Pathwise Smooth Splittable Congestion Games and Inefficiency

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Abstract

Very recently the inefficiency of Nash equilibria has been analyzed in the context of *Splittable Congestion Games*. These games are like the congestion games but allow the players to use convex combinations of subsets of resources. A new notion has been introduced in order to give bounds on the inefficiency or Price of Anarchy; such a notion has been termed the local smoothness. We present a unified framework where local smoothness and smoothness, a previously introduced notion, are presented as particular cases of a more general approach which we term pathwise smoothness. Such an approach is based partially on the Hadamard's Lemma, which shows that it is possible to present any function, linear or not, by means of families of linear functions.

Keywords: Atomic; Congestion; Correlated equilibrium; Price of Anarchy; Splittable; Hadamard's Lemma

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1 Introduction

In a typical strategic game it is true that Nash equilibria are socially inefficient (see [6]). It is particularly interesting to study this inefficiency in the case of congestion games. In a few words we can describe a congestion game as a game where there is a ground of resources and each player can select a subset of them. Each resource has a cost which depends on the load induced by the players who use it. Of course, each player tries to minimize the sum of the resources' costs used in his/her strategy. This type of games is simple but rich enough to capture situations very diverse and very interesting such as oligopoly models, network design, routing and even migrations of species. The original paper on this type of games is due to Rosenthal (see [12]), while the further formalization (with more results and a simplified proof) is due to Monderer and Shapley (see [10]), who introduced the notion of Potential Games. The most important result for this class of games is that there is always a Nash equilibrium in pure strategies. A simple, but very interesting, example was given by Pigou. It shows that Nash equilibria are suboptimal even in simple congestion games and the inefficiency grows with the degree of nonlinearity of the cost functions. For this class of games it is very useful, as a measure of inefficiency, the notion of the price of anarchy ($\mathcal{P}o\mathcal{A}$). This notion was introduced by Koutsoupias and Papadimitrou in (see [8]). They defined, for a system in which noncooperative agents share a common resource, the ratio between the worst possible Nash equilibrium and the social optimum as a measure of the effectiveness of the system and they computed upper and lower bounds for this ratio in a model in which several agents share a very simple network. In particular, along this line, in the class of congestion games the work of Roughgarden and Tardos has been very influential and surprising (see [14]) since, by building on the work of Koutsoupias and Papadimitrou, they studied more general networks and they computed specific bounds for different classes of cost functions. Many developments followed and very recently Roughgarden and Schoppmann have analyzed a class of games called the *Splittable Congestion Games* where it is possible to quantify, in the worst possible scenario, such an inefficiency (PoA) with the help of a supplementary condition known as the local (λ, μ)-smoothness condition (see [18] and [2]). The notion of local (λ, μ)-smoothness is very useful in order to give such measures for the $\mathcal{P}o\mathcal{A}$. For this reason, this notion has been presented in two different versions: a global version and a local one.

The main point for the introduction of the local version is to exploit to the fullest extent the local behavior. We think, given the importance of congestion games and the study of inefficiency, that it is of some interest to find new conditions to quantify the $\mathcal{P}o\mathcal{A}$.

The are two main contributions in this short paper. The first is to introduce a unified way to present local and global versions as special cases of a more general notion, that we suggest calling *pathwise smoothness*. The advantage of this new approach is twofold: not only does it shed some new light on previously developed constructions, but it also shows that we can use in a very systematic way a well-defined and easily computable family of linear maps in order to investigate the inefficiency of Nash equilibria in congestion games. This is particularly useful since, after all, we need to study the nonlinearity of cost functions because, as shown by Roughgarden, this nonlinearity is the crucial ingredient for the inefficiency, and not the topology of the network, as previously assumed. Of course, the use of linear maps is quite relevant when applicable in the study of nonlinearity. The family of linear maps depends on a family of paths which are to be chosen. It is shown that the previous known constructions due to Roughgarden and his coauthors are associated, in a certain sense, with the extreme points of the set of possible choices. Therefore, they are presented as different instances of the same construction. The second contribution is to present a simple mathematical criteria that allows us to easily compute an upper bound for $\mathcal{P}o\mathcal{A}$ when players play a correlated equilibrium.

The paper is organized as follows. In the next section we will present the relevant definitions of congestion games and a new proposed notion of path structure. In the last section we prove a general theorem that will allow us to recover the previous notions and quantify the size of the $\mathcal{P}o\mathcal{A}$.

2 Splittable Congestion Games and Paths

In order to better position our work we start by briefly talking about congestion games. In an (*atomic unsplittable*) congestion game a finite ground of resources is given and a finite number of players must share them. Each player i commits to use one particular subset of them $\mathcal{S}_i$, out of subsets of the resources. Such a subset can vary across the players. The contribution of a player to the load or congestion is determined by the player's weight $w \in (0, \infty)$. In the special case where all the players have $w = 1$ we say that the congestion game is *unweighted* otherwise we say that the game is *weighted*. Using a resource e is costly and we assume that the cost is given by a function c_e. A very concrete and useful way to think about congestion is via networks. A network congestion game is specified by a graph (V, E), together with a pair $o_i, d_i \in V$ for each player i. Each edge in the graph is a resource and the strategies of each player i are the set of paths from o_i to d_i.

We study the splittable variant of congestion games, where each player has a weight w_i and a list of available strategies (each a subset of resources), and each player chooses how to split fractionally its weight across its strategies. The splittable model is more appropriate than the traditional "unsplittable" model in some applications, such as multipath routing in networks. So, if we think of the previous network congestion game given by a graph (V, E), together with a pair $o_i, d_i \in V$ for each player i, then the strategies are given by sets of possible paths from o_i to d_i.

Now we give the definition of a Splittable Congestion Game in a more formal way. There is a set of *resources* $\mathcal{E}$ which must be shared among N players and each resource $e \in \mathcal{E}$ has a load-dependent cost given by a function $\ell_e : \mathbb{R}_+ \to \mathbb{R}_+$. Each player has a subset of resources $\mathcal{P}_i \subseteq 2^{\mathcal{E}} \backslash \varnothing$ available and the set of strategies is $S_i = \left\{ x^i \in \mathbb{R}_+^{\mathcal{P}_i} \middle| \sum_{p \in \mathcal{P}_i} x_p^i = w_i \right\}$ where w_i is the ratio of resources the agent can use, this is also called the set of fractional strategies and, from now on, we shall write $S = S_1 \times ... \times S_N$. Such a share is fixed at the beginning of the game and it is endogenous and we will write $G = \left(N; (S_i)_{i \in N}; (c_i)_{i \in N}\right)$. The set of cost functions is denoted by $\mathcal{L}$ and we assume that if $\ell \in \mathcal{L}$ then the function $x\ell(x)$ is convex, in this case we say that ℓ is semi-convex. We also assume that each element of $\mathcal{L}$ is differentiable, non-decreasing and semi-convex. We say that the set $\mathcal{L}$ is non-trivial if it contains a function ℓ_1 that is not identically zero and there exists a function $\ell \in \mathcal{L}$ which is scale invariant i.e. $\forall a, b > 0$ the function $a\ell(bx) \in \mathcal{L}$. Given a strategy profile $x = \left(x^i\right)_{i \in N}$ and a resource $e \in \mathcal{E}$ we call

$$\sum_{p \in \mathcal{P}_i; e \in p} x_p^i \stackrel{def}{=} x_e^i \qquad \text{the load of player } i \text{ on } e$$

and

$$\sum_{i \in N} x_e^i \stackrel{def}{=} x_e \qquad \text{the total load on } e$$

Given a strategy profile $x = \left(x^i\right)_{i \in N}$ the cost of player $i \in N$ is defined as

$$c_i(x) = \sum_{e \in \mathcal{E}} x_e^i \cdot \ell_e(x_e)$$

and player i tries to minimize $c_i(x)$. In this setting, we define the social cost in this way

$$SC(x) = \sum_{i \in N} c_i(x).$$

A pure strategy Nash equilibrium is a strategy profile $x = \left(x^i\right)_{i \in N}$ such that for each agent

$$c_i(x) \leq c_i(y^i, x^{-i}) \qquad \text{for every } y^i \in S_i$$

A mixed Nash equilibrium is a set of stochastically independent probability $P_1, ..., P_N$ on $S_1, ..., S_N$ such that

$$\int_S c_i(x) dP_1(x^1)...dP_1(x^N) \leq \int_S c_i(y^i, x^{-i}) dP_1(x^1)...dP_1(x^N) \qquad \text{for every } y^i \in S_i$$

A probability measure P on S is called a correlated equilibrium for the game if for each measurable function $f : S_i \to S_i$ we have

$$\int_S c_i(x) dP(x) \leq \int_S c_i(f(x^i), x^{-i}) dP(x).$$

It is important to keep in mind that for this class of games it is possible to characterize a Nash equilibrium in a very elegant way. In fact it has been proven that $x = \left(x^i\right)_{i \in N}$ is a Nash equilibrium if and only if the variational inequality

$$\sum_{e \in \mathcal{E}} \ell_e^i\left(x_e\right)\left(y_e^i - x_e^i\right) \geq 0$$

holds for any player i and for any strategy $y^i \in S_i$ where, by definition, we have $\ell_e^i(x) = \ell_e(x) + x\ell_e^{'}(x)$ (see [4] and [5]).

Now we introduce the path-structure for this type of games. A family of paths for the game is a selected family of paths $\varphi = \left(\left\{\varphi_{x^i,y^i}\right\}_{x^i,y^i \in S_i}\right)_{i\in N}$ where, for each $x^i, y^i \in S_i$, φ_{x^i,y^i} is a differentiable function $\varphi_{x^i,y^i} : [0,1] \to S_i$ with $\varphi_{x^i,y^i}(0) = x^i$.

Definition 1 *A path structure for the game is a selected family of paths* $\varphi = \left(\left\{\varphi_{x^i,y^i}\right\}_{x^i,y^i \in S_i}\right)_{i\in N}$ *and we write* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ *if a path structure is given.*

From now on we will use the symbol $\nabla_i c_i$ for

$$\nabla_i c_i = \left(\frac{\partial c_i}{\partial x_1^i}, ..., \frac{\partial c_i}{\partial x_{m_i}^i}\right)$$

i.e. to denote the gradient of c_i w.r.t. $x^i = \left(x_1^i, ..., x_{m_i}^i\right)$. Moreover, in the rest of the paper, if V to W are two vector spaces we will use the symbol $\mathbf{L}(V,W)$ to denote the set of linear transformations from V to W, this means

$$\mathbf{L}(V,W) = \{A : V \to W \,|A \text{ linear}\}.$$

Definition 2 *Given* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ *then for any* $x \in S = S_1 \times ... \times S_N$ *we define* $L_x^\varphi : S \to \mathbf{L}(\mathbb{R}^{m_1+...+m_N}, \mathbb{R}^N)$ *in the following way*

$$L_x^\varphi(y) = \begin{bmatrix} L_{x^1}^\varphi\left(y^1\right) & 0 & \cdots & 0 \\ 0 & L_{x^2}^\varphi\left(y^2\right) & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & \cdots & L_{x^N}^\varphi\left(y^N\right) \end{bmatrix}$$

where

$$L_{x^i}^\varphi\left(y^i\right) = \int_0^1 \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i})dt.$$

The next definition is very important and it will play a crucial role

Definition 3 *A cost minimization game* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ *is said to be* $\varphi-$*pathwise* (λ,μ)*-smooth with respect to the outcome* $y = \left(y^i\right)_{i\in N}$ *if for every outcome* $x = \left(x^i\right)_{i\in N}$

$$\sum_{i=1}^N \left(c_i(x) + L_{x^i}^\varphi\left(y^i\right)\left(y^i - x^i\right)\right) \leq \lambda SC(y) + \mu SC(x)$$

The next definition is an important one from our point of view.

Definition 4 *Given* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ *we say that* φ *satisfies the coherency condition for the correlated equilibrium* P *if the following inequality*

$$\int_S \langle \mathbf{1}_N L_x^{\varphi}(y), (y-x)\rangle \, dP(x) \geq 0.$$

holds for any outcome y *where the game is* $\varphi-$*pathwise* (λ, μ)*-smooth and* $\mathbf{1}_N = (1, ..., 1) \in \mathbb{R}^N$. *If the condition is satisfied for every correlated equilibrium of* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ *we say that* φ *satisfies the coherency condition.*

Of course, as usual, we assume that the *Price of Anarchy,* for a system in which noncooperative agents share common resources, is the ratio between the worst possible Nash equilibrium and the social optimum. More formally we can write that

Definition 5 *Given* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}\right)$ *we define the Price of Anarchy (*$\mathcal{P}o\mathcal{A}$ *for short)*

$$\mathcal{P}o\mathcal{A} = \sup_{P \in \mathcal{CE}(G)} \frac{\int_S SC(x) dP(x)}{SC(s^*)}.$$

where $\mathcal{CE}(G)$ *is the set of correlated equilibria of* G *and* $s^* \in S = S_1 \times ... \times S_N$ *is a minimum-cost outcome.*

In a general setting we can say that with the notion of $\mathcal{P}o\mathcal{A}$ we try to quantify the loss of optimality. In particular, this notion is useful if we wish to answer the question: How good is the worst equilibrium from a social perspective?

There are several reasons why the study of the price of anarchy is important. First, the notion of the price of anarchy goes beyond the simple but important fact that Nash equilibria are not efficient and it gives an exact quantification of this inefficiency. Moreover, understanding the price of anarchy in a system is very useful to the system designer. In many systems implementing the most efficient outcome might be infeasible or too costly. Even when regulation is tolerated, the cost of implementing the regulation might be too high. For example, if we think of a network congestion game as given by a graph (V, E) and a set of cost functions then the designer could have the power to add or delete edges in the graph but this would be way too complex. Therefore understanding the inefficiency of different designs allows the system designer to choose among the design options more precisely.

In order to present this concept in a very simple way and to understand its relationship to the nonlinearity of the cost functions we can consider the following example, due to Pigou, where there is a network with only two-edges and two-vertices

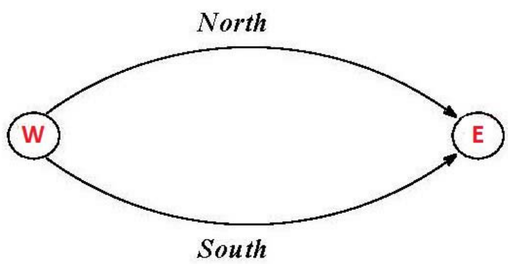

Figure 1. A standard Pigou Network

We assume that the cost of travelling, from west to east, via the northern road is $c_{North}(x) = 1$ i.e. constant and the cost of travelling via the southern road is $c_{South}(x) = (\rho(x))^k$ with $k \in \mathbb{N}$, where $\rho : [0, \infty) \to [0, \infty)$ is such that $\rho([0, 1)) \subseteq [0, 1)$ and $\rho(1) = 1$[1]. We also assume that there is a large number of players and the total number is normalized to 1. In this case an allocation between the northern road and southern road is a strategy and if we assume, for the moment, that $k = 1$ i.e. $c_{South}(x) = \rho(x)$ and each agent has a very small individual weight, then choosing to travel via the southern road is a Nash equilibrium and the total cost is 1. But we can see readily that if the players would split equally across the two roads than the cost would be $\frac{1}{2}\rho\left(\frac{1}{2}\right) + \frac{1}{2}$ therefore we must have

$$\mathcal{P}o\mathcal{A}(k = 1) \geq \frac{2}{\rho\left(\frac{1}{2}\right) + 1} > 1$$

Now we consider the case where $c_{South}(x) = (\rho(x))^k$ with $k \in \mathbb{N}$; again every player travelling via the northern road is a Nash equilibrium but we can consider

[1]The reader should keep in mind that the assumptions for the function ρ are very mild. Moreover, the special case $\rho(x) = x$ is well-known (see [18]).

the case where a small percentage, let's say Δ, travel north and the rest, $1-\Delta$ choose the southern road. In this case we would have

$$(1-\Delta)\cdot c_{South}\,(1-\Delta)+\Delta\cdot c_{North}\,(\Delta)=(1-\Delta)\,(\rho(1-\Delta))^{k}+\Delta$$

and this fact would imply that

$$\mathcal{P}o\mathcal{A}(k)\geq\frac{1}{(\rho(1-\Delta))^{k}+\Delta}$$

therefore we can write that

$$\lim_{k\to\infty}\mathcal{P}o\mathcal{A}(k)\geq\lim_{k\to\infty}\frac{1}{(\rho(1-\Delta))^{k}+\Delta}=\frac{1}{\Delta}$$

Of course Δ can be chosen arbitrarily small hence we can claim that $\mathcal{P}o\mathcal{A}(k)$, the price of anarchy for the congestion Pigou game with $c_{North}(x)=1$ and $c_{South}(x)=(\rho(x))^{k}$, goes up as the cost function $c_{South}(x)=(\rho(x))^{k}$ gets more and more nonlinear. It is clear than that the nonlinearity is crucial to quantify the price of anarchy.

3 Bounds on the inefficiency of the Nash equilibrium

In the previous section we introduced a path structure and we gave the most important definitions. The reason why we introduced those definitions is that two special instances of them have been successfully used in the study of the inefficiency of Nash equilibria. The two instances are the local smoothness and the smooth case.

In this section we aim to explore the newly introduced notions and to present several results which are new and they include, as special cases, the previous known results. Of course the main goal is to show that under suitable conditions it is possible to obtain bounds on the inefficiency by using the notions presented in the previous section. It is important to keep in mind that the computation of the $\mathcal{P}o\mathcal{A}$ can be quite challenging since the search for equilibria and the computation of their social cost can be a non-trivial problem. For this reason it is of interest to find an approach that allows us to bypass the search of equilibria but still delivers bounds on the $\mathcal{P}o\mathcal{A}$. In this section, following in part what has already been used, but also introducing new tools, we wish to add to this line of research. It is important to stress that our contribution is possible thanks to the representation of functions by means of a family of linear functions. This type of representation has played quite a big role in the study of nonlinear functions and we find this to be particularly important as we have showed, in the last section, with the help of a class of congestion games, that the nonlinearity plays a huge role in the computation of $\mathcal{P}o\mathcal{A}$.

We start with the simple but important

Theorem 6 *Let P be a correlated equilibrium of a cost minimization game. If the game is $\varphi-$pathwise locally (λ, μ)-smooth with respect to the outcome $y = \left(y^i\right)_{i \in N}$ with $\mu < 1$ and φ satisfies the coherency condition for P then*

$$\int_S SC(x)dP \leq \frac{\lambda}{1-\mu} SC(y).$$

Proof. We begin by observing that

$$\begin{aligned}
\int_S SC(x)dP &= \int_S \left(\sum_{i=1}^N c_i(x) \right) dP(x) \\
&\leq \int_S \left(\sum_{i=1}^N c_i(x) \right) dP(x) + \int_S \langle \mathbf{1}_N L_x^{\varphi}(y), (y-x) \rangle \, dP(x) \\
&= \int_S \left(\sum_{i=1}^N c_i(x) \right) dP(x) \\
&\quad + \int_S \sum_{i=1}^N \left\langle \int_0^1 \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i}), (y^i - x^i) \right\rangle dP(x) \\
&= \int_S \sum_{i=1}^N \left(c_i(x) + L_{x^i}^{\varphi}\left(y^i\right)\left(y^i - x^i\right) \right) dP(x) \\
&\leq \int_S \left(\lambda SC(y) + \mu SC(x) \right) dP(x) \\
&= \lambda SC(y) + \mu \int_S SC(x) dP(x).
\end{aligned}$$

The last inequality implies that

$$\int_S SC(x)dP - \mu \int_S SC(x)dP \leq \lambda SC(y)$$

or

$$\int_S SC(x)dP \leq \frac{\lambda}{1-\mu} SC(y)$$

as we claimed. ■

We can use this result to obtain an interesting consequence. In fact we readily get the following

Proposition 7 *Let P be a correlated equilibrium of a cost minimization game $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$. If the game G is $\varphi-$pathwise (λ, μ)-smooth with respect to the outcome $y = \left(y^i\right)_{i\in N}$ and*

$$\int_S \left\langle \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i}), (y^i - x^i)\right\rangle dP \geq 0$$

holds then

1. The path structure φ satisfies the coherency condition for P;

2. Moreover, if $\mu < 1$ we have

$$\int SC(x)dP \leq \frac{\lambda}{1-\mu} SC(y).$$

Proof. The assumption

$$\int_S \left\langle \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i}), (y^i - x^i)\right\rangle dP \geq 0$$

implies that

$$\int_0^1 \int_S \left\langle \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i}), (y^i - x^i)\right\rangle dPdt \geq 0.$$

Using this inequality and Fubini's Theorem we can write that

$$\begin{aligned} 0 &\leq \int_0^1 \int_S \left\langle \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i}), (y^i - x^i)\right\rangle dPdt \\ &= \int_S \int_0^1 \left\langle \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i}), (y^i - x^i)\right\rangle dtdP \\ &= \int_S \left\langle \int_0^1 \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i})dt, (y^i - x^i)\right\rangle \end{aligned}$$

hence it follows that $\int_S L^{\varphi}_{x^i}\left(y^i\right)(y^i - x^i)dP \geq 0$ and this implies that

$$\int_S \left\langle \mathbf{1}_N L^{\varphi}_x(y), (y-x)\right\rangle dP \geq 0.$$

The second claim is an immediate consequence of what we proved in Theorem 4. ■

At this point, as corollary, we can recover a central result previously established in the cited work. In fact we can prove the following

Corollary 8 *If the game* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ *is* $\varphi-$*pathwise locally* (λ, μ)*-smooth with respect to the outcome* $y = \left(y^i\right)_{i\in N}$ *and* $\varphi_{x^i,y^i}(t) = x^i$ *and* $\mu < 1$ *then*

1. The structure φ *satisfies the coherency condition;*

2. Moreover, since we assume $\mu < 1$*, we have*

$$\int SC(x)dP \le \frac{\lambda}{1-\mu} SC(y).$$

Proof. To prove the first claim we observe that it is sufficient to show that $\int_S \left\langle \nabla_i c_i(x), (y^i - x^i)\right\rangle dP$ must be nonnegative[2]. To prove this we assume, by way of contradiction, that

$$\int_S \left\langle \nabla_i c_i(x), (y^i - x^i)\right\rangle dP < 0$$

then we can choose a sequence $\{r_n\} \subset (0, 1]$ with $r_n \to 0$ and, by using Lebesgue Dominated Convergence Theorem, this implies that

$$\begin{aligned}
0 &> \int_S \left\langle \nabla_i c_i(x), (y^i - x^i)\right\rangle dP \\
&= \int_S \lim_{n\to\infty} \frac{1}{r_n} \left(c_i(x^i + r_n(y^i - x^i), x^{-i}) - c_i(x)\right), dP(x) \\
&= \lim_{n\to\infty} \int_S \frac{1}{r_n} \left(c_i(x^i + r_n(y^i - x^i), x^{-i}) - c_i(x)\right), dP(x) \\
&= \lim_{n\to\infty} \frac{1}{r_n} \int_S c_i(x^i + r_n(y^i - x^i), x^{-i}) - c_i(x) dP(x)
\end{aligned}$$

therefore, when n is big enough, we have

$$\frac{1}{r_n} \int_S \left(c_i(x^i + r_n(y^i - x^i), x^{-i}) - c_i(x)\right) dP(x) < 0$$

and since by hypothesis $\{r_n\} \subset (0, 1]$ we can write that

$$\int_S c_i(x^i + r_n(y^i - x^i), x^{-i}) P(x) < \int_S c_i(x) dP(x)$$

which is impossible since by hypothesis P is a correlated equilibrium. Therefore we have

$$\int_S \left\langle \nabla_i c_i(x), (y^i - x^i)\right\rangle dP \ge 0$$

This inequality, together with Theorem 1 and Proposition 2, implies that 1. and 2. must hold. ■

[2] The strategy we use to prove this inequality is essentially due to Neyman. However, for the sake of completeness, we spell out the details.

In order to show the wide applicability of the introduced construction we are going to investigate its applicability in the context of a cost minimization game G which is (λ, μ)-smooth. We shall show that this notion can be, from the formal point of view, derived from our general definition with the help of a classical result known as Hadamard's Lemma. This classical result allows us to present a general function by using only linear functions and, among other things, can be even used to prove other important and fundamental results in Analysis such as the Morse Lemma. The reader should see [1] for a very nice treatment of this important result and its applications. However, since in the cited book a very general version is presented (i.e. using Banach spaces and related notions) we prove it explicitly in the case of finite dimensional spaces.

Proposition 9 (Finite Dimensional Hadamard's Lemma) *Let $g : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^r map with $r \geq 1$ and let $x^0 \in U$. If U is convex and $g(x^0) = 0$ then there exists a C^{r-1}map $L_{x^0} : U \to \mathbf{L}(\mathbb{R}^n, \mathbb{R}^m)$ such that*

$$g(x) = L_{x^0}(x)\left(x - x^0\right)$$

where $\mathbf{L}(\mathbb{R}^n, \mathbb{R}^m) = \{A : \mathbb{R}^n \to \mathbb{R}^m \,| A \text{ linear}\}$.

Proof. To start we observe that the convexity of U implies that for any $x \in U$ we have $x^0 + tx \in U$ and if we write $g(x_1, ..., x_n) = (g_1(x_1, ..., x_n), ..., g_m(x_1, ..., x_n))$ than we can write that

$$\begin{aligned} g(x) &= \int_0^1 \frac{d}{dt} g(x_1^0 + t(x_1 - x_1^0), ..., x_n^0 + t(x_n - x_n^0))dt \\ &= \int_0^1 \left(\sum_{j=1}^n \frac{\partial g_i}{\partial x_j}(x_1^0 + t(x_1 - x_1^0), ..., x_n^0 + t(x_n - x_n^0)) \left(x_j - x_j^0\right) \right)_{i=1,...,m} dt \\ &= \left(\sum_{j=1}^n \left(x_j - x_j^0\right) \int_0^1 \frac{\partial g_i}{\partial x_j}(x_1^0 + t(x_1 - x_1^0), ..., x_n^0 + t(x_n - x_n^0))dt \right)_{i=1,...,m} \end{aligned}$$

If we define

$$L_{x^0}(x) = \left[\int_0^1 \frac{\partial g_i}{\partial x_j}(x_1^0 + t(x_1 - x_1^0), ..., x_n^0 + t(x_n - x_n^0))dt \right]_{\substack{i=1,...,m \\ j=1,..,n}}$$

clearly $L_{x^0}(x) \in \mathbf{L}(\mathbb{R}^n, \mathbb{R}^m)$, each component is C^{r-1} and the following

$$g(x) = L_{x^0}(x)\left(x - x^0\right)$$

holds. ■

We are ready to prove the next result but, before we do that, we need to recall a few definitions which are important in the study of the $\mathcal{P}o\mathcal{A}$ of an equilibrium in a congestion game. We already anticipated that by definition the $\mathcal{P}o\mathcal{A}$ is the biggest possible ratio between the expected social cost of the equilibrium and that of a cost minimization strategy profile or social optimum. Of course it is not a trivial task to quantify by direct computations. It comes out extremely useful, for bounding the $\mathcal{P}o\mathcal{A}$, to introduce the notion of *smoothness*. The central ideal of the use of the smoothness is as follows: start by using the properties of the game to prove a smoothness inequality which means to find a pair λ, μ, and then use an extension theorem that is based on the equilibrium condition to obtain a bound on the $\mathcal{P}o\mathcal{A}$ itself.

Definition 10 ((λ, μ)-smoothness) *A cost minimization game is (λ, μ)-smooth if for any outcome x, for a fixed outcome y, the following inequality holds for some constants $\lambda > 0$ and $\mu < 1$*

$$\sum_{i=1}^{N} c_i(y^i, x^{-i}) \leq \lambda SC(y) + \mu SC(x).$$

The importance of this notion stems from the fact that it allows us to compute upper bounds for $\mathcal{P}o\mathcal{A}$ in a relatively easy fashion. Our next result shows that our general construction captures, as a special case, the notion of *smoothness*. In fact we can prove the following

Theorem 11 *If the game is φ–pathwise (λ, μ)-smooth with respect to the outcome $y = \left(y^i\right)_{i \in N}$ with the family $\varphi = \left(\left\{\varphi_{x^i,y^i}\right\}_{x^i,y^i}\right)_{i \in N}$ where $\varphi_{x^i,y^i} = x^i + t(y^i - x^i)$ then the following hold*

1. The strong inequality holds

$$\sum_{i=1}^{N} c_i(y^i, x^{-i}) \leq \lambda SC(y) + \mu SC(x)$$

i.e. the game is (λ, μ)-smooth.

2. The inequality

$$\frac{1}{1-\mu} \sum_{i=1}^{N} L_{x^i}^{\varphi}\left(y^i\right)\left(y^i - x^i, x^{-i}\right) \leq \frac{\lambda}{1-\mu} SC(y) - SC(x)$$

holds

3. For every correlated equilibrium P with $\mu < 1$ $\int_S SC(x)dP \leq \frac{\lambda}{1-\mu} SC(y)$.

Proof. To prove the first claim we notice that if we define

$$\widetilde{c_i}(y^i, x^{-i}) = c_i(y^i, x^{-i}) - c_i(x^i, x^{-i})$$

then $\widetilde{c_i}(x^i) = 0$ and $\nabla_i c_i = \nabla_i \widetilde{c_i}$ since $c_i(x^i, x^{-i})$ is a constant. Hence we can apply Hadamard's Lemma to $\widetilde{c_i}$ and this gives that for any $i = 1, ..., N$

$$\begin{aligned}
L^{\varphi}_{x^i}\left(y^i\right)(y^i - x^i) &= \left\langle \int_0^1 \nabla_i c_i(\varphi_{x^i,y^i}(t), x^{-i}), (y^i - x^i) \right\rangle dt \\
&= \left\langle \int_0^1 \nabla_i c_i(x^i + t(y^i - x^i), x^{-i}), (y^i - x^i) \right\rangle dt \\
&= \left\langle \int_0^1 \nabla_i \widetilde{c_i}(x^i + t(y^i - x^i), x^{-i}), (y^i - x^i) \right\rangle dt \\
&= \widetilde{c_i}(y^i, x^{-i})
\end{aligned}$$

hence

$$\begin{aligned}
\lambda SC(y) + \mu SC(x) &\geq \sum_{i=1}^{N} c_i(x) + \sum_{i=1}^{N} L^{\varphi}_{x^i}\left(y^i\right)(y^i - x^i) \\
&= \sum_{i=1}^{N} \left[c_i(x) + \widetilde{c_i}(y^i)\right] \\
&= \sum_{i=1}^{N} \left[c_i(x) + c_i(y^i, x^{-i}) - c_i(x^i, x^{-i})\right] \\
&= \sum_{i=1}^{N} c_i(y^i, x^{-i})
\end{aligned}$$

To prove 2. we observe that

$$\begin{aligned}
\sum_{i=1}^{N} c_i(y^i, x^{-i}) &= \sum_{i=1}^{N} c_i(x^i, x^{-i}) + \left(c_i(y^i, x^{-i}) - c_i(x^i, x^{-i})\right) \\
&= \sum_{i=1}^{N} \left(c_i(x^i, x^{-i}) + L^{\varphi}_{x^i}\left(y^i\right)(y^i - x^i, x^{-i})\right) \\
&= SC(x) + \sum_{i=1}^{N} L^{\varphi}_{x^i}\left(y^i\right)(y^i - x^i, x^{-i})
\end{aligned}$$

which implies

$$\frac{1}{1-\mu} \sum_{i=1}^{N} L^{\varphi}_{x^i}\left(y^i\right)(y^i - x^i, x^{-i}) \leq \frac{\lambda}{1-\mu} SC(y) - SC(x)$$

as claimed. To prove 3. we observe that

$$\int_S \langle \mathbf{1}_N L_x^{\varphi}(y), (y-x)\rangle \, dP \geq 0$$

holds since

$$\begin{aligned}
\int_S \langle \mathbf{1}_N L_x^{\varphi}(y), (y-x)\rangle \, dP &= \sum_{i=1}^{N} \int_S L_{x^i}^{\varphi}\left(y^i\right)(y^i - x^i) dP \\
&= \int_S \left(\sum_{i=1}^{N} \left[c_i(y^i, x^{-i}) - c_i(x^i, x^{-i}) \right] \right) dP \\
&= \int_S \sum_{i=1}^{N} c_i(y^i, x^{-i}) dP - \int_S \left(\sum_{i=1}^{N} c_i(x^i, x^{-i}) \right) dP \\
&\geq 0
\end{aligned}$$

hence, by what we proved in Theorem 1, it follows the thesis. ■

It is useful at this point to clarify what we briefly mentioned in the introduction. In the introduction we wrote that previously known constructions, due to Roughgarden and his coauthors, are associated, in a certain sense, to the extreme points of the set of possible choices. In fact when we need to obtain results about (λ, μ)-smooth we use the maps $\varphi_{x^i,y^i} = x^i + t(y^i - x^i)$, i.e. a path that goes from x^i to y^i. Instead, when we need to obtain results about locally (λ, μ)-smooth we use the maps $\varphi_{x^i,y^i}(t) = x^i$ i.e. a path that stays at x^i. Therefore, from the geometrical point of view, in the first case we travel along the path to reach the second extreme point of the path and in the second case we remain at the first extreme point of the path.

We are now in the position to present a method to get bounds for the $\mathcal{PoA}$ in games where a path structure has been selected. It is important to keep in mind that the construction we are going to present will allow us to recover previous known results once the appropriate family of paths is given. It is well-known that in previous work sharp bounds were obtained for games that have the family structure which was termed the local one. Also, more was established for a special class of game in the splittable and in the not splittable cases. The reader will find in [15] a very nice summary and discussion of the known cases. To proceed we need the following

Definition 12 *Given* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$, *we define* $\forall \ell \in \mathcal{L}$ *and*

$\forall y, x \in S_1 \times ... \times S_N$ *the functions* $\widetilde{\ell}, \mathcal{H}, H$ *and* k *as it follows*

$$\widetilde{\ell}\left(x^{-i}, \varphi_{x^i,y^i}(t)\right) \stackrel{def}{=} \ell\left(\sum_{j\neq i} x^j + \varphi_{x^i,y^i}(t)\right)$$

and

$$\mathcal{H}(x,y;\varphi,\ell) \stackrel{def}{=} \sum_{i=1}^{N} y^i \int_0^1 \widetilde{\ell}\left(x^{-i}, \varphi_{x^i,y^i}\right)$$

and

$$H(x,y;\varphi,\ell) \stackrel{def}{=} \sum_{i=1}^{N} x^i \left(\ell\left(\sum_{i=1}^{N} x^i\right) - \int_0^1 \widetilde{\ell}\left(x^{-i}, \varphi_{x^i,y^i}\right)\right)$$

and

$$k(x,y;\varphi,\ell) \stackrel{def}{=} \sum_{i=1}^{N} \int_0^1 \widetilde{\ell}'\left(x^{-i}, \varphi_{x^i,y^i}\right) \varphi_{x^i,y^i}(t)(y^i - x^i).$$

and we give also the following

Definition 13 *We say that the path structure* φ *is uniform if there exists, for every* $\ell \in \mathcal{L}$, *a function* $K(x,y;\varphi,\ell)$ *such that*

$$H(x,y;\varphi,\ell) + k(x,y;\varphi,\ell) \leq K(x,y;\varphi,\ell)$$

with $K(x,y;\varphi,\ell) > 0$ *when* $x > \alpha y$ *where* $\alpha \in (0,1)$ *is fixed.*

With these definitions in mind we are now ready to prove the simple but useful

Theorem 14 *Given* $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ *with* $\mathcal{L}$ *as class of allowable cost functions. If the path structure is uniform via* $K(x,y;\varphi,\ell)$ *and the following*

$$\mathcal{H}(x,y;\varphi,\ell) + K(x,y;\varphi,\ell) \leq \lambda y \ell(y) + \mu x \ell(x)$$

holds then the splittable congestion game is $\varphi-$*pathwise locally* (λ,μ)*-smooth.*

Proof. Note that we have the following

$$\begin{aligned}
& \sum_{i=1}^{N}\left(c_i(x)+L_{x^i}^{\varphi}\left(y^i\right)\left(y^i-x^i\right)\right)\\
= & \sum_{i=1}^{N}\left[c_i(x)+\left\langle\int_0^1 \nabla_i c_i(\varphi_{x^i,y^i}(t),x^{-i}),(y^i-x^i)\right\rangle dt\right]\\
= & \sum_{i=1}^{N}\sum_{e\in\mathcal{E}}\begin{pmatrix} x_e^i\ell_e\left(x_e\right)+\int_0^1\widetilde{\ell}_e\left(x_e^{-i},\varphi_{x_e^i,y_e^i}(t)\right)(y_e^i-x_e^i)+\\ \int_0^1\varphi_{x_e^i,y_e^i}(t)\widetilde{\ell}'_e\left(x_e^{-i},\varphi_{x_e^i,y_e^i}(t)\right)(y_e^i-x_e^i)\end{pmatrix}\\
= & \sum_{e\in\mathcal{E}}\begin{pmatrix}\sum_{i=1}^{N}y_e^i\int_0^1\widetilde{\ell}_e\left(x_e^{-i},\varphi_{x_e^i,y_e^i}(t)\right)+\\ \sum_{i=1}^{N}x_e^i\left(\ell_e\left(x_e\right)-\int_0^1\widetilde{\ell}_e\left(x_e^{-i},\varphi_{x_e^i,y_e^i}(t)\right)\right)\end{pmatrix}\\
& +\sum_{e\in\mathcal{E}}\left(\sum_{i=1}^{N}\int_0^1\widetilde{\ell}'_e\left(x_e^{-i},\varphi_{x_e^i,y_e^i}\right)\varphi_{x_e^i,y_e^i}(t)(y_e^i-x_e^i)\right)\\
= & \sum_{e\in\mathcal{E}}\left(\mathcal{H}(x_e,y_e;\varphi,\ell_e)+H(x_e,y_e;\varphi,\ell_e)+k(x_e,y_e;\varphi,\ell_e)\right)
\end{aligned}$$

therefore it follows that

$$\begin{aligned}
& \sum_{e\in\mathcal{E}}\left(\mathcal{H}(x_e,y_e;\varphi,\ell_e)+H(x_e,y_e;\varphi,\ell_e)+k(x_e,y_e;\varphi,\ell_e)\right)\\
\leq & \sum_{e\in\mathcal{E}}\left(\mathcal{H}(x_e,y_e;\varphi,\ell_e)+K(x_e,y_e;\varphi,\ell_e)\right)\\
\leq & \sum_{e\in\mathcal{E}}\left(y_e\ell(y_e)+\mu x_e\ell(x_e)\right)
\end{aligned}$$

which implies that

$$\sum_{i=1}^{N}\left(c_i(x)+L_{x^i}^{\varphi}\left(y^i\right)\left(y^i-x^i\right)\right)\leq\lambda SC(y)+\mu SC(x)$$

as we claimed. ■

In order to get the bounds we need to introduce a few definitions. To begin with we set

$$g_{\ell,x,y}^{(\varphi)}\left(\mu\right)=\frac{\mathcal{H}(x,y;\varphi)+K(x,y)-\mu x\ell(x)}{y\ell(y)\left(1-\mu\right)}$$

for any *admissible triple* ℓ,x,y meaning $\ell\in\mathcal{L}$ and $x,y\geq 0$ with $\ell(y)>0$. We have

$$\sup_{\substack{\ell\in\mathcal{L}\\ x\geq 0,y\geq 0}} g_{\ell,x,y}^{(\varphi)}\left(\mu\right)\leq\frac{\lambda}{1-\mu}$$

hence we set $\gamma_{\varphi}(\mathcal{L})$ as follows

$$\gamma_{\varphi}(\mathcal{L}) = \inf_{\mu \in [0,1)} \sup_{\substack{\ell \in \mathcal{L} \\ x \geq 0, y \geq 0}} g^{(\varphi)}_{\ell,x,y}(\mu).$$

At this point we can observe that it is possible that for the same game to have a set Φ of path structures such that for any $\varphi \in \Phi$ the game is $\varphi-$*pathwise* (λ, μ)-*smooth.* In this case, it follows immediately that

Theorem 15 *For every non-trivial set $\mathcal{L}$ of cost functions and for every splittable congestion game with cost functions in $\mathcal{L}$ which is $\varphi-$pathwise locally (λ, μ)-smooth with respect to every outcome, with $\mu < 1$ and φ satisfying the coherency condition, the price of anarchy $\mathcal{POA}$ of the correlated equilibria satisfies the following inequality*

$$\mathcal{POA} \leq \gamma_{\Phi}(\mathcal{L})$$

where, by definition, $\gamma_{\Phi}(\mathcal{L}) = \inf_{\varphi \in \Phi} \gamma_{\varphi}(\mathcal{L})$ and Φ is the family of path structures which are uniform.

Proof. The proof is quite immediate since for any $\varphi \in \Phi$ where Φ is the family of path structures which are uniform then, by the last theorem, it must follow that

$$\sum_{i=1}^{N} \left(c_i(x) + L^{\varphi}_{x^i}\left(y^i\right)\left(y^i - x^i\right)\right) \leq \lambda SC(y) + \mu SC(x)$$

which in turn implies, by the first theorem in this section, that

$$\int_S SC(x)dP \leq \frac{\lambda}{1-\mu} SC(y)$$

and therefore, given the definition of γ_{φ}, we can claim, for any $\varphi \in \Phi$, that the inequality $\mathcal{POA} \leq \gamma_{\varphi}(\mathcal{L})$ must hold which implies that

$$\mathcal{POA} \leq \gamma_{\Phi}(\mathcal{L}) = \inf_{\varphi \in \Phi} \gamma_{\varphi}(\mathcal{L}).$$

■

At this point we have established an important upper bound for the loss of optimality in the game. Finally we observe that if we take for the game the local path-structure $\varphi = \left(\left\{\varphi_{x^i,y^i}\right\}_{x^i,y^i}\right)_{i \in N}$ with $\varphi_{x^i,y^i}(t) = x^i$ and Φ which contains only that path structure and we use the function $K(x, y; \varphi, \ell) =$

$K(x,y;\varphi)\ell'$ where $K(x,y;\varphi)$ is defined as $y^2/4$ if $x \geq y/2$ and $x(y-x)$ $x < y/2$ then we recover what has been done.

We wish to present an application of what we have developed so far. To start we remind the reader that one of the most important type of results in this line of research is to quantify the exact bounds for the $\mathcal{P}o\mathcal{A}$ i.e. $\gamma_{\Phi}(\mathcal{L})$. To answer this question it means to find a congestion game G with cost functions in $\mathcal{L}$ that achieves $\gamma(\mathcal{L})$ or a sequence of congestion games with a Price of Anarchy which converge to $\gamma(\mathcal{L})$. We want to show that if our starting point is a game $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ where the paths are such that $\varphi_{x^i,y^i}(t)$ is not a constant function but there exists a sufficiently small $\delta > 0$ such that $\sup\left\{\left\|\varphi_{x^i,y^i}(t) - x^i\right\| | t \in [0,1]\right\} \leq \delta$ for every $x^i, y^i \in S_i$ and $\forall i = 1,..,N$, we call this type $\delta-$*short paths, then* we are at a good starting point, assuming of course that coherence and so on are satisfied. In fact, under the $\delta-$*short paths* assumption, it follows that there exist $\epsilon_1 = \epsilon_1(x,y,\ell), \epsilon_2 = \epsilon_2(x,y,\ell), \epsilon_3 = \epsilon_3(x,y,\ell)$[3] such that

$$\mathcal{H}(x,y;\varphi,\ell) \overset{def}{=} \epsilon_1 \sum_{i=1}^{N} y^i \ell\left(\sum_j x^j\right) = \epsilon_1 \ell\left(\sum_j x^j\right)\sum_{i=1}^{N} y^i$$

and

$$H(x,y;\varphi,\ell) \overset{def}{=} \epsilon_2 \ell\left(\sum_{j\neq i} x^j\right)\sum_{i=1}^{N} x^i$$

and

$$k(x,y;\varphi,\ell) \overset{def}{=} \epsilon_3 \sum_{i=1}^{N} \widetilde{\ell'}\left(\sum_{j\neq i} x^j\right) x^i(y^i - x^i).$$

In this case we have

$$H(x,y;\varphi,\ell)+k(x,y;\varphi,\ell) \overset{def}{=} \epsilon_3 \ell'\left(\sum_{j\neq i} x^j\right)\sum_{i=1}^{N} x^i(y^i-x^i)+\epsilon_2\ell\left(\sum_j x^j\right)\sum_{i=1}^{N} x^i$$

and we can write, for some $\theta \in [0,1]$,

$$H(x,y;\varphi,\ell)+k(x,y;\varphi,\ell) = \epsilon_3 \ell'\left(\sum_{j\neq i} x^j\right)\sum_{i=1}^{N} x^i(y^i-x^i)+\epsilon_2\ell'\left(\theta\sum_j x^j\right)\left(\sum_{i=1}^{N} x^i\right)^2$$

[3] The intuition is clear: if the paths are very short then $\epsilon_1 = \epsilon_1(x,y,\ell)$ is very close to 1, $\epsilon_2 = \epsilon_2(x,y,\ell)$ is very close to 0, $\epsilon_3 = \epsilon_3(x,y,\ell)$ is very close to 1.

which implies

$$H(x,y;\varphi,\ell)+k(x,y;\varphi,\ell)\le \ell'\left(\sum_{j\neq i}x^j\right)\left(\epsilon_3\sum_{i=1}^N x^i(y^i-x^i)+\epsilon_2\left(\sum_{i=1}^N x^i\right)^2\right)$$

and, if we define the function $K(x,y)$ in this way

$$K(x,y)=\begin{cases}\epsilon_3 x(y-x)+\epsilon_2 x^2 & x\le \frac{y}{2}\\ \\ \epsilon_3 \frac{y}{4}^2+\epsilon_2 x^2 & x>\frac{y}{2}\end{cases}$$

we get

$$H(x,y;\varphi,\ell)+k(x,y;\varphi,\ell)\le \ell'\left(\sum_{i=1}^N x^i\right)K(x,y).$$

Now we consider

$$g^{(\varphi)}_{\ell,x,y}(\mu)=\frac{\mathcal{H}(x,y;\varphi,\ell)+K(x,y)\ell'(x)-\mu x\ell(x)}{y\ell(y)(1-\mu)}$$

we assume that the following holds

$$\mathcal{H}(x,y;\varphi,\ell)+K(x,y;\varphi,\ell)\le \lambda y\ell(y)+\mu x\ell(x)$$

To proceed we introduce the following function

$$h^{(\varphi)}_{\ell,x,y}=(\mathcal{H}(x,y;\varphi,\ell)-x\ell(x))+K(x,y;\varphi,\ell)=(\epsilon_1\ell(x)y-x\ell(x))+K(x,y;\varphi,\ell)$$

and we note that if $g^{(\varphi)}_{\ell,x,y}(\mu)$ gives the sup then we have

$$\begin{aligned}\frac{\lambda}{1-\mu} &= g^{(\varphi)}_{\ell,x,y}(\mu)\\ &= \frac{\mathcal{H}(x,y;\varphi,\ell)+K(x,y;\varphi,\ell)-\mu x\ell(x)}{y\ell(y)(1-\mu)}\\ &= \frac{h^{(\varphi)}_{\ell,x,y}-\mu x\ell(x)+x\ell(x)}{y\ell(y)(1-\mu)}\end{aligned}$$

and, by a simple calculation, we get immediately that the sign of $h^{(\varphi)}_{\ell,x,y}$ tells us about the monotonicity of $g^{(\varphi)}_{\ell,x,y}(\mu)$ since

$$\frac{\partial g^{(\varphi)}_{\ell,x,y}}{\partial\mu}(\mu)=\frac{h^{(\varphi)}_{\ell,x,y}(\mu)}{y\ell(y)(1-\mu)^2}$$

The reader should also notice that since

$$g^{(\varphi)}_{\ell,x,y}(\mu)=\frac{(\epsilon_1\ell(x)y-x\ell(x))+K(x,y;\varphi)}{y\ell(y)(1-\mu)}$$

if we try to match the upper bound we must have

$$
\begin{aligned}
\frac{\lambda}{1-\mu} &= \gamma_{\varphi}(\mathcal{L}) = \inf_{\mu\in[0,1)} \sup_{\substack{\ell\in\mathcal{L} \\ x\geq 0, y\geq 0}} g^{(\varphi)}_{\ell,x,y}(\mu) \\
&= \inf_{\mu\in[0,1)} \sup_{\substack{\ell\in\mathcal{L} \\ x\geq 0, y\geq 0}} \frac{(\epsilon_1 \ell(x) y - x\ell(x)) + K(x,y;\varphi)\ell(x)}{y\ell(y)(1-\mu)} \\
&= \inf_{\mu\in[0,1)} \sup_{\substack{\ell\in\mathcal{L} \\ x\geq 0, y\geq 0}} \begin{cases} \frac{(\epsilon_1\ell(x)y - x\ell(x)) + \epsilon_3 x(y-x) + \epsilon_2 x^2}{y\ell(y)(1-\mu)} \ell'(x) & x \leq \frac{y}{2} \\ \frac{(\epsilon_1\ell(x)y - x\ell(x)) + \epsilon_3 \frac{y}{4}^2 + \epsilon_2 x^2}{y\ell(y)(1-\mu)} \ell'(x) & x > \frac{y}{2} \end{cases}
\end{aligned}
$$

therefore we have three new parameters we can use, namely $\epsilon_1, \epsilon_2, \epsilon_3$. This is an important point since the search of exact values for the Price of Anarchy is extremely hard and limited to a few cases.

In order to proceed we need a simple technical result which is very useful. We assume that the following holds

Condition 16 (C1) *We always assume that* $\forall \widehat{\gamma} \in \left(0, \gamma_{\varphi}(\mathcal{L})\right)$

$$
\mathcal{A}_{\widehat{\gamma}} = \left\{ (\ell, x, y) \middle| g^{(\varphi)}_{\ell,x,y}(\mu) \geq \widehat{\gamma} \ \text{ and } \ g^{(\varphi)}_{\ell,x,y}(\mu) \text{ is strictly increasing} \right\} \neq \varnothing
$$

We observe that if δ is small enough the condition is certainly satisfied. We prove

Proposition 17 *Let $\mathcal{L}$ be a set of non trivial cost functions. If we assume there is an admissible ℓ, x, y for a splittable congestion game $\varphi-$pathwise locally (λ, μ)-smooth $G = \left(N; (S_i)_{i\in N}; (c_i)_{i\in N}; \varphi\right)$ with $\delta-$short paths such that $h^{(\varphi)}_{\ell,x,y} < 0$ then for every $\widehat{\gamma} < \gamma_{\varphi}(\mathcal{L})$ there are $\mu < 1$ and admissible ℓ_1, x_1, y_1 and ℓ_2, x_2, y_2 such that*

$$
\begin{aligned}
g^{(\varphi)}_{\ell_1,x_1,y_1}(\mu) &= g^{(\varphi)}_{\ell_2,x_2,y_2}(\mu) \geq \widehat{\gamma} \\
& \quad\text{and} \\
sgn\left(h^{(\varphi)}_{\ell_1,x_1,y_1}\right) &= -sgn\left(h^{(\varphi)}_{\ell_2,x_2,y_2}\right).
\end{aligned}
$$

Proof. If we set

$$
\mu^* = \inf\left\{ \mu \in [0,1) \middle| g^{(\varphi)}_{\ell,x,y} \in \mathcal{A}_{\widehat{\gamma}} \right\}
$$

then $\mu^* < 1$. We claim that there exists $\widehat{\mu} < 1$ and admissible triples ℓ_1, x_1, y_1 and ℓ_2, x_2, y_2 such that $g^{(\varphi)}_{\ell_1,x_1,y_1}(\widehat{\mu}) \geq g^{(\varphi)}_{\ell_2,x_2,y_2}(\widehat{\mu}) \geq \widehat{\gamma}$ with $g^{(\varphi)}_{\ell_1,x_1,y_1}$ strictly increasing and $g^{(\varphi)}_{\ell_2,x_2,y_2}$ strictly decreasing. In fact if the $g^{(\varphi)}_{\ell_1,x_1,y_1}$ strictly increasing with $g^{(\varphi)}_{\ell_1,x_1,y_1}(\mu^*) \geq \widehat{\gamma}$ then $\widehat{\mu} < \mu^*$for which $g^{(\varphi)}_{\ell_2,x_2,y_2}(\widehat{\mu}) > \widehat{\gamma}$ then it follows immediately that $\mu^* = 0$ and if we use the fact that there is $h^{(\varphi)}_{\ell,x,y}$ with $h^{(\varphi)}_{\ell,x,y} < 0$ then we must have

$$h^{(\varphi)}_{\ell,x,y} = (\mathcal{H}(x,y) - x\ell(x)) + K(x,y;\varphi) = (\epsilon_1 \ell(x)\, y - x\ell(x)) + K(x,y)\ell'(x) < 0$$

which implies, since $K(x,y) > 0$, that $0 < \epsilon_1 y < x$. Therefore it follows that $g^{(\varphi)}_{\ell,x,y}(\widehat{\mu}) > \frac{-\widehat{\mu}x\ell(x)}{y\ell(y)(1-\widehat{\mu})} \to \infty$ as $y\ell(y) \to 0$ if $y \to \sup\{y \geq 0\, | y\ell(y) = 0|\}$ since $\widehat{\mu} < 0$. If we set $(\ell_2, x_2, y_2) = (\ell, x_1, y_2)$ where y_2 is such that $g^{(\varphi)}_{\ell_2,x_2,y_2}(\widehat{\mu}) > g^{(\varphi)}_{\ell_1,x_1,y_1}(\widehat{\mu})$ we establish the claim in this case. If instead for every $g^{(\varphi)}_{\ell,x,y}$ strictly increasing we have $g^{(\varphi)}_{\ell,x,y}(\mu^*) < \widehat{\gamma}$ then the claim follows immediately by using the simple fact that $\Gamma_{\mathcal{L},\varphi}(\mu^*) \geq \gamma_{\varphi}(\mathcal{L}) > \widehat{\gamma}$ because in this case there must be a strictly continuous decreasing function with $g^{(\varphi)}_{\ell_2,x_2,y_2}(\widehat{\mu}) > \widehat{\gamma}$.

Finally we observe that since in both cases $g^{(\varphi)}_{\ell_1,x_1,y_1}(\widehat{\mu}) \geq g^{(\varphi)}_{\ell_2,x_2,y_2}(\widehat{\mu}) \geq \widehat{\gamma}$ with $g^{(\varphi)}_{\ell_1,x_1,y_1}$ strictly increasing and $g^{(\varphi)}_{\ell_2,x_2,y_2}$ strictly decreasing and they are both unbounded in a small neighborhood of $\mu = 1$ they must intersect. ■

In order to understand why the last result is important we observe that, by definition,

$$h^{(\varphi)}_{\ell,x,y} = (\mathcal{H}(x,y) - x\ell(x)) + K(x,y;\varphi)$$

and we also observe that if $g^{(\varphi)}_{\ell,x,y}(\mu)$ is such that $\frac{\lambda}{1-\mu} = g^{(\varphi)}_{\ell,x,y}(\mu)$ then it must follow

$$\begin{aligned} \frac{\lambda}{1-\mu} &= g^{(\varphi)}_{\ell,x,y}(\mu) \\ &= \frac{\mathcal{H}(x,y) + K(x,y;\varphi) - \mu x\ell(x)}{y\ell(y)\,(1-\mu)} \\ &= \frac{h^{(\varphi)}_{\ell,x,y} - \mu x\ell(x) + x\ell(x)}{y\ell(y)\,(1-\mu)} \end{aligned}$$

hence we get

$$x\ell(x) = \lambda y\ell(y) + \mu x\ell(x) - h^{(\varphi)}_{\ell,x,y}$$

so, since $h^{(\varphi)}_{\ell,x,y}$ it appears as an "error" term w.r.t. $x\ell(x) = \lambda y\ell(y) + \mu x\ell(x)$, we need to minimize it. Using the last proposition, if we work only with two resources, we can find two $g^{(\varphi)}_{\ell_i,x_i,y_i}$with opposite $h^{(\varphi)}_{\ell_i,x_i,y_i}$ and we can get the exact bound since the sum of them is zero. That's exactly what the last proposition does for us. Since a similar version of the last proposition was the main tool in the case of local smoothness to construct a sequence of games to achieve the desired bound of cost of anarchy, we think that by using the last proposition we just proved we have covered a major step in extending what has been done in the local (λ, μ)-smooth case to an interesting more general case namely the *short paths* case.

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