MULTIPLICATIVE CONSTANTS AND MAXIMAL MEASURABLE COCYCLES IN BOUNDED COHOMOLOGY

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Abstract. Inspired by the theory of maximal representations via bounded cohomology, we introduce the notion of both multiplicative constant and maximal measurable cocycle. Maximal cocycles satisfy usually a trivialization property, since under suitable hypothesis they are cohomologous to a preferred representation.

The main application of this paper is the definition and the study of the Cartan invariant of a measurable cocycle associated to a complex hyperbolic lattice. We first extend the classic Cartan invariant of representations to measurable cocycles. Then, built on our fibered multiplicative formula, we completely describe the relation between totally real cocycles and the vanishing of the Cartan invariant. In this way we get an extension of a result of Burger and Iozzi [BI12, Theorem 1.1] to the case of measurable cocycles. Finally, applying our theory of multiplicative constants, we completely characterize measurable cocycles with maximal Cartan invariant as the ones which can be trivialized, possibly modulo a compact subgroup when $m > n$.

Contents

Date: 20th September 2020. The first author was supported by the CRC 1085 *Higher Invariants* (Universität Regensburg, funded by the DFG).

Key words and phrases. lattice, Zimmer cocycle, boundary map, bounded cohomology, transfer map, fibered product, maximal cocycle.

1. Introduction

1.1. **Historical background.** A fruitful approach to the study of geometric structures on a topological space X is to introduce a function whose critical values detect some nice structures on X, for instance the ones with many symmetries. Typically, one defines a bounded function and then either its maximum or its minimum corresponds to the desired structure. One of the easiest example is given by the elementary problem to find the rectangle with largest area among the ones with a fixed perimeter. In this case, the maximum of the area function corresponds to the square. More generally, if we consider a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary, the isoperimetric inequality show that the minimum of the ratio between the square of the length of the boundary and the area of Ω is attained if and only if the domain is a disk.

The key point of the previous approach is that the above inequalities are all sharp. The interest in such inequalities has grown in the years finding many applications. One of the most striking applications is the study of lattices in semisimple Lie groups. More precisely, given two simple Lie groups of non-compact type G, G' and a lattice $\Gamma \leq G$, Burger and Iozzi [BI09] described a nice technique to construct numerical invariants associated to any representation $\rho\colon \Gamma \to G'$. Using the induced map by ρ in continuous bounded cohomology $\mathrm{H}^\bullet_{cb}(\rho)$, the main idea to define the desired invariant is to pullback a preferred class $\Psi' \in \mathrm{H}^\bullet_{cb}(G';\mathbb{R})$ along $\mathrm{H}^\bullet_{cb}(\rho)$ and then evaluate it on the fundamental class of $\Gamma\backslash X$ via the Kronecker pairing and the comparison map. Here X is the Riemannian symmetric space associated to G and $\Gamma\backslash X$ denotes the quotient manifold (notice that the fundamental class is in fact a *relative* fundamental class when the manifold $\Gamma\backslash X$ is not compact).

The construction of such invariants allows to suitably realize them as real *multiplicative constants*, i.e. real numbers appearing in a integral formula. Burger and Iozzi themselves refer to the latter as *useful formula* [BI09]. Indeed, they explain [BI09] that the useful formula encodes all the information about the multiplicative constant, and hence about the chosen invariant. For instance, one can usually show that those numerical invariants are bounded and their maximum corresponds precisely to those representations induced by representations of the ambient group.

The first example of this construction dates back to the study of representations $\rho\colon \Gamma_g \to \mathrm{Homeo}^+(\mathbb{S}^1)$ of the surface group $\Gamma_g = \pi_1(\Sigma_g)$, where Σ_g is a closed surface of genus $g \geq 2$. Indeed, since Ghys [Ghy87] proved that the Euler class is bounded, i.e. $\mathrm{e}_b \in \mathrm{H}^2_b(\mathrm{Homeo}^+(\mathbb{S}^1); \mathbb{Z})$, one can pullback it and define the *Euler invariant* $\mathrm{eu}(\rho)$ of the representation. This numerical invariant is constant along the semiconjugacy class of ρ. Moreover, Milnor [Mil58] and Wood [Woo71] proved indipendently that the absolute value of the Euler invariant is bounded from above by the modulus of the Euler characteristic $\chi(\Sigma_g)$. Following the idea of maximal representations described above, Matsumoto [Mat87] proved that the maximal value of the Euler invariant corresponds to those representations that are semiconjugated to a hyperbolization $\pi_0\colon \Gamma_g \to \mathrm{PSL}(2,\mathbb{R})$. The analogous result using bounded cohomology has been developed by Iozzi [Ioz02].

During the years, similar techniques have been also extended to the context of representations of higher dimensional hyperbolic lattices, both in the real and in the complex cases. For instance, let us consider a torsion-free lattice $\Gamma \leq \mathrm{PO}^\circ(n,1) \cong \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$ with $n \geq 3$. In this setting Mostow-Prasad Rigidity Theorem [Mos68, Pra73] implies that any lattice isomorphic to Γ is actually conjugated to it via an element of $\mathrm{PO}(n,1)$. The original proof by Mostow was based on quasi-conformal maps of spheres, but later Thurston [Thu79] and Gromov [Gro82] proposed an alternative proof via ℓ^1-homology and simplicial volume. In a dual way, since the group with twisted real coefficients $\mathrm{H}^n_{cb}(\mathrm{PO}(n,1); \mathbb{R}_\varepsilon)$ is generated by the volume class, Bucher, Burger and Iozzi [BBI13] introduced the notion of *volume* $\mathrm{Vol}(\rho)$ of a representation $\rho\colon \Gamma \to \mathrm{PO}^\circ(n,1)$ via the pullback of such cohomological class. As in the case of the Euler invariant, also the volume of representations is constant along the $\mathrm{PO}^\circ(n,1)$-conjugacy classes and it satisfies a Milnor-Wood type inequality $|\mathrm{Vol}(\rho)| \leq \mathrm{Vol}(\Gamma \backslash \mathbb{H}^n_{\mathbb{R}})$, where $\Gamma\backslash\mathbb{H}^n_{\mathbb{R}}$ is the real hyperbolic manifold with fundamental group Γ. The study of the maximality of this invariant shows that the maximum is attained if and only if the representation is $\mathrm{PO}(n,1)$-conjugated to the standard lattice embedding $i\colon \Gamma \to \mathrm{PO}(n,1)$ [BBI13]. This result formalizes a dual statement of Mostow-Prasad Rigidity Theorem in the context of representations.

Dealing with complex hyperbolic lattices, something analogous happens. If we assume that $\Gamma \leq \mathrm{PU}(n,1)$ is a torsion-free lattice, with $n \geq 2$, we can associate a numerical invariant to representations $\rho\colon \Gamma \to \mathrm{PU}(m,1)$, with $m \geq n \geq 2$. In this setting the Kähler form ω_m on the complex hyperbolic space $\mathbb{H}^m_{\mathbb{C}}$ determines a bounded cohomology class $\kappa^b_m \in \mathrm{H}^2_{cb}(\mathrm{PU}(m,1); \mathbb{R})$ which is also a generator of the one-dimensional real vector space. Then, Burger and Iozzi [BI07b] defined the *Cartan invariant* $i(\rho)$ via the pullback of the previous class κ^b_m. Given a measurable ρ-equivariant map $\varphi\colon \mathbb{H}^n_{\mathbb{C}} \to \mathbb{H}^m_{\mathbb{C}}$, the interpretation of the latter invariant as a multiplicative constant leads to the study the incidence structure of chains with respect to φ (see Section 5.1). As in the previous examples, the Cartan invariant satisfies the following inequality $|i_\rho| \leq 1$, where the equality $i_\rho = 1$ holds if and only if ρ is conjugated to the standard lattice embedding $i\colon \Gamma \to \mathrm{PU}(n,1) \leq \mathrm{PU}(m,1)$ (the group $\mathrm{PU}(n,1)$ is seen as a subgroup of $\mathrm{PU}(m,1)$ via the upper-left corner injection). In the case of *non-uniform* lattices, an analogous result was obtained by Koziarz-Maubon [KM08] via an alternative approach involving harmonic maps.

For sake of completeness we mention that the study of representations of lattices of rank-one Lie groups has been systematically investigated via equivariant maps. Dunfield [Dun99], Francaviglia and Klaff [Fra04, FK06] and Kim and Kim [KK14] gave a different definition of volume of representations $\rho\colon \Gamma \to \mathrm{PO}(m,1)$, where $\Gamma \leq \mathrm{PO}(n,1)$ is a torsion-free lattice and $m \geq n \geq 2$. Some of these results rely on the notion of natural maps introduced by Besson, Courtois and Gallot [BCG95, BCG96, BCG98] and more precisely on the sharp estimate on the Jacobian of such maps. Actually the volume rigidity can be extended even at ideal points of the representations space, as proved by both Francaviglia and one of the authors [FS18, Sav18].

In the particular case of 3-manifold groups, i.e. when Γ is a torsion-free lattice in $\mathrm{PO}^\circ(3,1)$, one could also study representations into $\mathrm{PSL}(n,\mathbb{C})$. Following the work of Goncharov [Gon93], Bucher, Burger and Iozzi [BBI18] proved that the Borel class $\beta_b(n) \in \mathrm{PSL}(n,\mathbb{C})$ is a generator. This allows them to pullback $\beta_b(n)$ along a representation and define the *Borel invariant* $\beta_n(\rho)$. Again, this numerical invariant is bounded from above by a constant proportional to the volume $\mathrm{Vol}(\Gamma\backslash\mathbb{H}^3_{\mathbb{R}})$. Here the maximality of the Borel invariant detects the $\mathrm{PSL}(n,\mathbb{C})$-conjugacy class of the *geometric representation*, i.e. the composition of the standard lattice embedding $i\colon \Gamma \to \mathrm{PSL}(2,\mathbb{C})$ with the irreducible representation $\pi_n\colon \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PSL}(n,\mathbb{C})$. It is still an open question proposed by Guilloux [Gui17, Conjecture 1] if the Borel function is rigid also at the ideal points of the character variety $X(\Gamma,\mathrm{PSL}(n,\mathbb{C}))$. One of the authors partially answered to the question when Γ is the tetrahedral reflection lattice [Savb] (in the same spirit we mention also the ω-Borel invariant defined in [Sav19]).

The importance of multiplicative constants becomes even more clear in the study of representations of either surface groups or complex hyperbolic lattices into the connected component of the identity $G = \mathrm{Isom}^\circ(X)$ of isometries of a Hermitian symmetric space. In the case of surface groups it is worth mentioning the Toledo invariant studied by Burger-Iozzi-Wienhard [BIW10], who gave a great contribution to the comprehension of higher Tëichmuller theory and maximal representations. When $G = \mathrm{SU}(m,n)$, for complex lattices we refer to the superrigidity result of Pozzetti [Poz15], which should be thought of as an adaptation of Margulis [Mar75] to context of rank-one lattices.

1.2. **Measurable cocycles and multiplicative constants.** Recently, inspired by the Bader, Furman and Sauer's definition of Euler number for self-couplings via bounded cohomology [BFS13], the authors extended several numerical invariants of representations to the theory of measurable cocycles. First, one of the authors investigated the Borel invariant of measurable cocycles of 3-manifold groups with values into $\mathrm{PSL}(n,\mathbb{C})$ [Sava]. Then, the authors [MS] extended the notion of both the volume and the Euler number of representations to measurable cocycles of real hyperbolic lattices $\Gamma \leq \mathrm{PO}^\circ(n,1)$, with $n \geq 2$.

One of the aims of this paper is to present a uniform introduction to the theory of multiplicative constants and maximal measurable cocycles via bounded cohomology. We consider the following general setting. Let G, G' be two locally compact second countable groups and let $L, Q \leq G$ be two closed subgroups. Assume that Q is *amenable* and that $L\backslash G$ admits a G-invariant probability measure. Let (X, μ_X) be a *standard Borel probability L-space* and let Y be a measurable G'-space. Moreover, given a measurable cocycle $\sigma\colon L \times X \to G'$ we assume there exists a *generalized boundary map* $\phi\colon G/Q \times X \to Y$, i.e. a measurable σ-equivariant map. Under this hypothesis, one can construct a *pullback map* $\mathrm{C}^\bullet(\Phi^X)$ at the level of bounded measurable cochains with real coefficients (see Section 3.1) which induces a well-defined map in cohomology. This allows to consider the pullback of a chosen bounded cohomology class (for instance, the Euler class, the volume class and the Borel class, referring to the examples mentioned above).

Remarkably, this approach *always* extends the construction of the ordinary numerical invariant of representations. Indeed, given any continuous representation $\rho\colon L \to G'$, one can always define a *measurable cocycle* σ_ρ associated to it. Then, since by Burger and Iozzi [BI02] the map $\mathrm{H}^\bullet_{cb}(\rho)$ can be implemented at the level of cochains using measurable boundary maps, we show in Proposition 3.14 that our pullback map associated to σ_ρ agrees with the one associated to ρ described by Burger and Iozzi. Moreover, our pullback remains unchanged along the G'-cohomology class of the cocycle σ (see Proposition 3.12).

The general theory of pullbacks along measurable cocycles (and their respective boundary maps) leads to the following *easy multiplicative formula*, which allows us to study the maximality of the invariants of measurable cocycles (it may be interpreted as an extension of [BI09, Proposition 2.44, Principle 3.1]):

Proposition 1 (Easy multiplicative formula). *In the situation described above, we have the following results:*

(1) Let $\psi' \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ be an everywhere defined G'-invariant cocycle. Let $\psi \in \mathrm{L}^\infty((G/Q)^{\bullet+1})^G$ be a G-invariant cocycle. Denote by $\Psi \in \mathrm{H}^\bullet_{cb}(G;\mathbb{R})$ the class of ψ. If we suppose that $\Psi = \mathrm{trans}^\bullet_{G/Q}[\mathrm{C}^\bullet(\Phi^X)(\psi')]$, then we have

$$\int_{L\backslash G}\int_X \psi'(\phi(\overline{g}.\eta_1,x),\ldots,\phi(\overline{g}.\eta_{\bullet+1},x)d\mu_X(x)d\mu(\overline{g}) = \psi(\eta_1,\ldots,\eta_{\bullet+1}) + \text{cobound.}\ ,$$

for almost every $(\eta_1,\ldots,\eta_{\bullet+1}) \in (G/Q)^{\bullet+1}$.

(2) Suppose that $\mathrm{H}^\bullet_{cb}(G;\mathbb{R}) \cong \mathbb{R}\,\Psi(=\mathbb{R}[\psi])$. Then, there exists a real constant $\lambda_{\psi',\psi}(\sigma) \in \mathbb{R}$ depending on σ, ψ', ψ such that

$$\int_{L\backslash G}\int_X \psi'(\phi(\overline{g}.\eta_1,x),\ldots,\phi(\overline{g}.\eta_{\bullet+1},x)d\mu_X(x)d\mu(\overline{g}) = \lambda_{\psi',\psi}(\sigma)\cdot\psi(\eta_1,\ldots,\eta_{\bullet+1}) + \text{cobound.}\ ,$$

for almost every $(\eta_1,\ldots,\eta_{\bullet+1}) \in (G/Q)^{\bullet+1}$.

This integral formula which may appear quite complicated at first sight, enables us to introduce the fundamental notion of *multiplicative constant* $\lambda_{\psi',\psi}(\sigma)$ associated to a measurable cocycle σ and two given bounded cocycles ψ, ψ' (see Definition 3.16). When there is no coboundary term in the formula, we show that the multiplicative constant appearing in Proposition 1 is always bounded (see Proposition 3.17) and its maximal value is attained if and only if the cocycle σ can be trivialized, i.e. it is cohomologous to a cocycle induced by a representation $G \to G'$ (see Theorem 3.20). Several instances of this general theory are presented in Section 3.5, where we collect some previous works of the authors.

Unfortunately this approach does not work if we need to evaluate essentially bounded functions over subsets of measure zero (this is the case for instance when we consider chains in $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$, see Section 5.1). This obstruction leads us to introduce a suitable extension of our pullback map by considering L^∞-coefficients instead of real ones. Using the notion of *fibered product space* and the associated L^∞-resolution, we implement a pullback map which factors through the standard pullback map (see Proposition 4.6). This allows to compute multiplicative constants via a *fibered multiplicative formula* (see Proposition 4.10). This result extends the easy multiplicative formula mentioned above (Proposition 1) and it will be fundamental for the study of the Cartan invariant of measurable cocycles. However, since it is rather technical, we prefer to not state it in the Introduction.

1.3. **Application: Cartan invariant of measurable cocycles.** Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice, with $n \geq 2$. The interest in the study of representations of Γ into $\mathrm{PU}(m,1)$, where $m \geq n \geq 2$ dates back to the work of Goldman-Millson [GM87], Corlette [Cor88] and Toledo [Tol89]. Given such a representation $\rho\colon \Gamma \to \mathrm{PU}(m,1)$, as mentioned before, Burger and Iozzi [BI07b] introduce the notion of *Cartan invariant* i_ρ associated to ρ.

In this paper, as the main application of our general fibered multiplicative formula 4.10, we introduce and study the *Cartan invariant* associated to measurable cocycles. More precisely, let (X, μ_X) be a standard Borel probability Γ-space and let $\sigma\colon \Gamma \times X \to \mathrm{PU}(m,1)$ be a measurable cocycle. If we assume the existence of an essentially unique boundary map $\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$ associated to σ, then we can apply the machinery developed in Section 3.1 to define the pullback of a multiple of the *Cartan cocycle* $c_m : (\partial_\infty \mathbb{H}^m_{\mathbb{C}})^3 \to \mathbb{R}$. Since πc_n is a representative of the bounded Kähler class $\kappa^b_m \in \mathrm{H}^2_{cb}(\mathrm{PU}(m,1);\mathbb{R})$, the pullback of such a cochain along ϕ determines canonically a bounded Γ-invariant differential form $\omega(\sigma)$ on $\mathbb{H}^n_{\mathbb{C}}$ [BI07a].

Being bounded, the form $\omega(\sigma)$ is also square-integrable (this is always true when the quotient manifold $M \coloneqq \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ is compact) and, using the natural scalar product on $\mathrm{H}^2_{(2)}(M)$ induced by the Riemannian structure, we orthogonally project it on the subspace generated by the Kähler form ω_M. The real number we obtain in this way is the *Cartan invariant* $i(\sigma)$.

We verify that this numerical invariant generalizes the one introduced by Burger and Iozzi for representations in Proposition 5.3. Additionally by Proposition 5.4 the invariant is constant on the $\mathrm{PU}(m,1)$-cohomology classes.

Remarkably, the Cartan invariant can be expressed in the language of multplicative constants introduced in Section 3.4, from which follows directly its boundedness (see also Proposition 5.5 and Corollary 5.7).

The possibility to express the Cartan invariant as a multiplicative constant leads us to the study of *totally real cocycles*, in the spirit of the definition given by Burger and Iozzi [BI12]. When X is Γ-ergodic, we prove in Theorem 2 that totally real cocycles are characterized by the vanishing of the pullback of the Cartan cocycle. This extends the main result of the Burger and Iozzi's work [BI12] to this setting.

Theorem 2. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice and and let (X,μ_X) be a standard Borel probability Γ-space. Given a measurable cocycle $\sigma\colon \Gamma\times X \to \mathrm{PU}(m,1)$ with $m \geq n \geq 2$, assume that there exists an essentially unique boundary map $\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$. Then, we have the followings*

(1) If σ is totally real, then $i(\sigma)=0$;
(2) If X is Γ-ergodic and $\mathrm{C}^2(\phi)=0$, then σ is totally real.

Finally, we show that our Cartan invariant of measurable cocycles satisfies a rigidity result, which extends the one by Burger and Iozzi for representations [BI07b, Theorem 2].

Theorem 3. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice, with $n \geq 2$. Let (X,μ_X) be a standard Borel probability Γ-space . Given a measurable cocycle $\sigma\colon \Gamma \times X \to \mathrm{PU}(m,1)$ with essentially unique boundary map $\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$, we have*

$$|i(\sigma)| \leq 1 \ .$$

In particular, $i(\sigma)=1$ if and only if σ is cohomologous to the cocycle σ_i induced by the standard lattice embedding $i\colon \Gamma \to \mathrm{PU}(n,1) \leq \mathrm{PU}(m,1)$, possibly modulo a compact subgroup when $m>n$. Here $\mathrm{PU}(n,1)$ is seen as a subgroup of $\mathrm{PU}(m,1)$ via the upper-left corner injection.

The proof of the previous theorem crucially relies to the interpretation of the *fibered multiplicative formula* in this specific context. In order to apply that formula, a fundamental role is played by the configuration space $\mathcal{C}_n^{[3]}$ of triples of points on chains of $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$. Given the boundary map ϕ, this induces canonically a measurable map on the configuration space, that is $\phi^{[3]}\colon \mathcal{C}_n^{[3]} \times X \to (\partial_\infty \mathbb{H}^m_{\mathbb{C}})^3$. This map can be thought of as a fibered boundary map. With this notation, we prove the following:

Proposition 4. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice and let (X,μ_X) be a standard Borel probability Γ-space . Consider a measurable cocycle $\sigma\colon \Gamma \times X \to \mathrm{PU}(m,1)$ with $m \geq n \geq 2$. Assume there exists an essentially unique boundary map $\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$. Denote by $\phi^{[3]}\colon \mathcal{C}_n^{[3]} \times X \to (\partial_\infty \mathbb{H}^m_{\mathbb{C}})^3$ the map induced on the configuration space $\mathcal{C}_n^{[3]}$. Then, we have*

$$\int_{\Gamma\backslash \mathrm{PU}(n,1)} \int_X c_m(\phi^{[3]}((\overline{g}.C,\overline{g}\xi_1,\overline{g}.\xi_2,\overline{g}.\xi_3),x)) d\mu_X(x) d\mu(\overline{g}) = i(\sigma) c_n(\xi_1,\xi_2,\xi_3) \ ,$$

for almost every $C \in \mathcal{C}_n$ and $\xi_1,\xi_2,\xi_3 \in C$. Here μ is the $\mathrm{PU}(n,1)$-invariant probability measure on $\Gamma\backslash \mathrm{PU}(n,1)$.

Since (X, μ_X) is standard Borel space, the slice $\phi_x \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$ is measurable for almost every $x \in X$ by [FMW04, Lemma 2.6]. Considering a triple (ξ_1, ξ_2, ξ_3) lying on a chain and assuming the maximality of the Cartan invariant $i(\sigma)$, we have that the slice ϕ_x respects the chain geometry and hence by [BI07b, Theorem 1] it coincides almost everywhere with a totally geodesic embedding. The conclusion then follows from a result of Bader, Furman and Sauer [BFS13, Proposition 3.2], up to introducing a compact subgroup of $\mathrm{PU}(m,1)$ when $m > n$.

1.4. **Plan of the paper.** In Section 2, we recall basic terminology about bounded cohomology and measurable cocycles. More precisely, one can find the definition of amenable actions, measurable cocycles and boundary maps (Section 2.1), Burger and Monod's approach to bounded cohomology (Section 2.2), transfer maps in bounded cohomology (Section 2.3), bounded differential forms (Section 2.4) and an introduction to fibered products and L^∞-resolutions (Section 2.5).

We describe the general setting in which one can study multiplicative constants associated to measurable cocycles in Section 3. Here, we first define the pullback along generalized boundary maps (Section 3.1). Then, we compare our pullback with the usual one for representation in Section 3.2. After having described our easy multiplicative formula (Proposition 1) in Section 3.3, we introduce the notion of multiplicative constants associated to a measurable cocycle and we study the notion of maximality (Section 3.4). In Section 3.5, we show some applications of the previous results.

Section 4 is mainly devoted to the extension of the previous results to fibered products and resolutions with L^∞-coefficients. To this end, we describe a new fibered pullback (Section 4.1) and then we show that it factors through the standard pullback defined above (Section 4.2). This allows us to prove the fibered multiplicative formula in Proposition 4.10.

Finally, Section 5 contains the main application of our constructions. Here, we introduce and study the Cartan invariant of measurable cocycles (Section 5.1). After having proved that it extends the Cartan invariant of representations (Section 5.1), we provide the proofs of Theorems 2 and 3 in Section 5.2. Here, we also discuss the proof of Proposition 4.

2. Preliminary definitions and results

2.1. **Amenable actions, measurable cocycles and boundary maps.** In this section we are going to introduce the notion of *amenable groups and amenable actions*. A particular case of the latter will be the natural action of a locally compact second countable group G on the quotient G/Q, where Q is amenable. This example will be crucial to construct a suitable resolution for computing the bounded cohomology groups of G or, more generally, of any of its closed subgroup (see Section 2.2). Quotients of the form G/Q where Q is amenable will be also the building blocks of *generalized boundary maps* associated to *measurable cocycles*. We are going to introduce them along this section.

The notion of amenable spaces was introduced by Zimmer [Zim78]. We refer the reader to Zimmer's book [Zim84, Section 4.3] for a wider discussion on this topic (see also [Mon01, Section 5.3] and [BI09, Section 2.5]). On the other hand, measurable cocycles and boundary maps are discussed in details by both Furstenberg [Fur73, Fur81] and Zimmer [Zim, Zim84].

We begin by fixing some notation that we will need along the section. Let G be a locally compact second countable group and consider its associated Haar measure and σ-algebra. Let (X, μ) be a *standard Borel measurable G-space*, i.e. a standard Borel measurable space endowed with a measure-preserving G-action. When μ is a probability measure without atoms, we say that (X, μ) is a *standard Borel probability G-space*. If (Y, ν) is another measurable space, we denote by $\mathrm{Meas}(X, Y)$ the space of Borel measurable functions endowed with its natural topology.

In the following definition, we assume that G acts on $\mathrm{L}^\infty(G; \mathbb{R})$ in the following way

$$g.f(g_0) = f(g^{-1}g_0) \ ,$$

for all $g, g_0 \in G$ and $f \in \mathrm{L}^\infty(G; \mathbb{R})$.

Definition 2.1. A *mean* on $\mathrm{L}^\infty(G; \mathbb{R})$ is a continuous linear functional

$$m \colon \mathrm{L}^\infty(G; \mathbb{R}) \to \mathbb{R} \ ,$$

such that $m(f) \geq 0$ whenever $f \geq 0$ and $m(\chi_G) = 1$. We say that G is *amenable* if $\mathrm{L}^\infty(G; \mathbb{R})$ admits a G-invariant mean (i.e. $m(g.f) = m(f)$ for all $g \in G$ and $f \in \mathrm{L}^\infty(G; \mathbb{R})$).

Some examples of amenable groups are given by Abelian, compact and solvable groups. Similarly, extensions of amenable groups by amenable groups are still amenable and the same holds for the inductive limits of amenable groups. A typical example of amenable group is given by any minimal parabolic subgroup P of a Lie group G. Indeed, P is a compact extension of a solvable group and hence amenable.

There are many different ways for characterizing amenability. We recall here the one defined in terms of a certain fixed-point property. More precisely, let E be a Banach G-module, i.e. a separable Banach space with a continuous action of G via linear isometries. The latter condition means that there exists a representation $\pi \colon G \to \mathrm{Isom}(E)$ such that the action map $\theta_\pi \colon G \times E \to E$, $\theta_\pi(g, e) = \pi(g)(e)$ is continuous, where $G \times E$ is endowed with the product topology. Consider now the dual action of G on the dual Banach space E^* with the weak-* topology. Then, G is amenable if and only if there exists a point fixed by G in any G-invariant compact convex subset of the unit ball in E^* [Pie84, Section 1.4, Section 5.4].

This relation between actions and amenable groups leads to the definition of *amenable actions*, which provides a generalization of the notion of amenable groups.

Definition 2.2. Let G be a locally compact second countable group and let (S, μ) be a G-space with a quasi-invariant measure (i.e. it is G-invariant up to a set of measure zero). We say that the action of G on (S, μ) is *amenable*, or equivalently (S, μ) is an *amenable G-space*, if there exists a norm-one G-equivariant projection

$$p \colon \mathrm{L}^\infty(G \times S; \mathbb{R}) \to \mathrm{L}^\infty(S; \mathbb{R}) \ ,$$

which is $\mathrm{L}^\infty(S)$-linear, such that $p(\chi_{G \times S}) = \chi_S$ and $p(f) \geq 0$ whenever f is non-negative.

Since action by amenable groups are amenable, the previous definition extends Definition 2.1. Moreover, the converse also holds: any group acting on a space with finite invariant measure is amenable [Zim84, Proposition 4.3.3]. Before reinterpreting amenability of actions in terms of a fixed-point property, we introduce the definition of measurable cocycle.

Definition 2.3. Let G, H be locally compact second countable groups and let (X, μ) be a standard Borel probability G-space. A *measurable cocycle* (or *Zimmer's cocycle*) is a measurable map $\sigma\colon G \times X \to H$ such that

$$\sigma\colon G \to \mathrm{Meas}(X, H),\ \ g \mapsto \sigma(g, \cdot)\ , \tag{1}$$

is continuous and σ satisfies the following

$$\sigma(g_1 g_2, x) = \sigma(g_1, g_2.x)\sigma(g_2, x)\ ,$$

for every $g_1, g_2 \in G$ and almost every $x \in X$. The notation $g_2.x$ denotes the action of G on X.

Let (S, μ) be a G-space, where μ is (quasi)-invariant. Given a separable Banach space E, we consider a measurable cocycle $\sigma\colon G \times S \to \mathrm{Isom}(E)$. This allows us to define an action on the space $\mathrm{L}^\infty(S, E^*)$ via σ as follows

$$(g.\varphi)(s) \coloneqq \sigma(g, s)^* \varphi(g^{-1}.s)\ ,$$

for every $\varphi \in \mathrm{L}^\infty(S, E^*)$. The notation $\sigma(g, s)^*$ stands for the adjoint of the isometry $\sigma(g, s)$. Denote by E^*_1 the unit ball in E^* and suppose now that for each $s \in S$, we have a compact convex subset $A_s \subset E^*_1$. Assume that A_s varies in a Borel measurable way, that is the set $\{(s, A_s) | s \in S\} \subset S \times E^*_1$ is a Borel subset. Moreover, suppose that the latter set is G-*invariant*:

$$A_{g.s} = \sigma(g, s)^* A_s\ ,$$

for almost every $s \in S$. An *affine G-space over S*, denoted by $\mathrm{Meas}(S, \{A_s\})$, is the set of measurable functions $f\colon S \to E^*_1$ such that $f(s) \in A_s$ for almost every $s \in S$. If we endow $\mathrm{L}^\infty(S, E^*)$ with the weak-* topology, then it is immediate to check that $\mathrm{Meas}(S, \{A_s\})$ is a compact convex subset of $\mathrm{L}^\infty_{\mathrm{w}^*}(S, E^*)$. Moreover, an affine G-space over S is clearly G-invariant with the respect to the dual action induced by σ and it is closed in the unit ball of $\mathrm{L}^\infty_{\mathrm{w}^*}(S, E^*)$.

In this situation, we can characterize amenable actions as follows: G acts amenably on S if and only if every affine G-space over S has a point fixed by the action of G [Zim84, Section 4.3]. In fact, amenable actions do not only characterize amenable groups but also subgroups. Indeed a subgroup $Q \leq G$ is amenable if and only if the G-action on the quotient G/Q is amenable [Zim84, Proposition 4.3.2]. For instance, this result applies to any (minimal) parabolic subgroup of a Lie group. In Section 2.2, we will explain how amenable actions are useful for computing bounded cohomology via suitable resolutions.

Quotients of the form G/Q, where Q is amenable, may be also used for defining the notion of boundary maps.

Definition 2.4. Let G, H be two locally compact second countable groups and suppose that $Q \leq G$ is a closed amenable subgroup. Let (X, μ) be a standard Borel probability G-space and let Y be a measurable H-space. Given a measurable cocycle $\sigma\colon G \times X \to H$, we say that a measurable map $\phi\colon G/Q \times X \to Y$ is *σ-equivariant* if we have

$$\phi(g.\eta, g.x) = \sigma(g,x)\phi(\eta,x) \ ,$$

for every $g \in G$ and almost every $\eta \in G/Q, x \in X$.

A *(generalized) boundary map* for σ is a σ-equivariant measurable map.

The importance of generalized boundary maps in our context is discussed in Section 3.1. There, we will show that they provide a meaningful way for defining a pullback in continuous bounded cohomology.

Since Equation (1) can be suitably interpreted as the Einleberg-MacLane condition on σ for being a Borel 1-cocycle in $\mathrm{Meas}(G, \mathrm{Meas}(X,H))$ (see [FM77],[Zim]), it is natural to define cohomologous cocycles.

Definition 2.5. Let $\sigma\colon G \times X \to H$ be a measurable cocycle and let $f\colon X \to H$ be a measurable map. The *twisted cocycle associated to σ and f* is defined as

$$f.\sigma\colon G \times X \to H, \ (f.\sigma)(g,x) := f(g.x)^{-1}\sigma(g,x)f(x) \ ,$$

for every $g \in G$ and almost every $x \in X$. Two different cocycles $\sigma_1, \sigma_2\colon G \times X \to H$ are *cohomologous* if there exists a measurable function $f\colon X \to H$ such that

$$\sigma_2 = f.\sigma_1 \ .$$

Similarly, we say that σ_1 and σ_2 are *cohomologous modulo a subgroup $C \leq H$* if

$$\sigma_2 = f.\sigma_1 \mod C \ .$$

It is worth mentioning that the generalized boundary map associated to a twisted cocycle can be easily described.

Definition 2.6. Let $\sigma\colon G \times X \to H$ be a measurable cocycle with generalized boundary map $\phi\colon G/Q \times X \to Y$. Given a measurable function $f\colon X \to H$ the *twisted boundary map associated to f and ϕ* is defined as

$$f.\phi\colon G/Q \times X \to Y, \ (f.\phi)(\eta,x) := f(x)^{-1}\phi(\eta,x) \ ,$$

for every $g \in G$ and almost every $\eta \in G/Q, x \in X$.

We conclude the section by recalling how representation theory may suitably sits inside the world of measurable cocycles.

Definition 2.7. Let $\rho\colon G \to H$ be a continuous representation and let (X,μ) be a standard Borel probability G-space. The *cocycle associated to the representation ρ* is defined as

$$\sigma_\rho\colon G \times X \to H, \ \sigma_\rho(g,x) = \rho(g) \ ,$$

for every $g \in G$ and almost every $x \in X$.

This comparison between representations and measurable cocycles will be crucial in Section 3.2 to show that our method to pullback classes along measurable cocycles actually extends the ordinary pullback for representations.

2.2. **Bounded cohomology and its functorial approach.** In this section we are going to recall the definitions and the properties of both continuous and continuous bounded cohomology that we will need in the sequel.

We first introduce continuous (bounded) cohomology via the homogeneous resolution and then, following the work of Burger and Monod [Mon01, BM02], we compute it in terms of strong resolutions by relatively injective modules. We will mainly focus our attention on resolutions given by essentially bounded functions on amenable spaces in the sense of Definition 2.2.

Let G be a locally compact second countable group and let E be a Banach G-module whose G-action is realized via a representation $\pi\colon G \to \mathrm{Isom}(E)$. If we assume that E is the dual of some Banach space, there exists a natural way to endow E with the weak-* topology and hence with associated weak-* Borel structure.

We denote the space of *E-valued continuous functions* by

$$\mathrm{C}_c^\bullet(G;E) \coloneqq \{f\colon G^{\bullet+1} \to E \mid f \text{ is continuous}\}$$

and we define the standard homogeneous coboundary operator as follows

$$\delta^\bullet\colon \mathrm{C}_c^\bullet(G;E) \to \mathrm{C}_c^{\bullet+1}(G;E)$$

$$\delta^\bullet(f)(g_1,\ldots,g_{\bullet+2}) \coloneqq \sum_{j=1}^{\bullet+2}(-1)^{j-1}f(g_1,\ldots,g_{j-1},g_{j+1}\ldots,g_{\bullet+2}) \ .$$

In this way we obtain a complex $(\mathrm{C}_c^\bullet(G;E),\delta^\bullet)$ which is exact and hence it has trivial cohomology. For this reason we are going to restrict the coboundary operator to the subspace of G-invariant functions $\mathrm{C}_c^\bullet(G;E)^G$. More precisely, a continuous function $f\colon G^{\bullet+1} \to E$ is *G-invariant* if $g.f = f$ for every element $g \in G$, where the G-action on the space $\mathrm{C}_c^\bullet(G;E)$ is defined by

$$(g.f)(g_1,\ldots,g_{\bullet+1}) = \pi(g)f(g^{-1}g_1,\ldots,g^{-1}g_{\bullet+1}) \ , \tag{2}$$

for every $g,g_1,\ldots,g_{\bullet+1} \in G$. As said previously, we can restrict the coboundary operator to the subspace of G-invariant functions to obtain the following subcomplex

$$(\mathrm{C}_c^\bullet(G;E)^G,\delta^\bullet) \ .$$

Definition 2.8. The *continuous cohomology* of G with coefficients in E, denoted by $\mathrm{H}_c^\bullet(G;E)$, is the cohomology of the complex $(\mathrm{C}_c^\bullet(G;E)^G,\delta^\bullet)$.

Since E has a Banach structure by assumption, we can use it to define on $\mathrm{C}_c^\bullet(G;E)$ an L^∞-norm given by

$$\|f\|_\infty \coloneqq \sup\{\|f(g_1,\ldots,g_{\bullet+1})\|_E \mid g_1,\ldots,g_{\bullet+1} \in G\} \ ,$$

where $f \in \mathrm{C}_c^\bullet(G;E)$ and $\|\cdot\|_E$ denotes the norm in E. A continuous function is *bounded* if its L^∞-norm is finite. Let $\mathrm{C}_{cb}^\bullet(G;E) \subset \mathrm{C}_c^\bullet(G;E)$ be the subspace of continuous bounded functions. Since the image of a bounded function through the coboundary operator $\delta^\bullet$ is still bounded, we can restrict $\delta^\bullet$ to the space of continuous bounded G-invariant functions $\mathrm{C}_{cb}^\bullet(G;E)^G$ to obtain a complex, that is

$$(\mathrm{C}_{cb}^\bullet(G;E)^G,\delta^\bullet) \ .$$

Definition 2.9. The *continuous bounded cohomology* of G with coefficients in E, denoted by $\mathrm{H}^\bullet_{cb}(G;E)$, is the cohomology of the complex $(\mathrm{C}^\bullet_{cb}(G;E)^G, \delta^\bullet)$.

Notice that the L^∞-norm previously defined on $\mathrm{C}^\bullet_{cb}(G;E)$ canonically determines an L^∞-seminorm in cohomology given by

$$\|f\|_\infty \coloneqq \inf\{\|\psi\|_\infty \mid [\psi] = f\} \ .$$

We say that an isomorphism between seminormed cohomology groups is *isometric* if the corresponding seminorms are preserved.

Beyond the difference determined by the induced seminorm on continuous bounded cohomology, there is usually a gap between continuous cohomology and continuous bounded cohomology that can be detected as follows. Given the natural inclusion

$$i\colon\ \mathrm{C}^\bullet_{cb}(G;E)^G \to \mathrm{C}^\bullet_{c}(G;E)^G \ ,$$

the *comparison map*

$$\mathrm{comp}^\bullet_G\colon\ \mathrm{H}^\bullet_{cb}(G;E) \to \mathrm{H}^\bullet_{c}(G;E)$$

is the induced map by i in cohomology.

We now recall an alternative definition of continuous bounded cohomology of G with coefficients in E in terms of strong resolutions via relatively injective Banach G-modules. Since we will not explicitly work with these technical notions in the sequel, we omit them and we refer the reader to Monod's book [Mon01, Section 4.1 and 7.1].

Given the Banach G-module E, let $(E^\bullet, d^\bullet)$ be a strong resolution of E via relatively injective Banach G-modules. Let us denote by $g.v$ the action of $g \in G$ on an element $v \in E^\bullet$. We define $((E^\bullet)^G, d^\bullet)$ the complex of G-invariant vectors of E with the restriction of the coboundary operator. Following the work by Burger and Monod [BM02, Corollary 1.5.3] (see also [Mon01, Theorem 7.2.1]), one can show that the continuous bounded cohomology of G with E-coefficients can be computed considering the cohomology of the complex $((E^\bullet)^G, d^\bullet)$. Nevertheless, it is worth noticing that the isomorphism $\mathrm{H}^k_{cb}(G;E) \cong \mathrm{H}^k((E^\bullet)^G)$ is not isometric in general. Beyond the obvious case given by $(\mathrm{C}^\bullet_{cb}(G,E)^G, \delta^\bullet)$, we are going to exhibit another crucial example of strong resolution of E given by relatively injective Banach G-modules for which the previous isomorphism is actually isometric.

Let Q be an amenable closed subgroup of G. As already mentioned in Section 2.1, the natural G-action on the quotient G/Q is amenable. We define the complex $(\mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1};E), \delta^\bullet)$ of essentially bounded weak-* measurable functions on $(G/Q)^{\bullet+1}$ together with the standard homogeneous coboundary operator. If we complete the latter complex by adding the inclusion of constant functions $E \to \mathrm{L}^\infty_{\mathrm{w}^*}(G/Q;E)$, we obtain a resolution of E. Endowing $\mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1};E)$ with a structure of Banach G-module induced by the action described in Equation (2), Burger and Monod [BM02, Theorem 1] proved that $(\mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1};E), \delta^\bullet)$ provides a strong resolution of E via relatively injective Banach G-modules. As mentioned above this readily implies that the subcomplex of G-invariant vectors computes the continuous bounded cohomology of G with coefficients in E. A striking result, is that this isomorphism is in fact isometric with respect to the natural seminorms [BM02, Theorem 2].

We will often consider the resolution of E via the Banach G-submodules of alternating cochains. Recall that an essentially bounded weak-* measurable function $f\colon (G/Q)^{\bullet+1} \to E$ is *alternating* if

$$\varepsilon(\tau) f(g_1, \dots, g_{\bullet+1}) = f(g_{\tau(1)}, \dots, g_{\tau(\bullet+1)}) \ ,$$

where $\tau \in S_{\bullet+1}$ is a permutation and ε denotes its sign. As proved by Burger and Monod [BM02, Theorem 2], the resolution of essentially bounded weak-* measurable alternating functions $(\mathrm{L}^\infty_{\mathrm{w}^*,\mathrm{alt}}((G/Q)^{\bullet+1}; E), \delta^\bullet)$ still computes isometrically the continuous bounded cohomology $\mathrm{H}^\bullet_{cb}(G; E)$.

In some cases, it may be convenient to work directly with $\mathcal{B}^\infty((G/Q)^{\bullet+1}; E)$, the space of bounded weak-* measurable functions on $(G/Q)^{\bullet+1}$ with respect to the supremum norm. By considering the homogeneous coboundary operator, we can construct the complex $(\mathcal{B}^\infty((G/Q)^{\bullet+1}; E), \delta^\bullet)$. Adding E to the complex via the inclusion of constant functions, we get a resolution of E which is only strong [BI02, Proposition 2.1]. For that reason we cannot conclude that the cohomology of the subspace of invariant vectors computes the continuous bounded cohomology $\mathrm{H}^\bullet_{cb}(G; E)$. Nevertheless, there always exists a canonical map [BI02, Corollary 2.2]

$$\mathfrak{c}^k\colon\ \mathrm{H}^k(\mathcal{B}^\infty((G/Q)^{\bullet+1}; E)^G) \to \mathrm{H}^k(\mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1}; E)^G) \cong \mathrm{H}^k_{cb}(G; E)$$

for every $k \in \mathbb{N}$. This shows that each bounded weak-* measurable G-invariant function canonically determines a cohomology class in $\mathrm{H}^k_{cb}(G; E)$. The same result still holds in the case of alterning functions.

In Section 3.1, we will tacitly use the previous result for showing that the pullback of a bounded weak-* measurable G-invariant function lies in fact in $\mathrm{L}^\infty_{\mathrm{w}^*}$.

2.3. **Transfer maps.** In this section we briefly recall the notion of *transfer maps*. These maps will be crucial to introduce the notion of multiplicative constant associated to a pair of cocycles. We refer the reader to Monod's book [Mon01] for a broader discussion on this topic. Let G be a locally compact second countable group and let $i\colon L \to G$ be the inclusion of a closed subgroup L into G. By functoriality of continuous bounded cohomology, the inclusion induces a pullback in continuous bounded cohomology

$$\mathrm{H}^\bullet_{cb}(i)\colon\ \mathrm{H}^\bullet_{cb}(G; \mathbb{R}) \to \mathrm{H}^\bullet_{cb}(\Gamma; \mathbb{R}) \ .$$

A transfer map is a map which provides a cohomological left inverse of $\mathrm{H}^\bullet_{cb}(i)$. In order to define it, we assume that $L\backslash G$ admits a G-invariant probability measure μ. For instance, this is the case when L is a lattice of G.

Definition 2.10. We define the *transfer cochain map* to be the following map

$$\widehat{\mathrm{trans}}^\bullet_L\colon\ \mathrm{C}^\bullet_{cb}(G; \mathbb{R})^L \to \mathrm{C}^\bullet_{cb}(G; \mathbb{R})^G$$

$$\widehat{\mathrm{trans}}^\bullet_L(\psi)(g_1, \dots, g_{\bullet+1}) \coloneqq \int_{L\backslash G} \psi(\overline{g}.g_1, \dots, \overline{g}.g_{\bullet+1}) d\mu(\overline{g}) \ ,$$

for every $(g_1, \ldots, g_{\bullet+1}) \in G^{\bullet+1}$ and $\psi \in \mathrm{C}^\bullet_{cb}(G;\mathbb{R})^L$. Here $\overline{g}$ is the equivalence class of g in the quotient $L\backslash G$.

Remark 2.11. Notice that in the definition of the transfer map we evaluated ψ on the points $\overline{g}.g_1, \ldots \overline{g}.g_\bullet$, which is not precise since the latter are equivalence classes in the quotient. However, the L-invariance of ψ induces a well-defined function on the quotient $L\backslash G$. In order to avoid heavy notation, when we work with transfer maps we will always keep the previous notation.

The *transfer map* $\mathrm{trans}^\bullet_L$ is the map induced in cohomology by $\widehat{\mathrm{trans}}^\bullet_L$

$$\mathrm{trans}^\bullet_L \colon \mathrm{H}^\bullet_{cb}(L;\mathbb{R}) \to \mathrm{H}^\bullet_{cb}(G;\mathbb{R}) \ .$$

Since we defined the transfer map only on the subcomplex of invariant vectors, when we deal with resolutions we cannot use the same expression to compute the cohomological transfer map. However, there exists a standard procedure for adapting the previous construction to resolutions of L^∞-functions on amenable spaces [Mon01]. For the convenience of the reader, we briefly recall here how to perform this modification.

Assume that $P, L \leq G$ are closed subgroups of a locally compact second countable group. As before, suppose that $L\backslash G$ admits a G-invariant probability measure μ. Then, if P is amenable, by Section 2.1 we know that the action of G on the quotient G/P is amenable (this happens for instance when P is a minimal parabolic subgroup of a Lie group G). As mentioned in Section 2.2 we know that the resolution of L-invariant essentially bounded functions on G/P computes the continuous bounded cohomology $\mathrm{H}^\bullet_{cb}(L;\mathbb{R})$. Hence, we can define a suitable version of *transfer map*

$$\mathrm{trans}^\bullet_{G/P} \colon \mathrm{H}^\bullet_{cb}(L;\mathbb{R}) \to \mathrm{H}^\bullet_{cb}(G;\mathbb{R})$$

as the one induced in cohomology by the following cochain map

$$\widehat{\mathrm{trans}}^\bullet_{G/P} \colon \mathrm{L}^\infty((G/P)^{\bullet+1};\mathbb{R})^L \to \mathrm{L}^\infty((G/P)^{\bullet+1};\mathbb{R})^G$$

$$\widehat{\mathrm{trans}}^\bullet_{G/P}(\psi)(\xi_1, \ldots, \xi_{\bullet+1}) := \int_{L\backslash G} \psi(\overline{g}.\xi_1, \ldots, \overline{g}.\xi_{\bullet+1}) d\mu(\overline{g}) \ ,$$

for almost all $(\xi_1, \ldots, \xi_{\bullet+1}) \in (G/P)^{\bullet+1}$ and $\psi \in \mathrm{L}^\infty((G/P)^{\bullet+1};\mathbb{R})^L$.

The relation between the two transfer maps $\mathrm{trans}^\bullet_L$ and $\mathrm{trans}^\bullet_{G/P}$ can be explained through the following commutative diagram [Mon01, Section 8.6]

$$\begin{array}{ccc} \mathrm{H}^\bullet_{cb}(L;\mathbb{R}) & \xrightarrow{\mathrm{trans}^\bullet_L} & \mathrm{H}^\bullet_{cb}(G;\mathbb{R}) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{H}^\bullet_{cb}(L;\mathbb{R}) & \xrightarrow[\mathrm{trans}^\bullet_{G/P}]{} & \mathrm{H}^\bullet_{cb}(G;\mathbb{R}) \ , \end{array} \tag{3}$$

where the vertical arrows are the canonical isomorphisms obtained by extending the identity $\mathbb{R} \to \mathbb{R}$ to the complex of continous bounded and essentially bounded functions, respectively.

2.4. **Bounded differential forms.** Let G be a semisimple rank-one Lie group of non-compact type and denote by X the associated Riemannian symmetric space. Since the boundary at infinity $\partial_\infty X$ is an amenable G-space, given any closed subgroup $L \leq G$, the cohomology of the complex $(\mathrm{L}^\infty_{\mathrm{alt}}(\partial_\infty X^{\bullet+1}); \mathbb{R})^L, \delta^\bullet)$ naturally computes the continuous bounded cohomology groups $\mathrm{H}^\bullet_{cb}(L;\mathbb{R})$ (see Section 2.2). Here, we are going to describe a cochain map between the complex of essentially bounded alternating functions on $\partial_\infty X$ and the space of smooth bounded differential forms $\Omega^\bullet_\infty(X)$. This cochain map will be useful in the definition of the Cartan invariant of measurable cocycles associated to complex hyperbolic lattices (Section 5).

Denote by $\Omega^k(X)$ the space of smooth differential forms on X. Recall that, given any $\omega \in \Omega^k(X)$, the natural Riemannian structure on X determines a norm function as follows

$$x \mapsto \|\omega_x\| \coloneqq \sup_{u_1,\ldots,u_k \in T_xX} \omega_x(u_1,\ldots,u_k) \ ,$$

where $u_1,\ldots,u_k \in T_xX$ is an orthonormal k-frame in the tangent space T_xX.

The previous norm allows us to define the space $\Omega^k_\infty(X)$ of smooth bounded differential forms as follows. Let $\omega \in \Omega^k(X)$ be a smooth k-form, where $k \geq 0$. We say that ω is *bounded* if both the functions

$$x \mapsto \|\omega_x\|, \quad x \mapsto \|(d\omega)_x\| \ ,$$

are elements of $\mathrm{L}^\infty(X;\mathbb{R})$, where we considered the natural G-invariant measure on the symmetric space X. By definition the differential operator $d^\bullet$ restricts to the subcomplex of smooth bounded differential forms $\Omega^\bullet_\infty(X)$ and hence we have a well-defined cohomology $\mathrm{H}^\bullet(\Omega^\bullet_\infty(X)^L)$ computed on the L-invariant subcomplex.

Burger and Iozzi [BI07a, Lemma 3.3] proved that for each $k \in \mathbb{N}$ there exists a G-equivariant cochain map

$$\widehat{\delta}^\bullet_\infty \colon \mathrm{L}^\infty_{\mathrm{alt}}((\partial_\infty X)^{\bullet+1};\mathbb{R}) \to \Omega^\bullet_\infty(X) \ ,$$

whose norm is bounded by a suitable power of the volume entropy of X. By restricting this map to the subcomplexes of L-invariants of both $\mathrm{L}^\infty_{\mathrm{alt}}((\partial_\infty X)^{\bullet+1};\mathbb{R})$ and $\Omega^\bullet_\infty(X)$ we obtain the following well-defined map in cohomology

$$\delta^\bullet_\infty \colon \mathrm{H}^\bullet_{cb}(L;\mathbb{R}) \to \mathrm{H}^\bullet(\Omega^\bullet_\infty(X)^L) \ .$$

Let us now specialize to the case in which $L = \Gamma \leq G$ is a lattice. If $1 \leq p \leq \infty$, we denote by $\Omega^k_p(X)^\Gamma$ the space of Γ-invariant differential k-forms ω such that both the functions

$$x \mapsto \|\omega_x\|, \quad x \mapsto \|(d\omega)_x\| \ ,$$

are elements of $\mathrm{L}^p(M;\mathbb{R})$, where M is the quotient manifold $\Gamma\backslash X$ endowed with the natural G-invariant measure class. Note that we are tacitly using the invariance of the form ω to obtain a differential form on M. As above, the restriction of the differential operator $d^\bullet$ to $\Omega^\bullet_p(X)^\Gamma$ promotes it to a cochain complex. Let $\mathrm{H}^\bullet(\Omega^\bullet_p(X)^\Gamma)$ be its cohomology. Since for every $1 \leq p \leq \infty$ we have a natural inclusion

$$\Omega^\bullet_\infty(X)^\Gamma \to \Omega^\bullet_p(X)^\Gamma \ ,$$

we can precompose it with $\widehat{\delta}^\bullet_\infty$ and get a cochain map

$$\widehat{\delta}^\bullet_p \colon \mathrm{L}^\infty_{\mathrm{alt}}((\partial_\infty X)^{\bullet+1};\mathbb{R})^\Gamma \to \Omega^\bullet_p(X)^\Gamma \ .$$

The latter cochain map induces in cohomology the following map [BI07a, Section 3]

$$\delta^\bullet_p \colon \mathrm{H}^\bullet_b(\Gamma;\mathbb{R}) \to \mathrm{H}^\bullet(\Omega^\bullet_p(X)^\Gamma) \ .$$

Since for G-invariant differential forms the norm does not depend on the chosen point, we have $\Omega^\bullet(X)^G \subset \Omega^\bullet_\infty(X)$. The latter result together with the finiteness of the volume of M provides a well-defined restriction map

$$\Omega^\bullet(X)^G \to \Omega^\bullet_p(X)^\Gamma \ ,$$

which admits a left-inverse $j^\bullet_p$ defined as follows

$$j^\bullet_p(\omega) := \int_{\Gamma\backslash G} (L^*_g\omega)\mu(\overline{g}) \ ,$$

for any $\omega \in \Omega^\bullet_p(X)^\Gamma$. Here μ is the G-invariant probability measure induced on the quotient $\Gamma\backslash G$, L_g is the left translation by $g \in G$ and $\overline{g}$ is the class of the element g in the quotient. A remarkable fact proved by Burger and Iozzi [BI07a, Proposition 3.2] is that we have the following commutative diagram

(4)
$$\begin{array}{ccccc}
\mathrm{H}^\bullet_b(\Gamma;\mathbb{R}) & \xrightarrow{\ \delta^\bullet_p\ } & \mathrm{H}^\bullet(\Omega^\bullet_p(X)^\Gamma) & \xrightarrow{\ j^\bullet_p\ } & \Omega^\bullet(X)^G \\
\Big\downarrow{\scriptstyle \mathrm{trans}^\bullet_\Gamma} & & & & \Big\downarrow{\scriptstyle \mathcal{V}\mathcal{E}^\bullet_G} \\
\mathrm{H}^\bullet_{cb}(G;\mathbb{R}) & & \xrightarrow[\ \mathrm{comp}^\bullet_G\ ]{} & & \mathrm{H}^\bullet_c(G;\mathbb{R}) \ ,
\end{array}$$

where $\mathrm{trans}^\bullet_\Gamma$ is the transfer map introduced in Section 2.3 and $\mathcal{V}\mathcal{E}^\bullet$ is the Van Est isomorphism [Gui80, Corollary 7.2]. Notice that, since by Cartan's Lemma every G-invariant differential forms is closed, the complex $\Omega^\bullet(X)^G$ coincides with its cohomology. The importance of Diagram (4) will be clear in the proof of Proposition 5.5 in order to express the Cartan invariant of measurable cocycles as a multiplicative constant.

We conclude this section by recalling that, when X is a Hermitian symmetric space and Γ is a lattice of the isometry group $G = \mathrm{Isom}^\circ(X)$, then $j^\bullet_2$ is an orthogonal projector on the subspace $\mathbb{R}\,\omega_M$ generated by the Kähler form ω_M (see [BI07a, Lemma 5.2]). Notice that, since the Kähler form ω_M is actually induced by the natural Kähler form ω_X on X, then $\mathbb{R}\,\omega_M$ can be identified with the space $\Omega(X)^G$.

2.5. **Fibered products and resolution of L^∞-coefficients.** In this section we are going to define the notion of fibered products. Fibered products are spaces that will be needed in the sequel in order to construct a resolution which computes continuous bounded cohomology groups with coefficients in suitable L^∞-spaces. This will provide the main formula (see Proposition 4.10). Our presentation of fibered products mainly follows Burger and Iozzi's approach [BI09, Section 4.1.1, 4.1.2].

Let G be a locally compact second countable group and let $P \leq H \leq G$ be two closed subgroups of G. We denote by $p\colon G/P \to G/H$ the canonical projection. With the help of the latter map, for every $n \geq 1$ we can define the *n-fold fibered product* $(G/P)^n_f$ as the closed subset of $(G/P)^n$ given by

$$(G/P)^n_f := \{(\xi_1, \ldots, \xi_n) \in (G/P)^n \,|\, p(\xi_1) = \ldots = p(\xi_n)\} \ ,$$

and we set $(G/P)^n_f = G/H$ for the value $n = 0$. Since p is a G-equivariant map by definition, it easily follows that the diagonal action of G on $(G/P)^n$ leaves invariant the subset $(G/P)^n_f$. In this way we get a canonical projection

$$p_n\colon (G/P)^n_f \to G/H \ ,$$

whose fiber may be homeomorphically identified with H/P.

We now present an alternative description of the space $(G/P)^n_f$ that we will need in the sequel. Notice first that the product space $G \times (H/P)^n$ comes with a natural right H-action given by

$$h.(g, \xi_1, \ldots, \xi_n) := (gh, h^{-1}\xi_1, \ldots, h^{-1}\xi_n) \ , \tag{5}$$

for every $g \in G$, $\xi_1, \ldots, \xi_n \in H/P$ and $h \in H$. The inclusion $H \leq G$ as subgroup allows us to interpret the quotient H/P as a subspace of G/P. Hence, we get a map

$$q_n\colon G \times (H/P)^n \to (G/P)^n_f, \quad q_n(g, \xi_1, \ldots, \xi_n) := (g\xi_1, \ldots, g\xi_n) \ .$$

If we consider the right G-action on the first component of $G \times (H/P)^n$ and the diagonal G-action on $(G/P)^n_f$, one can check that the map above is well-defined, surjective, G-equivariant. Moreover, it is H-invariant with respect to the natural H-action on $G \times (H/P)^n$ and hence induces a map

$$\overline{q}_n\colon (G \times (H/P)^n)/H \to (G/P)^n_f$$

which is a homeomorphism. In this way we can think of $(G/P)^n_f$ as quotient space.

Denote by μ and ν two Borel probability measures on G and on H/P, respectively. Assume that μ lies in the same measure class of the Haar measure on G and ν is contained in the H-invariant measure class of H/P. We can consider the pushforward measure $\nu_n := (q_n)_*(\mu \times \nu^n)$ induced by the projection map q_n. In this way we get a Borel probability measure ν_n on $(G/P)^n_f$ whose class is G-invariant (by the G-equivariance of q_n). Hence we are allowed to speak about the G-modules $\mathrm{L}^\infty((G/P)^n_f)$ of essentially bounded functions on $(G/P)^n_f$. Here and in the sequel, we will often omit the coefficients of maps in L^∞ if they are real.

In order to construct a cochain complex $(\mathrm{L}^\infty(G/P)^\bullet_f, d^\bullet)$, we have to define suitable coboundary operators

$$d^n\colon \mathrm{L}^\infty((G/P)^n_f) \to \mathrm{L}^\infty((G/P)^{n+1}_f)$$

for $n \geq 0$. For every $n \geq 1$ and $i \in \{1, \ldots, n\}$, we can define the map

$$p_{n,i}\colon (G/P)^{n+1}_f \to (G/P)^n_f, \ p_{n,i}(x_1, \ldots, x_{n+1}) := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \ . \tag{6}$$

Set now

$$d^0\colon \mathrm{L}^\infty(G/H) \to \mathrm{L}^\infty((G/P)_f), \ d^0(f)(\xi) := f(p(\xi)),$$

and for $n \geq 1$ define

$$d^n \colon \mathrm{L}^\infty((G/P)^n_f) \to \mathrm{L}^\infty((G/P)^{n+1}_f) \ ,$$

$$d^n f(\xi_1, \ldots, \xi_{n+1}) \coloneqq \sum_{i=1}^{n+1} (-1)^{i-1} f(p_{n,i}(\xi_1, \ldots, \xi_{n+1})) \ .$$

Since we have $(p_{n,i})_*(\nu_{n+1}) = \nu_n$, the operator d^n is actually a well-defined map between L^∞-spaces. As showed by Burger and Iozzi [BI09, Proposition 4.1], for every closed subgroup $L \leq G$ the cochain complex $(\mathrm{L}^\infty((G/P)^\bullet_f), d^\bullet)$ is a strong resolution of the L-module $\mathrm{L}^\infty(G/H)$ via Banach L-modules. Moreover, when P is amenable and $n \geq 1$, the G-action on the fibered product $(G/P)^n_f$ is amenable in the sense of Definition 2.2. Hence, the L-modules $\mathrm{L}^\infty((G/P)^n_f)$ are relatively injective, for $n \geq 1$. As explained in Section 2.2, the restriction of the coboundary operators to the L-invariant submodules provides a subcomplex $(\mathrm{L}^\infty((G/P)^\bullet_f)^L, d^\bullet)$, whose cohomology computes the bounded cohomology groups of L with coefficients into $\mathrm{L}^\infty(G/H)$, that is

$$\mathrm{H}^n(\mathrm{L}^\infty((G/P)^\bullet_f)^L) \cong \mathrm{H}^n_{cb}(L; \mathrm{L}^\infty(G/H)) \ . \tag{7}$$

Remark 2.12. It is worth noticing that when $G = H$ the fibered product $(G/P)^n_f$ reduces to the standard cartesian product $(G/P)^n$. In the same way the complex of essentially bounded functions on the fibered products $(G/P)^\bullet_f$ reduces to the complex $(\mathrm{L}^\infty((G/P)^\bullet), \delta^\bullet)$, where $\delta^\bullet$ denotes the usual coboundary operator. Moreover, if we assume that the group P is amenable, Equation (7) implies that the restriction to L-invariant cochains of the complex $(\mathrm{L}^\infty((G/P)^\bullet), d^\bullet)$ computes the continuous bounded cohomology groups $\mathrm{H}^\bullet_{cb}(L; \mathbb{R})$ with trivial coefficients (see Section 2.2).

The previous construction allows us to explicitly implement transfer maps with L^∞-coefficients. Assume that P is amenable so that the isomorphism stated in Equation (7) holds for every $n \in \mathbb{N}$. Given $\psi \in \mathrm{L}^\infty((G/P)^\bullet_f)^L$ with $\bullet \geq 1$, we can consider the following map

$$\widehat{\tau}^\bullet_{G/P} \colon \mathrm{L}^\infty((G/P)^{\bullet+1}_f)^L \to \mathrm{L}^\infty((G/P)^{\bullet+1}_f)^G \tag{8}$$

$$\widehat{\tau}^\bullet_{G/P}(\psi)(\xi_1, \ldots, \xi_{\bullet+1}) \coloneqq \int_{L\backslash G} \psi(\overline{g}.\xi_1, \ldots, \overline{g}.\xi_{\bullet+1}) d\mu(\overline{g}) \ ,$$

where $\xi_1, \ldots, \xi_{\bullet+1} \in (G/P)^{\bullet+1}_f$ and μ is a probability measure in the measure class of the G-invariant measure of $L\backslash G$. It is easy to check that $\widehat{\tau}^\bullet_{G/P}$ is a cochain map which induces a well-defined map in cohomology

$$\tau^\bullet_{G/P} \colon \mathrm{H}^\bullet_{cb}(L; \mathrm{L}^\infty(G/H)) \to \mathrm{H}^\bullet_{cb}(G; \mathrm{L}^\infty(G/H)) \ .$$

We will refer in the sequel to the previous map as the *transfer map with coefficients*. A shown by Burger and Iozzi [BI09, Lemma 4.4], the above map fits in the following commutative diagram

$$\begin{array}{ccc} \mathrm{H}^{\bullet}_{cb}(L;\mathbb{R}) & \xrightarrow{\ \mathrm{trans}^{\bullet}_{L}\ } & \mathrm{H}^{\bullet}_{cb}(G;\mathbb{R}) \\ \Big\downarrow{\scriptstyle \mathrm{H}^{\bullet}_{cb}(\kappa_L)} & & \Big\downarrow{\scriptstyle \mathrm{H}^{\bullet}_{cb}(\kappa_G)} \\ \mathrm{H}^{\bullet}_{cb}(L;\mathrm{L}^{\infty}(G/H)) & \xrightarrow[\ \tau^{\bullet}_{G/P}\ ]{} & \mathrm{H}^{\bullet}_{cb}(G;\mathrm{L}^{\infty}(G/H)) \end{array} \tag{9}$$

where $\mathrm{trans}^{\bullet}_{L}$ is the transfer map defined in Section 2.3 and $\mathrm{H}^{\bullet}_{cb}(\kappa_L)\colon \mathrm{H}^{\bullet}_{cb}(L;\mathbb{R}) \to \mathrm{H}^{\bullet}(L;\mathrm{L}^{\infty}(G/H))$ is the map induced by the inclusion of coefficients $\kappa\colon \mathbb{R} \to \mathrm{L}^{\infty}(G/H)$ given by constant functions (the same also holds for the map $\mathrm{H}^{\bullet}_{cb}(\kappa_G)$).

3. Pullback maps, multiplicative constants and maximal measurable cocycles

The main goal of this section is to define a pullback of classes in bounded cohomology via generalized boundary maps. This general procedure will be specialized to several different examples (see for instance [Sava, MS]). Thanks to this construction we will be able to extend Burger and Iozzi's *easy formula* for representations [BI09, Proposition 2.44] to the wider setting of measurable cocycles. This formula will allow us to introduce the notion of multiplicative constants. Under suitable assumptions, we will show that these numerical invariants have bounded absolute value and when this upper bound is attained, we can deduce some rigidity properties of the associated measurable cocycles. More precisely, we will show that maximal cocycles may be trivialized.

Setup 3.1. *Along this section we assume the following:*

- *G, G' are two locally compact second countable groups;*
- *G' acts measurably on a measurable space Y;*
- *L, Q are closed subgroups of G such that Q is amenable and the quotient $L\backslash G$ admits a G-invariant probability measure μ;*
- *(X, μ_X) is a standard Borel probability L-space;*
- *$\sigma\colon L \times X \to G'$ is a measurable cocycle with an essentially unique generalized boundary map $\phi\colon G/Q \times X \to Y$.*

3.1. Pullback along generalized boundary maps.

In this section we are going to introduce a way to pullback cocycles along generalized boundary maps. This construction takes inspiration from a work of Bader, Furman and Sauer [BFS13, Proposition 4.2] which was suitably extended by the two authors in several different directions, see for instance [Sava, Proposition 3.1] and [MS, Section 4]. Here, we propose to adapt the previous construction to Setup 3.1, providing the proof of this principle in this wider setting. Thanks to the generality of our assumptions we get two advantages. First, we can finally treat the study of maximal cocycles in a unified theory (see the current section and Sections 3.4, 3.5). Secondly, this general setting provides a suitable setup for dealing later with fibered products. This will allow us both to prove the fibered multiplicative formula (Proposition 4.10) and to study the Cartan invariant of measurable cocycles (Proposition 4).

Assume now to be in the situation of Setup 3.1. Given a measurable cocycle $\sigma\colon L\times X \to G'$ with an essentially unique generalized boundary map $\phi\colon G/Q\times X \to Y$, we would like to *pullback* a cocycle $\psi \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ along ϕ, providing a new cocycle in the space $\mathrm{L}^\infty((G/Q)^{\bullet+1};\mathbb{R})^L$. To this end, we recall that the Banach space $\mathrm{L}^\infty(X)$ has a natural L-module structure given by:

$$\gamma.f = f(\gamma^{-1}.x) \ ,$$

where we assume that $\mathbb{R}$ is endowed with the trivial Γ-action.

We are now ready to define the *pullback* along a generalized boundary map ϕ.

Definition 3.2. In the assumption of Setup 3.1, we define the $\mathrm{L}^\infty(X)$*-valued pullback along* ϕ as the following cochain map

$$\mathrm{C}^\bullet(\phi)\colon \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'} \to \mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1};\mathrm{L}^\infty(X))^L$$
$$\mathrm{C}^\bullet(\phi)(\psi)(\eta_1,\ldots,\eta_{\bullet+1}) \coloneqq (x\mapsto \psi(\phi(\eta_1,x),\ldots,\phi(\eta_{\bullet+1},x))) \ ,$$

where $\psi \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$, $\eta_1,\ldots,\eta_{\bullet+1}\in G/Q$ and $x\in X$.

We begin by showing that $\mathrm{C}^\bullet(\phi)$ is a well-defined cochain map (compare with [MS, Lemma 4.2]).

Lemma 3.3. *The map* $\mathrm{C}^\bullet(\phi)$ *is a well-defined norm non-increasing cochain map.*

Proof. The map $\mathrm{C}^\bullet(\phi)$ is norm non-increasing since it is a pullback. Moreover, it is easy to check that it is a cochain map. Let us show that $\mathrm{C}^\bullet(\phi)$ is well defined. More precisely, we have to show that for every $\psi\in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$, the cocycle $\mathrm{C}^\bullet(\phi)(\psi)$ is L-invariant. First of all, notice that we can identify

$$\mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1};\mathrm{L}^\infty(X))^L \cong \mathrm{L}^\infty((G/Q)^{\bullet+1}\times X)^L \ ,$$

where the latter space is endowed with its natural diagonal L-action. Let $x\in X$, $\gamma\in L$ and $\eta_1,\ldots,\eta_{\bullet+1}\in G/Q$, we have that

$$\begin{aligned}\gamma\cdot \mathrm{C}^\bullet(\phi)(\psi)(\eta_1,\ldots,\eta_{\bullet+1})(x) &= \mathrm{C}^\bullet(\phi)(\psi)(\gamma^{-1}.\eta_1,\ldots,\gamma^{-1}.\eta_{\bullet+1},\gamma^{-1}.x) =\\ &= \psi(\phi(\gamma^{-1}.\eta_1,\gamma^{-1}.x),\ldots,\phi(\gamma^{-1}.\eta_{\bullet+1},\gamma^{-1}.x)) =\\ &= \psi(\sigma(\gamma^{-1},x)\phi(\eta_1,x),\ldots,\sigma(\gamma^{-1},x)\phi(\eta_{\bullet+1},x)) =\\ &= \psi(\phi(\eta_1,x),\ldots,\phi(\eta_{\bullet+1},x))\\ &= \mathrm{C}^\bullet(\phi)(\psi)(\eta_1,\ldots,\eta_{\bullet+1})(x) \ ,\end{aligned}$$

where in the first line we used the definition of diagonal action, in the second line we applied the definition of generalized boundary map and the last line comes from the G'-invariance of ψ. □

Remark 3.4. Recall that when we deal with real coefficient there is no difference between L^∞-functions and the $\mathrm{L}^\infty_{\mathrm{w}^*}$-ones. For this reason, in the previous proof we could identify the two spaces of functions.

As explained above our final goal is to pullback a cocycle $\psi \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ along ϕ in such a way that we get a new cocycle in $\mathrm{L}^\infty((G/Q)^{\bullet+1};\mathbb{R})^L$. To that end we will compose the $\mathrm{L}^\infty(X)$-pullback along ϕ with the integration map (compare with [BFS13, Sava, MS]).

Definition 3.5. We define the *integration map* $\mathrm{I}^\bullet_X$ as the following cochain map

$$\mathrm{I}^\bullet_X \colon \mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1};\mathrm{L}^\infty(X))^L \to \mathrm{L}^\infty((G/Q)^{\bullet+1};\mathbb{R})^L$$
$$\mathrm{I}^\bullet_X(\psi)(\eta_1,\dots,\eta_{\bullet+1}) \coloneqq \int_X \psi(\eta_1,\dots,\eta_{\bullet+1})(x)d\mu_X(x) \ ,$$

where $\psi \in \mathrm{L}^\infty((G/Q)^{\bullet+1};\mathrm{L}^\infty(X))^L$, $\eta_1,\dots,\eta_{\bullet+1} \in G/Q$ and μ_X is the probabilty measure on the standard Borel L-space X.

The fact that the integration map is a well-defined norm non-increasing cochain map is the content of the following lemma (compare [MS, Lemma 4.2]).

Lemma 3.6. *The integration map* $\mathrm{I}^\bullet_X$ *is a well-defined norm non-increasing cochain map.*

Proof. Given a cocycle $\psi \in \mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1};\mathrm{L}^\infty(X))^L$, it is easy to check that $\mathrm{I}^\bullet_X(\psi)$ is L-invariant. Indeed, given $\eta_1,\dots,\eta_{\bullet+1} \in G/Q$ and $\gamma \in L$, we have

$$\begin{aligned}\gamma.\, \mathrm{I}^\bullet_X(\psi)(\eta_1,\dots,\eta_{\bullet+1}) &= \int_X \psi(\gamma^{-1}.\eta_1,\dots,\gamma^{-1}.\eta_{\bullet+1})(x)d\mu_X(x) = \\ &= \int_X \psi(\eta_1,\dots,\eta_{\bullet+1})(\gamma.x)d\mu_X(x) \\ &= \int_X \psi(\eta_1,\dots,\eta_{\bullet+1})(x)d\mu_X(x) = \mathrm{I}^\bullet_X(\psi)(\eta_1,\dots,\eta_{\bullet+1}) \ ,\end{aligned}$$

where we used the fact that both ψ and μ_X are L-invariant.

Since it is immediate to check that the integration map is also a norm non-increasing cochain map, the statement is proved. □

Remark 3.7. As already noticed by the authors [MS, Remarks 4.5 and 5.3], the previous construction via integration is only possible working with bounded cocycles. Indeed, there is no hope to extend this map to the case of unbounded cochains.

We are now ready to define the *pullback map along* ϕ.

Definition 3.8. In the situation of Setup 3.1, the *pullback map along* ϕ is the cochain map

$$\mathrm{C}^\bullet(\Phi^X) \colon \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'} \to \mathrm{L}^\infty((G/Q)^{\bullet+1};\mathbb{R})^L$$
$$\mathrm{C}^\bullet(\Phi^X) \coloneqq \mathrm{I}^\bullet_X \circ \mathrm{C}^\bullet(\phi) \ .$$

Remark 3.9. It is worth mentioning that the pullback along ϕ can be restricted to the subcomplexes of alternating cochains.

The fact that the pullback map just defined induces a well-defined map in cohomology is the content of the following:

Proposition 3.10. *The pullback map* $\mathrm{C}^\bullet(\Phi^X)$ *is a norm non-increasing cochain map, hence it induces a well-defined map*

$$\mathrm{H}^\bullet(\Phi^X)\colon \mathrm{H}^\bullet(\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}) \to \mathrm{H}^\bullet_{cb}(L;\mathbb{R}),\ \mathrm{H}^\bullet(\Phi^X)([\psi]) := [\mathrm{C}^\bullet(\Phi^X)(\psi)]\ .$$

Proof. As a consequence of both Lemmas 3.3 and 3.6 it follows immediately that the pullback $\mathrm{C}^\bullet(\Phi^X)$ is a norm non-increasing cochain map, being the composition of two such maps $\mathrm{C}^\bullet(\phi)$ and $\mathrm{I}^\bullet_X$.

Recall that Q is an amenable group by Setup 3.1, and so G acts amenably on the quotient G/Q. This property is inherited by L being a closed subgroup of G. Since the subcomplex of L-invariant essentially bounded functions $\mathrm{L}^\infty((G/Q)^{\bullet+1};\mathbb{R})^L$ computes the continuous bounded cohomology $\mathrm{H}^\bullet_{cb}(L;\mathbb{R})$, we get the thesis. □

Remark 3.11. Note that in full generality we might construct a pullback map in cohomology using any measurable σ-equivariant map $\phi\colon S\times X\to Y$, where S is any amenable L-space.

We conclude this section by showing that given two cohomologous measurable cocyles, then they produce the same pullback (compare with [MS, Proposition 5.7 and Proposition 7.5]):

Proposition 3.12. *In the situation of Setup 3.1, let* $f.\sigma\colon L\times X\to G'$ *be cocycle cohomologous to* σ *with respect to a measurable map* $f\colon X\to G'$*. Then, for all* $\psi\in\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$*, we have*

$$\mathrm{C}^\bullet(\Phi^X)(\psi)=\mathrm{C}^\bullet(f.\Phi^X)(\psi)\ ,$$

where $\mathrm{C}^\bullet(\Phi^X)$ *and* $\mathrm{C}^\bullet(f.\Phi^X)$ *denote the pullback maps along the associated boundary maps* ϕ *and* $f.\phi$*, respectively.*

Proof. Recall by Definitions 2.5 and 2.6 that the boundary map $f.\phi$ associated to $f.\sigma$ is given by

$$f.\phi\colon G/Q\times X\to Y\ ,\quad (f.\phi)(\eta,x)=f^{-1}(x)\phi(\eta,x)\ ,$$

for almost every $\eta\in G/Q$ and $x\in X$. Then, we have

$$\begin{aligned}\mathrm{C}^\bullet(f.\Phi^X)(\psi)(\eta_1,\ldots,\eta_{\bullet+1}) &= \int_X \psi((f.\phi)(\eta_1,x),\ldots,(f.\phi)(\eta_{\bullet+1},x))d\mu_X(x)=\\ &=\int_X \psi(f^{-1}(x)\phi(\eta_1,x),\ldots,f^{-1}(x)\phi(\eta_{\bullet+1},x))d\mu_X(x)=\\ &=\int_X \psi(\phi(\eta_1,x),\ldots,\phi(\eta_{\bullet+1},x))d\mu_X(x)=\\ &=\mathrm{C}^\bullet(\Phi^X)(\psi)(\eta_1,\ldots,\eta_{\bullet+1})\ ,\end{aligned}$$

for almost every $\eta_1,\ldots,\eta_{\bullet+1}\in G/Q$. This finishes the proof. □

Remark 3.13. Sometimes it is natural to consider the G'-module $\mathbb{R}$ with a twisted action. For instance if G' admits a sign homomorphism, we can use it to twist the real coefficients. In that situation the previous equality will be true only up to a sign (see for instance [MS, Proposition 5.7]).

3.2. **Pullback along generalized boundary maps vs. pullback of representations.** Let us assume the setting described in Setup 3.1. By Definition 2.7, given any standard Borel probability L-space (X, μ_X) and any continuous representation $\rho\colon L \to G'$, there exists an associated measurable cocycle $\sigma_\rho\colon L \times X \to G'$ defined by $\sigma_\rho(\gamma, x) = \rho(\gamma)$ for every $\gamma \in L$ and $x \in X$. Let us assume that the representation ρ admits an essentially unique ρ-equivariant measurable map $\varphi\colon G/Q \to Y$. It is immediate to construct a generalized boundary map ϕ associated to σ_ρ as follows

$$\phi\colon G/Q \times X \to Y, \ \ \phi(\eta, x) = \varphi(\eta) \ ,$$

for almost every $\eta \in G/Q$ and $x \in X$.

As explained by Burger and Iozzi [BI02, BI09], the pullback map

$$\mathrm{H}^\bullet_{cb}(\rho)\colon \ \mathrm{H}^\bullet_{cb}(G';\mathbb{R}) \to \mathrm{H}^\bullet_{cb}(L;\mathbb{R})$$

can be implemented using the measurable map φ. We are going to show now that in our setting the pullback associated to ρ via φ agrees with the pullback map along ϕ. This property turns out to be fundamental to coherently extend numerical invariants of representations to measurable cocycles (see [Sava, Proposition 3.4] and [MS, Propositions 5.4 and 7.4]).

Proposition 3.14. *In the situation of Setup 3.1, let $\rho\colon L \to G'$ be a continuous representation which admits an essentially unique ρ-equivariant measurable map $\varphi\colon G/Q \to Y$. Then, we have*

$$\mathrm{C}^\bullet(\Phi^X) = \mathrm{C}^\bullet(\varphi) \ .$$

Proof. Since the boundary map ϕ associated to σ_ρ does not depend on the second variable, it is immediate to check that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'} & \xrightarrow{\mathrm{C}^\bullet(\phi)} & \mathrm{L}^\infty_{\mathrm{w}^*}((G/Q)^{\bullet+1};\mathrm{L}^\infty(X))^L \\ & {\scriptstyle \mathrm{C}^\bullet(\varphi)}\searrow \qquad \swarrow{\scriptstyle \mathrm{I}^\bullet_X} & \\ & \mathrm{L}^\infty((G/Q)^{\bullet+1};\mathbb{R})^L & \end{array} \ .$$

This finishes the proof. ☐

Remark 3.15. Note that the existence of a cocycle of the form $\sigma\colon L \times X \to G'$ required in Setup 3.1 is irrelevant in the previous result.

3.3. **Easy multiplicative formula.** In this section we show how to deduce an easy version of the fibered multiplicative formula stated in Proposition 4.10. Since the latter is rather technical formula and involves fibered products, we prefer to first discuss a simplified version. Moreover, as we will show in Section 3.5, the easier version already contains useful information. We refer the reader to some works of the authors in that direction [Sava, MS].

Assuming Setup 3.1, the existence of a transfer map $\mathrm{trans}^\bullet_{G/Q}$ implies the following:

Proposition 1. *In the situation of Setup 3.1, we have the following results:*

(1) *Let $\psi' \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ be an everywhere defined G'-invariant cocycle. Let $\psi \in \mathrm{L}^\infty((G/Q)^{\bullet+1})^G$ be a G-invariant cocycle. Denote by $\Psi \in \mathrm{H}^\bullet_{cb}(G;\mathbb{R})$ the class of ψ. If we suppose that $\Psi = \mathrm{trans}^\bullet_{G/Q}[\mathrm{C}^\bullet(\Phi^X)(\psi')]$, then we have*

$$\int_{L\backslash G}\int_X \psi'(\phi(\overline{g}.\eta_1,x),\ldots,\phi(\overline{g}.\eta_{\bullet+1},x)d\mu_X(x)d\mu(\overline{g}) = \psi(\eta_1,\ldots,\eta_{\bullet+1}) + \text{cobound.} \ ,$$

for almost every $(\eta_1,\ldots,\eta_{\bullet+1}) \in (G/Q)^{\bullet+1}$.

(2) *Suppose that $\mathrm{H}^\bullet_{cb}(G;\mathbb{R}) \cong \mathbb{R}\,\Psi(=\mathbb{R}[\psi])$. Then, there exists a real constant $\lambda_{\psi',\psi}(\sigma) \in \mathbb{R}$ depending on σ,ψ',ψ such that*

$$\int_{L\backslash G}\int_X \psi'(\phi(\overline{g}.\eta_1,x),\ldots,\phi(\overline{g}.\eta_{\bullet+1},x)d\mu_X(x)d\mu(\overline{g}) = \lambda_{\psi',\psi}(\sigma)\cdot\psi(\eta_1,\ldots,\eta_{\bullet+1}) + \text{cobound.} \ ,$$

for almost every $(\eta_1,\ldots,\eta_{\bullet+1}) \in (G/Q)^{\bullet+1}$.

Proof. Ad 1. Since Setup 3.1 ensures the existence of the transfer map $\mathrm{trans}^\bullet_{G/Q}$, the first formula is easily true.

Ad 2. Since $\mathrm{H}^\bullet_{cb}(G;\mathbb{R})$ is one-dimensional and generated by $\Psi = [\psi]$ as an $\mathbb{R}$-vector space, we have that $\mathrm{trans}^\bullet_{G/Q}[\mathrm{C}^\bullet(\Phi^X)(\psi')]$ must be a real multiple of Ψ. This finishes the proof. □

3.4. **Multiplicative constants and maximal measurable cocycles.** In this section we are going to introduce the notions of multiplicative constant and maximal (measurable) cocycle. The importance of maximal cocycles relies on the fact that they are usually cohomologous to a preferred representation and hence they can be trivialized. This trivialization property suggests a rigid behaviour of measurable cocycles and it is sometimes translated in terms of properties of the ambient group, such as its tautness (see [BFS13, Theorem A]).

In the situation of Setup 3.1, let $\psi' \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ and consider the class $\Psi = [\psi] \in \mathrm{H}^\bullet_{cb}(G;\mathbb{R})$ represented by a bounded Borel cocycle $\psi\colon (G/Q)^{\bullet+1} \to \mathbb{R}$. Assuming that $\mathrm{H}^\bullet_{c}(G;\mathbb{R}) \cong \mathrm{H}^\bullet_{cb}(G;\mathbb{R}) = \mathbb{R}\,\Psi$, Proposition 1 implies that

$$\int_{L\backslash G}\int_X \psi'(\phi(\overline{g}.\eta_1,x),\ldots,\phi(\overline{g}.\eta_{\bullet+1},x))d\mu_X(x)d\mu(\overline{g}) = \\ = \lambda_{\psi',\psi}(\sigma)\psi(\eta_1,\ldots,\eta_{\bullet+1}) + \text{cobound.} \ . \tag{10}$$

Definition 3.16. The real number $\lambda_{\psi',\psi}(\sigma) \in \mathbb{R}$ which appears in Equation (10) is called the *multiplicative constant associated to σ,ψ',ψ.*

Suppose now that in Equation (10) there is no coboundary term. This assumption is in general not too restrictive, since there exist several such examples (for instance when G acts ergodically on $(G/P)^\bullet$ or when $G = \mathrm{PO}(n,1)$ and $G/Q = \mathbb{S}^{n-1}$).

Without the coboundary term, Equation (10) reduces to

$$\int_{L\backslash G}\int_X \psi'(\phi(\overline{g}.\eta_1,x),\ldots,\phi(\overline{g}.\eta_{\bullet+1},x))d\mu_X(x)d\mu(\overline{g}) = \tag{11}$$
$$=\lambda_{\psi',\psi}(\sigma)\psi(\eta_1,\ldots,\eta_{\bullet+1}) \ ,$$

which is equivalent to the following equation in terms of cochains

$$\widehat{\mathrm{trans}}_L^\bullet \circ \mathrm{C}^\bullet(\Phi^X)(\psi') = \lambda_{\psi',\psi}(\sigma)\psi \ .$$

In this context, we are ready to show an explicit upper bound for the multiplicative constant $\lambda_{\psi',\psi}(\sigma)$ in terms of L^∞-norms of the cocycles ψ and ψ'.

Proposition 3.17. *In the situation of Setup 3.1, let $\psi' \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ and consider the class $\Psi = [\psi] \in \mathrm{H}^\bullet_{cb}(G;\mathbb{R})$ represented by a bounded Borel cocycle $\psi\colon (G/Q)^{\bullet+1} \to \mathbb{R}$. Assume that it holds*

$$\widehat{\mathrm{trans}}_L^\bullet \circ \mathrm{C}^\bullet(\Phi^X)(\psi') = \lambda_{\psi',\psi}(\sigma)\psi \ .$$

Then, we have

$$|\lambda_{\psi',\psi}(\sigma)| \leq \frac{\|\psi'\|_\infty}{\|\psi\|_\infty} \ .$$

Proof. By hypothesis we know that

$$\widehat{\mathrm{trans}}_L^\bullet \circ \mathrm{C}^\bullet(\Phi^X)(\psi') = \lambda_{\psi',\psi}(\sigma)\psi \ .$$

If we consider the left-hand side, we can write the following estimate

$$\|\widehat{\mathrm{trans}}_L^\bullet \circ \mathrm{C}^\bullet(\Phi^X)(\psi')\|_\infty \leq \|\psi'\|_\infty \ ,$$

since both $\widehat{\mathrm{trans}}_L^\bullet$ and $\mathrm{C}^\bullet(\Phi^X)$ are norm non-increasing maps (see Proposition 3.10). Hence it follows that

$$|\lambda_{\psi',\psi}(\sigma)|\|\psi\|_\infty \leq \|\psi'\|_\infty \ ,$$

as claimed. □

In virtue of Proposition 3.17 we can give the following:

Definition 3.18. Assuming the situation of Proposition 1, we say that a measurable cocycle $\sigma\colon L\times X \to G'$ is *maximal* if its multiplicative constant $\lambda_{\psi',\psi}(\sigma)$ attains the maximum value, that is

$$\lambda_{\psi',\psi}(\sigma) = \frac{\|\psi'\|_\infty}{\|\psi\|_\infty} \ .$$

The importance of maximal cocycles relies on the fact that they can be usually trivialized, being all cohomologous to a suitable continuous representation $\pi\colon L \to G'$. To explain how one can prove such result, we conside the following:

Setup 3.19. *In the situation of Proposition 1, we also assume the following:*

- *Both ψ' and ψ are everywhere defined cocycles and attain their essential supremum, that is there exist $\eta_1,\ldots,\eta_{\bullet+1} \in G/Q$ and $y_1,\ldots,y_{\bullet+1} \in Y$ such that*

$$\psi'(y_1,\ldots,y_{\bullet+1}) = \|\psi'\|_\infty,\ \ \psi(\eta_1,\ldots,\eta_{\bullet+1}) = \|\psi\|_\infty \ .$$

- *A* maximal *measurable map* $\varphi \colon G/Q \to Y$ *is a map such that*
$$\psi'(\varphi(g\eta_1), \dots, \varphi(g\eta_{\bullet+1})) = \|\psi'\|_\infty \ ,$$
for almost every $g \in G$ *and for* $\eta_1, \dots, \eta_{\bullet+1} \in G/Q$ *such that*
$$\psi(\eta_1, \dots, \eta_{\bullet+1}) = \|\psi\|_\infty \ .$$
- *There exists a continuous representation* $\pi \colon G \to G'$ *and continuous* π*-equivariant map* $\Pi \colon G/Q \to Y$ *which satisfies the following: given a* maximal *measurable map* $\varphi \colon G/Q \to Y$*, there exists a unique element* $g' \in G'$ *such that*
$$\varphi(\eta) = g'\Pi(\eta) \ ,$$
for almost every $\eta \in G/Q$*.*
- *The stabilizer of the map* Π *is trivial, that is the only element* $g' \in G'$ *such that* $g' \circ \Pi = \Pi$ *is the neutral element of* G'*. We denote the previous stabilizer by* $\mathrm{Stab}_{G'}(\Pi)$*.*

Theorem 3.20. *In the situation of Setup 3.19, suppose that* $\sigma \colon L \times X \to G'$ *is maximal. Then, it is cohomologous to the restriction of the representation* π *to* L*. In other words, there exists a measurable map* $f \colon X \to G'$ *such that for all* $\gamma \in L$ *and almost every* $x \in X$*, we have*
$$\pi(\gamma) = f(\gamma.x)^{-1}\sigma(\gamma, x)f(x) \ .$$

Proof. Since the cocycle σ is maximal, by definition the multiplicative constant $\lambda_{\psi',\psi}(\sigma)$ associated to σ, ψ', ψ satisfies
$$\lambda_{\psi',\psi}(\sigma) = \frac{\|\psi'\|_\infty}{\|\psi\|_\infty} \ .$$
If we now substitute this value in Equation (11) we get
$$\int_{L\backslash G}\int_X \psi'(\phi(\overline{g}.\eta_1, x), \dots, \phi(\overline{g}.\eta_{\bullet+1}, x))d\mu_X(x)d\mu(\overline{g}) = \frac{\|\psi'\|_\infty}{\|\psi\|_\infty}\psi(\eta_1, \dots, \eta_{\bullet+1}) \ .$$
Since ψ attains its essential supremum by assumption, there exist $\eta_1, \dots, \eta_{\bullet+1} \in G/Q$ such that
$$\psi(\eta_1, \dots, \eta_{\bullet+1}) = \|\psi\|_\infty \ . \tag{12}$$

Notice that a priori Equation (11) only holds almost everywhere. However, following either Bucher-Burger-Iozzi [BBI13, Section 4] or Monod [Mon15], one can show that the equality actually holds everywhere. This allows us to evaluate it on specific points of $(G/Q)^{\bullet+1}$. Let us pick $\eta_1, \dots, \eta_{\bullet+1} \in G/Q$ such that Equation (12) is satisfied. Then, we have
$$\int_{L\backslash G}\int_X \psi'(\phi(\overline{g}.\eta_1, x), \dots, \phi(\overline{g}.\eta_{\bullet+1}, x))d\mu_X(x)d\mu(\overline{g}) = \|\psi'\|_\infty \ . \tag{13}$$
It follows that
$$\psi'(\phi(\overline{g}.\eta_1, x), \dots, \phi(\overline{g}.\eta_{\bullet+1}, x)) = \|\psi'\|_\infty \ ,$$

for almost every $\overline{g} \in L\backslash G$ and almost every $x \in X$. Since ϕ is σ-equivariant, the previous equality

$$\psi'(\phi(g.\eta_1, x), \ldots, \phi(g.\eta_{\bullet+1}, x)) = \|\psi'\|_\infty \ , \tag{14}$$

actually holds for almost every $g \in G$ and almost every $x \in X$.

We can now set $\phi_x \colon G/Q \to Y$, $\phi_x(\eta) \coloneqq \phi(\eta, x)$. By Equation (14), we have that ϕ_x is maximal for almost every $x \in X$. By the assumptions of Setup 3.19, there must exist an element $f(x) \in G'$ such that

$$\phi_x(\eta) = f(x)\Pi(\eta) \ ,$$

for almost every $\eta \in G/Q$. In this way we obtain a map $f \colon X \to G'$. Since X is a standard Borel probability space and $\hat{\phi} \colon X \to \mathrm{Meas}(G/Q, Y)$ is measurable, the measurability of f follows by a result of Fisher, Morris and Whyte [FMW04, Lemma 2.6].

The thesis now follows applying a result of Bader, Furman and Sauer [BFS13, Proposition 3.2]. More precisely, given $\gamma \in L$, on the one hand we have

$$\phi(\gamma.\eta, \gamma.x) = \sigma(\gamma, x)\phi(\eta, x) = \sigma(\gamma, x)f(x)\Pi(\eta) \ ,$$

and on the other

$$\phi(\gamma.\eta, \gamma.x) = f(\gamma.x)\Pi(\gamma.x) = f(\gamma.x)\pi(\gamma)\Pi(\eta) \ .$$

Notice that in the second equality, we used the π-equivariance of the map Π. The fact that $\mathrm{Stab}_{G'}(\Pi)$ is trivial implies that

$$\pi(\gamma) = f(\gamma.x)^{-1}\sigma(\gamma, x)f(x) \ ,$$

which concludes the proof. □

3.5. **Applications of the easy multiplicative formula.** As we mentioned in the introduction, one of the main results of this paper is to extend the multiplicative formula discussed in Proposition 1 to the case involving fibered products. Indeed, one of the greatest disadvantage of the latter is that it does not provide any useful information about the values of the measurable map ϕ over sets of zero measure. Unfortunately, this is precisely the case when we will investigate rigidity of complex hyperbolic lattices in Section 5. Indeed, we will have to evaluate our measurable map ϕ on a *chain*, i.e. a subset of the boundary of the complex hyperbolic space of zero measure (see Section 5.1 for a precise definition).

Nevertheless, the easy multiplicative formula turns out to be useful in many situations. For convenience of the reader we show here some examples of applications of Proposition 1. More precisely, we show how to deduce rigidity results from the maximality of some multiplicative constants. It is worth mentioning that all the applications presented in the current section have been already proved and discussed elsewhere by the authors [Sava, MS].

It is worth noticing that in all the examples of this section, the hypothesis of Proposition 3.17 are satisfied. Indeed, we are going to present examples in which there are no coboundaries appearing in Proposition 1.

3.5.1. *Mostow Rigidity for measurable cocycles.* Mostow Rigidity Theorem says that in dimension $n \geq 3$ any two homotopy equivalent finite-volume hyperbolic manifolds are isometric [Mos68, Pra73]. Dually, this result can be stated in terms of volume of representations [FK06, BBI13]. Let Γ be a torsion-free non-uniform lattice of $\mathrm{PO}^\circ(n,1)$ and let $\rho\colon \Gamma \to \mathrm{PO}^\circ(n,1)$ be a representation. Bucher, Burger and Iozzi [BBI13] proved that ρ is maximal if and only if it is conjugated to the standard lattice embedding (the same result also holds in the uniform case).

We show here how to extend the rigidity of representations to measurable cocycles via the general theory introduced so far. Let M be a complete finite-volume hyperbolic n-manifold with $n \geq 3$. We denote by $L \leq \mathrm{PO}^\circ(n,1)$ the fundamental group of M, which is by hypothesis a torsion-free non-uniform lattice. According to Setup 3.1, we set $G = G' = \mathrm{PO}(n,1)$. Then, we choose $Y = G/Q = \partial \mathbb{H}^n_{\mathbb{R}} \cong \mathbb{S}^{n-1}$, where Q is a (minimal) parabolic subgroup of G. Finally, we set $\psi = \psi' = \mathrm{Vol}_n$ to be the alternating G-invariant *volume cocycle* defined as

$$\mathrm{Vol}_n\colon (\mathbb{S}^{n-1})^{n+1} \to \mathbb{R}\ ,$$

$$\mathrm{Vol}_n(\xi_1,\ldots,\xi_{n+1}) \coloneqq \text{signed volume of the hyperbolic convex hull of } \xi_1,\ldots,\xi_{n+1}\ .$$

It is worth mentioning that in this case we need to assume that $\mathbb{R}$ is endowed with a G-module structure induced by the twist given by the sign of isometries in $\mathrm{PO}(n,1) = \mathrm{Isom}(\mathbb{H}^n_{\mathbb{R}})$.

Under the previous assumptions, one can define a numerical invariant called *volume of measurable cocycles* [MS]. According to Setup 3.1, let $\sigma\colon L \times X \to \mathrm{PO}^\circ(n,1)$ be a measurable cocycle, where (X,μ_X) is a standard Borel probability L-space. Moreover, σ admits an essentially unique boundary map $\phi\colon \mathbb{S}^{n-1} \times X \to \mathbb{S}^{n-1}$. Since M is a complete finite-volume hyperbolic manifold, it is known that M is homotopy equivalent to its core, i.e. a compact subset of M obtained by removing horocyclic neighbourhoods of the cusps. Then, we have a well-defined isometric isomorphism [MS, Section 3.4]

$$\mathrm{J}^n\colon\ \mathrm{H}^n_b(L;\mathbb{R}) \to \mathrm{H}^n_b(M;\mathbb{R}) \to \mathrm{H}^n_b(M, M\setminus N;\mathbb{R}) \to \mathrm{H}^n_b(N,\partial N;\mathbb{R})\ , \tag{15}$$

where the first map is given by Gromov's Mapping Theorem [Gro82, Iva87, FM], the second one comes from the long exact sequence in bounded cohomology [BBF$^+$14] and the last one is induced by the homotopy equivalence of the pair $(M, M\setminus N) \simeq (N,\partial N)$. We are now able to define the volume of σ.

Definition 3.21. Given a measurable cocycle $\sigma\colon L\times X \to \mathrm{PO}^\circ(n,1)$ with essentially unique boundary map $\phi\colon \mathbb{S}^{n-1}\times X \to \mathbb{S}^{n-1}$, we define the *volume* of σ to be the following numerical invariant

$$\mathrm{Vol}(\sigma) \coloneqq \langle \mathrm{comp}^n \circ \mathrm{J}^n \left[\mathrm{C}^n(\Phi^X)(\mathrm{Vol}_n)\right], [N,\partial N]\rangle\ ,$$

where $\mathrm{C}^n(\Phi^X)$ is the pullback along ϕ and $[N,\partial N]$ denotes the relative fundamental class of N.

As the authors proved [MS, Proposition 1.2], the multiplicative constant in this setting is given by

$$\lambda_{\psi',\psi}(\sigma) = \frac{\mathrm{Vol}(\sigma)}{\mathrm{Vol}(M)} \ .$$

Since the are no coboundaries appearing in Proposition 1 (see [BBI13, Proposition 2]), Proposition 3.17 shows that the following Milnor-Wood inequality holds [MS, Proposition 5.10]

$$|\,\mathrm{Vol}(\sigma)| \leq \mathrm{Vol}(M) \ .$$

Finally, working in this setting, by [BBI13, Proposition 4.7] it is easy to check that also Setup 3.19 is satisfied. Hence, Theorem 3.20 implies that if σ is maximal, then σ is cohomologous to the cocycle associated to the standard lattice embedding $L \to G$. In fact, one can strengthen this result: being maximal is equivalent to being cohomologous to the cocycle associated to the standard lattice embedding (see [MS, Theorem 1.1]).

3.5.2. *Matsumoto's Theorem for measurable cocycle.* Since in dimension $n = 2$ Mostow Rigidity does not hold, we consider now $\mathrm{Homeo}^+(\mathbb{S}^1)$ instead of $\mathrm{PSL}(2,\mathbb{R})$. We briefly recall the content of Mastumoto's Theorem [Mat87]. Consider the Euler class $e \in \mathrm{H}^2(\mathrm{Homeo}^+(\mathbb{S}^1);\mathbb{Z})$ determined by any section $s\colon \mathrm{Homeo}^+(\mathbb{S}^1) \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ of the central extension

$$0 \to \mathbb{Z} \to \mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R}) \to \mathrm{Homeo}^+(\mathbb{S}^1) \to 0 \ ,$$

where $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$ denotes the group of homeomorphisms of $\mathbb{R}$ commuting with integer translations. Let Σ_g be a closed surface of genus $g \geq 2$. It is immediate to check that any representation $\rho\colon \pi_1(\Sigma_g) \to \mathrm{Homeo}^+(\mathbb{S}^1)$ allows to pullback the Euler class $\rho^*(e) \in \mathrm{H}^2(\pi_1(\Sigma_g);\mathbb{Z}) \cong \mathrm{H}^2(\Sigma_g;\mathbb{Z})$, where the latter isomorphism is due to the asphericity of Σ_g. We define the *Euler number of a representation* ρ to be

$$\mathrm{eu}(\rho) \coloneqq \langle \rho^*(e), [\Sigma_g] \rangle \ ,$$

where $[\Sigma_g]$ denotes the fundamental class of Σ_g. The works of both Milnor [Mil58] and Wood [Woo71] show that $|\,\mathrm{eu}(\rho)| \leq |\chi(\Sigma_g)|$. We say that ρ is *maximal* if the previous upper bound is attained. Matsumoto's Theorem states that a maximal representation ρ must be semiconjugated to a hyperbolization. Recall that a semiconjugacy is an element of $\mathrm{Homeo}^+(\mathbb{S}^1)$ induced by a monotone increasing map of $\mathrm{Homeo}_{\mathbb{Z}}(\mathbb{R})$.

Later Matsumoto's result was reproved via bounded cohomology techniques by Iozzi [Ioz02]. We describe here how to extend Matsumoto's result to measurable cocycles via the techniques exposed in Section 3.4.

We adapt Setup 3.1 as follows. Let $G = \mathrm{PSL}(2,\mathbb{R})$ and let $G' = \mathrm{Homeo}(\mathbb{S}^1)^+$. If Σ_g is a closed surface of genus $g \geq 2$, a hyperbolization $\pi_0\colon \pi_1(\Sigma_g) \to \mathrm{PSL}(2,\mathbb{R})$ allows to realize the fundamental group $\pi_1(\Sigma_g)$ as a closed subgroup $L \leq \mathrm{PSL}(2,\mathbb{R})$. Given a (minimal) parabolic subgroup of $Q < G$, we can define $G/Q = Y = \mathbb{S}^1$. Before introducing the cocycles ψ and ψ', we recall the definition of the *orientation* *cocycle*. Once we have fixed an orientation of $\mathbb{S}^1$, the orientation cocycle is the bounded $\mathrm{Homeo}^+(\mathbb{S}^1)$-invariant alternating cocycle on $\mathbb{S}^1$ defined as follows

$$\mathrm{or}\colon (\mathbb{S}^1)^3 \to \mathbb{R}, \quad \mathrm{or}(\xi_0,\xi_1,\xi_2) \coloneqq \begin{cases} +1 & \text{if } (\xi_0,\xi_1,\xi_2) \text{ are positively oriented },\\ -1 & \text{if } (\xi_0,\xi_1,\xi_2) \text{ are negatively oriented },\\ 0 & \text{otherwise }. \end{cases}$$

Given the orientation cocycle, we set $\psi = \psi' = \mathrm{or}$. We can define the Euler number of a measurable cocycle as follows.

Definition 3.22. Let $\sigma\colon L \times X \to \mathrm{Homeo}(\mathbb{S}^1)^+$ be a measurable cocycle with essentially unique generalized boundary map $\phi\colon \mathbb{S}^1 \times X \to \mathbb{S}^1$, where (X,μ_X) is a standard Borel probability L-space. The *Euler number of* σ is defined by

$$\mathrm{eu}(\sigma) = \langle \mathrm{comp}^2 \circ g_{\Sigma_g}\left[\mathrm{C}^2(\Phi^X)(\epsilon)\right], [\Sigma_g]\rangle \ ,$$

where $\mathrm{C}^2(\Phi^X)$ is the pullback along ϕ, $g_{\Sigma_g}\colon \mathrm{H}^2_b(\Gamma;\mathbb{R}) \to \mathrm{H}^2_b(\Sigma_g;\mathbb{R})$ is Gromov's Mapping Theorem isometric isomorphism and $\epsilon = -\mathrm{or}/2 \in \mathcal{B}^\infty((\mathbb{S}^1)^3;\mathbb{R})^{\mathrm{Homeo}^+(\mathbb{S}^1)}$.

According with the definition above, one can prove [MS, Proposition 1.6] that the multiplicative constant is given by

$$\lambda_{\psi',\psi}(\sigma) = \frac{\mathrm{eu}(\sigma)}{\chi(\Sigma_g)} \ .$$

Since the action of L on S^1 is ergodic and the orientation cocycle is alternating, it is easy to check there are no coboundaries appearing in the formula of Proposition 1. This implies that the hypothesis of Proposition 3.17 are satisfied. Therefore, we obtain that also the Euler number of a measurable cocycle satisfies a Milnor-Wood inequality [MS, Proposition 7.7]:

$$|\,\mathrm{eu}(\sigma)| \leq |\chi(\Sigma_g)| \ .$$

Finally, the techniques developed along Section 3.4 leads to the study of *maximal* cocycles. By Theorem 3.20, we can conclude that if σ is maximal, then it is cohomologous to the cocycle associated to a fixed hyperbolization $\pi_0\colon L \to \mathrm{PSL}(2,\mathbb{R})$ via a measurable map $f\colon X \to \mathrm{Homeo}^+(\mathbb{S}^1)$ [MS, Theorem 1.5]. Note the Setup 3.19 is satisfied here as shown for instance by [Ioz02, Proposition 5.5].

3.5.3. *Borel invariant of measurable cocycles.* There also exists a rigidity result similar to the ones described above [Ioz02, BBI13], when we consider representations of a hyperbolic lattice of $\mathrm{PSL}(2,\mathbb{C})$ into the group $\mathrm{PSL}(n,\mathbb{C})$. Indeed, Bucher, Burger and Iozzi [BBI18] defined a numerical invariant for such representations using the combinatorics of the space $\mathscr{F}(n,\mathbb{C})$ of full flags into $\mathbb{C}^n$. More precisely, following a work by Goncharov [Gon93], they showed that there exists a measurable function

$$B_n\colon \mathscr{F}(n,\mathbb{C})^4 \to \mathbb{R} \ ,$$

called *Borel function*, which is a $\mathrm{PSL}(n,\mathbb{C})$-invariant Borel measurable cocycle defined everywhere. Moreover, its absolute value is bounded by $\binom{n+1}{3}\nu_3$. Here ν_3 denotes the hyperbolic volume of the regular ideal tetrahedron in $\mathbb{H}^3_{\mathbb{R}}$. This function agrees with the volume function when $n = 2$, but for $n \geq 3$ its definition becomes quite technical and we refer the reader to [BBI18, Savb] for more details. Following Section 2.2, the Borel function determines a class $\beta_b(n) \in \mathrm{H}^3_{cb}(\mathrm{PSL}(n,\mathbb{C});\mathbb{R})$ which generates the group. Hence, we can pullback it along the map induced by the representation ρ in cohomology and suitably define a numerical invariant $\beta_n(\rho)$, called Borel invariant. Bucher, Burger and Iozzi showed that the absolute value of the Borel invariant is bounded. Moreover, its maximum is attained if and only if the representation ρ is $\mathrm{PSL}(n,\mathbb{C})$-conjugated to the composition of the irreducible representation $\pi_n \colon \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PSL}(n,\mathbb{C})$ with the standard lattice embedding or to the complex conjugated of the latter composition.

We show here how to adapt Setup 3.1 in order to exted the notion of Borel invariant to measurable cocycles. Let $G = \mathrm{PSL}(2,\mathbb{C})$ and $G' = \mathrm{PSL}(n,\mathbb{C})$. Suppose that $L \leq \mathrm{PSL}(2,\mathbb{C})$ is a torsion-free lattice and denote by $M := L\backslash \mathbb{H}^3_{\mathbb{R}}$ the complete hyperbolic 3-manifold of finite-volume associated to L. Let us take any compact core $N \subseteq M$ and we set $Y = \mathscr{F}(n,\mathbb{C})$ and $G/Q = \mathbb{P}^1(\mathbb{C})$, where $Q \leq G$ is any minimal parabolic subgroup of G. Finally, we take $\psi' = B_n$ and $\psi = \mathrm{Vol}_3$, where we identify $\mathbb{S}^2 \cong \mathbb{P}^1(\mathbb{C})$ in order to define properly the volume function Vol_3. With the previous notation, we can give the following:

Definition 3.23. Assume that a measurable cocycle $\sigma \colon L \times X \to \mathrm{PSL}(n,\mathbb{C})$ admits an essentially unique boundary map $\phi\colon \mathbb{P}^1(\mathbb{C}) \times X \to \mathscr{F}(n,\mathbb{C})$, where (X,μ_X) is a standard Borel probability L-space. Then the *Borel invariant associated to* σ is defined by

$$\beta_n(\sigma) = \langle \mathrm{comp}^3 \circ \mathrm{J}^3 \left[\mathrm{C}^3(\Phi^X)(B_n)\right], [N,\partial N] \rangle \ ,$$

where $\mathrm{C}^3(\Phi^X)$ is the pullback along ϕ, J^3 is the composition introduced in Equation (15) and $[N,\partial N]$ denotes the relative fundamental class of the compact core N.

Also in this case one may study the Borel invariant via multiplicative constants. Indeed, as shown by one of the authors [Sava, Proposition 4.1], we have

$$\lambda_{\psi',\psi}(\sigma) = \frac{\beta_n(\sigma)}{\mathrm{Vol}(M)} \ .$$

Moreover, since L acts doubly ergodically on the sphere $\mathbb{P}^1(\mathbb{C})$, it easy to verify that no coboundary appears in the formula of Proposition 1. Hence we can apply Proposition 3.17 and obtain the following inequality [Sava, Proposition 3.7]:

$$|\beta_n(\rho)| \leq \binom{n+1}{3} \mathrm{Vol}(M) \ .$$

The previous inequality allows us to refer to maximal cocycles. Since Setup 3.19 is satisfied by [BBI18, Proposition 31], by Theorem 3.20 we can conclude that σ is maximal if and only if it is cohomologous to the cocycle induced by $\pi_n \circ i$ (or its complex conjugated), where $\pi_n \colon \mathrm{PSL}(2,\mathbb{C}) \to \mathrm{PSL}(n,\mathbb{C})$ is an irreducible representation and $i\colon L \to \mathrm{PSL}(2,\mathbb{C})$ the standard lattice embedding (see [Sava, Theorem 1]).

4. Multiplicative constants and change of coefficients

In this section we describe a more general multiplicative formula which extends the one proved in Proposition 1. Inspired by the previous computations with real coefficients, we extend our result by changing our coefficients into L^∞-functions on suitable homogenous spaces. Our construction closely follows Burger and Iozzi's approach [BI09, Section 4]. The resolution via fibered product spaces introduced in Section 2.5 will allow us to define a fibered pullback map in bounded cohomology along generalized boundary maps associated to measurable cocycles. A key property of the latter pullback is that it factors through the standard pullback map (Definition 3.8). This factorization will enlighten the relation between fibered pullback maps and multiplicative constants associated to cocycles.

Setup 4.1. *Along this section we consider the following set of assumptions:*

- G, G' *are two locally compact second countable groups;*
- G' *acts measurably on a measurable space* Y*;*
- P, H, Q *are closed subgroups of* G *such that* $P \leq H \cap Q$ *and both* P *and* Q *are amenable;*
- $L \leq G$ *is a closed subgroup of* G *such that the quotient* $L\backslash G$ *admits a* G*-invariant probability measure* μ*;*
- (X, μ_X) *is a standard Borel probability* L*-space;*
- $\sigma\colon L \times X \to G'$ *is a measurable cocycle with an essentially unique generalized boundary map* $\phi\colon G/Q \times X \to Y$.

4.1. **Fibered pullback maps.** The aim of this section is to describe the fibered pullback map induced in bounded cohomology via generalized boundary maps. In the situation of Setup 4.1, there exists a natural multiplication map

$$\widetilde{m}^n\colon G \times (H/P)^n \to (G/Q)^n,\ \ \widetilde{m}^n(g, h_1P, \ldots, h_nP) = (gh_1Q, \ldots, gh_nQ)\ , \tag{16}$$

which is clearly G-equivariant with respect to the natural right G-action on the first factor of $G \times (H/P)^n$ and the diagonal right G-action on $(G/Q)^n$, respectively. Moreover, $\widetilde{m}^n$ is H-invariant with respect to the H-action defined by Equation (5), that is

$$\widetilde{m}^n(gh, h^{-1}h_1P, \ldots, h^{-1}h_nP) = \widetilde{m}^n(g, h_1P, \ldots, h_nP)$$

for every $g \in G$ and every $h, h_1, \ldots, h_n \in H$. Thus, it induces a well-defined map on the quotient

$$\overline{m}^n\colon (G \times (H/P)^n)/H \to (G/Q)^n\ . \tag{17}$$

Then, we can precompose $\overline{m}^n$ with

$$\overline{q}_n^{-1}\colon (G/P)^n_f \to (G \times (H/P)^n)/H\ ,$$

which is the inverse of the homeomorphism introduced in Section 2.5. This produces a map

$$m^n := \overline{m}^n \circ \overline{q}_n^{-1}\colon (G/P)^n_f \to (G/Q)^n\ , \tag{18}$$

called *multiplication map on fibered products.* On the other hand, the boundary map $\phi\colon G/Q \to Y$ allows us to define the following product map

$$\phi^n\colon (G/Q)^n \times X \to Y^n,\ \ \phi^n(\eta_1,\dots,\eta_n,x) \coloneqq (\phi(\eta_1,x),\dots,\phi(\eta_n,x))\ .$$

Now, for every $n \geq 1$, one can construct the following measurable map

$$\phi^n_f \coloneqq \phi^n \circ (m^n \times \mathrm{id}_X)\colon (G/P)^n_f \times X \to Y^n\ , \tag{19}$$

$$\phi^n_f(\xi_1,\dots,\xi_n,x) \coloneqq \phi^n(m^n(\xi_1,\dots,\xi_n),x) = (\phi(m(\xi_1),x),\dots,\phi(m(\xi_n),x))\ ,$$

where $\mathrm{id}_X\colon X \to X$ is the identity map. Here, we interpret $m^n = (m^1,\dots,m^1)$ and we drop the apex 1 from the notation. Since ϕ^n inherits the σ-equivariant from ϕ just by taking the composition of ϕ with the product between the multiplication map on fibered products and the identity, we have

$$\begin{aligned}\phi^n_f(\gamma.\xi_1,\dots,\gamma.\xi_n,\gamma.x) &= (\phi(m(\gamma.\xi_1),\gamma.x),\dots,\phi(m(\gamma.\xi_n),\gamma.x)) = \\ &= (\phi(\gamma.m(\xi_1),\gamma.x),\dots,\phi(\gamma.m(\xi_n),\gamma.x)) = \\ &= \sigma(\gamma,x)(\phi(m(\xi_1),x),\dots,\phi(m(\xi_n),x)) = \\ &= \sigma(\gamma,x)\phi^n_f(\xi_1,\dots,\xi_n,x)\ ,\end{aligned}$$

for every $\gamma \in L$ and almost every $(\xi_1,\dots,\xi_n) \in (G/P)^n_f$ and $x \in X$.

Our aim is now to describe the pullback of a cocycle $\psi \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ into the space $\mathrm{L}^\infty((G/P)^{\bullet+1}_f)^L$ via the maps above. First, notice that there exists a natural pullback map given by

$$\mathrm{C}^\bullet(\phi_f)\colon\ \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'} \to \mathrm{L}^\infty((G/P)^{\bullet+1}_f \times X)^L$$

$$\mathrm{C}^\bullet(\phi_f)(\psi)(\xi_1,\dots,\xi_{\bullet+1},x) \coloneqq \psi(\phi^{\bullet+1}_f(\xi_1,\dots,\xi_{\bullet+1},x))\ ,$$

where $(\xi_1,\dots,\xi_{\bullet+1}) \in (G/P)^{\bullet+1}_f$ and $x \in X$. Here an element $\gamma \in L$ acts on $\psi \in \mathrm{L}^\infty((G/P)^{\bullet+1}_f \times X)$ diagonally. By construction $\mathrm{C}^\bullet(\phi_f)$ is norm non-increasing and the proof of the well-definedness follows verbatim the proof of Lemma 3.3.

Since we want to obtain a cocycle which does not depend on the variable $x \in X$, we can compose the map $\mathrm{C}^\bullet(\phi_f)$ with the integration map $\mathrm{I}^\bullet_X$ introduced in Definition 3.5. In this way, we get the following:

Definition 4.2. In the situation of Setup 4.1, we define the *fibered pullback map along* ϕ as follows

$$\mathrm{C}^\bullet(\Phi^X_f)\colon\ \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'} \to \mathrm{L}^\infty((G/P)^{\bullet+1}_f)^L\ ,\ \ \mathrm{C}^\bullet(\Phi^X_f) \coloneqq \mathrm{I}^\bullet_X \circ \mathrm{C}^\bullet(\phi_f)\ ,$$

$$\mathrm{C}^\bullet(\Phi^X_f)(\psi)(\xi_1,\dots,\xi_{\bullet+1}) \coloneqq \int_X \psi(\phi^{\bullet+1}_f(\xi_1,\dots,\xi_{\bullet+1},x))d\mu_X(x)\ .$$

Remark 4.3. As explained in Remark 2.12, when $G = H$ the fibered product $(G/P)^n_f$ becomes the standard product $(G/P)^n$. Similarly the complex $(\mathrm{L}^\infty((G/P)^\bullet_f), d^\bullet)$ reduces to the complex $(\mathrm{L}^\infty((G/P)^\bullet), \delta^\bullet)$. Hence, when $H = G$ the definition of fibered pullback map gives us back the standard pullback map along ϕ (see Definition 3.8).

The following proposition shows that the fibered pullback map induces a well-defined map at a cohomological level.

Proposition 4.4. *In the situation of Setup 4.1, the fibered pullback map along* ϕ $\mathrm{C}^\bullet(\Phi_f^X)$ *is a well-defined, norm non-increasing cochain map. Hence, it induces a well-defined map in cohomology*

$$\mathrm{H}^\bullet(\Phi_f^X)\colon\ \mathrm{H}^\bullet(\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}) \to \mathrm{H}^\bullet_{cb}(L;\mathrm{L}^\infty(G/H)),\ \ \mathrm{H}^\bullet(\Phi_f^X)([\psi]) \coloneqq [\mathrm{C}^\bullet(\Phi_f^X)(\psi)]\ .$$

Proof. Since $\mathrm{C}^\bullet(\Phi_f^X)$ is the composition of two norm non-increasing maps $\mathrm{C}^\bullet(\phi_f)$ and $\mathrm{I}^\bullet_X$, it is also so. Moreover, $\mathrm{C}^\bullet(\Phi_f^X)$ sends G'-invariant cochains to L-invariant ones. Indeed, it is the composition of the map $\mathrm{C}^\bullet(\phi_f)$ which restricts to invariants and an equivariant map $\mathrm{I}^\bullet_X$ (see Lemma 3.6).

The only thing we have to show is that $\mathrm{C}^\bullet(\Phi_f^X)$ is a cochain map. Since $\mathrm{I}^\bullet_X$ is a cochain map, it is sufficient to show that $\mathrm{C}^\bullet(\phi_f)$ is also so. Notice that for every $n \in \mathbb{N}$ and every $i \in \{1, \ldots, n+1\}$ we have the following commutative diagram

$$\begin{array}{ccc}
(G/P)_f^{n+1} \times X & \xrightarrow{\ \phi_f^{n+1}\ } & Y^{n+1} \\
\Big\downarrow{\scriptstyle p_{n,i}\times \mathrm{id}_X} & & \Big\downarrow{\scriptstyle r_i} \\
(G/P)_f^{n} \times X & \xrightarrow[\ \phi_f^{n}\ ]{} & Y^{n}\ ,
\end{array}$$

where $p_{n,i}\colon (G/P)_f^{n+1} \to (G/P)_f^n$ is the face map defined in Equation (6), the function $r_i\colon Y^{n+1} \to Y^n$ is defined as $r_i(y_1, \ldots, y_{n+1}) \coloneqq (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_{n+1})$ and $\mathrm{id}_X\colon X \to X$ is the identity map. Given an element $\psi \in \mathcal{B}^\infty(Y^n;\mathbb{R})^{G'}$, thanks to the commutativity of the diagram above, we have

$$\begin{aligned}
d^n(\mathrm{C}^{n-1}(\phi_f)(\psi))(\xi_1, \ldots, \xi_{n+1}, x) &= \sum_{i=1}^{n+1} (-1)^{i-1}\, \mathrm{C}^{n-1}(\phi_f)(\psi)((p_{n,i} \times \mathrm{id}_X)(\xi_1, \ldots, \xi_{n+1}, x)) = \\
&= \sum_{i=1}^{n+1} (-1)^{i-1} (\psi)((\phi_f^n \circ (p_{n,i} \times \mathrm{id}_X))(\xi_1, \ldots, \xi_{n+1}, x)) = \\
&= \sum_{i=1}^{n+1} (-1)^{i-1} (\psi)((r_i \circ \phi_f^{n+1})(\xi_1, \ldots, \xi_{n+1}, x)) = \\
&= \mathrm{C}^n(\phi_f)(\delta^n\psi)(\xi_1, \ldots, \xi_{n+1}, x)\ ,
\end{aligned}$$

where $\delta^n\colon\ \mathcal{B}^\infty(Y^n;\mathbb{R})^{G'} \to \mathcal{B}^\infty(Y^{n+1};\mathbb{R})^{G'}$ is the usual coboundary operator. The statement now follows by a result of Burger and Iozzi [BI09, Proposition 4.1] since P is amenable and the complex $\mathrm{L}^\infty((G/P)_f^\bullet)^L$ computes the continuous bounded cohomology groups $H^\bullet_{cb}(L;\mathrm{L}^\infty(G/H))$, as shown in Equation (7). □

Remark 4.5. If we assume that $H = G$, we have already noticed in Remark 2.12 that the fibered product $(G/P)_f^n$ coincides with the standard product space $(G/P)^n$.

Thus, as a consequence of Remark 4.3, when $H = G$ the content of Proposition 4.4 reduces to Proposition 3.10.

4.2. **Factorization of fibered pullback maps and multiplicative constants.** In this section we show that the fibered pullback map along ϕ, $\mathrm{C}^\bullet(\Phi_f^X)$, can be factored through the standard pullback map $\mathrm{C}^\bullet(\Phi^X)$ along ϕ. Here we assume again to be in the situation of Setup 4.1. This result allows us to describe the relation between $\mathrm{C}^\bullet(\Phi_f^X)$ and the multiplicative constant $\lambda_{\psi',\psi}(\sigma)$.

First of all, notice that the multiplication map defined by Equation (18) allows us to implement the map induced in cohomology by the change of coefficients $\kappa\colon \mathbb{R} \to \mathrm{L}^\infty(G/H)$. Indeed, suppose that in Setup 4.1 we have $L = G = G'$, $Y = G/Q$ and $\sigma\colon G \times X \to G$ is the projection on the first factor, i.e. the cocycle induced by the identity $\mathrm{id}_G\colon G \to G$. Then, the fibered map $\phi_f^{\bullet+1}\colon (G/P)_f^{\bullet+1} \times X \to (G/Q)^{\bullet+1}$ reduces to the multiplication $m^{\bullet+1}$ of Equation (18):

$$\phi_f^{\bullet+1}(\xi_1, \dots, \xi_{\bullet+1}, x) = m^{\bullet+1}(\xi_1, \dots, \xi_{\bullet+1}) \ .$$

Consider now a class $\Psi \in \mathrm{H}^\bullet(G;\mathbb{R})$ represented by a bounded strict G-invariant Borel cocycle $\psi\colon (G/Q)^{\bullet+1} \to \mathbb{R}$ which is everywhere defined. By construction, the fibered pullback of the cocycle ψ along ϕ reduces to the standard pullback of ψ along m, that is

$$\mathrm{C}^\bullet(\Phi_f^X)(\psi) = \mathrm{C}^\bullet(m)(\psi) \in \mathrm{L}^\infty((G/P)_f^{\bullet+1})^G \ .$$

Since the above map is a morphism of strong resolutions by relatively injective G-modules which extends the inclusion $\kappa : \mathbb{R} \to \mathrm{L}^\infty(G/H)$, by functoriality the image

$$\mathrm{H}^\bullet_{cb}(\kappa_G)(\Psi) \in \mathrm{H}^\bullet_{cb}(G; \mathrm{L}^\infty(G/H))$$

of the class $\Psi \in \mathrm{H}^\bullet_{cb}(G;\mathbb{R})$ via the map induced by the change of coefficients, admits as representative the bounded strict G-invariant Borel cocycle

$$\mathrm{C}^\bullet(m)(\psi)\colon (G/P)_f^{\bullet+1} \to \mathbb{R},\ \ \mathrm{C}^\bullet(m)(\psi)(\xi_1, \dots, \xi_{\bullet+1}) \coloneqq \psi(m(\xi_1), \dots, m(\xi_{\bullet+1})) \ .$$

Hence, we get the desired factorization of the fibered pullback map along ϕ through the standard pullback map along ϕ.

Proposition 4.6 (Factorization fibered pullback). *In the situation of Setup 4.1, the fibered pullback cocycle $\mathrm{C}^\bullet(\Phi_f^X)(\psi)$ of a cocycle $\psi \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ can be expressed as the image of the composition*

$$\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'} \xrightarrow{\mathrm{C}^\bullet(\Phi^X)} \mathrm{L}^\infty((G/Q)^{\bullet+1};\mathbb{R})^G \xrightarrow{\mathrm{C}^\bullet(m)} \mathrm{L}^\infty((G/P)_f^{\bullet+1})^G \ .$$

In other words, we have

$$\mathrm{C}^\bullet(\Phi_f^X)(\psi) = \mathrm{C}^\bullet(m) \circ \mathrm{C}^\bullet(\Phi^X)(\psi)$$

and

$$[\mathrm{C}^\bullet(\Phi_f^X)(\psi)] = \mathrm{H}^\bullet_{cb}(\kappa_G)[\mathrm{C}^\bullet(\Phi^X)(\psi)] \ .$$

Proof. Note that we have the following commutative diagram

$$
\begin{array}{ccc}
\mathrm{L}^\infty((G/Q)^{\bullet+1}\times X)^L & \xrightarrow{\mathrm{C}^\bullet(m\times \mathrm{id}_X)} & \mathrm{L}^\infty((G/P)_f^{\bullet+1}\times X)^L \\
\Big\downarrow{\scriptstyle \mathrm{I}^\bullet_X} & & \Big\downarrow{\scriptstyle \mathrm{I}^\bullet_X} \\
\mathrm{L}^\infty((G/Q)^{\bullet+1})^L & \xrightarrow[\mathrm{C}^\bullet(m)]{} & \mathrm{L}^\infty((G/P)_f^{\bullet+1})^L \ ,
\end{array}
\tag{20}
$$

where id_X denotes the identity map on X. Given $\psi\in\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$, we have

$$
\begin{aligned}
\mathrm{C}^\bullet(\Phi^X_f)(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) &= (\mathrm{I}^\bullet_X\circ \mathrm{C}^\bullet(\phi_f))(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) = \\
&= (\mathrm{I}^\bullet_X\circ \mathrm{C}^\bullet(m\times \mathrm{id}_X)\circ \mathrm{C}^\bullet(\phi))(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) = \\
&= (\mathrm{C}^\bullet(m)\circ \mathrm{I}^\bullet_X\circ \mathrm{C}^\bullet(\phi))(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) = \\
&= (\mathrm{C}^\bullet(m)\circ \mathrm{C}^\bullet(\Phi^X))(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) \ ,
\end{aligned}
$$

where we pass from the first to the second line using the definition of $\phi_f^{\bullet+1}$ as the composition of $\phi^\bullet\circ(m^\bullet\times\mathrm{id}_X)$. Moreover, we also need the commutative Diagram (20) for moving from the second to the third line. This finishes the proof. ☐

Our goal is now to show that our approach extends Burger and Iozzi's results [BI09, Section 4.1] for representations. To this end, let $\rho\colon L\to G'$ be a continuous representation with associated a measurable ρ-equivariant map $\varphi\colon G/Q\to Y$. Following [BI09, Section 4.1] we can use the measurable map φ to define a fibered measurable map

$$\varphi_f^\bullet := m^\bullet\circ\varphi^\bullet\colon (G/P)_f^\bullet\to Y^\bullet \ ,$$

which induces a cochain map

$$\mathrm{C}^\bullet(\varphi_f)\colon\ \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}\to \mathrm{L}^\infty((G/P)_f^{\bullet+1})^L \ ,$$

Hence, there exists a well-defined pullback map in cohomology

$$\mathrm{H}^\bullet(\varphi_f)\colon\ \mathrm{H}^\bullet(\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'})\to \mathrm{H}^\bullet_{cb}(L;\mathrm{L}^\infty(G/H)) \ .$$

Let (X,μ_X) be a standard Borel probability L-space . We already know that ρ naturally induces a cocycle $\sigma_\rho\colon L\times X\to G'$ (see Definition 2.7). Moreover, the measurable map φ can be used to define a generalized boundary map as follows

$$\phi\colon G/Q\times X\to Y,\ \ \phi(\eta,x):=\varphi(\eta)\ ,$$

for almost every $\eta\in G/Q$. Similarly, the fibered map induced by ϕ coincides with the one induced by φ

$$\phi_f^\bullet\colon (G/P)_f^\bullet\times X\to Y^\bullet,\ \ \phi_f^\bullet(\xi,x)=\varphi_f^\bullet(\xi)\ . \tag{21}$$

Proposition 4.7. *In the situation of Setup 4.1, let $\rho\colon L\to G'$ be a continuous representation and suppose there exists an essentially unique measurable ρ-equivariant map $\varphi\colon G/Q\to Y$. Then, given $\psi\in\mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$, we have*

$$\mathrm{C}^\bullet(\Phi^X_f)(\psi)=\mathrm{C}^\bullet(\varphi_f)(\psi)\ .$$

Proof. The proof follows the line of Proposition 3.14 and it is an easy consequence of the commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'} & \xrightarrow{\mathrm{C}^\bullet(\phi_f)} & \mathrm{L}^\infty((G/P)_f^{\bullet+1}\times X)^L \\ & {\scriptstyle \mathrm{C}^\bullet(\varphi_f)}\searrow \quad \swarrow{\scriptstyle \mathrm{I}_X^\bullet} & \\ & \mathrm{L}^\infty((G/P)_f^{\bullet+1})^L & \end{array}.$$

Indeed given $\psi \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$, we have

$$\begin{aligned} \mathrm{C}^\bullet(\Phi_f^X)(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) &= (\mathrm{I}_X^\bullet \circ \mathrm{C}^\bullet(\phi_f))(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) = \\ &= (\mathrm{I}_X^\bullet \circ \mathrm{C}^\bullet(\varphi_f))(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) = \\ &= \int_X \psi(\varphi_f^{\bullet+1}(\xi_1,\ldots,\xi_{\bullet+1}))d\mu_X(x) = \\ &= \mathrm{C}^\bullet(\varphi_f)(\xi_1,\ldots,\xi_{\bullet+1}) \ . \end{aligned}$$

This proves the commutativity of the diagram, whence the thesis. □

Remark 4.8. Notice that in Setup 4.1, we also assume the existence of a measurable cocycle. However, we do not need this assumption in the previous result, since we construct σ_ρ from the given representation ρ.

Remark 4.9. Let $\rho\colon L \to G'$ be a continuous representation and suppose there exists a ρ-equivariant measurable map $\varphi\colon G/Q \to Y$. By Proposition 3.14, we know that

$$\mathrm{C}^\bullet(\Phi^X)(\psi) = \mathrm{C}^\bullet(\varphi)(\psi) \ .$$

Hence, as a consequence of both Proposition 4.6 and Proposition 4.7, we obtain

$$\mathrm{C}^\bullet(\varphi_f) = \mathrm{C}^\bullet(\Phi_f^X)(\psi) = \mathrm{C}^\bullet(m)\circ \mathrm{C}^\bullet(\Phi^X)(\psi) = \mathrm{C}^\bullet(m)\circ\mathrm{C}^\bullet(\varphi)(\psi) \ .$$

This shows that Proposition 4.6 can be interpreted as a generalization of the factorization result in the case of representations stated by Burger and Iozzi [BI09, Proposition 4.6].

Now we are finally ready to state and prove the *fibered multiplicative formula* which extends both our easy multiplicative formula (Proposition 1) to the fibered setting and Burger and Iozzi's results for representations [BI09, Proposition 4.9, Principle 4.11] to the wider setting of measurable cocycles.

Proposition 4.10 (Fibered multiplicative formula)**.** *In the situation of Setup 4.1, we have the following:*

(1) Let $\psi' \in \mathcal{B}^\infty(Y^{\bullet+1};\mathbb{R})^{G'}$ be an everywhere defined G'-invariant cocycle and let $\psi \in \mathrm{L}^\infty((G/Q)^{\bullet+1})^G$ be a G-invariant cocycle . Denote by $\Psi \in \mathrm{H}_{cb}^\bullet(G;\mathbb{R})$ the class of ψ. If we suppose that $\Psi = \mathrm{trans}_L^\bullet[\mathrm{C}^\bullet(\Phi^X)(\psi')]$, then we have

$$\int_{L\backslash G}\int_X \psi'(\phi_f^{\bullet+1}(\overline{g}.\xi_1,\ldots,\overline{g}.\xi_{\bullet+1},x))d\mu_X(x)d\mu(\overline{g}) = \mathrm{C}^\bullet(m)(\psi)(\xi_1,\ldots,\xi_{\bullet+1}) + \text{cobound.}\ ,$$

for almost every $(\xi_1, \ldots, \xi_{\bullet+1}) \in (G/P)_f^{\bullet+1}$.

(2) *Suppose that* $\mathrm{H}_{cb}^\bullet(G;\mathbb{R}) \cong \mathbb{R}\,\Psi (= \mathbb{R}[\psi])$ *and let* $\lambda_{\psi',\psi}(\sigma) \in \mathbb{R}$ *be the multiplicative constant associated to* σ, ψ', ψ. *Then, we have*

$$\int_{L\backslash G}\int_X \psi'(\phi_f^{\bullet+1}(\overline{g}.\xi_1, \ldots, \overline{g}.\xi_\bullet, x))d\mu_X(x)d\mu(\overline{g}) = \lambda_{\psi',\psi}(\sigma)\,\mathrm{C}^\bullet(m)(\psi)(\xi_1, \ldots, \xi_{\bullet+1}) + \text{cobound.}\ ,$$

for almost every $(\xi_1, \ldots, \xi_{\bullet+1}) \in (G/P)_f^{\bullet+1}$.

Proof. Ad 1. The proof is based on the commutativity of Diagram (9). Recall that we have

$$\mathrm{H}_{cb}^\bullet(\kappa_G) \circ \mathrm{trans}_L^\bullet = \tau_{G/P}^\bullet \circ \mathrm{H}_{cb}^\bullet(\kappa_G)\ ,$$

where $\mathrm{trans}_L^\bullet$ is the transfer map defined in Section 2.3 and $\tau_{G/P}^\bullet$ is the transfer map with coefficients defined in Section 2.5.

Since by hypothesis we know that

$$\Psi = \mathrm{trans}_L^\bullet[\mathrm{C}^\bullet(\Phi^X)(\psi')]\ ,$$

we can apply to both sides the map $\mathrm{H}_{cb}^\bullet(\kappa_G)$ induced by the change of coefficients:

$$\mathrm{H}_{cb}^\bullet(\kappa_G)(\Psi) = \mathrm{H}_{cb}^\bullet(\kappa_G) \circ \mathrm{trans}_L^\bullet[\mathrm{C}^\bullet(\Phi^X)(\psi')] = \tau_{G/P}^\bullet \circ \mathrm{H}_{cb}^\bullet(\kappa_G)[\mathrm{C}^\bullet(\Phi^X)(\psi')]\ .$$

Moreover, by Proposition 4.6 we have that

$$[\mathrm{C}^\bullet(\Phi_f^X)(\psi')] = \mathrm{H}_{cb}^\bullet(\kappa_G)[\mathrm{C}^\bullet(\Phi^X)(\psi')]\ ,$$

which implies

$$\mathrm{H}_{cb}^\bullet(\kappa_G)(\Psi) = \tau_{G/P}^\bullet[\mathrm{C}^\bullet(\Phi_f^X)(\psi')]\ .$$

The previous equality can be rewritten at the level of cochains

$$\mathrm{C}^\bullet(m)(\psi) + \delta^\bullet\theta = \widehat{\tau}_{G/Q}^\bullet(\mathrm{C}^\bullet(\Phi_f^X)(\psi'))\ ,$$

where $\theta \in \mathrm{L}^\infty((G/P)_f^\bullet)^G$. This finishes the proof.

Ad 2. If we have the isomorphism $\mathrm{H}_{cb}^\bullet(G;\mathbb{R}) \cong \mathbb{R}\,\Psi$, then there must exist a multiplicative constant $\lambda_{\psi',\psi}(\sigma) \in \mathbb{R}$ such that

$$\mathrm{trans}_L^\bullet \circ [\mathrm{C}^\bullet(\Phi^X)(\psi')] = \lambda_{\psi',\psi}(\sigma)\Psi\ .$$

The claim now follows as a consequence of *Ad 1.* □

Remark 4.11. It is immediate to check that when $H = G$ the content of Proposition 4.10 reduces to the easy multiplicative formula (Proposition 1).

Remark 4.12. Sometimes one can assume that there is no coboundary term in both the equations which appear in Proposition 4.10. This assumptions is not too restrictive. Indeed, in presence of an ergodic action of H on the quotient $(H/P)^\bullet$, the group G acts ergodically on the fibered product $(G/P)_f^\bullet$ (see [BI09, Remark 4.10]). For instance this may happen when $\bullet = 2$ and so $\mathrm{L}^\infty((G/P)_f^2) \cong \mathbb{R}$. Hence, there cannot be any coboundary term.

5. Cartan invariant of measurable cocycles of complex hyperbolic lattices

Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice with $n \geq 2$ and let (X,μ_X) be a standard Borel probability Γ-space. Consider a cocycle $\sigma\colon \Gamma \times X \to \mathrm{PU}(m,1)$, where we suppose $m \geq n$. In this section we are going to define the *Cartan invariant* $i(\sigma)$ associated to σ. To this end, we have to assume the existence of an essentially unique boundary map ϕ associated to σ. Since the pullback class along ϕ determines a square integrable smooth 2-form on the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, we will follow the construction performed by Burger and Iozzi [BI07b]. Indeed, our numerical invariant will extend the classic Cartan invariant associated to representations and it will satisfy a rigidity result which may be interpreted as a Cartan theorem for measurable cocycle. More precisely, we will show that the Cartan invariant of any cocycle satisfies $|i(\sigma)| \leq 1$. Moreover, we will prove that the maximum is attained if and only if σ is cohomologous to the cocycle induced by the standard lattice embedding $i\colon \Gamma \to \mathrm{PU}(n,1) \leq \mathrm{PU}(m,1)$. To prove the latter result we will use our fibered multiplicative formula described in Proposition 4.10.

5.1. Cartan invariant of measurable cocycles.

Let us consider $\mathbb{C}^{n,1}$, i.e. the complex vector space $\mathbb{C}^{n+1}$ endowed with the standard Hermitian form of signature $(n,1)$ defined by

$$h\colon\ \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \to \mathbb{C},\ \ h(v,w) = \sum_{i=1}^{n} v_i \overline{w}_i - v_{n+1}\overline{w}_{n+1}\ .$$

If we denote by V_- the cone of negative vectors

$$V_- \coloneqq \{v \in \mathbb{C}^{n,1}\,|h(v,v) < 0\}\ ,$$

the *complex n-dimensional hyperbolic space* $\mathbb{H}^n_{\mathbb{C}}$ is the projectivization of the negative cone $\mathbb{P}(V_-)$ equipped with the unique distance d satisfying

$$\cosh^2 d([v],[w]) \coloneqq \frac{h(v,w)h(w,v)}{h(v,v)h(w,w)}\ ,$$

for every $v,w \in \mathbb{C}^{n,1}$. The complex n-dimensional hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ endowed with this metric structure becomes a simply connected Riemannian manifold whose sectional curvature varies between -4 and -1. Since its isometry group is $\mathrm{PU}(n,1)$, the complex hyperbolic space can be alternatively interpreted as the global symmetric space naturally associated to $\mathrm{PU}(n,1)$.

For any $k \in \{0,\ldots,n\}$, a *k-plane* is a totally geodesic isometric copy of $\mathbb{H}^k_{\mathbb{C}}$ holomorphically embedded in $\mathbb{H}^n_{\mathbb{C}}$. Of course, when $k=1$ we find the usual notion of *complex geodesic*.

For our purposes we will be mainly interested in the boundary at infinity $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$ of the complex hyperbolic space. This boundary can be identified with a $(2n-1)$-dimensional sphere corresponding to the projectivization of the null cone

$$V_0 = \{v \in \mathbb{C}^{n+1} \mid h(v,v) = 0\}\ .$$

Notice that the boundary of a k-plane in $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$ consists of an embedded copy of $\partial_\infty \mathbb{H}^k_{\mathbb{C}}$. We refer to the latter as a *k-chain*, or simply *chain* if $k = 1$. Since any chain is completely determined by any two points lying on it, two distinct chains are either disjoint or they meet exactly in one point.

Consider now the Hermitian triple product

$$\langle \cdot, \cdot, \cdot \rangle \colon (\mathbb{C}^{n,1})^3 \to \mathbb{C},\ \ \langle z_1, z_2, z_3 \rangle := h(z_1, z_2)h(z_2, z_3)h(z_3, z_1)\ .$$

If we denote by $(\partial_\infty \mathbb{H}^n_{\mathbb{C}})^{(3)}$ the set of triples of distinct points on the boundary at infinity, the triple product allows us to define the following function

$$c_n \colon (\partial_\infty \mathbb{H}^n_{\mathbb{C}})^{(3)} \to [-1, 1],\ \ c_n(\xi_1, \xi_2, \xi_3) := \frac{2}{\pi} \arg(z_1, z_2, z_3)\ ,$$

where $\xi_i = [z_i]$ and we choose the branch of the argument function such that $\arg(z) \in [-\pi/2, \pi/2]$. It can be seen that c_n extends to a $\mathrm{PU}(n, 1)$-invariant alternating Borel cocycle on the whole $(\partial_\infty \mathbb{H}^n_{\mathbb{C}})^3$. This produces an element

$$c_n \in \mathcal{B}^\infty_{\mathrm{alt}}((\partial_\infty \mathbb{H}^n_{\mathbb{C}})^3; \mathbb{R})^{\mathrm{PU}(n,1)}\ .$$

The previous cocycle c_n is called *Cartan cocycle*. As explained in Section 2.2, the Cartan cocycle naturally determines a class in the one-dimensional bounded cohomology group $\mathrm{H}^2_{cb}(\mathrm{PU}(n, 1); \mathbb{R})$ (see for instance [BI07a, Section 5]).

Let $\omega_n \in \Omega^2(\mathbb{H}^n_{\mathbb{C}})$ be the Kähler 2-form, which is $\mathrm{PU}(n, 1)$-invariant. By the Van Est isomorphism [Gui80, Corollary 7.2] we know that $\Omega^2(\mathbb{H}^n_{\mathbb{C}})^{\mathrm{PU}(n,1)}$ is naturally isomorphic to $\mathrm{H}^2_c(\mathrm{PU}(n, 1); \mathbb{R})$ and hence ω_n determines a continuous cohomology class κ_n, called *Kähler class*. Since the Kähler class is bounded, it lies in the image of the comparison map

$$\mathrm{comp}^2 \colon\ \mathrm{H}^2_{cb}(\mathrm{PU}(n, 1); \mathbb{R}) \to \mathrm{H}^2_c(\mathrm{PU}(n, 1); \mathbb{R})\ .$$

Hence, it comes from a class $\kappa^b_n \in \mathrm{H}^2_{cb}(\mathrm{PU}(n, 1); \mathbb{R})$ which can be considered as a generator. The latter element k^b_n is called *bounded Kähler class*. One can express the relation between the class determined by the Cartan cocycle and the bounded Kähler class as follows

$$[c_n] = \frac{\kappa^b_n}{\pi} \in \mathrm{H}^2_{cb}(\mathrm{PU}(n, 1); \mathbb{R})\ .$$

The previous formula shows that the cocycle πc_n is a natural representative for the bounded Kähler class.

We are now ready to define the *Cartan invariant* associated to a measurable cocycle. Let $\Gamma \leq \mathrm{PU}(n, 1)$ be a torsion-free lattice, with $n \geq 2$. Let (X, μ_X) be a standard Borel probability Γ-space. Consider a measurable cocycle $\sigma \colon \Gamma \times X \to \mathrm{PU}(m, 1)$, where $m \geq n$. Assume that σ admits an essentially unique boundary map $\phi \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$. Following Definition 3.8 we consider the pullback along the boundary map ϕ of the cocycle πc_m:

$$\mathrm{C}^2(\Phi^X)(\pi c_m) \in \mathrm{L}^\infty_{\mathrm{alt}}((\partial_\infty \mathbb{H}^n_{\mathbb{C}})^3; \mathbb{R})^\Gamma\ .$$

Thus, as explained in Section 2.4, we know that the previous cocycle canonically determines a bounded Γ-invariant differential forms via the map

$$\widehat{\delta}^2_\infty \colon \mathrm{L}^\infty_{\mathrm{alt}}((\partial_\infty \mathbb{H}^n_{\mathbb{C}})^3;\mathbb{R})^\Gamma \to \Omega^2_\infty(\mathbb{H}^n_{\mathbb{C}})^\Gamma \ ,$$

restricted to Γ-invariants. Moreover, recall that the space of bounded differential forms naturally injects into the space of invariant L^2-forms via the injection $i^2_2 \colon \Omega^2_\infty(\mathbb{H}^n_{\mathbb{C}})^\Gamma \to \Omega^2_2(\mathbb{H}^n_{\mathbb{C}})^\Gamma$, and hence we obtain a map

$$\widehat{\delta}^2_2 \colon \mathrm{L}^\infty_{\mathrm{alt}}((\partial_\infty \mathbb{H}^n_{\mathbb{C}})^3;\mathbb{R})^\Gamma \to \Omega^2_2(\mathbb{H}^n_{\mathbb{C}})^\Gamma \ .$$

The map above allows to us define the following differential 2-form

$$\omega_n(\sigma) \coloneqq \widehat{\delta}^2_2(\mathrm{C}^2(\Phi^X)(\pi c_m)) \in \Omega^2_2(\mathbb{H}^2_{\mathbb{C}})^\Gamma \ .$$

Let now $M \coloneqq \Gamma \backslash \mathbb{H}^n_{\mathbb{C}}$ be the complex hyperbolic manifold of finite volume associated to Γ. Being bounded and $\mathrm{PU}(n,1)$-invariant, the Kähler form ω_n descends naturally to a differential 2-forms ω_M which lies in $\mathrm{H}^2_{(2)}(M;\mathbb{R}) \coloneqq \mathrm{H}^2(\Omega^\bullet_2(M;\mathbb{R}))$. Similarly, the Γ-invariance of $\omega(\sigma)$ implies that we may interpret it as a differential 2-form on M and hence it determines an element in $\mathrm{H}^2_{(2)}(M;\mathbb{R})$.

Definition 5.1. Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice. Let (X,μ_X) be a standard Borel probability Γ-space. Consider a measurable cocycle $\sigma \colon \Gamma \times X \to \mathrm{PU}(m,1)$, with $m \geq n \geq 2$. Suppose that there exists an essentially unique boundary map $\phi \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$. We define the *Cartan invariant associated to the cocycle* σ to be the following numerical invariant

$$i(\sigma) \coloneqq \frac{\langle \omega_n(\sigma), \omega_M \rangle}{\langle \omega_M, \omega_M \rangle} = \frac{\langle \widehat{\delta}^2_2(\mathrm{C}^2(\Phi^X)(\pi c_n)), \omega_M \rangle}{\langle \omega_M, \omega_M \rangle} \ ,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on the space of L^2-forms.

Remark 5.2. It is worth noticing that when Γ is a *uniform* lattice (and so $M = \Gamma / X$ is compact) it follows that $\mathrm{H}^2_{(2)}(M;\mathbb{R}) \cong \mathrm{H}^2_{\mathrm{dR}}(M;\mathbb{R})$ as a consequence of Hodge theory. In particular, in this context both $\omega(\sigma)$ and ω_M can be interpreted as elements of $\mathrm{H}^2_{\mathrm{dR}}(M;\mathbb{R})$.

Since any representation $\rho \colon \Gamma \to \mathrm{PU}(m,1)$ naturally determines a measurable cocycle $\sigma_\rho \colon \Gamma \times X \to \mathrm{PU}(m,1)$, a natural question is whether there exists a relation between the Cartan invariant associated to ρ defined by Burger and Iozzi [BI07b] and our Cartan invariant of σ_ρ just introduced. Assuming ρ non-elementary, we are going to answer affirmatively to the latter question in the following:

Proposition 5.3. *Let* $\Gamma \leq \mathrm{PU}(n,1)$ *be a torsion-free lattice and consider a non-elementary representation* $\rho \colon \Gamma \to \mathrm{PU}(m,1)$*, where* $m \geq n \geq 2$*. For any standard Borel probability* Γ*-space* (X,μ_X)*, given the measurable cocycle associated to* ρ*,* $\sigma_\rho \colon \Gamma \times X \to \mathrm{PU}(m,1)$*, we have*

$$i(\sigma_\rho) = i_\rho \ .$$

Proof. Since the representation ρ is non-elementary, we know that there exists a boundary map $\varphi \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$ which is essentially unique by the doubly ergodic action of Γ on $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$ (see for instance [BM96]).

Recall that, given any standard Borel probability Γ-space (X, μ_X), the measurable cocycle σ_ρ associated to the representation ρ is defined by

$$\sigma_\rho \colon \Gamma \times X \to \mathrm{PU}(m,1),\ \ \sigma_\rho(\gamma, x) := \rho(\gamma)\ ,$$

for every $\gamma \in \Gamma$ and almost every $x \in X$. Moreover, the associated boundary map is defined as

$$\phi \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}},\ \ \phi(\xi, x) \coloneqq \varphi(\xi)\ ,$$

for almost every $\xi \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ and almost every $x \in X$. Recall now that $\widehat{\delta}^2_2(\mathrm{C}^2(\varphi)(\pi c_m))$ is a natural representative for the class $\rho^2_{(2)}(\kappa_m) \in \mathrm{H}^2_{(2)}(M;\mathbb{R})$, where the map

$$\rho^2_{(2)} \colon\ \mathrm{H}^2_c(\mathrm{PU}(m,1);\mathbb{R}) \to \mathrm{H}^2_{(2)}(M;\mathbb{R})$$

is the one defined by Burger and Iozzi [BI07a, Corollary 6]. Since Proposition 3.14 implies

$$\mathrm{C}^2(\Phi^X)(\pi c_m) = \mathrm{C}^2(\varphi)(\pi c_m)\ ,$$

we get the following chain of equalities

$$i(\sigma) = \frac{\langle \widehat{\delta}^2_2(\mathrm{C}^2(\Phi^X)(\pi c_m)), \omega_M\rangle}{\langle \omega_M, \omega_M\rangle} = \frac{\langle \widehat{\delta}^2_2(\mathrm{C}^2(\varphi)(\pi c_m)), \omega_M\rangle}{\langle \omega_M, \omega_M\rangle} = \frac{\langle \rho^2_2(\kappa_m), \omega_M\rangle}{\langle \omega_M, \omega_M\rangle} = i_\rho\ .$$

This finishes the proof. □

We conclude this section by showing that the Cartan invariant is constant along the $\mathrm{PU}(m,1)$-cohomology class of the cocycle $\sigma \colon \Gamma \times X \to \mathrm{PU}(m,1)$. This result generalizes the invariance of the classic Cartan invariant of representations with respect to the action of $\mathrm{PU}(m,1)$ given by conjugacy.

Proposition 5.4. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice and let (X,μ_X) be a standard Borel probability Γ-space. Given $m \geq n \geq 2$, we consider a measurable cocycle $\sigma \colon \Gamma \times X \to \mathrm{PU}(m,1)$. Assume that σ admits an essentially unique boundary map $\phi \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$. For any measurable map $f \colon X \to \mathrm{PU}(m,1)$, we have*

$$i(f.\sigma) = i(\sigma)\ ,$$

where $f.\sigma$ is the cocycle twisted by f.

Proof. By Definition 2.5 and 2.6, recall that the twisted cocycle

$$f.\sigma \colon \Gamma \times X \to \mathrm{PU}(m,1),\ \ (f.\sigma)(\gamma,x) \coloneqq f(\gamma.x)^{-1}\sigma(\gamma,x)f(x)\ ,$$

admits as essentially unique boundary map

$$f.\phi \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}},\ \ (f.\phi)(\xi,x) = f(x)^{-1}\phi(\xi,x)\ ,$$

for almost every $\xi \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ and almost every $x \in X$.

Since thanks to Proposition 3.12 we know that

$$\mathrm{C}^2(f.\Phi^X)(\pi c_m) = \mathrm{C}^2(\Phi^X)(\pi c_m)\ ,$$

it is immediate to check that

$$i(f.\sigma)=\frac{\langle\widehat{\delta}_2^2(\mathrm{C}^2(f.\Phi^X)(\pi c_m)),\omega_M\rangle}{\langle\omega_M,\omega_M\rangle}=\frac{\langle\widehat{\delta}_2^2(\mathrm{C}^2(\Phi^X)(\pi c_m)),\omega_M\rangle}{\langle\omega_M,\omega_M\rangle}=i(\sigma)\ .$$

This finishes the proof. □

5.2. **The Cartan invariant as a multiplicative constant.** As discussed so far we can interpret the Cartan invariant associated to measurable cocycles as an extension of the ordinary Cartan invariant for representations. In this section we show that the Cartan invariant is in fact a multiplicative constant in the sense of Definition 3.16. Thanks to this interpretation, we will get a Milnor-Wood type inequality for the Cartan invariant (Corollary 5.7). Moreover, we are going to investigate the relation between the vanishing of the Cartan invariant and the totally real condition of the associated cocycle (see Definition 5.8). This will provide an extension of a result by Burger and Iozzi [BI12, Theorem 1.1] to the setting of measurable cocycle. We conclude this section by showing that maximal measurable cocycles can be trivialized (modulo a compact group when $m>n$). The proof of the latter statement is based on Proposition 4.10.

Let $\Gamma\leq\mathrm{PU}(n,1)$ be a torsion-free lattice and let (X,μ_X) be a standard Borel probability Γ-space . Suppose that $\sigma\colon\Gamma\times X\to\mathrm{PU}(m,1)$ is a measurable cocycle with essentially unique boundary map $\phi\colon\partial_\infty\mathbb{H}^n_{\mathbb{C}}\times X\to\partial_\infty\mathbb{H}^m_{\mathbb{C}}$. Here, we always assume $m\geq n\geq 2$.

Recall that the (bounded) Kähler class κ_n is a generator of the (bounded) continuous cohomology group $\mathrm{H}^2_{(c)b}(\mathrm{PU}(n,1);\mathbb{R})$ [BI07a, Section 5]. This means that

$$\mathrm{H}^2_c(\mathrm{PU}(n,1);\mathbb{R})\cong\mathrm{H}^2_{cb}(\mathrm{PU}(n,1);\mathbb{R})=\mathbb{R}\,\kappa_n=\mathbb{R}[\pi c_n]\ .$$

Hence, our setting satisfies the hypothesis of Proposition 1.2. Thus, given the Borel cocycles $c_m\in\mathcal{B}^\infty((\partial_\infty\mathbb{H}^m_{\mathbb{C}})^3;\mathbb{R})^{\mathrm{PU}(m,1)}$ and $c_n\in\mathrm{L}^\infty((\partial_\infty\mathbb{H}^n_{\mathbb{C}})^3;\mathbb{R})^{\mathrm{PU}(n,1)}$, we can consider the *multiplicative constant* $\lambda_{c_m,c_n}(\sigma)$ *associated to* σ,c_m,c_n. For ease of notation, we will simply denote by $\lambda_{m,n}(\sigma)$ the previous multiplicative constant. The following result shows that the Cartan invariant agrees with the multiplicative constant $\lambda_{m,n}(\sigma)$.

Proposition 5.5. *Let $\Gamma\leq\mathrm{PU}(n,1)$ be a torsion-free lattice and let (X,μ_X) be a standard Borel probability Γ-space. Given a measurable cocycle $\sigma\colon\Gamma\times X\to\mathrm{PU}(m,1)$ with $m\geq n\geq 2$, assume that it admits an essentially unique boundary map $\phi\colon\partial_\infty\mathbb{H}^n_{\mathbb{C}}\times X\to\partial_\infty\mathbb{H}^m_{\mathbb{C}}$. Then, we have*

$$i(\sigma)=\lambda_{m,n}(\sigma)\ .$$

Proof. As already mentioned in Section 2.4, recall that the space $\Omega^2(\mathbb{H}^n_{\mathbb{C}})^{\mathrm{PU}(n,1)}$ is isomorphic to the subspace $\mathbb{R}\,\omega_M$ of the space $\mathrm{H}^2(\Omega^\bullet_2(\mathbb{H}^n_{\mathbb{C}})^\Gamma)$. In particular, the previous isomorphism sends the generator ω_M to the Kähler form ω_n. According to this observation, we can adapt Diagram (4) to our context. This produces the following commutative diagram

$$
\begin{array}{ccccc}
\mathrm{H}^2(\mathcal{B}^\infty_{\mathrm{alt}}((\partial_\infty \mathbb{H}^m_{\mathbb{C}})^{\bullet+1};\mathbb{R})^{\mathrm{PU}(m,1)}) & & & & \\
\Big\downarrow{\scriptstyle \mathrm{H}^2(\Phi^X)} & & & & \\
\mathrm{H}^2_b(\Gamma;\mathbb{R}) & \xrightarrow{\delta^2_{2,\Gamma}} & \mathrm{H}^2(\Omega^\bullet_2(\mathbb{H}^n_{\mathbb{C}})^\Gamma) & \xrightarrow{j^2_2} & \mathbb{R}\,\omega_M \\
\Big\downarrow{\scriptstyle \mathrm{trans}^2_\Gamma} & & & & \Big\downarrow{\scriptstyle \cong} \\
\mathrm{H}^2_{cb}(\mathrm{PU}(n,1);\mathbb{R}) & \xrightarrow[\mathrm{comp}^2_{\mathrm{PU}(n,1)}]{} & \mathrm{H}^2_c(\mathrm{PU}(n,1);\mathbb{R}) & \xrightarrow[(\mathcal{V}\,\mathcal{E}^2_{\mathrm{PU}(n,1)})^{-1}]{} & \Omega^2(\mathbb{H}^n_{\mathbb{C}})^{\mathrm{PU}(n,1)} \ .
\end{array}
$$

By Definition 3.16 we know that

$$\mathrm{comp}^2_{\mathrm{PU}(n,1)}\big(\mathrm{trans}^2_\Gamma[\mathrm{C}^2(\Phi^X)(\pi c_m)]\big) = \lambda_{m,n}(\sigma)\mathrm{comp}^2_{\mathrm{PU}(n,1)}(\kappa^b_n) = \lambda_{m,n}(\sigma)\kappa_n \ .$$

Applying the inverse of the Van Est isomorphism [Gui80, Corollary 7.2] we obtain

$$(\mathcal{V}\,\mathcal{E}^2_{\mathrm{PU}(n,1)})^{-1}(\lambda_{m,n}(\sigma)(\kappa_n)) = \lambda_{m,n}(\sigma)\omega_n \ .$$

Since we know that the map $j^2_2 \colon \mathrm{H}^2(\Omega^\bullet_2(\mathbb{H}^n_{\mathbb{C}})^\Gamma) \to \mathbb{R}\,\omega_M$ is an orthogonal projector, we know that

$$j^2_2(\delta^2_{2,\Gamma}[\mathrm{C}^2(\Phi^X)(\pi c_m)]) = j^2_2(\omega(\sigma)) = \frac{\langle \omega(\sigma), \omega_M\rangle}{\langle \omega_M, \omega_M\rangle}\omega_M = i(\sigma)\omega_M \ .$$

Then, the commutativity of the diagram above implies

$$\lambda_{m,n}(\sigma) = i(\sigma) \ ,$$

as claimed. □

Remark 5.6. When σ is the cocycle associated to a non-elementary representation $\rho\colon \Gamma \to \mathrm{PU}(m,1)$, it is easy to verify that the multiplicative constant $\lambda_{m,n}(\sigma)$ becomes the *bounded Toledo invariant* $\mathrm{t}_b(\rho)$. This allows us to interpret the previous proposition as a generalization of [BI07a, Lemma 5.3].

Thanks to Proposition 5.5, we easily obtain a Milnor-Wood type inequality for the Cartan invariant.

Corollary 5.7. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice and let (X,μ_X) be standard Borel probability Γ-space. When $m \geq n \geq 2$, consider a measurable cocycle $\sigma\colon \Gamma \times X \to \mathrm{PU}(m,1)$ which admits an essentially unique boundary map $\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$. Then, we have*

$$|i(\sigma)| \leq 1 \ .$$

Proof. By Proposition 5.5 we know that the Cartan invariant $i(\sigma)$ is equal to the multiplicative constant associated to σ, c_m, c_n:

$$i(\sigma) = \lambda_{m,n}(\sigma) \ .$$

Using the definition of multiplicative constant, the equation above implies

$$i(\sigma)\kappa_n^b = \mathrm{trans}_\Gamma^2[\mathrm{C}^2(\Phi^X)(\pi c_m)] \ ,$$

where κ_n^b is the bounded Kähler class.

Recall that $\|c_m\|_\infty = \|c_n\|_\infty = 1$ [Gol99, Chapter 7]. Then, since the action of Γ on $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$ is doubly ergodic, we have

$$i(\sigma)c_n = \widehat{\mathrm{trans}}_\Gamma^2 \left(\mathrm{C}^2(\Phi^X)(c_m)\right)$$

at the level of alternating cochains. Hence, the setup of Proposition 3.17 is satisfied and we get the thesis. ☐

Before studying the rigidity property of the Cartan invariant, we investigate now for which conditions the Cartan invariant of a measurable cocycles vanishes. To this end, we need to introduce the notion of *totally real* measurable cocycle.

Definition 5.8. Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice and let (X,μ_X) be a standard Borel probability Γ-space. A measurable cocycle $\sigma\colon \Gamma \times X \to \mathrm{PU}(m,1)$ is said to be *totally real* if it is cohomologous to a cocycle

$$\sigma_{\mathrm{real}}\colon \Gamma \times X \to G_0 \ ,$$

where G_0 is the stabilizer of some sphere $\mathcal{S}_0 \subset \partial_\infty \mathbb{H}^m_{\mathbb{C}}$ which is the boundary of a totally geodesic copy $\mathbb{H}^k_{\mathbb{R}} \subset \mathbb{H}^m_{\mathbb{C}}$.

Remark 5.9. Following the same terminology used by Zimmer [Zim84, Chapter 9.2], we may say that a measurable cocycle $\sigma : \Gamma \times X \to \mathrm{PU}(m,1)$ is totally real if its *algebraic hull* is the stabilizer G_0 described above.

The notion of totally real cocycles is strictly related with the vanishing of the Cartan invariant. The relation is completely described by the following result which extends to measurable cocycles a work by Burger and Iozzi [BI12, Theorem 1.1] for representations.

Theorem 2. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice and and let (X,μ_X) be a standard Borel probability Γ-space. Given a measurable cocycle $\sigma\colon \Gamma\times X \to \mathrm{PU}(m,1)$ with $m \geq n \geq 2$, assume that there exists an essentially unique boundary map $\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$. Then, we have the followings*

(1) If σ is totally real, then $i(\sigma) = 0$;

(2) If X is Γ-ergodic and $\mathrm{C}^2(\phi) = 0$, then σ is totally real.

Proof. Ad 1. Since by Proposition 5.4 the Cartan invariant is invariant along the $\mathrm{PU}(m,1)$-cohomology class of σ, we can suppose without loss of generality that σ takes values in G_0, where

$$G_0 = \mathrm{Stab}_{\mathrm{PU}(m,1)}(\mathcal{S}_0)$$

is the stabilizer of some sphere $\mathcal{S}_0$ which is the boundary of a totally geodesic copy $\mathbb{H}^k_{\mathbb{R}} \subset \mathbb{H}^m_{\mathbb{C}}$. In this case, it is easy to check that we can restrict the image of the boundary map ϕ to the $(k-1)$-dimensional sphere $\mathcal{S}_0$, that is

$$\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \mathcal{S}_0 \ .$$

For almost every $x \in X$, we define $\phi_x \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \to \mathcal{S}_0$ by $\phi_x(\xi) := \phi(\xi, x)$. Since X is a standard Borel space, the map ϕ_x is measurable for almost every $x \in X$ [FMW04, Lemma 2.6].

Recall by Proposition 5.5 that $i(\sigma) = \lambda_{m,n}(\sigma)$ and as already shown in its proof, we have

$$\widehat{\mathrm{trans}}^2_\Gamma \left(\mathrm{C}^2(\Phi^X)(c_m)\right) = i(\sigma)c_n$$

at the level of cochains. By rewriting explicitly the equality above, we get

$$\int_{\Gamma\backslash \mathrm{PU}(n,1)} \int_X c_m(\phi_x(\overline{g}.\xi_1), \phi_x(\overline{g}.\xi_2), \phi_x(\overline{g}.\xi_3)) d\mu_X(x) d\mu(\overline{g}) = i(\sigma)c(\xi_1, \xi_2, \xi_3) \ ,$$

for almost every $\xi_1, \xi_2, \xi_3 \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$. Here μ is the $\mathrm{PU}(n,1)$-invariant probability measure on $\Gamma\backslash \mathrm{PU}(n,1)$. Since ϕ_x takes values into the sphere $\mathcal{S}_0$ for almost every $x \in X$, by [BI12, Corollary 3.1] we have that

$$c_m(\phi_x(\overline{g}.\xi_1), \phi_x(\overline{g}.\xi_2), \phi_x(\overline{g}.\xi_3)) = 0 \ ,$$

for almost every $x \in X$ and almost every $\overline{g} \in \Gamma\backslash \mathrm{PU}(n,1)$. Hence, the equality $i(\sigma) = 0$, as desired.

Ad 2. Suppose that $\mathrm{C}^2(\phi)(c_m) = 0$. This means that

$$c_m(\phi_x(\xi_1), \phi_x(\xi_2), \phi_x(\xi_3)) = 0 \ ,$$

for almost every $x \in X$ and almost every $\xi_1, \xi_2, \xi_3 \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$. If we consider the essential image

$$E_x := \mathrm{EssIm}(\phi_x) \ ,$$

we immediately notice that the σ-equivariance of the map ϕ implies that

$$E_{\gamma.x} = \sigma(\gamma, x) E_x \ ,$$

for every $\gamma \in \Gamma$ and almost every $x \in X$. Since the Cartan cocycle vanishes identically on $\mathrm{EssIm}(\phi_x)$, by [BI12, Corollary 3.1] it follows that for almost every $x \in X$, there exists an integer $1 \leq k(x) \leq m$ and a suitable $(k(x)-1)$-sphere $\mathcal{S}_x$ embedded in $\partial_\infty \mathbb{H}^m_{\mathbb{C}}$, such that

$$E_x \subseteq \mathcal{S}_x \ ,$$

for almost every $x \in X$. By the σ-equivariance of the family $\{E_x\}_{x \in X}$ we have that

$$\mathcal{S}_{\gamma.x} = \sigma(\gamma, x)\, \mathcal{S}_x \ ,$$

for every $\gamma \in \Gamma$ and almost every $x \in X$. By keeping the notation $k(x) = \dim_{\mathbb{R}} \mathcal{S}_x$, the previous condition implies that the dimension essentially constant and we denote it by k.

Denote by $\mathrm{Sph}^{k-1}(\partial_\infty \mathbb{H}^m_{\mathbb{C}})$ the space of $(k-1)$-spheres embedded in the boundary at infinity $\partial_\infty \mathbb{H}^m_{\mathbb{C}}$. Notice that $\mathrm{Sph}^{k-1}(\partial_\infty \mathbb{H}^m_{\mathbb{C}})$ is a $\mathrm{PU}(m,1)$-homogeneous space, since the action of $\mathrm{PU}(m,1)$ on $(k-1)$-spheres is transitive. Hence there exists a measurable function $f \colon X \to \mathrm{PU}(m,1)$ such that

$$\mathcal{S}_x = f(x)\, \mathcal{S}_{x_0} \ ,$$

for a fixed $x_0 \in X$ and almost every $x \in X$. It easy to verify that $f.\sigma$ has image contained into $G_0 = \mathrm{Stab}_{\mathrm{PU}(m,1)}(\mathcal{S}_{x_0})$, and the claim is proved. □

After having characterized the totally real measurable cocycles, we discuss now the maximal ones. More precisely, we are going to show that if a measurable cocycle has maximal Cartan invariant, then it is cohomologous to the cocycle associated to the standard lattice embedding $i\colon \Gamma \to \mathrm{PU}(n,1) \leq \mathrm{PU}(m,1)$ (see Theorem 3). To this end, we have to rewrite the fibered multiplicative formula (Proposition 4.10) in this setting. First, we introduce some notations. Denote by $\mathcal{C}_n$ the set of all the possible chains in $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$:

$$\mathcal{C}_n \coloneqq \{C \subset \partial_\infty \mathbb{H}^n_{\mathbb{C}} \,|C \text{ is a chain}\} \ .$$

Then, we define the *configuration space of k-tuples of points on a chain* as follows

$$\mathcal{C}_n^{[k]} \coloneqq \{(C,\xi_1,\ldots,\xi_k) \in \mathcal{C}_n \times (\partial_\infty \mathbb{H}^n_{\mathbb{C}})^k | C \in \mathcal{C}_n, \xi_1,\ldots,\xi_k \in C\} \ .$$

Notice that $\mathcal{C}_n$ can be realized as homogeneous space of the group $\mathrm{PU}(n,1)$. Indeed, denote the stabilizer of a fixed chain $C_0 \in \mathcal{C}_n$ by

$$H \coloneqq \mathrm{Stab}_{\mathrm{PU}(n,1)}(C_0) \ .$$

The group H is isomorphic to the group $\mathrm{P}(\mathrm{U}(1,1) \times \mathrm{U}(n-1))$ and since the action of $\mathrm{PU}(n,1)$ on $\mathcal{C}_n$ is transitive (being transitive on pair of distinct points of $\partial_\infty \mathbb{H}^n_{\mathbb{C}}$), we immediately get a $\mathrm{PU}(n,1)$-equivariant measure-class-preserving diffeomorphism

$$\mathrm{ev}_{C_0}\colon \ \mathrm{PU}(n,1)/H \to \mathcal{C}_n, \ \ \mathrm{ev}_{C_0}(gH) = g.C_0 \ .$$

Similarly, let $\xi_0 \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$ be a fixed basepoint and let

$$Q = \mathrm{Stab}_{\mathrm{PU}(n,1)}(\xi_0)$$

be the associated stabilizer in $\mathrm{PU}(n,1)$. Set $P \coloneqq Q \cap H$. We can define the map

$$\mathrm{ev}_{C_0,\xi_0}\colon \ \mathrm{PU}(n,1)/P \to \mathcal{C}_n^{[1]}, \ \ \mathrm{ev}_{C_0,\xi_0}(gP) \coloneqq (g.C_0, g.\xi_0) \ .$$

The above map is a measure class preserving diffeomorphism which is $\mathrm{PU}(n,1)$-equivariant with respect to the natural $\mathrm{PU}(n,1)$-action on $\mathrm{PU}(n,1)/P$ and the diagonal action of $\mathrm{PU}(n,1)$ on $\mathcal{C}_n^{[1]}$, respectively. Since $\mathrm{PU}(n,1)/P$ coincides with the fibered product $(\mathrm{PU}(n,1)/P)_f$ associated to the projection

$$\pi\colon \ \mathcal{C}_n^{[1]} \to \mathcal{C}_n, \ \ \pi(C,\xi) = C \ ,$$

the construction described above can be extended to every $k \geq 1$. More precisely, for every $k \geq 1$ and $(C_0,\xi_1,\ldots,\xi_k) \in \mathcal{C}_n^{[k]}$, the map

$$\mathrm{ev}_{C_0,\xi_1,\ldots,\xi_k}\colon \ \mathrm{PU}(n,1) \times (H/P)^k \to \mathcal{C}_n^{[k]} \ ,$$

$$\mathrm{ev}_{C_0,\xi_1,\ldots,\xi_k}(g,h_1P,\ldots,h_kP) \coloneqq (g.C_0, gh_1.\xi_1,\ldots,gh_k.\xi_k) \ ,$$

is H-invariant with respect to the action defined by Equation (5) and hence it induces a measure-class-preserving diffeomorphism between $\mathcal{C}_n^{[k]}$ and the k-fold fibered product:

$$\overline{\mathrm{ev}}_{C_0,\xi_1,\ldots,\xi_k}\colon (\mathrm{PU}(n,1)/P)^k_f \to \mathcal{C}_n^{[k]} \ .$$

Consider now $\Gamma \leq \mathrm{PU}(n,1)$ a torsion-free lattice, with $n \geq 2$. Let (X,μ_X) be a standard Borel probability Γ-space. Let $\sigma\colon \Gamma \times X \to \mathrm{PU}(m,1)$ be a measurable cocycle with essentially unique boundary map $\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$, where $m \geq$

n. If we now restrict the boundary map ϕ to a chain $C \in \mathcal{C}_n$, by Fubini's Theorem we have that the map

$$\phi_C \colon C \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}} \ ,$$

is measurable and σ-equivariant

$$\phi_{\gamma.C}(\gamma.\xi, \gamma.x) = \sigma(\gamma, x)\phi_C(\xi, x) \ ,$$

for every $\gamma \in \Gamma$ and almost every $C \in \mathcal{C}_n, \xi \in C, x \in X$. The map above allows us to define the following map

$$\phi^{[3]} \colon \mathcal{C}_n^{[3]} \times X \to (\partial_\infty \mathbb{H}^m_{\mathbb{C}})^3 \ ,$$

$$\phi^{[3]}((C, \xi_1, \xi_2, \xi_3), x) \coloneqq (\phi_C(\xi_1, x), \phi_C(\xi_2, x), \phi_C(\xi_3, x)) \ .$$

We are now ready to rewrite the fibered multiplicative formula (Proposition 4.10.2) in this specific context.

Proposition 4. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice and let (X, μ_X) be a standard Borel probability Γ-space . Consider a measurable cocycle $\sigma \colon \Gamma \times X \to \mathrm{PU}(m,1)$ with $m \geq n \geq 2$. Assume there exists an essentially unique boundary map $\phi \colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$. Denote by $\phi^{[3]} \colon \mathcal{C}_n^{[3]} \times X \to (\partial_\infty \mathbb{H}^m_{\mathbb{C}})^3$ the map induced on the configuration space $\mathcal{C}_n^{[3]}$. Then, we have*

$$\int_{\Gamma \backslash \mathrm{PU}(n,1)} \int_X c_m(\phi^{[3]}((\overline{g}.C, \overline{g}\xi_1, \overline{g}.\xi_2, \overline{g}.\xi_3), x)) d\mu_X(x) d\mu(\overline{g}) = i(\sigma) c_n(\xi_1, \xi_2, \xi_3) \ ,$$

for almost every $C \in \mathcal{C}_n$ and $\xi_1, \xi_2, \xi_3 \in C$. Here μ is the $\mathrm{PU}(n,1)$-invariant probability measure on $\Gamma \backslash \mathrm{PU}(n,1)$.

Proof. Let $P, H, Q \leq \mathrm{PU}(n,1)$ be as in the discussion above. Since Q is the parabolic stabilizer of a point $\xi_0 \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$, it is an amenable group. The same also holds for P being a closed subgroup of the amenable group Q. Notice that H acts ergodically on the product $(H/P)^2$ and hence by Remark 4.12, there are no coboundary terms appearing in the fibered multiplicative formula (Proposition 4.10.2).

In order to satisfy Setup 4.1 and apply Proposition 4.10.2, we set

$$L = \Gamma, \ G = \mathrm{PU}(n,1), \ G' = \mathrm{PU}(m,1) \text{ and } Y = \partial_\infty \mathbb{H}^m_{\mathbb{C}} \ .$$

Moreover, we take

$$\psi' = c_m \in \mathcal{B}^\infty((\partial_\infty \mathbb{H}^m_{\mathbb{C}})^3; \mathbb{R})^{\mathrm{PU}(m,1)} \text{ and } \psi = c_n \in \mathrm{L}^\infty((\partial_\infty \mathbb{H}^n_{\mathbb{C}})^3; \mathbb{R})^{\mathrm{PU}(n,1)} \ .$$

Then, since

$$\lambda_{m,n}(\sigma)\kappa^b_n = \lambda_{m,n}(\sigma)[\pi c_n] = \mathrm{trans}^2_\Gamma[\mathrm{C}^2(\Phi^X)(\pi c_m)] \ ,$$

we have

$$\int_{\Gamma \backslash \mathrm{PU}(n,1)} \int_X c_m(\phi^3_f(\overline{g}.\xi_1, \overline{g}.\xi_2, \overline{g}.\xi_3, x)) d\mu_X(x) d\mu(\overline{g}) = \lambda_{m,n}(\sigma) c_n(\xi_1, \xi_2, \xi_3) \ .$$

The thesis now follows from Proposition 5.5, which implies $\lambda_{m,n}(\sigma) = i(\sigma)$. Notice that the discussion above shows that $\phi^3_f = \phi^{[3]}$. □

As a consequence of Proposition 4, we obtain our main rigidity theorem.

Theorem 3. *Let $\Gamma \leq \mathrm{PU}(n,1)$ be a torsion-free lattice, with $n \geq 2$. Let (X,μ_X) be a standard Borel probability Γ-space . Given a measurable cocycle $\sigma\colon \Gamma \times X \to \mathrm{PU}(m,1)$ with essentially unique boundary map $\phi\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \times X \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$, we have*

$$|i(\sigma)| \leq 1 \ .$$

In particular, $i(\sigma) = 1$ if and only if σ is cohomologous to the cocycle σ_i induced by the standard lattice embedding $i\colon \Gamma \to \mathrm{PU}(n,1) \leq \mathrm{PU}(m,1)$, possibly modulo a compact subgroup when $m > n$. Here $\mathrm{PU}(n,1)$ is seen as a subgroup of $\mathrm{PU}(m,1)$ via the upper-left corner injection.

Proof. Assume that σ is maximal, i.e. $i(\sigma) = 1$. If we substitute this value in the formula of Proposition 4, we get that

$$\int_{\Gamma\backslash \mathrm{PU}(n,1)} \int_X c_m(\phi^{[3]}((\overline{g}.C, \overline{g}.\xi_1, \overline{g}.\xi_2, \overline{g}.\xi_3), x)) d\mu_X(x) d\mu(\overline{g}) = c_n(\xi_1,\xi_2,\xi_3) \ ,$$

for almost every $C \in \mathcal{C}_n$ and $\xi_1,\xi_2,\xi_3 \in C$. Let us take a chain $C \in \mathcal{C}_n$ and a triple of points $\xi_1,\xi_2,\xi_3 \in C$ such that $c_n(\xi_1,\xi_2,\xi_3) = 1$. Then, the previous formula implies

$$\int_{\Gamma\backslash \mathrm{PU}(n,1)} \int_X c_m(\phi^{[3]}((\overline{g}.C, \overline{g}.\xi_1, \overline{g}.\xi_2, \overline{g}.\xi_3), x)) d\mu_X(x) d\mu(\overline{g}) = 1 \ . \tag{22}$$

As a consequence of Equation (22), we obtain

$$c_m(\phi^{[3]}((\overline{g}.C, \overline{g}.\xi_1, \overline{g}.\xi_2, \overline{g}.\xi_3), x)) = 1 \ ,$$

for almost every $\overline{g} \in \Gamma\backslash \mathrm{PU}(n,1)$ and almost every $x \in X$. The σ-equivariance of the map $\phi^{[3]}$ implies that

$$c_m(\phi^{[3]}((g.C, g.\xi_1, g.\xi_2, g.\xi_3), x)) = 1 \tag{23}$$

still holds for almost every $g \in \mathrm{PU}(n,1)$ and almost every $x \in X$.

Now for almost every $x \in X$ we define $\phi_x\colon \partial_\infty \mathbb{H}^n_{\mathbb{C}} \to \partial_\infty \mathbb{H}^m_{\mathbb{C}}$ by $\phi_x(\xi) \coloneqq \phi(\xi,x)$. By construction the functions ϕ_x are measurable for almost every $x \in X$ [FMW04, Lemma 2.6]. If we restrict these measurable functions to a chain $C \in \mathcal{C}_n$, we obtain for almost every $C \in \mathcal{C}_n$ a measurable map

$$\phi_{x,C}\colon C \to \partial_\infty \mathbb{H}^m_{\mathbb{C}} \ .$$

Hence, we can consider the following measurable map

$$\phi^{[3]}_x\colon \mathcal{C}^{[3]}_n \to (\partial_\infty \mathbb{H}^m_{\mathbb{C}})^3, \ \ \phi^{[3]}_x(C,\xi_1,\xi_2,\xi_3) \coloneqq (\phi_{x,C}(\xi_1), \phi_{x,C}(\xi_2), \phi_{x,C}(\xi_3)) \ .$$

If we rewrite Equation (23) via $\phi^{[3]}_x$, we get

$$c_m(\phi^{[3]}_x(g.C, g.\xi_1, g.\xi_2, g.\xi_3)) = c_m(\phi_{x,g.C}(g.\xi_1), \phi_{x,g.C}(g.\xi_2), \phi_{x,g.C}(g.\xi_3)) = 1$$

for almost every $g \in \mathrm{PU}(n,1)$ and almost every $x \in X$. The latter equation implies that ϕ_x satisfies the hypothesis of [BI07b, Theorem 2.1] and hence it is induced by an isometric holomorphic embedding of $\mathbb{H}^n_{\mathbb{C}}$ in $\mathbb{H}^m_{\mathbb{C}}$. More precisely, let us consider the upper-left corner injection given by

$$i_{n,m}\colon \mathrm{PU}(n,1) \to \mathrm{PU}(m,1),\ \ i_{n,m}(g) = \begin{pmatrix} g & 0 \\ 0 & \mathrm{I}_{m-n}, \end{pmatrix}$$

where I_{m-n} is the identity matrix of order $(m-n)$. This map induces a natural embedding $\jmath\colon \mathbb{H}^n_{\mathbb{C}} \to \mathbb{H}^m_{\mathbb{C}}$ which is $i_{n,m}$-equivariant and any other isometric holomorphic embedding is given by $g \circ \jmath$, where $g \in \mathrm{PU}(m,1)$. By [BI07b, Theorem 2.1] there exists $f(x) \in \mathrm{PU}(m,1)$ such that

$$\phi_x(\xi) = f(x)\jmath(\xi) \ ,$$

for almost every $\xi \in \partial_\infty \mathbb{H}^n_{\mathbb{C}}$. In this way we get a map $f\colon X \to \mathrm{PU}(m,1)$. Since by assumption X is a standard Borel space, the measurability of f follows by Fisher, Morris and Whyte [FMW04, Lemma 2.6]. Indeed the map $\widehat{\phi}\colon X \to \mathrm{Meas}(\partial_\infty \mathbb{H}^n_{\mathbb{C}}, \partial_\infty \mathbb{H}^m_{\mathbb{C}})$, $\widehat{\phi}(x) := \phi_x$ is measurable. The thesis now follows by imitating the proof of Bader, Furman and Sauer [BFS13, Proposition 3.2].

Let

$$C := \mathrm{Stab}_{\mathrm{PU}(m,1)}(\jmath)$$

the subgroup of $\mathrm{PU}(m,1)$ which fixes the image of $\jmath$ pointwise. The latter is the trivial group when $n=m$ and it is compact when $m>n$. Fix now $\gamma \in \Gamma$. On one hand it holds

$$\phi(\gamma.\xi, \gamma.x) = f(\gamma.x)\jmath(\gamma.\xi) = f(\gamma.x) i_{n,m}(\gamma)\jmath(\xi) \ ,$$

and on the other hand we get

$$\phi(\gamma.\xi, \gamma.x) = \sigma(\gamma.x)\phi(\xi, x) = \sigma(\gamma, x) f(x)\jmath(\xi) \ .$$

Thus it follows

$$i_{n,m}(\gamma) = f(\gamma.x)^{-1}\sigma(\gamma,x)f(x) \qquad \mathrm{mod}\ C \ ,$$

ad claimed. □

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