f \in L_X^1(m)$, whenever there exists a sequence f_n of \mathcal{A} simple functions which m -converges to f and such that $\lim_{m,n} \int \|f_m - f_n\| dm = 0$.² Measurability and integrability of f are qualified as *weak* or *norm* when the corresponding property characterizes the set $\{x^* f : x^* \in \mathbb{B}^*\}$ or the function $\|f\|$, respectively.

Although not strictly necessary, it will spare a considerable amount of repetitions to assume, as we shall do with no further mention, that f is weakly and norm measurable.

The definite Pettis integral of f over $A \in \mathcal{A}$, if it exists, is the unique element $x_A \in X$ satisfying $x^* x_A = \int_A x^* f dm$ for all $x^* \in X^*$. Then f is Pettis integrable if it admits a definite Pettis integral over every $A \in \mathcal{A}$.

Concerning function spaces, the symbol $\mathfrak{F}(S, T)$ (resp. $\mathfrak{F}(S)$) denotes the functions from S to T (resp. from S to \mathbb{R}) and if $A \subset S$ and $g \in \mathfrak{F}(S, T)$, the image of A under g is indicated by $g[A]$. If S is a topological space, \bar{F} denotes the closure of $F \subset S$, while $\mathcal{C}(S)$ (resp. $\mathcal{C}_u(S)$) indicates the family of real valued, continuous (resp. uniformly continuous) functions on S . The suffix b attached to a class of real valued function indicates the subclass consisting of bounded functions (e.g. $\mathcal{C}_{ub}(S)$). We shall use repeatedly the following version of a result of Hager [15, Theorem 4.2]: given a family $\mathcal{G} \subset \mathfrak{F}(X)$ of linear functionals, the collection of compositions $U(g_1, \dots, g_n)$ with $g_1, \dots, g_n \in \mathcal{G}$ and $U \in \mathfrak{F}_u(\mathbb{R}^n)$ is uniformly dense in $\mathcal{C}_u((X, \tau_{\mathcal{G}}))$ where $\tau_{\mathcal{G}}$ is the initial topology induced by \mathcal{G} .

We will often make reference to the normed linear spaces

$$\mathcal{L}(m, f) = \left\{ H \in \mathfrak{F}(\Omega \times X) : H(\cdot, f(\cdot)) \in L^1(m) \right\}, \tag{2.3}$$

² Some authors speak of functions measurable or integrable in the above defined sense as strongly measurable and Bochner integrable. We will not follow this terminology.

with $\|H\|_{\mathcal{L}(m, f)} = \|H(f)\|_{L^1(m)}$,³ and $\mathcal{L}_u(m, f) = \mathcal{C}_u(X) \cap \mathcal{L}(m, f)$.

Definition 1 Let S be a non empty set and $\mathcal{H} \subset \mathcal{L}(m, f)$. Then (\tilde{m}, \tilde{f}) is a representation of (m, f) relatively to \mathcal{H} and with support S if $\tilde{f} \in \mathfrak{F}(S, X)$, $\tilde{m} \in \mathbb{P}(S)$, $\mathcal{H} \subset \mathcal{L}(\tilde{m}, \tilde{f})$ and

$$\int h(f)dm = \int h(\tilde{f})d\tilde{m}, \quad h \in \mathcal{H}. \tag{2.4}$$

3 A general theorem on representations

We now prove a general result on representations which will be useful in the sequel. It gives an abstract characterization of this kind of problems and it is perhaps of its own interest. Given that this result only depends on uniform continuity and that both the strong and the weak topology make X into a uniformizable space, in the context of this section we will consider X only for its properties as a uniform space. We refer to [23] for terminology on uniform spaces.

Definition 2 Let \mathcal{D} be a diagonal uniformity on X . A set $K \subset X$ is an m -cover of (the range of) f (or (m, \mathcal{D}) -cover) if

$$\sup_{D \in \mathcal{D}} m^*(f \notin D(K)) = 0 \tag{3.1}$$

where $D(K) = \bigcup_{k \in K} \{x : (x, k) \in D\}$. We speak of an X valued function g as a m -cover of f if the range of g m -covers f .

If X is metric, then the sets $D(K)$ will just be the cover of K with open balls of given radius. In terms of Definition 2 a m -null function is simply an X valued function which is m -covered by the origin.

Theorem 1 Let \mathcal{D} be a diagonal uniformity on X and $\mathcal{H} = \mathcal{C}_u((X, \mathcal{D})) \cap \mathcal{L}(m, f)$. Let S be a non empty set, \mathcal{N} an ideal of its subsets and $\tilde{f} \in \mathfrak{F}(S, X)$. Consider the following properties:

- (a) $\tilde{f}[N^c]$ is an (m, \mathcal{D}) -cover of f for each $N \in \mathcal{N}$;
- (b) there exists $\tilde{m} \in \mathbb{P}(S)$ vanishing on \mathcal{N} such that (\tilde{m}, \tilde{f}) represents (m, f) relatively to \mathcal{H} .

Then (a) \Rightarrow (b) and, if \mathcal{D} is the weak uniformity generated by \mathcal{H} , (b) \Rightarrow (a).

³ Abusing notation we shall identify $\mathfrak{F}(X)$ with the corresponding subspace of $\mathfrak{F}(\Omega \times X)$. We shall also write for simplicity $H(f)$ in place of $H(\cdot, f(\cdot))$.

Proof Assume (a) and fix $h \in \mathcal{H}$. For each $N \in \mathcal{N}$, $D \in \mathcal{D}$ and $n \in \mathbb{N}$ there exists $A_n \in \mathcal{A}$ such that $m(A_n) > 1 - 2^{-n}$ and $f[A_n] \subset D(\tilde{f}[N^c])$. Thus,

$$\begin{aligned} \int h(f)dm &= \lim_n \int_{A_n} h(f)dm \\ &\geq \inf_{z \in D(\tilde{f}[N^c])} h(z) \\ &\geq \inf_{s \in N^c} h(\tilde{f})(s) - \sup_{x, z \in D} |h(z) - h(x)|. \end{aligned}$$

Given that $h \in \mathcal{C}_u((X, \mathcal{D}))$ we conclude

$$\sup_{N \in \mathcal{N}} \inf_{s \in N^c} h(\tilde{f})(s) \leq \int h(f)dm.$$

Claim (b) then follows from [4, Theorem 4.5].

Assume now that (\tilde{m}, \tilde{f}) is as in (b). Fix $N \in \mathcal{N}$ and let $K = \overline{\tilde{f}[N^c]}$. Of course, to check that $\tilde{f}[N^c]$ is an (m, \mathcal{D}) -cover of f it is enough to restrict attention to a base of the uniformity \mathcal{D} . If \mathcal{D} is the weak uniformity induced by \mathcal{H} , one such base consists of sets of the form

$$D_0 = \bigcap_{i=1}^n \{(x, y) \in X \times X : |h_i(x) - h_i(y)| < \varepsilon_i\}, \quad h_1, \dots, h_n \in \mathcal{H}. \tag{3.2}$$

Then, $y \notin D_0(\tilde{f}[N^c])$ if and only if

$$1 = \inf_{s \in N^c} \sup_{\{i=1, \dots, n\}} \left| \frac{h_i(\tilde{f}(s)) - h_i(y)}{\varepsilon_i} \right| \wedge 1 = H(y). \tag{3.3}$$

Notice that $H \in \mathcal{H}$, $H = 0$ on $\tilde{f}[N^c]$ and $H = 1$ on $D_0(\tilde{f}[N^c])^c$. But then

$$0 = \int_{N^c} H(\tilde{f})d\mu = \int H(\tilde{f})d\mu = \int H(f)dm \geq m^*(f \notin D_0(\tilde{f}[N^c])).$$

□

It is implicit in Theorem 1 that obtaining representations by embedding Ω or the range of f in some larger set is a trivial exercise which gains some interest only if it permits to obtain special properties on the representing measure \tilde{m} , such as countable additivity. This is the case of compactifications that we shall consider in the following sections.

Although Theorem 1 is very useful in proving existence of representations, it is also too general to provide detailed information on the representing pair. This will then require additional structure.

In applications of Theorem 1 we will omit reference to \mathcal{D} whenever this is the uniformity induced by the norm.

Corollary 1 *Let $f \in L_X^0(m)$ and fix a non empty set S . Then (m, f) admits a representation relative to $\mathcal{L}_u(m, f)$ and supported by S if and only if f has an m -cover of cardinality not greater than S .*

Proof Let $K \subset X$ be an m -cover of f . If the cardinality of K is not greater than that of S then there exists $\tilde{f} \in \mathfrak{F}(S, X)$ which is onto K . It is then clear from Theorem 1 that (\tilde{m}, \tilde{f}) is the claimed representation of (m, f) for some $\tilde{m} \in \mathbb{P}(S)$. For the converse observe that if $f \in L_X^0(m)$ then $\mathcal{C}_{ub}(X) \subset \mathcal{L}(m, f)$ and that the norm uniformity coincides with the weak uniformity generated by $\mathcal{L}_u(m, f)$ (see e.g. [23, 37.9]). \square

Corollary 1 clarifies the role of measurability in this class of problems. If f is measurable then all uniformly continuous transformations of f are measurable too and the set $\mathcal{L}_u(m, f)$ is sufficiently rich to generate the norm uniformity. With weak measurability we can only establish representations relative to the family $\mathcal{C}_u((X, weak)) \cap \mathcal{L}(m, f)$. The diagonal uniformity induced by this class of functions is the same as that induced by X^* . The sets $D(K)$ defining an m -cover would then just be the weakly open sets. See Corollary 5.

4 Representations over \mathbb{N}

By Theorem 1 if X is separable, then (m, f) admits a representation supported by \mathbb{N} . More interestingly, if $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence of \mathcal{A} simple functions m -converging to f , then $\bigcup_n f_n[\Omega]$ is an m -cover of f . Thus, measurable functions admit representations supported by \mathbb{N} .

The prominence of \mathbb{N} as a prototypical model for *finitely additive* integration was clearly noted, among others, by Maharam [20]. It appears from the preceding remark that it is a model for vector integration *in general*. Among other features of \mathbb{N} , the classical decomposition of Yosida and Hewitt takes an especially simple form since the set of countably additive measures on \mathbb{N} is isomorphic to ℓ^1 while a purely finitely additive set function is characterized by the property of vanishing on singletons (i.e. of being weightless, in Maharam’s terminology which we adopt). An example of weightless, atomless measures are limit frequencies, which are often suggested as a convenient statistical model and are the most appropriate model for the uniform distribution over the integers, see e.g. Kadane and O’Hagan [16].

In the following Theorem 2 we get, further to existence, a more precise description of the representation. Remarkably, we find that the representing probability may always be chosen to be weightless. This finding seems to contrast with the popular view which considers this family of set functions virtually useless.

Theorem 2 *Let $f \in L_X^0(m)$. There exist $\kappa \in \mathfrak{F}(\mathbb{N}, \Omega)$, a countable algebra $\mathcal{A}_0 \subset \mathcal{A}$ and $\mu \in \mathbb{P}(\mathbb{N})$ weightless such that, letting $F = f \circ \kappa$, the following is true: $f \in L_X^0(\mathcal{A}_0, m)$, $F \in L_X^0(\kappa^{-1}(\mathcal{A}_0), \mu)$, $\mathcal{L}_u(m, f) \subset \mathcal{L}(\mu, F)$ and*

$$\int_A h(f)dm = \int_{\kappa^{-1}(A)} h(F)d\mu, \quad A \in \mathcal{A}_0, h \in \mathcal{L}_u(m, f) \tag{4.1}$$

Moreover, if f is Bochner, norm or weakly integrable then so is F .

Proof Fix a sequence $\langle g_n \rangle_{n \in \mathbb{N}}$ of \mathcal{A} simple functions such that $m^*(\|f - g_n\| > 2^{-n}) \leq 2^{-(n+1)}$ and choose $D_n \in \mathcal{A}$ such that $D_n \subset \{\|f - g_n\| \leq 2^{-n}\}$ and $m(D_n) > 1 - 2^{-n}$. Fix the trivial partition π_0 and define inductively π_n as the partition induced by π_{n-1} , D_n and the sets supporting g_n . Denote by \mathcal{A}_n the algebra generated by π_n and $\mathcal{A}_0 = \bigcup_n \mathcal{A}_n$. For each $n \in \mathbb{N}$ and $E \in \pi_n$ choose $\omega_E^n \in E$ and define

$$f_n = \sum_{E \in \pi_n} f(\omega_E^n) \mathbb{1}_E, \quad n \in \mathbb{N}. \tag{4.2}$$

Of course, $\|f - f_n\| \leq 2^{-(n-1)}$ on D_n . Thus $\langle f_n \rangle_{n \in \mathbb{N}}$ is a sequence of \mathcal{A}_0 simple functions m -converging to f and $f \in L_X^0(\mathcal{A}_0, m)$.

Let $\kappa \in \mathfrak{F}(\mathbb{N}, \Omega)$ be an enumeration of $\{\omega_E^n : E \in \pi_n, n \in \mathbb{N}\}$ and set $F = f \circ \kappa$. Enumerate \mathcal{A}_0 as $\{A_1, A_2, \dots\}$. Fix $k \in \mathbb{N}$. For all $j \in \mathbb{N}$ sufficiently large and all $A_1, \dots, A_j \in \mathcal{A}_j$ then

$$(f_j, \mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_j})[\Omega] \subset (F, \mathbb{1}_{\kappa^{-1}(A_1)}, \dots, \mathbb{1}_{\kappa^{-1}(A_j)})[\{k, k + 1, \dots\}]. \tag{4.3}$$

Consider the $X \oplus c_0$ valued functions \hat{f} and \hat{F} implicitly defined as $\hat{f}(\omega) = (f(\omega), \dots, 2^{-i} \mathbb{1}_{A_i}(\omega), \dots)$ and $\hat{F}(n) = (F(n), \dots, 2^{-i} \mathbb{1}_{\kappa^{-1}(A_i)}(n), \dots)$. From the inequality

$$\begin{aligned} \|\hat{f}(\omega) - \hat{F}(n)\|_{X \oplus c_0} &= \|f(\omega) - F(n)\|_X + \sup_i 2^{-i} |\mathbb{1}_{A_i}(\omega) - \mathbb{1}_{\kappa^{-1}(A_i)}(n)| \\ &\leq \|f_j(\omega) - F(n)\|_X + \sup_{i \leq j} 2^{-i} |\mathbb{1}_{A_i}(\omega) - \mathbb{1}_{\kappa^{-1}(A_i)}(n)| \\ &\quad + 2^{-j} + \|f_j(\omega) - f(\omega)\|_X \end{aligned}$$

we conclude

$$\inf_{n \geq k} \|\hat{f}(\omega) - \hat{F}(n)\|_{X \oplus c_0} \leq 2^{-j} + \|f_j(\omega) - f(\omega)\|_X. \tag{4.4}$$

Consequently, $\hat{F}[\{k, k + 1, \dots\}]$ m -covers \hat{f} for each $k \in \mathbb{N}$ and, by Theorem 1, there exists $\mu \in \mathbb{P}(S)$ such that (μ, \hat{F}) represents (m, \hat{f}) relatively to $\mathcal{L}(m, \hat{f}) \cap \mathcal{C}_u(X)$ and that $\mu(\{1, \dots, k\}) = 0$ for any $k \in \mathbb{N}$. Thus μ is weightless. Notice that if $A \in \mathcal{A}_0$ and $h \in \mathcal{H}$ then $h(f) \mathbb{1}_A = \hat{h}(\hat{f})$ for some $\hat{h} \in \mathcal{L}(m, \hat{f}) \cap \mathcal{C}_u(X \oplus c_0)$. From this we deduce (4.1). The claim that $F \in L_X^0(\kappa^{-1}(\mathcal{A}_0), \mu)$ is clear from the first part of the proof.

Concerning the last claim, given that f and F are measurable, Bochner, norm or weak integrability depend on which functions are included in \mathcal{H} . The corresponding property thus carry over from f to F . □

The only explicit link between m and μ is the relation $\mu(\kappa^{-1}(A)) = m(A)$ for each $A \in \mathcal{A}_0$. Thus, in case m is countably additive, then so will be the restriction μ_0 of μ to $\kappa^{-1}(\mathcal{A}_0)$. However, by [1, Corollary 3], μ_0 has then a countably additive extension μ_1 to the power set of \mathbb{N} and such extension would definitely satisfy (4.1). We have

an example in which a countably additive and a purely finitely additive set function induce exactly the same representation.

Corollary 2 *Let (S, Σ, λ) be a countably additive probability space. Then $L^1_X(\lambda)$ is isometrically isomorphic with a closed subset of ℓ^1_X .*

Proof Choose $g \in L^1_X(\lambda)$. By Theorem 2 and the preceding remark there exists $\mu_1 \in \mathbb{P}_{ca}(\mathbb{N})$ and a representation G of g supported by \mathbb{N} such that,

$$\int h(g)d\lambda = \sum_n h(G)(n)\mu_1(n), \quad h \in \mathcal{C}_u(X) \cap \mathcal{L}(\lambda, g). \tag{4.5}$$

Since g is Bochner integrable, so is G . Let $T(g)_n = G(n)\mu_1(n)$, for any $n \in \mathbb{N}$. Clearly, $\|T(g)\|_{\ell^1_X} = \sum_n \|G(n)\| \mu_1(n) = \int \|g\| d\lambda = \|g\|_{L^1_X(\lambda)}$. If $\langle g_n \rangle_{n \in \mathbb{N}}$ is a sequence in $L^1_X(\lambda)$ such that $T(g^n)$ converges in ℓ^1_X then $\langle g_n \rangle_{n \in \mathbb{N}}$ is Cauchy in $L^1_X(\lambda)$. Its norm limit g_0 is such that $T(g_0)$ is the norm limit of $T(g_n)$. \square

Given the completely general nature of the Banach space X , Theorem 2 admits some easy extensions to the case in which X is the direct sum of a finite or a countable family of Banach spaces. These extensions imply that the representation obtained in Theorem 2 may be constructed so as to preserve m -convergence or convergence in $L^1(m)$.

Corollary 3 *For each $i \in \mathbb{N}$, let X_i be a Banach space and $f_i \in L^0_{X_i}(m)$. There exist $\kappa \in \mathfrak{F}(\mathbb{N}, \Omega)$, a countable algebra $\mathcal{A}_0 \subset \mathcal{A}$ and $\mu \in \mathbb{P}(\mathbb{N})$ weightless such that $f_i \in L^0_X(\mathcal{A}_0, m)$ and, letting $F_i = f_i \circ \kappa$,*

$$\int_A h(f_1, \dots, f_n)dm = \int_{\kappa^{-1}(A)} h(F_1, \dots, F_n)d\mu \tag{4.6}$$

for each $n \in \mathbb{N}$, $A \in \mathcal{A}_0$ and $h \in \mathcal{C}_u(\oplus_{i=1}^n X_i) \cap \mathcal{L}(m, \oplus_{i=1}^n f_i)$,

Proof Fix $N \in \mathbb{N}$. Of course $Y_N = \oplus_{i=1}^N X_i$ is a Banach space if endowed, e.g., with the norm $\|x_1, \dots, x_N\| = \sup_{1 \leq i \leq N} \|x_i\|_{X_i}$. Moreover, the function $\oplus_{i=1}^N f_i$ belongs to $L^0_{Y_N}(m)$. We can apply Theorem 2 and find a map $\kappa_N \in \mathfrak{F}(\mathbb{N}, \Omega)$ such that the range of $(\oplus_{i=1}^N f_i) \circ \kappa_N$ in restriction to all cofinite sets m -almost covers $(\oplus_{i=1}^N f_i)$. This conclusion remains valid if we replace κ_N with the enumeration κ of $\bigcup_{N \in \mathbb{N}} \kappa_N[\mathbb{N}]$. Thus we obtain the same change of variable for each N . The claim follows noting that, if $F_i = f_i \circ \kappa$, then $\oplus_{i=1}^N F_i = (\oplus_{i=1}^N f_i) \circ \kappa$. \square

Thus, if $X = X_1 = X_2 = \dots$ and f_n converges to f_1 either in measure or in $L^1_X(m)$ then F_n converges to F_1 in the corresponding topology. The extension to an infinite direct sum requires to specify the norm explicitly. We adopt the following definition:

$$\bigoplus_{i \in \mathbb{N}} X_i = \left\{ x \in \prod_{i \in \mathbb{N}} X_i : \sum_i 2^{-i} \|x_i\|_{X_i} < \infty \right\}. \tag{4.7}$$

Corollary 4 For each $i \in \mathbb{N}$ let X_i be a Banach space and $f_i \in L^1_{X_i}(m)$. Assume that $\sup_i \|f_i\|_{X_i} \in L^1(m)$. Then there exist $\kappa \in \mathfrak{F}(\mathbb{N}, \Omega)$ and a countable subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ such that, letting F_i and μ be as in Corollary 3, $F_i \in L^1_{X_i}(\mu)$ and

$$\int_A h(f_1, f_2, \dots) dm = \int_{\kappa^{-1}(A)} h(F_1, F_2, \dots) d\mu, \tag{4.8}$$

for every $A \in \mathcal{A}_0$ and $h \in \mathcal{C}_u(\oplus_i X_i) \cap \mathcal{L}(m, \oplus_i f_i)$.

Proof Put $b = \sup_i \|f_i\|_{X_i}$. There is no loss of generality in assuming $\int b dm = 1$ and $\{b = +\infty\} = \emptyset$. We claim that $f = \oplus_i f_i$ is an element of $L^1_X(m)$.

Indeed, $\|f(\omega)\|_{\oplus_i X_i} = \sum_i 2^{-i} \|f_i(\omega)\|_{X_i} \leq b(\omega)$ so that $f \in \mathfrak{F}(\Omega, X)$. For each $i \in \mathbb{N}$, let $\{g_i^n\}_{n \in \mathbb{N}}$ be a sequence of \mathcal{A} simple functions such that $\int \|f_i - g_i^n\|_{X_i} dm \leq 2^{-n}$. If we let π^n be the meet of the finite, \mathcal{A} measurable partitions associated with g_i^n for $i = 1, \dots, n$ then we can write

$$g_i^n = \sum_{D \in \pi^n} \chi_i^n(D) \mathbb{1}_D, \quad \text{with } \chi_i^n \in \mathfrak{F}(\pi^n, X_i) \quad i = 1, \dots, n \tag{4.9}$$

or else $\chi_i^n = g_i^n = 0$ if $i > n$. The function $g^n = \oplus_i g_i^n$ maps Ω into X and is \mathcal{A} simple. In fact if $\chi^n = \oplus_i \chi_i^n$ we may represent g^n as

$$g^n = \sum_{D \in \pi^n} \chi^n(D) \mathbb{1}_D. \tag{4.10}$$

Then, $\|f - g^n\|_{\oplus_i X_i} = \sum_{i \leq n} 2^{-i} \|f_i - g_i^n\|_{X_i} + \sum_{i > n} 2^{-i} \|f_i\|_{X_i} \leq \sum_{i \leq n} 2^{-i} \|f_i - g_i^n\|_{X_i} + 2^{-n} b$, so that

$$\int \|f - g^n\|_{\oplus_i X_i} dm \leq 2^{-(n-1)}. \tag{4.11}$$

The claim then follows from Theorem 2. □

The condition $f \in L^0_X(m)$ turns out to be quite restrictive in applications. For example, the identity map on a subset $A \subset X$ is measurable if and only if for each $\varepsilon > 0$ there is a finite collection of open balls of radius ε such that the part of A which cannot be covered by such family has probability less than ε . A case in which measurability may be disposed of is given next.

Corollary 5 Let f be norm integrable and let X^* be separable. Then Theorem 2 applies upon replacing $\mathcal{L}_u(m, f)$ with $\mathcal{C}_u((X, \text{weak})) \cap \mathcal{L}(m, f)$.

Proof Let $\{x_n^* : n \in \mathbb{N}\}$ be dense in \mathbb{B}^* . Define $\Lambda \in \mathfrak{F}(X, \ell^1)$ by letting

$$\Lambda(x)_n = x_n^*(x) 2^{-n}, \quad n \in \mathbb{N}. \tag{4.12}$$

Given that f is weakly and norm measurable, by our general assumption, then $\Lambda(f) \in L^0_{\ell^1}(m)$. It follows from Theorem 2 that there exists $\kappa \in \mathfrak{F}(\mathbb{N}, \Omega)$ and \mathcal{A}_0 such that, letting $F = f \circ \kappa$, one gets

$$j(\Lambda(F)) \in L^0_X(\mu) \quad \text{and} \quad \int_A j(\Lambda(f))dm = \int_{\kappa^{-1}(A)} j(\Lambda(F))d\mu, \quad (4.13)$$

for every $A \in \mathcal{A}_0$ and $j \in \mathcal{C}_{ub}(\ell^1)$. The compositions $j(\Lambda)$ include all functions of the form $\alpha(x_{n_1}^*, \dots, x_{n_k}^*)$, with $x_{n_1}^*, \dots, x_{n_k}^* \in \mathbb{B}^*$ and $\alpha \in \mathcal{C}_{ub}(\mathbb{R}^k)$ and are thus dense in $\mathcal{C}_{ub}((X, weak))$ with respect to the uniform topology. The extension to $\mathcal{C}_u((X, weak)) \cap \mathcal{L}(m, f)$ is obvious. \square

An alternative proof may be obtained by noting that if X^* is separable then the weak topology of \mathbb{B} is metrizable and the proof of Theorem 2 carries over with only minor modifications.

In anticipation of the following sections, a standard application of Stone space techniques delivers a countably additive representation.

Theorem 3 *Assume that $f \in L^1_X(m)$. Let χ be the Stone isomorphism of \mathcal{A} and the field \mathcal{F} of clopen sets of its Stone space S . Let $\lambda \in \mathbb{P}_{ca}(\sigma\mathcal{F})$ be the extension of $m \circ \chi^{-1}$. There exists $\tilde{f} \in L^1_X(\lambda)$ such that $\mathcal{L}_u(m, f) \subset \mathcal{L}_u(\lambda, \tilde{f})$ and that, for each $A \in \mathcal{A}$, $\lambda(\chi(A) \cap \{\tilde{f} \notin \overline{f[A]}\}) = 0$ and*

$$\int_A h(f)dm = \int_{\chi(A)} h(\tilde{f})d\lambda, \quad h \in \mathcal{L}_u(m, f). \quad (4.14)$$

If f has closed range there exists $\nu \in \mathbb{P}_{ca}(\sigma\{h(f) : h \in \mathcal{C}_u(X)\})$ such that

$$h(f) \in L^1(\nu) \quad \text{and} \quad \int h(f)dm = \int h(f)d\nu, \quad h \in \mathcal{L}_u(m, f). \quad (4.15)$$

Proof Clearly, the isomorphism χ extends to \mathcal{A} simple functions by letting

$$\chi\left(\sum_{i=1}^n x_i \mathbb{1}_{A_i}\right) = \sum_{i=1}^n x_i \mathbb{1}_{\chi(A_i)}. \quad (4.16)$$

If $\{g_n\}_{n \in \mathbb{N}}$ is a sequence of \mathcal{A} simple functions that converges in $L^1_X(m)$ to the origin then $\chi(g_n)$ forms a Cauchy sequence in $L^1_X(\lambda)$ and its limit cannot be but the origin. Thus we obtain a further extension of χ as a map of $L^1_X(m)$ into $L^1_X(\lambda)$. Write $\tilde{f} = \chi(f)$. Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of \mathcal{A} simple functions converging to f in $L^1_X(m)$. Fix $A \in \mathcal{A}$, $\delta > 0$ set $C_n = \{\|\tilde{f} - \chi(f_n)\| < \delta\}$ and choose $E_n \in \mathcal{A}$ such that $E_n \subset \{\|f - f_n\| < \delta\}$. Then

$$\begin{aligned} \tilde{f}[C_n \cap \chi(A \cap E_n)] &\subset \chi(f_n)[\chi(A \cap E_n)] + \delta\mathbb{B} \\ &= f_n[A \cap E_n] + \delta\mathbb{B} \\ &\subset f[A] + 2\delta\mathbb{B}. \end{aligned}$$

Thus, $\lambda^*(\chi(A) \cap \{\tilde{f} \notin f[A] + 2\delta\mathbb{B}\}) \leq \lambda(C_n^c) + m(E_n^c)$ and we conclude that $\lambda(\chi(A) \cap \{\tilde{f} \notin f[A]\}) = 0$.

Fix $h \in \mathcal{L}_u(m, f)$ and $A \in \mathcal{A}$. Then, $h(f_n)$ converges to $h(f)$ in $L^1(m)$ and $h(\chi(f_n))$ converges to $h(\tilde{f})$ in $L^1(\lambda)$. But then,

$$\begin{aligned} \int_A h(f)dm &= \lim_n \int_A h(f_n)dm \\ &= \lim_n \int_{\chi(A)} h(\chi(f_n))d\lambda \\ &= \int_{\chi(A)} h(\tilde{f})d\lambda. \end{aligned}$$

If f has closed range and $\langle h_n \rangle_{n \in \mathbb{N}}$ is a sequence in $\mathcal{L}_u(m, f)$ such that $h_n(f)$ decreases pointwise to 0, then h_n decreases pointwise to 0 on $f[\Omega]$ i.e. $h_n(\tilde{f})$ decreases to 0 λ a.s.. It follows from (4.14) that $\int h(f)dm$ is a Daniell integral over the vector lattice $\mathcal{L}_u(m, f)$ which contains the unit. □

Theorem 3 requires the assumption $f \in L^1_X(m)$ which is quite restrictive and is hardly satisfied in applications. The reason is that Stone isomorphism would otherwise not be sufficient to identify a representation \tilde{f} . A similar problem will be encountered in the next section. On the other hand if f is just Pettis integrable Theorem 3 is no longer valid.

5 Abstract standard representations

In the present and the following section we look for a countably additive representation without assuming that f is measurable and by exploiting the Stone-Ćech compactification. It turns out that this technique is less useful than one may expect. In fact the compactification $\beta\Omega$ of Ω does not permit to find a natural extension of f . In particular it is hardly possible to obtain a continuous extension that may serve as a representation with respect to some countably additive measure. This is however possible in the following special case

Theorem 4 *Let X be the dual space of some Banach space W and assume that f is norm integrable. Then there exists a Radon probability measure λ defined on $\mathcal{B}(\beta\Omega)$ and a map $\tilde{f} \in \mathfrak{F}(\beta\Omega, X)$ such that*

$$\int h(f)dm = \int h(\tilde{f})d\lambda, \quad h \in \mathcal{C}_u((X, weak^*) \cap \mathcal{L}(m, f)). \tag{5.1}$$

Proof Consider the family $\mathfrak{F}_b(\Omega, X)$ of functions whose range is contained in some multiple of \mathbb{B} (the unit sphere of X). Endowing X with the weak* topology and Ω with the discrete topology, every $g \in \mathfrak{F}_b(\Omega, X)$ is continuous with values in a compact, Hausdorff space. It admits a continuous extension $\tilde{g} \in \mathfrak{F}(\beta\Omega, X)$ (see e.g. [23, 19.5]).

There exists a Radon probability measure λ defined on $\mathcal{B}(\beta\Omega)$ such that

$$\int h(g)dm = \int h(\tilde{g})d\lambda \tag{5.2}$$

for all $g \in \mathfrak{F}_b(\Omega, X)$ and all $h \in \mathcal{C}((X, weak^*) \cap \mathcal{L}(m, g))$. Define

$$f_n = \frac{f}{1 + 2^{-n} \|f\|}. \tag{5.3}$$

Clearly, $f_n \in \mathfrak{F}_b(\Omega, X)$. Moreover, $\|f - f_n\|$ converges to 0 in $L^1(m)$. Then, letting A run across finite subsets of the unit sphere of W , and taking advantage of τ additivity of λ

$$\begin{aligned} 0 &= \limsup_{m,n} \int_A \sup_{w \in A} |(f_n - f_m)w| dm \\ &= \limsup_{m,n} \int_A \sup_{w \in A} |(\tilde{f}_n - \tilde{f}_m)w| d\lambda \\ &= \lim_{m,n} \int \|\tilde{f}_n - \tilde{f}_m\| d\lambda. \end{aligned}$$

We may find a subsequence (still indexed by n for convenience) such that

$$\infty > \lim_k \int \sum_{n=1}^k \|\tilde{f}_{n+1} - \tilde{f}_n\| d\lambda = \int \sum_n \|\tilde{f}_{n+1} - \tilde{f}_n\| d\lambda.$$

There exists therefore a λ null set outside of which \tilde{f}_n is convergent in X to some limit \tilde{f} such that $\|\tilde{f}\| \in L^1(\lambda)$ and that $\|\tilde{f}_n - \tilde{f}\|$ converges to 0 in $L^1(\lambda)$. It is also clear that \tilde{f} is independent of the chosen subsequence. If $U \in \mathcal{C}_{ub}(\mathbb{R}^p)$ and $w_1, \dots, w_p \in W$

$$\begin{aligned} \int U(w_1, \dots, w_p)(f)dm &= \lim_n \int U(w_1, \dots, w_p)(f_n)dm \\ &= \lim_n \int U(w_1, \dots, w_p)(\tilde{f}_n)d\lambda \\ &= \int U(w_1, \dots, w_p)(\tilde{f})d\lambda. \end{aligned}$$

Therefore (5.1) follows from Hager’s lemma. □

In the general case, however, the situation is more complicated. We found it convenient to compactify $\Omega \times X$. This has the double advantage of finding a natural extension $T(H)$ of the elements H of $\mathcal{L}(m, f)$ and, on the other hand, to arrive at the countably additive representation of the integrals of such extensions. We refer to this result as an *abstract standard representation* of (m, f) , see (5.4). Its proof is given in the following Lemma 1.

Nevertheless, the problem of expressing $T(H)$ in the form $H(\tilde{f})$ for some X valued function \tilde{f} proves to be quite difficult to solve. The more so given our choice to treat f just a weakly and strong measurable function. To this end we later consider several additional assumptions on the space X .

The following is essentially an application of Stone–Čech and Stone–Weierstrass to our setting. We need to spell out several properties.

Lemma 1 *There exist: (i) a countably additive probability space $(S, \mathcal{B}(S), \lambda)$ with S compact and Hausdorff and λ regular, (ii) a Boolean homomorphism χ of $\mathcal{A}(m)$ into $\text{clopen}(S)$ and (iii) an isometric, vector lattice homomorphism T of $\mathcal{L}(m, f)$ into $L^1(\lambda)$ such that*

$$\int_A H(f)dm = \int_{\chi(A)} T(H)d\lambda, \quad A \in \mathcal{A}, H \in \mathcal{L}(m, f). \tag{5.4}$$

In addition:

- (a) T is an algebraic isomorphism of $\mathcal{L}_b(m, f)$ onto $\mathcal{C}(S)$;
- (b) each $\tilde{f} \in \mathcal{C}(S)$ is $\sigma(\text{clopen}(S))$ measurable;
- (c) if $E \in \text{clopen}(S)$ then $\lambda(E \Delta \chi(A)) = 0$ for some $A \in \mathcal{A}(m)$;
- (d) $T[\mathcal{L}(m, f)]$ is norm dense in $L^1(\lambda)$;
- (e) for arbitrary $n \in \mathbb{N}$, $H_1, \dots, H_n \in \mathcal{L}(m, f)$ and $\alpha \in \mathcal{C}_b(\mathbb{R}^n)$

$$\alpha(T(H_1), \dots, T(H_n)) = T(\alpha(H_1, \dots, H_n)), \quad \lambda \text{ a.s.}; \tag{5.5}$$

- (f) for each $A \in \mathcal{A}(m)$ and $h \in \mathcal{L}(m, f)$, $h(f)$ is m -null on A if and only if $T(h)$ is λ -null on $\chi(A)$; moreover, $\lambda(\chi(A) \setminus \tilde{A}) = 0$ where

$$\tilde{A} = \chi(A) \cap \bigcap_{\{h(f) \text{ } m\text{-null on } A\}} \{T(h) = 0\}. \tag{5.6}$$

Proof Since $\mathcal{L}_b(m, f)$ is a closed subalgebra of $\mathfrak{F}_b(\Omega \times X)$ containing the unit as well as a vector lattice there exist [8, IV.6.20] a compact, Hausdorff space S and an isometry U between $\mathcal{L}_b(m, f)$ (endowed with the uniform norm) and $\mathcal{C}(S)$. The map U is also an isomorphism of algebras as well as of partially ordered sets. Define

$$\chi(A) = \{U(\mathbb{1}_{A \times X}) = 1\}, \quad A \in \mathcal{A}(m). \tag{5.7}$$

Of course, $0 \leq U(\mathbb{1}_A) = U(\mathbb{1}_A)^2 \leq 1$ so that $U(\mathbb{1}_A) = \mathbb{1}_{\chi(A)}$ and $\chi(A)$ is clopen.

The integral $\int H(f)dm$ for $H \in \mathcal{L}_b(m, f)$ may thus be rewritten as $\phi(UH)$ with ϕ a linear functional on $\mathcal{C}(S)$ with $\|\phi\| = \|m\| = 1$. By Riesz–Markov there exists $\lambda \in \mathbb{P}_{ca}(\mathcal{B}(S))$ regular and such that for any $A \in \mathcal{A}$ and any $H \in \mathcal{L}_b(m, f)$

$$\int_A H(f)dm = \int U(\mathbb{1}_A H)d\lambda = \int_{\chi(A)} U(H)d\lambda. \tag{5.8}$$

The extension T of U to $\mathcal{L}(m, f)$ is obtained by first letting $T_0(H) = \lim_k U(H \wedge k)$ for $H \geq 0$, then noting that T_0 is additive on $\mathcal{L}(m, f)_+$ (because $(G_1 + G_2) \wedge k \leq G_1 \wedge k + G_2 \wedge k \leq (G_1 + G_2) \wedge 2k$) and eventually defining, up to a λ null set,

$$T(G) = T_0(G^+) - T_0(G^-), \quad G \in \mathcal{L}. \tag{5.9}$$

Clearly, T is a linear map of $\mathcal{L}(m, f)$ into $L^1(\lambda)$ and its restriction to $\mathcal{L}_b(m, f)$ coincides with U and is therefore an isomorphism of algebras. This proves (a). To prove the other properties of T , using linearity and the fact that if $G_1, G_2 \in \mathcal{L}(m, f)$ have disjoint supporting sets, then so have $T(G_1)$ and $T(G_2)$, it is enough to restrict to $\mathcal{L}(m, f)_+$. Thus, for any choice of $G_1, G_2 \in \mathcal{L}(m, f)_+$ we see that

$$\begin{aligned} \int_A G_1(f) dm &= \lim_k \int_A [G_1(f) \wedge k] dm \\ &= \lim_k \int_{\chi(A)} U(G_1 \wedge k) d\lambda \\ &= \int_{\chi(A)} T(G_1) d\lambda \end{aligned}$$

(so that (5.4) holds) and that

$$\begin{aligned} T(G_1 \wedge G_2) &= \lim_k U(G_1 \wedge G_2 \wedge k) \\ &= \lim_k U(G_1 \wedge k) \wedge U(G_2 \wedge k) = T(G_1) \wedge T(G_2). \end{aligned}$$

It is clear from (5.4) that $\|T(H)\|_{L^1(\lambda)} = \|H\|_{\mathcal{L}(m, f)}$.

(b) Let $\tilde{f} \in \mathcal{C}(S)$ and $H \in \mathcal{L}_b(m, f)$ be such that $T(H) = \tilde{f}$. Given that $H(f)$ is the $L^1(m)$ limit of a sequence of \mathcal{A} simple functions, then \tilde{f} is the $L^1(\lambda)$ limit of a sequence of $\chi(\mathcal{A})$ simple functions.

(c) Let $E \in \text{clopen}(S)$. Define $E_1 = \{U^{-1}(\mathbb{1}_E) = 1\}$ and $E_2 = \{\omega : (\omega, f(\omega)) \in E_1\}$. Since $\mathbb{1}_E \in \mathcal{C}(S)$, then by (a) $\mathbb{1}_{E_1} \in \mathcal{L}_b(m, f)$. Moreover, $\mathbb{1}_{E_1}(\omega, f(\omega)) = \mathbb{1}_{E_2}(\omega)$ so that $E_2 \in \mathcal{A}(m)$ and $\lambda(E \Delta \chi(E_2)) = 0$.

(d) The claim follows from the inclusion $L^\infty(\lambda) \subset \overline{T[\mathcal{L}(m, f)]}^{L^1(\lambda)}$. Fix $b \in L^\infty(\lambda)$. By Fremlin extension of Lusin’s Theorem [13, Theorem 2.b] (see also [3, 7.1.13]) there exists a sequence $\langle b_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{C}(S)$ that converges to b in $L^1(\lambda)$. However, by property (a), for each $n \in \mathbb{N}$ there exists $h_n \in \mathcal{L}_b(m, f)$ such that $T(h_n) = b_n$.

(e) Let $n \in \mathbb{N}$, $\alpha \in \mathcal{C}_b(\mathbb{R}^n)$ and $H_1, \dots, H_n \in \mathcal{L}(m, f)_+$. Consider the following a.s. equalities

$$\begin{aligned} \alpha(T(H_1), \dots, T(H_n)) &= \lim_k \alpha(U(H_1 \wedge k), \dots, U(H_n \wedge k)) \\ &= \lim_k U(\alpha(H_1 \wedge k, \dots, H_n \wedge k)) \\ &= U(\alpha(H_1, \dots, H_n)) \\ &= T(\alpha(H_1, \dots, H_n)). \end{aligned}$$

The first one follows from continuity of α ; the second is certainly true for fixed k when α is a polynomial and follows from uniform approximation by polynomials on the cube $[0, k]^n$; the third one is a consequence of the fact that $\alpha(H_1 \wedge k, \dots, H_n \wedge k) \in \mathcal{L}_b(m, f)$ and converges to $\alpha(H_1, \dots, H_n)$ in $L^1(m)$; the last one is a consequence of α being bounded.

(f) Fix $A \in \mathcal{A}(m)$ and let

$$\mathcal{N}_A = \left\{ h \in \mathcal{L}(m, f) : 1 \geq h \geq 0, h(f) \text{ is } m\text{-null on } A \right\}. \tag{5.10}$$

It is easy to see that $h \in \mathcal{N}_A$ is equivalent to

$$0 = \int_B h(f) dm = \int_{\chi(B)} T(h) d\lambda, \quad B \in \mathcal{A}, B \subset A. \tag{5.11}$$

By countable additivity the equality $0 = \int_{\chi(B)} T(h) d\lambda$ extends from $\chi[\mathcal{A}]$ to $\sigma(\text{clopen}(S))$ measurable subsets of $\chi(A)$, and given that, by (b), $T(h)$ is $\sigma(\text{clopen}(S))$ measurable, we conclude that $T(h) = 0$ a.s. on $\chi(A)$. On the other hand this last property and (5.11) imply $h \in \mathcal{N}_A$.

The set $\chi(A) \cap \bigcup_{h \in \mathcal{N}_A} \{T(h) > 0\}$ is open. Letting a run over the directed family of all finite subsets of \mathcal{N}_A , it follows from τ additivity of λ that,

$$\lambda\left(\chi(A) \cap \bigcup_{h \in \mathcal{N}_A} \{T(h) > 0\}\right) = \lim_a \lambda\left(\chi(A) \cap \bigcup_{h \in a} \{T(h) > 0\}\right) = 0.$$

□

By Lemma 1(f) the distribution of $h(f)$ under m and of $T(h)$ under λ coincide save, possibly, on a countable set of points. In fact $h(f) \leq t$ is equivalent to $(h(f) - t)^+ = 0$ so that, if $h \in \mathcal{L}(m, f)$ and if $\{h(f) \leq t\} \in \mathcal{A}(m)$, then

$$m(h(f) \leq t) = \lambda(T(h) \leq t). \tag{5.12}$$

Moreover, $\{h(f) \leq t\} \in \mathcal{A}(m)$ for all but countably many values of t .

Given its repeated use, we shall refer to the triple (χ, λ, T) satisfying the conditions of Lemma 1 as an *abstract standard representation* of (m, f) . It is implicit in the term *standard* that λ is a regular probability defined on the Borel subsets of a compact, Hausdorff space.

Definition 3 The triple $(\chi, \lambda, \tilde{f})$ is a standard representation of (m, f) relatively to $\mathcal{H} \subset \mathfrak{F}(X) \cap \mathcal{L}(m, f)$ if (χ, λ, T) is an abstract standard representation of (m, f) and $T(h) = h(\tilde{f})$ for all $h \in \mathcal{H}$.

When f is weakly and norm integrable the associated standard operator T has additional properties, in particular its restriction T_0 of T to X^* . In fact, $T_0 \in \mathfrak{F}(X^*, L^1(\lambda))$ is weakly compact and so is then its adjoint, $T_0^* \in \mathfrak{F}(L^\infty(\lambda), X^{**})$. The restriction of T_0^* to the indicators of sets in $\mathcal{B}(S)$ gives rise to the following object.

Definition 4 Let (χ, λ, T) be the abstract standard representation of (m, f) . The standard vector measure induced by (m, f) is the unique X^{**} valued, countably additive set function of bounded variation \tilde{F} defined on $\mathcal{B}(S)$ and satisfying⁴

$$\tilde{F}(\chi(A))x^* = \int_A x^* f dm, \quad A \in \mathcal{A}. \tag{5.13}$$

A number of useful properties may be proved easily.⁵

Lemma 2 Let f be weakly and norm integrable with abstract standard representation (χ, λ, T) . Let T_0 be the restriction of T to X^* and \tilde{F} the associated standard vector measure. Then f is Pettis integrable if and only if either one of the following equivalent conditions holds:

- (i) the range of T_0^* belongs to the natural embedding of X into X^{**} ;
- (ii) \tilde{F} takes its values in the natural embedding of X into X^{**} ;
- (iv). T_0 is weak*-to-weak continuous;
- (v) T_0 is bounded weak*-to-weak continuous.⁶

Proof Assume that f is Pettis integrable. The set function $F = \tilde{F} \circ \chi$ on \mathcal{A} takes its values in X . By linearity F may be extended to an X valued linear operator defined on the class of all \mathcal{A} simple functions (endowed with the supremum norm) and such extension has its norm dominated by $\int \|f\| dm$. This permits a further extension, F_1 , to the closure $\mathfrak{B}(\mathcal{A})$ of \mathcal{A} simple functions in the topology of uniform convergence. Fix $b \in L^\infty(\lambda)$. By Lemma 1(d) there exists a sequence $\langle h_n \rangle_{n \in \mathbb{N}}$ in $\mathcal{L}_b(m, f)$ such that $T(h_n)$ converges to b in $L^1(\lambda)$ and, with no loss of generality, such that $T(h_n)$ is bounded by $\|b\|_{L^\infty(\lambda)}$. But then

$$\begin{aligned} \|F_1(h_n(f)) - F_1(h_{n+k}(f))\| &\leq \sup_{x^* \in \mathbb{B}^{**}} \int |x^* f| |h_n(f) - h_{n+k}(f)| dm \\ &\leq \int T(\|\cdot\|) |T(h_n - h_{n+k})| d\lambda. \end{aligned}$$

The sequence $F_1(h_n(f))$ is thus Cauchy in X and

$$x^* \lim_n F_1(h_n(f)) = \lim_n x^* F_1(h_n(f)) = \int bT(x^*)d\lambda = T_0^*(b)x^*.$$

Thus (i) holds. The implication (i) \Rightarrow (ii) is obvious while its converse follows from continuity of T^* and density of simple $\mathcal{B}(S)$ measurable functions into $L^\infty(\lambda)$. Assume (i). Then we get

$$x^*x(b) = \int bT(x^*)d\lambda, \quad b \in L^\infty(\lambda), \quad x^* \in X^*$$

⁴ The composition $\tilde{F} \circ \chi$ is sometimes referred to as Dunford definite integral of f at A .

⁵ A partial analogue of the equivalence of Pettis integrability with (iv). was proved by Edgar [11, Proposition 4.1].

⁶ See [8, V.5.3] for a definition of the bounded weak* topology of X^* .

where $x(b)$ is the element of X corresponding to $T_0^*(b)$ through the natural embedding of X into X^{**} . From this it is immediate that T_0 is weak*-to-weak continuous and that f is Pettis integrable. The equivalence of (iv) with (v) follows from the definition of weak topology and from [8, V.5.6]. \square

Because of the equivalence with (v) and of [8, V.5.1], if X is separable the property of weak*-to-weak continuity of T_0 may be proved solely in terms of bounded sequences. In other words separable Banach spaces satisfy the condition of Mazur [11, p. 563]. In this special case, and with the additional assumption that m is countably additive, the conclusion that all weakly and norm integrable (or norm bounded) functions are Pettis integrable follows easily from the Lebesgue dominated convergence. Another obvious case in which Pettis integrability is guaranteed is when X is reflexive. This is again immediate from (i) and does not require countable additivity of m .

6 The Radon–Nikodým property

In this section we consider the implications of the Radon–Nykodým property (RNP) on our results.

Definition 5 X possesses (RNP) whenever for any countably additive, probability space (S, Σ, ν) and any measure of finite variation $G \in \mathfrak{F}(\Sigma, X)$ the relation $G \ll \mu$ implies the existence of $g \in \mathfrak{F}(S, X)$ Pettis integrable such that $G(E) = \int_E g d\nu$ for each $E \in \Sigma$.

Theorem 5 Let X possess (RNP) and let f be norm and weakly integrable. Then, f is Pettis integrable if and only if (m, f) admits a standard representation $(\chi, \lambda, \tilde{f})$ relative to $\mathcal{C}_u((X, weak)) \cap \mathcal{L}(m, f)$ with $\tilde{f} \in L^1_X(\lambda)$. Moreover, for each $A \in \mathcal{A}$ there exists a closed subset $\tilde{A} \subset \chi(A)$ such that $\lambda(\chi(A) \setminus \tilde{A}) = 0$ and $\tilde{f}[\tilde{A}] \subset \overline{f[A]}^{weak}$.

Proof f is Pettis integrable if and only if the standard vector measure \tilde{F} induced by (m, f) takes value in X . Given that $\tilde{F} \ll \lambda$ and that X has (RNP), this is in turn equivalent to the existence of $\tilde{f} \in L^1_X(\lambda)$ such that

$$x^* \tilde{F}(E) = \int_E x^* \tilde{f} d\lambda, \quad E \in \mathcal{B}(S). \tag{6.1}$$

By (5.5), the a.s. equality $T(x^*) = x^* \tilde{f}$ that follows from (6.1) extends from X^* to $\{\alpha(x^*_1, \dots, x^*_n) : n \in \mathbb{N}, \alpha \in \mathcal{C}_b(\mathbb{R}^n)\}$ so that $(\chi, \lambda, \tilde{f})$ is a standard representation of (m, f) relatively to $\mathcal{C}_u((X, weak)) \cap \mathcal{L}(m, f)$. The last claim follows immediately from Theorem 1 once noted that the restriction of f to any $A \in \mathcal{A}$ is a representation of the restriction of \tilde{f} to $\chi(A)$ relatively to $\mathcal{C}_{ub}((X, weak))$. \square

An easy implication of Theorem 5 is the equality

$$\lambda(\tilde{f} \in E) = m(f \in E), \quad E \subset X, E \text{ weakly closed}, f^{-1}(E) \in \mathcal{A} \tag{6.2}$$

which extends the remark on Lemma 1 from the distribution of $h(f)$ to that of f , over weakly open subsets of X .

A much more interesting representation obtains under a convenient assumption on the range of f .

Theorem 6 *Let X possess (RNP). Let f be norm and Pettis integrable and have essentially weakly closed range.⁷ Define $\mathcal{H} = \mathcal{C}_u((X, weak)) \cap \mathcal{L}(m, f)$. There exists $\mu \in \mathbb{P}_{ca}(f^{-1}[\mathcal{B}(X)])$ such that*

$$h(f) \in L^1(\mu) \quad \text{and} \quad \int h(f)dm = \int h(f)d\mu, \quad h \in \mathcal{H}. \tag{6.3}$$

Proof Let $(\chi, \lambda, \tilde{f})$ be the standard representation of (m, f) relative to \mathcal{H} , established in Theorem 5. For each $n \in \mathbb{N}$, let A_n be as in the claim and let \tilde{A}_n be related with A_n in the same way as is \tilde{A} with A in Theorem 5. Then, $\tilde{f}[\bigcup_n \tilde{A}_n] \subset f[\bigcup_n A_n]$. Fix $\omega_0 \in \Omega$ and let $\tau(s) = \omega_0$ for each $s \notin \bigcup_n \tilde{A}_n$ or else let $\tau(s) \in \{f = \tilde{f}(s)\}$ when $s \in \bigcup_n \tilde{A}_n$. It is then clear that

$$\lambda^*(\tilde{f} \neq f \circ \tau) \leq \lambda\left(\bigcap_n \tilde{A}_n^c\right) \leq \lambda(\tilde{A}_n^c) = \lambda(\chi(A_n)^c) = m(A_n^c) \leq 2^{-n}.$$

Thus $f \circ \tau \in L^1_X(\lambda)$ and, necessarily, $\tau^{-1}(E) \in \mathcal{B}(S)(\lambda)$ for every $E \in f^{-1}[\mathcal{B}(X)]$. It follows that

$$\mu = \lambda \circ \tau^{-1} \in \mathbb{P}_{ca}(f^{-1}[\mathcal{B}(X)]). \tag{6.4}$$

Notice that $h(f)$ is $(f^{-1}[\mathcal{B}(X)], \mathcal{B}(\mathbb{R}))$ measurable when $h \in \mathcal{C}_u((X, weak))$. The claim then follows from an application of the change of variable formula. \square

7 The Pettis integrability property

We return on this property which will be useful in our version of Choquet Theorem.

Definition 6 Given a finitely additive probability space (S, Σ, ν) , a Banach space X has $(\nu$ -PIP) if every weakly and norm ν -integrable function is Pettis ν integrable too. A Banach space satisfies (PIP) if the preceding property holds for any finitely additive probability space.

This property, which we briefly considered after Lemma 2, has been discussed extensively in the literature, in particular by Edgar [10, 11] and by Fremlin and Talagrand [14] (but see also [21, 22]), although just for the countably additive case in which it is, intuitively, a less restrictive property.

⁷ By this we mean that for each $n \in \mathbb{N}$ there exists $A_n \in \mathcal{A}$ with $m(A_n^c) < 2^{-n}$ such that $f[A_n]$ is weakly closed.

Apart from the obvious cases mentioned after Lemma 2, Edgar [11, Proposition 3.1] proved that if X is ν -measure compact (for example if the weak topology of X is Lindelöf) then it satisfies the (ν -PIP). A result in the negative was obtained by Fremlin and Talagrand [14, Theorem 2B] who proved that $X = \ell^\infty$ fails to possess the countably additive (PIP).⁸

The following result focuses on separability of X and establishes only a partial analogue of the countably additive case. Nevertheless it permits to decompose T into a Pettis representable part and a purely non representable part. Of course by Theorem 2 separability may be replaced with the assumption that f is measurable.

Theorem 7 *Let X be separable and f norm integrable with abstract, standard representation (χ, λ, T) . There exist $\eta \in L^1(\lambda)_+$ and $\tilde{f} \in L^1_X(\lambda)$ such that*

$$T(h) = h(\tilde{f}) + T^\perp(h), \quad h \in \mathcal{L}(m, f) \tag{7.1}$$

where $\tilde{f} = 0$ on $\{\eta > 0\}$ and

$$T^\perp(x^*) \geq \eta + \inf_{\sigma \in \mathfrak{S}(x^*)} \limsup_n T^\perp(\sigma_n), \quad x^* \in X^* \tag{7.2}$$

with $\mathfrak{S}(x^*)$ denoting bounded sequences in X^* which converge weakly* to x^* .

Proof Since X is separable, \mathbb{B}^* is metrizable in the weak* topology and X^* admits a countable, rational vector space X^*_0 which is weakly* dense in X^* . Let $\mathbb{B}^*_0 = \mathbb{B}^* \cap X^*_0$. Select a subset $S_0 \subset S$ of full λ measure such that the functionals T_s are linear on X^*_0 and that $\sup_{x^* \in \mathbb{B}^*_0} |T_s(x^*)| \leq T_s(\|\cdot\|) < +\infty$ for each $s \in S_0$.

Define $\eta \in \mathfrak{F}(S)$ by setting $\eta_s = 0$ if $s \notin S_0$ or else

$$\eta_s = \sup_\sigma \inf_n T_s(\sigma_n), \quad s \in S_0 \tag{7.3}$$

where the sup is computed over all sequences $\sigma = \langle \sigma_n \rangle_{n \in \mathbb{N}}$ in \mathbb{B}^*_0 which converge weakly* to 0. Clearly, $\eta \geq 0$.

Fix $a \in \mathbb{R}$. If $\langle n_k \rangle_{k \in \mathbb{N}}$ is a sequence in \mathbb{N} , define

$$A^a_{n_1, \dots, n_k} = \bigcap_{i=1}^k \{T(x^*_{n_i}) \geq a\} \quad \text{if } d(x^*_{n_i}, 0) \leq 2^{-i}, \quad i = 1, \dots, k \tag{7.4}$$

or else $A^a_{n_1, \dots, n_k} = \emptyset$. The inequality $T(x^*_{n_i}) \geq a$ is unaltered if we replace $T(x^*_{n_i})$ with its truncation $T(x^*_{n_i}) \wedge (a + 1) \vee (a - 1)$. However, by Lemma 1, such truncation is just the image under T of $x^*_{n_i} \wedge (a + 1) \vee (a - 1) \in \mathcal{L}_b(m, f)$ and is thus continuous. Thus the sets $A^a_{n_1, \dots, n_k}$ are actually closed and define a Souslin scheme so that

$$A^a = \bigcup_{\langle n_k \rangle_{k \in \mathbb{N}}} \bigcap_{i=1}^\infty A^a_{n_1, \dots, n_i} \tag{7.5}$$

⁸ Edgar [10] studies conditions under which weakly measurable functions are weakly equivalent to measurable ones.

is a Souslin set. Moreover,

$$\{\eta \geq a\} \cap S_0 = \bigcap_n A^{a-2^{-n}} \cap S_0 \tag{7.6}$$

so that η is universally measurable.

It is easily seen that for every $s \in S_0 \cap \{\eta = 0\}$ $\lim_n T_s(x_n^*)$ exists for every bounded sequence $\langle x_n^* \rangle_{n \in \mathbb{N}}$ in X_0^* which weakly* converges to some $x^* \in X^*$ and that the limit is independent of the intervening sequence. We may thus define

$$q_s(x^*) = \lim_n T_s(x_n^*) \quad s \in S_0 \cap \{\eta = 0\} \tag{7.7}$$

Then, q_s is a linear on X^* .

Fix $t > 0$ and let $\langle z_n^* \rangle_{n \in \mathbb{N}}$ be a weakly* convergent sequence in $t\mathbb{B}^* \cap q_s^{-1}(0)$ with limit $z^* \in t\mathbb{B}^*$. For each $n \in \mathbb{N}$ we can find a sequence $\langle y_{n,k}^* \rangle_{k \in \mathbb{N}}$ in $t\mathbb{B}_0^*$ weakly* convergent to z_n^* and such that $|q_s(y_{n,k}^*)| < 2^{-n}$. Using a diagonal argument we can then extract a sequence $\langle y_i^* \rangle_{i \in \mathbb{N}}$ again in $t\mathbb{B}_0^*$ which converges weakly* to z^* and with $|q_s(y_i^*)| < 2^{-i}$. This implies that $t\mathbb{B}^* \cap q_s^{-1}(0)$ is weakly* closed for each $t > 0$ and, by [8, V.5.6-7], that q_s is continuous in the weak* topology. But then there exists $\tilde{f}_s \in X$ such that $q_s(x^*) = x^* \tilde{f}_s$ for each $s \in S_0 \cap \{\eta = 0\}$ and $x^* \in X^*$ and thus such that

$$x^* \tilde{f}_s = T_s(x^*), \quad x^* \in X_0^*, \quad s \in S_0 \cap \{\eta = 0\}. \tag{7.8}$$

Define $\tilde{f}_s = 0$ when $s \notin S_0 \cap \{\eta = 0\}$. Clearly, \tilde{f} is weakly measurable and with separable range and thus measurable. Moreover, $\|\tilde{f}\| \leq \sup_{x^* \in \mathbb{B}^*} |T(x^*)| \leq T(\|\cdot\|)$ so that \tilde{f} is norm integrable and therefore $\tilde{f} \in L^1_X(\lambda)$.

Define $T^\perp(h) = T(h) - h(\tilde{f})$. If $\langle x_n^* \rangle_{n \in \mathbb{N}}$ is a bounded sequence in X_0^* weakly* converging to 0 and $x^* \in X^*$ then,

$$\begin{aligned} T^\perp(x^*) &\geq \inf_k T^\perp(x_k^*) + T^\perp(x^* - x_n^*) \\ &= \mathbb{1}_{\{\eta > 0\}} \inf_k T(x_k^*) + \limsup_n T^\perp(x^* - x_n^*) \\ &\geq \eta + \inf_{\sigma \in \mathfrak{S}(x^*)} \limsup_n T^\perp(\sigma_n). \end{aligned}$$

□

8 A non compact and non convex Choquet theorem

To close, we apply our techniques to Choquet integral representation on non compact sets. The main result in this direction was proved by Edgar [9] under the assumption that the Banach space X is separable and satisfies the (RNP) (and, as a consequence, the Krein–Milman property (KMP) and, as noted above, the countably additive (PIP)).

In addition, it is assumed that the intervening set is closed, bounded and convex. One key point in Choquet Theorem is proving that the set of extreme points is measurable in some appropriate sense. In our approach we shall not assume convexity and at the same time we may disregard measurability issues to some extent.

Proposition 1 *Let X be a Banach space satisfying (PIP) and (RNP) and C a closed and bounded subset of X . Fix $\Phi \subset \mathcal{C}((X, weak))$. Assume that $D \subset C$ satisfies either one of the following properties:*

- (i) *($D, weak$) is a normal topological space⁹ or*
- (ii) *each $z \in D$ is a point of continuity of C .¹⁰*

Then the condition

$$|\phi(x)| \leq \sup_{z \in D} |\phi(z)| < +\infty, \quad x \in C, \phi \in \Phi \tag{8.1}$$

is satisfied if and only if for each $x \in C$ there exists a unique $\mu_x \in \mathbb{P}_{ca}(\mathcal{B}(D))$ such that

$$\phi(x) = \int_D \phi(z) d\mu_x, \quad \phi \in \Phi. \tag{8.2}$$

Proof Of course (8.2) implies (8.1). Assume (8.1). Then each $x \in C$ acts as a continuous linear functional on the linear subspace of $\mathcal{C}_b(D)$ spanned by the elements of Φ . The norm preserving extension of each such functional to the whole of $\mathcal{C}_b(D)$ may be represented via a unique regular probability $m_x \in \mathbb{P}(\mathcal{B}(D))$ [8, IV.6.2]. In other words we obtain

$$\phi(f) \in L^1(m_x) \text{ and } \phi(x) = \int_D \phi(f) dm_x, \quad x \in C, \phi \in \Phi \tag{8.3}$$

where $f \in \mathfrak{F}(D, X)$, the identity map, is weakly integrable with respect to m_x (because $\mathbb{B}^* \subset \mathcal{C}_b(D)$) and bounded in norm. Therefore, by (PIP), f is Pettis integrable as well. Under (i) the restriction of m_x to the Borel σ algebra $\mathcal{B}((D, weak))$ generated by the weakly open sets is itself regular so that f has approximately weakly closed range. On the other hand, if D consists of the points of continuity of C , then the weak and the norm topology coincide in restriction to D and again f has approximately weakly closed range. Moreover, $f^{-1}[\mathcal{B}(X)] = \mathcal{B}(D)$. By Theorem 6 there exists $\mu_x \in \mathbb{P}_{ca}(\mathcal{B}(D))$ such that for every $h \in \mathcal{C}_u((X, weak))$

$$h(f) \in L^1(\mu_x) \text{ and } \int h(f) dm_x = \int h(f) d\mu_x \tag{8.4}$$

which implies (8.2). Given that $\{h(f) : h \in \mathcal{C}_u((X, weak))\}$ is a lattice containing the constant and that the σ algebra induced by it coincides with $f^{-1}[\mathcal{B}(X)]$, uniqueness of μ_x follows from Daniell Theorem. □

⁹ This condition is satisfied if X is weakly Lindelöf. More examples are contained in the classical work of Corson [5].

¹⁰ For example, if X has Kadec norm and D is the intersection of C with the unit sphere.

By comparison with the original result proved by Edgar, Proposition 1 may appear significantly more general but its generality is hindered by the need to explicitly assume (*PIP*). Notice that if C is convex then D may be the set of its extremal points or of its denting points. The existence of such sets is guaranteed since, as is well known, the (*RNP*) implies the (*KMP*) as well as dentability of bounded sets [6, Corollary]. In particular if C is bounded, closed and convex, and D is the set of its denting points, then property (*ii*) is satisfied, see [19, Theorem].

Funding Open access funding provided by Università degli Studi di Milano - Bicocca within the CRUI-CARE Agreement.

Data availability The author declares that this study constitutes purely theoretical research without any data association.

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References

1. Ascherl, A., Lehn, J.: Two principles for extending probability measures. *Manuscr. Math.* **21**(1), 43–50 (1977)
2. Berti, P., Regazzini, E., Rigo, P.: Modes of convergence in the coherent framework. *Sankhyā* **69**(2), 314–329 (2007)
3. Bogachev, V.I.: *Measure Theory*, vol. II. Springer, Berlin (2007)
4. Cassese, G.: Conglomerability and the representation of linear functionals. *J. Convex Anal.* **25**(3), 789–815 (2018)
5. Corson, H.H.: The weak topology of a Banach space. *Trans. Am. Math. Soc.* **101**(1), 1–15 (1961)
6. Davis, W.J., Phelps, R.R.: The Radon–Nikodym property and dentable sets in Banach spaces. *Proc. Am. Math. Soc.* **45**(1), 119–122 (1974)
7. Dubins, L.E., Savage, L.J.: *How to Gamble if You Must. Inequalities for Stochastic Processes*. Dover, New York (2014)
8. Dunford, N.J., Schwartz, J.T.: *Linear Operators. Part I*. Wiley, New York (1988)
9. Edgar, G.A.: A noncompact Choquet theorem. *Proc. Am. Math. Soc.* **49**(2), 354–358 (1975)
10. Edgar, G.A.: Measurability in a Banach space. *Indiana Univ. Math. J.* **26**(4), 663–677 (1977)
11. Edgar, G.A.: Measurability in a Banach space. II. *Indiana Univ. Math. J.* **28**(4), 559–579 (1979)
12. Fabian, M., Habala, P., Hájek, P., Montesinos, V., Zizler, V.: *Banach Space Theory*. Springer, New York (2010)
13. Fremlin, D.H.: Measurable functions and almost continuous functions. *Manuscr. Math.* **33**, 387–405 (1981)
14. Fremlin, D.H., Talagrand, M.: A decomposition theorem for additive set-functions, with applications to Pettis integrals and ergodic means. *Math. Zeit.* **168**(2), 117–142 (1979)
15. Hager, A.W.: Approximation theorems for uniformly continuous functions. *Trans. Am. Math. Soc.* **236**, 263–273 (1978)
16. Kadane, J.B., O'Hagan, A.: Using finitely additive probability: uniform distributions on the natural numbers. *J. Am. Stat. Assoc.* **90**(430), 626–631 (1995)
17. Karandikar, R.L.: A general principle for limit theorems in finitely additive probability. *Trans. Am. Math. Soc.* **273**, 541–550 (1982)

18. Karandikar, R.L.: A general principle for limit theorems in finitely additive probability: the dependent case. *J. Multivar. Anal.* **24**, 189–206 (1988)
19. Lin, B.-L., Lin, P.-K., Troyanski, S.L.: Characterizations of denting points. *Proc. Am. Math. Soc.* **102**(3), 526–528 (1988)
20. Maharam, D.: Finitely additive measures on the integers. *Sankhyā A* **38**(1), 44–59 (1976)
21. Musiał, K.: Pettis Integral. In: Pap, E. (ed.) *Handbook of Measure Theory*, pp. 531–586. North-Holland, Amsterdam (2002)
22. Stefansson, G.F.: Pettis integrability. *Trans. Am. Math. Soc.* **330**(1), 401–418 (1992)
23. Willard, S.: *General Topology*. Dover Publications, Mineola (2004)