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Research Paper

A note on unique continuation from the edge of a crack with no star-shapedness condition

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ABSTRACT

In the present paper, which aims at representing an improvement of De Luca and Felli (2021), we prove the validity of the strong unique continuation property for solutions to some second order elliptic equations from the edge of a crack via a description of their local behaviour. In particular we relax the star-shapedness condition on the complement of the crack considered in De Luca and Felli (2021) by applying a suitable diffeomorphism which straightens the boundary of the crack before performing an approximation of the fractured domain needed to derive a monotonicity formula

1. Introduction

In the present paper we provide a suitable monotonicity argument to prove local asymptotics and consequently a strong unique continuation principle at the origin for solutions to the second order elliptic equation

$$-\Delta u = fu \quad \text{in } \Omega \setminus \Gamma,$$

complemented with the partial homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \Gamma \cap \Omega,$$

where Γ is a closed set in $\mathbb{R}^N$ with $N \geq 2$ (see (1.1) below), Ω is a bounded open domain in $\mathbb{R}^{N+1}$ such that $0 \in \Omega \cap \partial\Gamma$ (as depicted in Fig. 1), and $f \in L^\infty_{\text{loc}}(\Omega \setminus \{0\})$ satisfies two sets of alternative assumptions (see (a1)–(a2)).

More precisely, we assume that Γ is of class C^2 around 0, namely up to rigid motions we assume that there exist $\bar{r} > 0$ and a function $g \in C^2(\mathbb{R}^{N-1}, \mathbb{R})$ such that, if $x = (x', x_N, x_{N+1}) \in \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R}$, we have that

$$\Gamma \cap B'_{\bar{r}} = \{x \in B'_{\bar{r}} : \ x_N \geq g(x')\}, \tag{1.1}$$

being $B'_{\bar{r}} := \{x = (x', x_N, 0) \in \mathbb{R}^{N+1} : |x| < \bar{r}\}$.

Furthermore without loss of generality we also may assume that

$$g(0) = 0 \qquad \text{and} \qquad \nabla g(0) = 0, \tag{1.2}$$

namely $\partial\Gamma \cap B'_{\bar{r}}$ is tangent at 0 to the hyperplane $\{x_N = 0\}$.

As for the potential f, we assume that either

$$f(x) = O(|x|^{-2+\delta}) \text{ as } |x| \to 0^+ \text{ for some } \delta > 0, \tag{a1}$$

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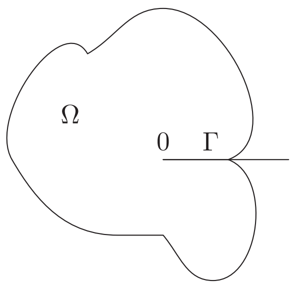

Fig. 1. The section of a possible configuration of the domain Ω with a fracture Γ.

or

$$f \in W^{1,p}(B_{\bar{r}}) \text{ for some } p > (N+1)/2, \tag{a2}$$

being $B_r := \{x = (x', x_N, x_{N+1}) \in \mathbb{R}^{N+1} : |x| < r\}$ for every $r > 0$.

Since our investigation is of local type, we thus focus our attention on the problem

$$\begin{cases} -\Delta u = fu & \text{in } B_{\bar{r}} \setminus \Gamma, \\ u = 0 & \text{on } \Gamma \cap B_{\bar{r}}. \end{cases} \tag{1.3}$$

We recall that the family of solutions to some elliptic equation is said to obey the strong unique continuation principle if no solution except for the trivial one has a zero of infinite order. One strategy to prove such a property consists of the monotonicity approach developed by Garofalo and Lin in [1]: typically one has to derive a Pohozaev-type identity in order to estimate from below the derivative of the so-called Almgren frequency function. To this aim, since sufficient regularity is not available for solutions of problem (1.3) as it is, we perform an approximation argument based on the study of a sequence of boundary value problems on certain approximating domains for which obtaining a family of Pohozaev-type identities (actually inequalities) is possible. The idea is then passing to the limit in order to recover a Pohozaev-type inequality for problem (1.3) as well. Afterwards, a fine blow-up analysis allows to classify the limit profiles of rescaled solutions and thus to prove the validity of the strong unique continuation principle, as desired.

We stress that the main novelty of the present paper lies on the ability of proving a strong unique continuation type result for solutions to problem (1.3) in a more general context compared to the setting considered in [2]. Indeed we are able to remove the additional assumption on Γ given by

$$g(x') - \nabla g(x') \cdot x' \geq 0 \quad \text{in a neighbourhood of the origin,} \tag{1.4}$$

which establishes that the complement of Γ in a neighbourhood of the origin is star-shaped with respect to the origin; in fact, the outward unit normal vector to Γ^c at any point on its edge in a neighbourhood of the origin is given by

$$\frac{(-\nabla g(x'), 1)}{\sqrt{1 + |\nabla g(x')|^2}}.$$

In [2] assumption (1.4) turns out to be crucial when proving the star-shapedness property of the approximating domains, since the left-hand side appears in the outer unit normal vector to their boundary (see [2, Lemma 2.4]). For this, at first one could think that (1.4) is a necessary condition and thus its remotion cannot be contemplated. Neverthless we overcome this difficulty by applying a local diffeomorphism which straightens the edge of the crack Γ around the origin before carrying out our approximation procedure, unlike what the authors do in [2], and which leads to study a problem in the presence of a non-identity matrix A (see (2.5)). This last occurrence in turn brings with it the necessity of using a different approximation, compared to that one used in [2], in order to make us able to get an "almost" star-shapedness (see (2.45)) for the approximating domains needed to estimate some boundary terms arising in the Pohozaev identities, thus producing a family of Pohozaev-type inequalities. To face this demand, we write a generalized version of the approximation employed in [3] (see the proof of Lemma 2.2).

We also want to point out that the choice of considering more restrictive assumptions on the potential f, compared to those ones taken in [2], is reasonable if one considers that they must be preserved even after applying the diffeomorphism. In particular under assumption (a1) the set of assumptions (H1-1)-(H1-3) in [2] is trivially satisfied. On the other hand, we claim that if f satisfies (a2) then the set of assumptions (H2-1)-(H2-5) is verified. Indeed, we recall that for every $r > 0$ and $v \in H^1(B_r)$ there holds that

$$\left(\int_{B_r} |v(x)|^{2^*} dx \right)^{\frac{2}{2^*}} \leq C_{N,p} \left(\int_{B_r} |\nabla v|^2 dx + \frac{N-1}{2r} \int_{\partial B_r} v^2 ds \right), \tag{1.5}$$

for some constant $C_{N,p} > 0$ depending only on N and p, being $2^* = 2(N+1)/(N-1)$ and ds the volume element on the sphere ∂B_r. Therefore, if ω_{N+1} denotes the $N+1$-dimensional Lebesgue measure of B_1, setting

$$S_{N,p} := \omega_{N+1}^{\frac{2p-N-1}{pN}}, \tag{1.6}$$

and thanks to the Hölder inequality, for all $r \in (0, \bar{r}]$ and $v \in H^1(B_r)$ we have that

$$\int_{B_r} |f| v^2\, dx \le S_{N,p}\, r^{\frac{2p-N-1}{p}} \left(\int_{B_{\bar{r}}} |f|^p dx \right)^{\frac{1}{p}} \left(\int_{B_r} |v(x)|^{2^*} dx \right)^{\frac{2}{2^*}}$$
$$\le S_{N,p} C_{N,p} \|f\|_{L^p(B_{\bar{r}})} r^{\frac{2p-N-1}{p}} \left(\int_{B_r} |\nabla v|^2\, dx + \frac{N-1}{2r} \int_{\partial B_r} v^2\, ds \right).$$

The claim is thereby proved by repeating the same argument with $\nabla f \cdot x$ in place of f.

In addition, concerning with the possibility of considering a crack which is contained in the whole $\mathbb{R}^{N+1}$ and is not merely a subset of $\mathbb{R}^N$, we strongly believe that it is possible to extend our result at least for sufficiently smooth fractures by considering a diffeomorphism which gives rise to a matrix depending also on the last component x_{N+1} (unlike what our diffeomorphism does) and having the same structure as A's one in (2.11), that is with all zeros in the last row and the last column except for the coefficient $a_{N+1,N+1}$. Such a study could place our results in a more general context and could therefore be the object of a future paper.

We are particularly motivated to remove the star-shapedness condition (1.4) not only to generalize the result in [2], but also because in this way the problem turns out to be directly related to that one investigated in [3] when $s = 1/2$, in virtue of a strong connection with the mixed Dirichlet–Neumann boundary value problem arising after applying the Caffarelli–Silvestre extension developed in [4] (see [2, Section 1] for more details about this). For the same reason, our result can be also seen as a generalization to any dimension greater than 2 of the results in [5], where the authors analyse a mixed boundary value problem in dimension 2 without any assumption of star-shapedness on the portion on which an homogeneous Dirichlet boundary condition is prescribed.

A strong unique continuation type result from the edge of a crack is also deduced in [6], with the difference that homogeneous Neumann boundary conditions on both sides of the crack are prescribed. As for the study of the strong unique continuation property at boundary points, we mention among others [7,8], where a strong unique continuation type result at corners under a zero Dirichlet boundary condition and a non-homogeneous Neumann boundary condition respectively is established via a classification of blow-up limit profiles. For a more detailed overview on papers in literature about this topic in a local setting, we strongly suggest the reader to consult [2, Section 1] and references therein. With regard to the study of the strong unique continuation property in a non local framework, it is worth citing [3,9,10].

In order to fully state the main result we introduce an eigenvalue problem on the unit N-dimensional sphere $\mathbb{S}^N := \partial B_1$ with a homogeneous Dirichlet boundary condition on the cut $\Theta := \tilde{\Gamma} \cap \partial B_1$ with

$$\tilde{\Gamma} = \{(x', x_N, 0) \in \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R} : x_N \ge 0\}, \tag{1.7}$$

given by

$$\begin{cases} -\Delta_{\mathbb{S}^N} \Psi = \mu \Psi & \text{in } \mathbb{S}^N \setminus \Theta, \\ \Psi = 0 & \text{on } \Theta. \end{cases} \tag{1.8}$$

Letting

$$\mathcal{H}_\Theta := \{\psi \in H^1(\mathbb{S}^N) : \psi = 0 \text{ on } \Theta \text{ in a trace sense}\}, \tag{1.9}$$

we say that $\mu \in \mathbb{R}$ is an eigenvalue of (1.8) if there exists a non-trivial $\Psi \in \mathcal{H}_\Theta$ such that

$$\int_{\mathbb{S}^N} \nabla_{\mathbb{S}^N} \Psi \cdot \nabla_{\mathbb{S}^N} \phi\, ds = \mu \int_{\mathbb{S}^N} \Psi \phi\, ds \quad \text{for all } \phi \in C_c^\infty(\mathbb{S}^N \setminus \Theta); \tag{1.10}$$

such a function Ψ is called an eigenfunction of problem (1.8) associated with μ. Classical spectral theory ensures the existence of an increasing and diverging sequence of real eigenvalues $\{\mu_k\}_{k\ge1}$ of problem (1.8) (counted with their finite multiplicities) which are explicitly given by the sequence

$$\left\{ \frac{k(k+2N+2)}{4} \right\}_{k\ge1} \tag{1.11}$$

(see [2, Appendix A] for the proof).

The main result of the present paper is the following: we essentially show that the asymptotics performed in [2, Theorem 1.1] is still valid without the star-shapedness condition (1.4).

Theorem 1.1. *Let $u \in H^1(B_{\bar{r}}) \setminus \{0\}$ be a weak solution to (1.3). Then there exist $k_0 \ge 1$ and an eigenfunction Ψ of problem (1.8) associated with the eigenvalue μ_{k_0} such that*

$$\frac{u(\lambda x)}{\lambda^{\frac{k_0}{2}}} \to |x|^{\frac{k_0}{2}} \Psi\left(\frac{x}{|x|}\right) \quad \text{in } H^1(B_1) \text{ as } \lambda \to 0^+.$$

This result allows us to deduce that the vanishing order of non-trivial solutions to (1.3) cannot exceed the limit of the Almgren function, which is shown to be equal to $k_0/2$ for some $k_0 \ge 1$ (see Lemma 5.9 below). Therefore, as a relevant consequence, we are able to establish that solutions to (1.3) impose the validity of a strong unique continuation property at 0.

Corollary 1.2. *If $u \in H^1(B_{\bar{r}})$ is a weak solution to (1.3) such that $u(x) = O(|x|^k)$ as $|x| \to 0^+$ for every $k \in \mathbb{N}$, then necessarily $u \equiv 0$ in $B_{\bar{r}}$.*

Organization of the paper

The rest of the paper is organized as follows. In Section 2 we introduce the straightened problem and we construct an ad-hoc approximation argument consisting in the study of some boundary value problems on approximating domains enjoying good properties to obtain a family of Pohozaev-type inequalities in Section 3. In Section 4 we develop Almgren's monotonicity approach and finally in Section 5 we carry out the so-called blow-up analysis which gives additional information on finite vanishing order of non-trivial solutions at the origin.

2. Removal of star-shapedness condition and approximation argument

2.1. A straightening of crack's edge

We briefly recall the construction and the main properties of the diffeomorphism introduced in [3, Section 2]. There exists $F : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ of class $C^{1,1}$ such that $\det DF(0) \neq 0$ and $F(0) = 0$, being DF the Jacobian matrix of F. Thus by the Inverse Function Theorem there exists $\tilde{r} > 0$ small enough such that

(i) $F|_{B_{\tilde{r}}} : B_{\tilde{r}} \to F(B_{\tilde{r}})$ is invertible, namely F is a local diffeomorphism;
(ii) $F(B_{\tilde{r}}) \subset B_{\tilde{r}}$.

In particular, letting $\tilde{\Gamma}$ as in (1.7), we have that

$$\begin{aligned} &F(B_{\tilde{r}} \setminus \tilde{\Gamma}) = F(B_{\tilde{r}}) \setminus \Gamma \subset B_{\tilde{r}} \setminus \Gamma, \\ &F(\tilde{\Gamma} \cap B_{\tilde{r}}) = \Gamma \cap F(B_{\tilde{r}}) \subset \Gamma \cap B_{\tilde{r}}. \end{aligned} \tag{2.1}$$

Moreover it holds that

$$F(x) = x + O(|x|^2) \quad \text{and} \quad F^{-1}(x) = x + O(|x|^2) \quad \text{as } |x| \to 0, \tag{2.2}$$

where $O(|x|^2)$ denotes a vector in $\mathbb{R}^{N+1}$ with all entries equal to $O(|x|^2)$ as $|x| \to 0$, and consequently

$$DF(x) = \mathrm{Id}_{N+1} + O(|x|) \quad \text{and} \quad DF^{-1}(x) = \mathrm{Id}_{N+1} + O(|x|) \quad \text{as } |x| \to 0, \tag{2.3}$$

where DF does not depend on x_{N+1}, Id_{N+1} is the identity matrix in $\mathbb{R}^{(N+1)\times(N+1)}$, $O(|x|)$ here stands for a matrix in $\mathbb{R}^{(N+1)\times(N+1)}$ having all elements being $O(|x|)$ as $|x| \to 0$. Additionally it holds that

$$\det DF(x) = 1 + O(|x'|^2) + O(x_N) \quad \text{as } |x'| \to 0 \text{ and } x_N \to 0. \tag{2.4}$$

In light of (2.1), if u is a weak solution to (1.3), then $U := u \circ F$ is a weak solution to

$$\begin{cases} -\mathrm{div}(A(x)\nabla U(x)) = \tilde{f}(x)U(x) & \text{in } B_{\tilde{r}} \setminus \tilde{\Gamma}, \\ U = 0 & \text{on } \tilde{\Gamma}, \end{cases} \tag{2.5}$$

where

$$A(x) := |\det DF(x)| DF(x)^{-1}(DF(x)^{-1})^T \quad \text{and} \quad \tilde{f}(x) := |\det DF(x)| f(F(x)). \tag{2.6}$$

By the $C^{1,1}$-regularity of F we have that

$$A \in C^{0,1}(B_{\tilde{r}}, \mathbb{R}^{(N+1)\times(N+1)}); \tag{2.7}$$

furthermore by (2.3), we have that $\tilde{f} \in L^\infty_{\mathrm{loc}}(B_{\tilde{r}} \setminus \{0\})$, and in addition either

$$\tilde{f}(x) = O(|x|^{-2+\delta}) \text{ as } |x| \to 0^+ \text{ for some } \delta > 0, \tag{2.8}$$

under assumption (a1), or

$$\tilde{f} \in W^{1,p}(B_{\tilde{r}}) \text{ for some } p > (N+1)/2, \tag{2.9}$$

under assumption (a2). We underline that by weak solution to (2.5) we mean that the admissible functional space for U is

$$H^1_{\tilde{\Gamma}}(B_{\tilde{r}}) := \overline{C^\infty(\overline{B_{\tilde{r}}} \setminus \tilde{\Gamma})}^{\|\cdot\|_{H^1(B_{\tilde{r}})}},$$

namely the closure with respect to the H^1-norm of the space of all $C^\infty(\overline{B_{\tilde{r}}})$-functions vanishing in a neighbourhood of $\tilde{\Gamma}$, and that U satisfies

$$\int_{B_{\tilde{r}}} A\nabla U \cdot \nabla \varphi \, dx = \int_{B_{\tilde{r}}} \tilde{f} U \varphi \, dx \quad \text{for all } \varphi \in C^\infty_c(B_{\tilde{r}} \setminus \tilde{\Gamma}). \tag{2.10}$$

For future purposes, we highlight that A can be written as the following block matrix

$$A = A(x', x_N) = \left(\begin{array}{c|c} B(x', x_N) & \underline{0} \\ \hline \underline{0} & \det DF(x', x_N) \end{array} \right), \tag{2.11}$$

where $\underline{0}$ is the null vector in $\mathbb{R}^N$ and

$$B(x', x_N) = \left(\begin{array}{c|c} \mathrm{Id}_{N-1} + O(|x'|^2) + O(x_N) & O(x_N) \\ \hline O(x_N) & 1 + O(|x'|^2) + O(x_N) \end{array} \right), \tag{2.12}$$

where Id_{N-1} denotes the identity matrix in $\mathbb{R}^{(N-1)\times(N-1)}$, in the top left block $O(|x'|^2)$ and $O(x_N)$ denote two matrices in $\mathbb{R}^{(N-1)\times(N-1)}$ with all elements being $O(|x'|^2)$ and $O(x_N)$ as $|x'| \to 0$ and $x_N \to 0$ respectively; in the top right block and in the lower left one, $O(x_N)$ stands for a vector in $\mathbb{R}^{(N-1)\times 1}$ and $\mathbb{R}^{1\times(N-1)}$ respectively, with all entries being $O(x_N)$ as $x_N \to 0$. Also, from (2.3), (2.4) and (2.6), we can deduce that

$$A(x) - \mathrm{Id}_{N+1} = O(|x|) \quad \text{as } |x| \to 0. \tag{2.13}$$

Thus, without loss of generality, we can assume that for every $x \in B_{\tilde{r}}$

$$A(x)y \cdot y \geq \frac{1}{2}|y|^2 \quad \text{for all } y \in \mathbb{R}^{N+1}, \tag{2.14}$$

and

$$\|A(x)\| \leq 2, \tag{2.15}$$

where $\|A(x)\|$ denotes the norm of $A(x)$ interpreted as the linear operator $y \in \mathbb{R}^{N+1} \mapsto A(x)y \in \mathbb{R}^{N+1}$. Now we define

$$\mu(x) := \begin{cases} (Ax \cdot x)/|x|^2 & \text{if } x \in B_{\tilde{r}} \setminus \{0\}, \\ 1 & \text{if } x = 0, \end{cases} \tag{2.16}$$

and

$$\beta(x) := \frac{Ax}{\mu(x)}. \tag{2.17}$$

We observe that from (2.13) and (2.16) we can conclude that

$$\mu(x) = 1 + O(|x|) \quad \text{as } |x| \to 0, \tag{2.18}$$

and also that

$$\nabla \mu(x) = O(1) \quad \text{as } |x| \to 0^+. \tag{2.19}$$

In particular μ is continuous on $B_{\tilde{r}}$ and, without loss of generality, we can assume that

$$\mu(x) \geq \frac{1}{2} \quad \text{for every } x \in B_{\tilde{r}}. \tag{2.20}$$

Furthermore by (2.17), (2.13) and (2.18), we have that as $|x| \to 0$

$$\begin{cases} \beta(x) & = x + O(|x|^2) = O(|x|), \\ D\beta(x) & = A(x) + O(|x|) = \mathrm{Id}_{N+1} + O(|x|), \\ \mathrm{div}\beta(x) & = N + 1 + O(|x|). \end{cases} \tag{2.21}$$

From this, again without loss of generality, we can deduce that for every $x \in B_{\tilde{r}}$

$$|\beta(x)| \leq \mathrm{const}|x|, \tag{2.22}$$

$$\|D\beta(x)\| \leq \mathrm{const}, \tag{2.23}$$

for some $\mathrm{const} > 0$ independent of x, and

$$|\mathrm{div}\beta(x)| \leq N + 2. \tag{2.24}$$

Also, using the notation $A = (a_{jk})_{j,k=1,\dots,N+1}$, we define for every $x \in B_{\tilde{r}}$ and $v_1, v_2 \in \mathbb{R}^{N+1}$

$$dA(x)v_1v_2 := \left(\sum_{j,k=1}^{N+1} \frac{\partial a_{jk}(x)}{\partial x_1} v_j v_k, \dots, \sum_{j,k=1}^{N+1} \frac{\partial a_{jk}(x)}{\partial x_N} v_j v_k, 0 \right) \in \mathbb{R}^{N+1}.$$

By direct computations, one can easily check that for every $x \in B_{\tilde{r}}$ and $v_1, v_2, v_2' \in \mathbb{R}^{N+1}$

$$dA(x)v_1v_2 = dA(x)v_2v_1, \tag{2.25}$$

$$dA(x)v_1v_2 - dA(x)v_1v_2' = dA(x)v_1(v_2 - v_2'), \tag{2.26}$$

and

$$|dA(x)v_1v_2| \leq \mathrm{const}|v_1||v_2|, \tag{2.27}$$

using (2.7), for some $\mathrm{const} > 0$ independent of x, v_1 and v_2.

2.2. Some crucial inequalities

We recall the following Hardy-type inequality with boundary terms (see [11, Theorem 1.1])

$$\left(\frac{N-1}{2}\right)^2 \int_{B_r} \frac{|U(x)|^2}{|x|^2}\, dx \le \int_{B_r} |\nabla U|^2\, dx + \frac{N-1}{2r} \int_{\partial B_r} U^2\, ds, \tag{2.28}$$

which will be often invoked throughout the whole paper.

Now we prove an adapted version of [2, Lemma 2.1], due to the presence of the matrix A.

Lemma 2.1. *Let f satisfy either (2.8) or (2.9). There exists $r_0 \in (0, \tilde{r})$ such that every $U \in H^1(B_r)$ with $r \in (0, r_0]$ satisfies the following inequality*

$$\int_{B_r} A\nabla U \cdot \nabla U\, dx - \int_{B_r} |\tilde{f}| U^2\, dx + C r^{-1+\epsilon} \int_{\partial B_r} \mu U^2\, ds \ge \frac{1}{4} \int_{B_r} |\nabla U|^2\, dx, \tag{2.29}$$

where $C > 0$ is a positive constant depending only on N under assumption (2.8) and depending on N, p and $\|\tilde{f}\|_{L^p(B_{\tilde{r}})}$ under assumption (2.9), and

$$\epsilon = \begin{cases} \delta & \text{under assumption (2.8)}, \\ \frac{2p-N-1}{p} & \text{under assumption (2.9)}. \end{cases} \tag{2.30}$$

Moreover it holds that

$$C r^{\epsilon} < \frac{N-1}{4} \quad \text{for every } r \in (0, r_0]. \tag{2.31}$$

Proof. Let $U \in H^1(B_r)$ with $r \in (0, \tilde{r})$ to be taken gradually smaller throughout the proof according to the needs.

We start by proving (2.29) under assumption (2.8). Using (2.20) and (2.28), we can estimate the second term on the left-hand side of (2.29) as follows

$$\int_{B_r} |\tilde{f}| U^2\, dx \le r^{\delta} \int_{B_r} \frac{|U(x)|^2}{|x|^2}\, dx \le \left(\frac{2}{N-1}\right)^2 r^{\delta} \int_{B_r} |\nabla U|^2\, dx + \frac{4}{N-1} r^{-1+\delta} \int_{\partial B_r} \mu U^2\, ds.$$

From this, choosing $r_0 \in (0, \tilde{r})$ sufficiently small in such a way that

$$r_0^{\delta} < \frac{(N-1)^2}{16}, \tag{2.32}$$

we have that for every $r \in (0, r_0]$

$$\int_{B_r} |\tilde{f}| U^2\, dx \le \frac{1}{4} \int_{B_r} |\nabla U|^2\, dx + \frac{4}{N-1} r^{-1+\delta} \int_{\partial B_r} \mu U^2\, ds. \tag{2.33}$$

Moreover, by (2.14) we have that for every $r \in (0, r_0]$

$$\int_{B_r} A\nabla U \cdot \nabla U\, dx \ge \frac{1}{2} \int_{B_r} |\nabla U|^2\, dx. \tag{2.34}$$

Thus, combining (2.33) and (2.34), we easily infer (2.29) under assumption (2.8) for every $U \in H^1(B_r)$ such that $r \in (0, r_0]$, with $C := 4/(N-1)$ and $\epsilon = \delta$. In particular (2.31) easily follows from (2.32).

Now we turn to prove the validity of (2.29) under assumption (2.9). To this aim, being $S_{N,p}$ as in (1.6), thanks to the Hölder inequality, (1.5) and (2.20), we have that

$$\begin{aligned} \int_{B_r} |\tilde{f}| U^2\, dx &\le S_{N,p}\, r^{\frac{2p-N-1}{p}} \left(\int_{B_{\tilde{r}}} |\tilde{f}|^p dx\right)^{\frac{1}{p}} \left(\int_{B_r} |U(x)|^{2^*} dx\right)^{\frac{2}{2^*}} \\ &\le S_{N,p} C_{N,p} \|\tilde{f}\|_{L^p(B_{\tilde{r}})} r^{\frac{2p-N-1}{p}} \left(\int_{B_r} |\nabla U|^2\, dx + \frac{N-1}{2r} \int_{\partial B_r} U^2\, ds\right) \\ &\le \frac{1}{4} \int_{B_r} |\nabla U|^2\, dx + (N-1) S_{N,p} C_{N,p} \|\tilde{f}\|_{L^p(B_{\tilde{r}})}\, r^{-1+\frac{2p-N-1}{p}} \int_{\partial B_r} \mu U^2\, ds, \end{aligned} \tag{2.35}$$

for every $r \in (0, r_0]$ with $r_0 \in (0, \tilde{r})$ such that

$$r_0^{\frac{2p-N-1}{p}} < (4 S_{N,p} C_{N,p} \|\tilde{f}\|_{L^p(B_{\tilde{r}})})^{-1}. \tag{2.36}$$

Thus, putting together (2.34) and (2.35), we get (2.29) under assumption (2.9) for every $U \in H^1(B_r)$ such that $r \in (0, r_0]$, with $C := (N-1) S_{N,p} C_{N,p} \|\tilde{f}\|_{L^p(B_{\tilde{r}})}$ and $\epsilon = (2p-N-1)/p$. In particular (2.31) immediately follows from (2.36). □

2.3. Approximation argument

We now really dive into the focal part of our discussion: it will turn out that the remotion of the star-shapedness condition (1.4) *downstairs* (since Γ lies on $\mathbb{R}^N$) makes proving an "almost" star-shapedness condition *upstairs* necessary, due to the presence of the matrix A.

We start the construction of the approximating domains by letting $\eta \in C^\infty([0,+\infty))$ be such that $0 \le \eta \le 1$, $\eta' \le 0$ and

$$\eta(t) = \begin{cases} 1 & \text{if } t \le 1/2, \\ 0 & \text{if } t \ge 1. \end{cases} \tag{2.37}$$

Then we fix any real $\alpha > 1$ and we introduce $f : [0,+\infty) \to \mathbb{R}$ such that

$$f(t) := \eta(t) + (1-\eta(t))t^{1/\alpha} \quad \text{for every } t \ge 0. \tag{2.38}$$

It holds that $f \in C^\infty([0,+\infty))$ and

$$f(t) - \alpha t f'(t) \ge 0 \quad \text{for every } t \ge 0. \tag{2.39}$$

Accordingly, we define a sequence of smooth functions given by

$$f_n(t) := f(nt)n^{-1/2\alpha} \quad \text{for every } n \ge 1 \text{ and } t \ge 0, \tag{2.40}$$

which inherits (2.39), namely we have that

$$f_n(t) - \alpha t f_n'(t) \ge 0 \quad \text{for every } t \ge 0. \tag{2.41}$$

From (2.37), (2.38) and (2.40), we can deduce that

$$f_n(0) = n^{-1/2\alpha}. \tag{2.42}$$

In order to suitably define the approximating domains, we introduce the following sequence of functions

$$\tilde{f}_n(t) := f_n(|t|) \quad \text{for every } n \ge 1 \text{ and } t \in \mathbb{R},$$

which, thanks to (2.41), satisfies

$$\tilde{f}_n(t) - \alpha t \tilde{f}_n'(t) \ge 0 \quad \text{for every } t \in \mathbb{R}. \tag{2.43}$$

Thus, for every $r \in (0, r_0]$ and $n \in \mathbb{N} \setminus \{0\}$, let

$$\mathcal{B}_{r,n} := B_r \cap \{(x', x_N, x_{N+1}) \in \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R} : x_N < \tilde{f}_n(x_{N+1})\},$$

see Fig. 2 below. The topological boundary of $\mathcal{B}_{r,n}$ can be written as follows

$$\partial\mathcal{B}_{r,n} = \overline{\mathcal{S}_{r,n} \cup \gamma_{r,n}}$$

where

$$\mathcal{S}_{r,n} := \partial B_r \cap \{(x', x_N, x_{N+1}) \in \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R} : x_N < \tilde{f}_n(x_{N+1})\}$$

and

$$\gamma_{r,n} := B_r \cap \{(x', x_N, x_{N+1}) \in \mathbb{R}^{N-1} \times \mathbb{R} \times \mathbb{R} : x_N = \tilde{f}_n(x_{N+1})\}. \tag{2.44}$$

The result in the following lemma will allow us to get rid of a boundary term over $\gamma_{r,n}$ in the Pohozaev identity, giving rise to a Pohozaev inequality.

Lemma 2.2. *For all $r \in (0, r_0]$ there exists $\bar{n} = \bar{n}(r) \in \mathbb{N} \setminus \{0\}$ sufficiently large such that for all $n \ge \bar{n}$*

$$Ax \cdot \nu(x) \ge 0 \quad \textit{for every } x \in \gamma_{r,n}, \tag{2.45}$$

where $\nu(x)$ denotes the unit outward normal vector at $x \in \partial\mathcal{B}_{r,n}$.

Proof. Let $r \in (0, r_0]$. We choose $\bar{n} \in \mathbb{N} \setminus \{0\}$ sufficiently large so that

$$\bar{n}^{1/2\alpha} > \frac{1}{r},$$

and we consider $n \ge \bar{n}$ in order to make sure that $\gamma_{r,n}$ is non-empty and consequently $\mathcal{B}_{r,n} \subsetneq B_r$, thanks to (2.42) (this is evident in Fig. 2). We immediately observe that if $x = (x', x_N, x_{N+1}) \in \gamma_{r,n}$, then by definition of $\gamma_{r,n}$ (given in (2.44))

$$x_N = \tilde{f}_n(x_{N+1}) \tag{2.46}$$

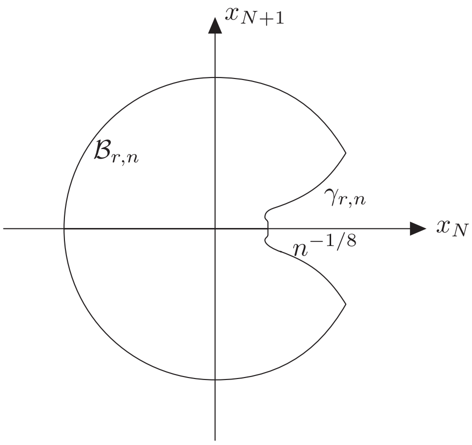

Fig. 2. A representation of the set $\mathcal{B}_{r,n}$ when $x' = 0$ and $\alpha = 1/4$.

and the outward unit normal vector to $\partial\mathcal{B}_{r,n}$ at x is given by

$$\nu(x) = \frac{(\underline{0}, 1, -\tilde{f}_n'(x_{N+1}))}{\sqrt{1 + [\tilde{f}_n'(x_{N+1})]^2}}.$$

Combining this with (2.11), (2.4) and (2.12), we obtain that

$$\sqrt{1 + [\tilde{f}_n'(x_{N+1})]^2}[Ax \cdot \nu(x)] = (1 + O(|x'|) + O(x_N))(x_N - x_{N+1}\tilde{f}_n'(x_{N+1})). \tag{2.47}$$

At this point we notice that possibly choosing r_0 smaller from the beginning, for every $x \in B_{r_0}$

$$C_1 \le 1 + O(|x'|) + O(x_N) \le C_2, \tag{2.48}$$

for some two positive constant $C_1 < 1$ and $C_2 > 1$ such that

$$\frac{C_2}{C_1} < \alpha; \tag{2.49}$$

so in particular (2.48) holds for every $x \in \gamma_{r,n}$. Now if x_{N+1} and $\tilde{f}_n'(x_{N+1})$ have the same sign, we put together (2.47) and (2.48) to find that

$$\begin{aligned}\sqrt{1 + [\tilde{f}_n'(x_{N+1})]^2}[Ax \cdot \nu(x)] &\ge C_1 x_N - C_2 x_{N+1}\tilde{f}_n'(x_{N+1}) \\ &\ge C_1(\tilde{f}_n(x_{N+1}) - \alpha x_{N+1}\tilde{f}_n'(x_{N+1})) \ge 0,\end{aligned}$$

using in addition (2.46), (2.43) and (2.49). Instead, if x_{N+1} and $\tilde{f}_n'(x_{N+1})$ have opposite signs, we simply take advantage of the first inequality in (2.48), which in turn also implies that $(1 + O(|x'|) + O(x_N))x_{N+1}\tilde{f}_n'(x_{N+1}) \le 0$, to conclude that

$$\sqrt{1 + [\tilde{f}_n'(x_{N+1})]^2}[Ax \cdot \nu(x)] \ge C_1 x_N = C_1\tilde{f}_n(x_{N+1}) \ge 0,$$

as a consequence of (2.46) and by construction of $\tilde{f}_n$. □

From now on, we fix a non-trivial solution $U \in H^1_{\tilde{\Gamma}}(B_{\tilde{r}})$ to (2.5). Then, inspired by the approximation technique developed in [2,3], we introduce a sequence of boundary value problems on the approximating domains and we prove that the resulting sequence of solutions $\{U_n\}$ converges to U. The goal of performing such a construction is to derive a Pohozaev-type identity (actually inequality by making use of Lemma 2.2) for each U_n which enjoys enough regularity, and consequently for U.

More precisely, let $r_0 \in (0, \tilde{r})$ be as in Lemma 2.1 and $\bar{n} = \bar{n}(r_0) \in \mathbb{N} \setminus \{0\}$ as in Lemma 2.2. For every $n \ge \bar{n}$ we consider the boundary value problem

$$\begin{cases} -\mathrm{div}\left(A\nabla U_n\right) = \tilde{f} U_n & \text{in } \mathcal{B}_{r_0,n}, \\ U_n = G_n & \text{on } \partial\mathcal{B}_{r_0,n}, \end{cases} \tag{2.50}$$

where $\{G_n\} \subseteq C^\infty(\overline{B_{\tilde{r}}} \setminus \tilde{\Gamma})$ and

$$G_n \to U \quad \text{in } H^1(B_{\tilde{r}}). \tag{2.51}$$

Without loss of generality we can suppose that G_n vanishes on $\gamma_{\tilde{r},n}$ for each fixed n. In particular, by a weak solution to (2.50) we mean a function $U_n \in H^1(\mathcal{B}_{r_0,n})$ such that

$$U_n = G_n \quad \text{on } \partial\mathcal{B}_{r_0,n} \text{ in the trace sense,}$$

and

$$\int_{\mathcal{B}_{r_0,n}} A\nabla U_n \cdot \nabla V\,dx = \int_{\mathcal{B}_{r_0,n}} \tilde{f} U_n V\,dx \quad \text{for every } V \in H_0^1(\mathcal{B}_{r_0,n}).$$

Proposition 2.3. *It holds that*

(i) for every $n \geq \bar{n}$ problem (2.50) admits a unique solution U_n;
(ii) $U_n \to U$ in $H^1(B_{r_0})$ after extending each U_n to zero in $B_{r_0} \setminus \mathcal{B}_{r_0,n}$.

Proof. To prove both points (i) and (ii), it is more convenient to study the following homogeneous boundary value problem

$$\begin{cases} -\operatorname{div}\left(A\nabla V_n\right) = \tilde{f} V_n + \tilde{f} G_n + \operatorname{div}\left(A\nabla G_n\right) & \text{in } \mathcal{B}_{r_0,n}, \\ V_n = 0 & \text{on } \partial\mathcal{B}_{r_0,n}. \end{cases} \tag{2.52}$$

We observe that U_n is a weak solution to (2.50) if and only if $V_n := U_n - G_n$ is a weak solution to (2.52) for each fixed $n \geq \bar{n}$, in the sense that $V_n \in H_0^1(\mathcal{B}_{r_0,n})$ and

$$\int_{\mathcal{B}_{r_0,n}} [A\nabla V_n \cdot \nabla\varphi - \tilde{f} V_n \varphi]\,dx = \int_{\mathcal{B}_{r_0,n}} [\tilde{f} G_n \varphi + \operatorname{div}(A\nabla G_n)\varphi]\,dx \quad \text{for every } \varphi \in H_0^1(\mathcal{B}_{r_0,n}). \tag{2.53}$$

First, we will prove that problem (2.52) admits a unique solution V_n for every $n \geq \bar{n}$. From this, it will follow that also problem (2.50) has a unique solution given by

$$U_n = V_n + G_n, \tag{2.54}$$

and thus (i) will be proved. For every $n \geq \bar{n}$, we set

$$Q_n : H_0^1(\mathcal{B}_{r_0,n}) \times H_0^1(\mathcal{B}_{r_0,n}) \to \mathbb{R},$$
$$Q_n(\psi,\varphi) := \int_{\mathcal{B}_{r_0,n}} [A\nabla\psi \cdot \nabla\varphi - \tilde{f}\psi\varphi]\,dx,$$

and

$$F_n : H_0^1(\mathcal{B}_{r_0,n}) \to \mathbb{R},$$
$$F_n(\varphi) := \int_{\mathcal{B}_{r_0,n}} [\tilde{f} G_n + \operatorname{div}(A\nabla G_n)]\varphi\,dx.$$

We observe that F_n is a linear and bounded operator on $H_0^1(\mathcal{B}_{r_0,n})$: indeed, after extending $\varphi \in H_0^1(\mathcal{B}_{r_0,n})$ to zero in $B_{r_0} \setminus \mathcal{B}_{r_0,n}$ using (2.8), (2.28), the boundedness of $\{G_n\}$ in $H^1(B_{r_0})$ by (2.51), and the continuity of the trace map

$$H^1(B_r) \to L^2(\partial B_r) \quad \text{for every } r > 0, \tag{2.55}$$

we have

$$\left|\int_{\mathcal{B}_{r_0,n}} \tilde{f} G_n \varphi\,dx\right| \leq \frac{4r_0^\delta}{(N-1)^2}\left(\int_{B_{r_0}} |\nabla G_n|^2\,dx + \frac{N-1}{2r_0}\int_{\partial B_{r_0}} G_n^2\,ds\right)^{1/2} \cdot \left(\int_{B_{r_0}} |\nabla\varphi|^2\,dx\right)^{1/2}$$
$$\leq \tilde{C}\|\varphi\|_{H_0^1(\mathcal{B}_{r_0,n})},$$

for some $\tilde{C} > 0$ independent of n; integrating by parts, using (2.15) and Hölder's inequality, we get

$$\left|\int_{\mathcal{B}_{r_0,n}} \operatorname{div}(A\nabla G_n)\varphi\,dx\right| \leq 2\left(\int_{\mathcal{B}_{r_0,n}} |\nabla G_n|^2\,dx\right)^{1/2}\left(\int_{\mathcal{B}_{r_0,n}} |\nabla\varphi|^2\,dx\right)^{1/2}$$
$$\leq \tilde{C}_1\|\varphi\|_{H_0^1(\mathcal{B}_{r_0,n})},$$

for some $\tilde{C}_1 > 0$ independent of n. Arguing as above, we can deduce that also the bilinear form Q_n is continuous on $H_0^1(\mathcal{B}_{r,n})$. Moreover, from Lemma 2.1, we can deduce that

$$Q_n(\psi,\psi) \geq \frac{1}{4}\|\psi\|^2_{H_0^1(\mathcal{B}_{r_0,n})} \quad \text{for every } \psi \in H_0^1(\mathcal{B}_{r_0,n}), \tag{2.56}$$

and thus Q_n is coercive on $H_0^1(\mathcal{B}_{r_0,n})$. By the Lax–Milgram Theorem problem (2.52) has a unique solution V_n for every $n \geq \bar{n}$, and thus (i) is proved. As for (ii), we notice that, since $\|V_n\|_{H_0^1(B_{r_0})} \leq 4(\tilde{C} + \tilde{C}_1)$ (up to extend to zero V_n in B_{r_0} outside of $\mathcal{B}_{r_0,n}$), $\{V_n\}$ turns out to be bounded in $H_0^1(B_{r_0})$. Thus there exists a subsequence $\{V_{n_k}\}$ such that

$$V_{n_k} \rightharpoonup V \quad \text{in } H_0^1(B_{r_0}) \tag{2.57}$$

for some $V \in H_0^1(B_{r_0})$. Furthermore V has null trace on $\tilde{\Gamma}$: indeed V_n has null trace on the set

$$\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N \geq \delta\} \cap B_{r_0}$$

for every $\delta > 0$, provided that n is sufficiently large (in dependence on δ); this is because V_n is identically zero in $B_{r_0} \setminus \mathcal{B}_{r_0,n}$ and

$$\{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N \geq \delta\} \cap B_{r_0} \subset B_{r_0} \setminus \mathcal{B}_{r_0,n}$$

for every $\delta > 0$, provided that n is sufficiently large (in dependence on δ). Hence, by a density argument, we are allowed to take $\varphi = V$ in identity (2.10), having that

$$\int_{B_{r_0}} A\nabla U \cdot \nabla V \, dx - \int_{B_{r_0}} \tilde{f} U V \, dx = 0. \tag{2.58}$$

On the other hand, using (2.53), integrating by parts, and exploiting (2.51) and (2.57), we have

$$\begin{aligned} Q_{n_k}(V_{n_k}, V_{n_k}) = F_{n_k}(V_{n_k}) &= \int_{B_{r_0}} [\tilde{f} G_{n_k} V_{n_k} + \operatorname{div}(A\nabla G_{n_k}) V_{n_k}] \, dx \\ &= \int_{B_{r_0}} \tilde{f} G_{n_k} V_{n_k} \, dx - \int_{B_{r_0}} A\nabla G_{n_k} \cdot \nabla V_{n_k} \, dx \\ &\to \int_{B_{r_0}} \tilde{f} U V \, dx - \int_{B_{r_0}} A\nabla U \cdot \nabla V \, dx \quad \text{as } k \to \infty. \end{aligned}$$

Combining this with (2.56) and (2.58), we deduce that $V_{n_k} \to 0$ in $H^1_0(B_{r_0})$. Repeating the same argument, one can find out that any other subsequence of $\{V_n\}$ always admits limit equal to 0. Hence, by Urysohn's subsequence principle we can conclude that $V_n \to 0$ in $H^1_0(B_{r_0})$, and therefore, putting together (2.51) and (2.54), we infer that $U_n \to U$ in $H^1(B_{r_0})$, thus proving (ii). □

3. A Pohozaev-type inequality

In this section we prove a Pohozaev-type inequality satisfied by any weak solution $U \in H^1_{\tilde{\Gamma}}(B_{\tilde{r}})$ to (2.5). The strategy of the proof consists in deriving first a "family" of Pohozaev-type identities and consequently inequalities (taking advantage of the result in Lemma 2.2) for the family of solutions $\{U_n\}$ to (2.50). Then we use the convergence of $\{U_n\}$ to U in an H^1-sense (proved in Proposition 2.3) to infer a Pohozaev-type inequality for U as well.

Proposition 3.1. *Let f satisfy either (2.8) or (2.9). Let $U \in H^1_{\tilde{\Gamma}}(B_{\tilde{r}})$ be a weak solution to problem (2.5) and let $\nu = \nu(x) = x/|x|$ for every $x \in \partial B_r$. Then, for a.e. $r \in (0, r_0)$*

$$\begin{aligned} r \int_{\partial B_r} (A\nabla U \cdot \nabla U) \, ds - 2r \int_{\partial B_r} \frac{1}{\mu} (A\nabla U \cdot \nu)^2 \, ds \geq{}& \int_{B_r} (\operatorname{div}\beta) A\nabla U \cdot \nabla U \, dx \\ &- 2 \int_{B_r} D\beta(A\nabla U) \cdot \nabla U \, dx + \int_{B_r} (dA\nabla U \nabla U) \cdot \beta \, dx \\ &+ 2 \int_{B_r} (\beta \cdot \nabla U) \tilde{f} U \, dx \end{aligned} \tag{3.1}$$

under assumption (2.8), and

$$\begin{aligned} r \int_{\partial B_r} (A\nabla U \cdot \nabla U) \, ds - 2r \int_{\partial B_r} \frac{1}{\mu} (A\nabla U \cdot \nu)^2 \, ds \geq{}& \int_{B_r} (\operatorname{div}\beta) A\nabla U \cdot \nabla U \, dx \\ &- 2 \int_{B_r} (D\beta)(A\nabla U) \cdot \nabla U \, dx + \int_{B_r} (dA\nabla U \nabla U) \cdot \beta \, dx \\ &+ r \int_{\partial B_r} \tilde{f} U^2 \, ds - \int_{B_r} (\tilde{f} \operatorname{div}\beta + \nabla \tilde{f} \cdot \beta) U^2 \, dx \end{aligned} \tag{3.2}$$

under assumption (2.9).

Proof. We start by observing that any solution U_n to problem (2.50) satisfies $U_n \in H^2(\mathcal{B}_{r,n} \setminus B_\delta)$ for all $r \in (0, r_0)$, $n \geq \bar{n}$, being $\bar{n} = \bar{n}(r)$ as in Lemma 2.2, and $\delta < 1/n^{2\alpha}$ (see Fig. 3). This descends from [12, Section 2.4], since by assumption $\tilde{f} \in L^\infty_{\mathrm{loc}}(B_{r_0} \setminus \{0\})$ and consequently $\tilde{f} U_n \in L^2_{\mathrm{loc}}(\mathcal{B}_{r_0,n} \setminus \{0\})$, the equation in (2.50) holds true in a smooth domain containing $\mathcal{B}_{r,n} \setminus B_\delta$ and in virtue of (2.7) and interior regularity.

In particular we deduce that

$$\operatorname{div}((A\nabla U_n \cdot \nabla U_n)\beta - 2A\nabla U_n(\beta \cdot \nabla U_n)) \in L^1(\mathcal{B}_{r,n} \setminus B_\delta).$$

So we can integrate the Rellich-Nečas identity

$$\begin{aligned} \operatorname{div}((A\nabla U_n \cdot \nabla U_n)\beta - 2A\nabla U_n(\beta \cdot \nabla U_n)) ={}& (\operatorname{div}\beta) A\nabla U_n \cdot \nabla U_n - 2D\beta(A\nabla U_n) \cdot \nabla U_n \\ &+ (dA\nabla U_n \nabla U_n) \cdot \beta - 2(\beta \cdot \nabla U_n)\operatorname{div}(A\nabla U_n) \end{aligned}$$

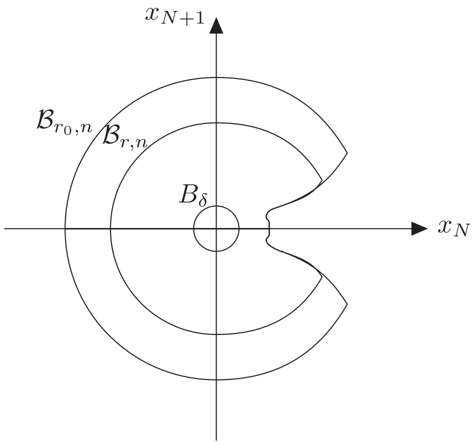

Fig. 3. A representation of the set $\mathcal{B}_{r,n} \setminus B_\delta$ with $r \in (0, r_0)$, $x' = 0$, $\alpha = 1/4$.

over $\mathcal{B}_{r,n} \setminus B_\delta$ and then, by applying [13, Proposition 2.7], we obtain that

$$\begin{aligned} &r\int_{S_{r,n}} (A\nabla U_n \cdot \nabla U_n)\, ds - 2r\int_{S_{r,n}} \frac{1}{\mu}(A\nabla U_n \cdot \nu)^2\, ds - \int_{\gamma_{r,n}} \frac{1}{\mu}\left|\frac{\partial U_n}{\partial \nu}\right|^2 (A\nu\cdot\nu)(Ax\cdot\nu)\, ds \\ &- \delta\int_{\partial B_\delta} (A\nabla U_n \cdot \nabla U_n)\, ds + 2\delta\int_{\partial B_\delta} \frac{1}{\mu}(A\nabla U_n\cdot\nu)^2\, ds = \int_{\mathcal{B}_{r,n}\setminus B_\delta} (\mathrm{div}\beta) A\nabla U_n\cdot\nabla U_n\, dx \\ &- 2\int_{\mathcal{B}_{r,n}\setminus B_\delta} D\beta(A\nabla U_n)\cdot\nabla U_n\, dx + \int_{\mathcal{B}_{r,n}\setminus B_\delta} (dA\nabla U_n\nabla U_n)\cdot\beta\, dx \\ &+ 2\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\beta\cdot\nabla U_n)\tilde f U_n\, dx, \end{aligned} \tag{3.3}$$

where ν denotes the outward unit normal vector to $\partial(\mathcal{B}_{r,n} \setminus B_\delta)$. To get (3.3), we used that $\beta \cdot x = |x|^2$ by (2.16) and (2.17); in addition, if $x \in S_{r,n}$ then $\nu = x/r$, if $x \in \partial B_\delta$ then $\nu = -x/\delta$; at last, the tangential component of ∇U_n is null on $\gamma_{r,n}$ since $U_n = G_n = 0$ on $\gamma_{\bar r,n} \supset \gamma_{r,n}$ in the trace sense, and hence $|\nabla U_n|^2 = |\frac{\partial U_n}{\partial \nu}|^2$.

Now we use Lemma 2.2 and (2.14) to infer that

$$\int_{\gamma_{r,n}} \frac{1}{\mu}\left|\frac{\partial U_n}{\partial \nu}\right|^2 (A\nu\cdot\nu)(Ax\cdot\nu)\, ds \geq 0,$$

and consequently (3.3) turns into the following inequality

$$\begin{aligned} &r\int_{S_{r,n}} (A\nabla U_n \cdot \nabla U_n)\, ds - 2r\int_{S_{r,n}} \frac{1}{\mu}(A\nabla U_n \cdot \nu)^2\, ds \\ &- \delta\int_{\partial B_\delta} (A\nabla U_n \cdot \nabla U_n)\, ds + 2\delta\int_{\partial B_\delta} \frac{1}{\mu}(A\nabla U_n\cdot\nu)^2\, ds \geq \int_{\mathcal{B}_{r,n}\setminus B_\delta} (\mathrm{div}\beta) A\nabla U_n\cdot\nabla U_n\, dx \\ &- 2\int_{\mathcal{B}_{r,n}\setminus B_\delta} D\beta(A\nabla U_n)\cdot\nabla U_n\, dx + \int_{\mathcal{B}_{r,n}\setminus B_\delta} (dA\nabla U_n\nabla U_n)\cdot\beta\, dx \\ &+ 2\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\beta\cdot\nabla U_n)\tilde f U_n\, dx. \end{aligned} \tag{3.4}$$

For the sake of convenience, from now on we split the proof into two steps.

Step one. In this step we will consider the limit as $\delta \searrow 0$ in (3.4). To such aim, we first observe that, since $U_n \in H^1(\mathcal{B}_{r_0,n})$ and $\tilde f \in L^\infty_{\mathrm{loc}}(\mathcal{B}_{r_0,n} \setminus \{0\})$, then for all $r \in (0, r_0)$ and $n \geq \bar n$ there exists a sequence $\{\delta_h\} \searrow 0$ such that

$$\delta_h \int_{\partial B_{\delta_h}} (|\nabla U_n|^2 + |\tilde f| U_n^2)\, ds \to 0 \quad \text{as } h \to \infty. \tag{3.5}$$

So, combining this with (2.15) and (2.20), we can deduce that

$$\delta_h \left[2\int_{\partial B_{\delta_h}} \frac{1}{\mu}(A\nabla U_n\cdot\nu)^2\, ds - \int_{\partial B_{\delta_h}} (A\nabla U_n\cdot\nabla U_n)\, ds \right] \to 0 \quad \text{as } h \to \infty. \tag{3.6}$$

Next, by (2.15), (2.24), (2.23), (2.27), (2.22), and the fact that $U_n \in H^1(B_r)$ after extending U_n to zero in $B_r \setminus \mathcal{B}_{r,n}$, applying the absolute continuity of the Lebesgue integral, we have that as $\delta \searrow 0$

$$\begin{aligned}
&\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\mathrm{div}\beta)A\nabla U_n \cdot \nabla U_n\,dx = \int_{B_r\setminus B_\delta} (\mathrm{div}\beta)A\nabla U_n \cdot \nabla U_n\,dx \to \int_{B_r} (\mathrm{div}\beta)A\nabla U_n \cdot \nabla U_n\,dx,\\
&\int_{\mathcal{B}_{r,n}\setminus B_\delta} D\beta(A\nabla U_n)\cdot \nabla U_n\,dx = \int_{B_r\setminus B_\delta} D\beta(A\nabla U_n)\cdot \nabla U_n\,dx \to \int_{B_r} D\beta(A\nabla U_n)\cdot \nabla U_n\,dx,\\
&\int_{\mathcal{B}_{r,n}\setminus B_\delta} (dA\nabla U_n \cdot \nabla U_n)\cdot \beta\,dx = \int_{B_r\setminus B_\delta} (dA\nabla U_n \cdot \nabla U_n)\cdot \beta\,dx \to \int_{B_r} (dA\nabla U_n \cdot \nabla U_n)\cdot \beta\,dx.
\end{aligned} \tag{3.7}$$

Moreover, under assumption (2.8), using (2.22) and applying the Hölder inequality to $|\nabla U_n|$ and $|U_n|/|x|$, we obtain that

$$\begin{aligned}
\int_{B_r} |(\beta\cdot\nabla U_n)\tilde f U_n|\,dx &\le \mathrm{const}\, r^\delta \int_{B_r} |\nabla U_n|\cdot \frac{|U_n|}{|x|}\,dx\\
&\le \mathrm{const}\, r^\delta \left(\int_{B_r} |\nabla U_n|^2\,dx\right)^{1/2} \left(\int_{B_r} \frac{|U_n(x)|^2}{|x|^2}\,dx\right)^{1/2},
\end{aligned}$$

for some const > 0 independent of r. From this, (2.28) and the fact that $\{U_n\}$ is bounded in $H^1(B_r)$ thanks to Proposition 2.3, we can infer that $(\beta\cdot\nabla U_n)\tilde f U_n \in L^1(B_r)$. Therefore by Lebesgue's dominated convergence theorem, we can conclude that as $\delta \searrow 0$

$$\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\beta\cdot\nabla U_n)\tilde f U_n\,dx = \int_{B_r\setminus B_\delta} (\beta\cdot\nabla U_n)\tilde f U_n\,dx \to \int_{B_r} (\beta\cdot\nabla U_n)\tilde f U_n\,dx. \tag{3.8}$$

Taking into account (3.6), (3.7) and (3.8), passing to the limit as $h\to\infty$ in (3.4) with $\delta=\delta_h$, we get that for all $r\in(0,r_0)$ and $n\ge\bar n$

$$\begin{aligned}
r\int_{\partial B_r} (A\nabla U_n\cdot\nabla U_n)\,ds - 2r\int_{\partial B_r} \frac{1}{\mu}(A\nabla U_n\cdot\nu)^2\,ds \ge{}& \int_{B_r} (\mathrm{div}\beta)A\nabla U_n\cdot\nabla U_n\,dx\\
&- 2\int_{B_r} D\beta(A\nabla U_n)\cdot\nabla U_n\,dx + \int_{B_r} (dA\nabla U_n\nabla U_n)\cdot\beta\,dx\\
&+ 2\int_{B_r} (\beta\cdot\nabla U_n)\tilde f U_n\,dx,
\end{aligned} \tag{3.9}$$

under assumption (2.8).

On the other hand, under assumption (2.9), since it holds that

$$(\beta\cdot\nabla U_n)\tilde f U_n = \tilde f(\beta\cdot\nabla(U_n^2/2)),$$

we use the divergence theorem to rewrite the last term in (3.4) as follows

$$\begin{aligned}
\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\beta\cdot\nabla U_n)\tilde f U_n\,dx ={}& \frac{r}{2}\int_{S_{r,n}} \tilde f U_n^2\,ds - \frac{\delta}{2}\int_{\partial B_\delta} \tilde f U_n^2\,ds\\
&- \frac12\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\nabla\tilde f\cdot\beta)U_n^2\,dx - \frac12\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\mathrm{div}\beta)\tilde f U_n^2\,dx.
\end{aligned} \tag{3.10}$$

In particular, we have that as $\delta\searrow 0$

$$\begin{aligned}
&\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\nabla\tilde f\cdot\beta)U_n^2\,dx = \int_{B_r\setminus B_\delta} (\nabla\tilde f\cdot\beta)U_n^2\,dx \to \int_{B_r} (\nabla\tilde f\cdot\beta)U_n^2\,dx,\\
&\int_{\mathcal{B}_{r,n}\setminus B_\delta} (\mathrm{div}\beta)\tilde f U_n^2\,dx = \int_{B_r} (\mathrm{div}\beta)\tilde f U_n^2\,dx,
\end{aligned} \tag{3.11}$$

by the absolute continuity of the Lebesgue integral, since $(\nabla\tilde f\cdot\beta)U_n^2$ and $(\mathrm{div}\beta)\tilde f U_n^2$ are both in $L^1(B_r)$, as a consequence of the Hölder inequality, the fact that $\tilde f\in W^{1,p}(B_r)$, (2.22) and (2.24). Thus, plugging (3.10) into (3.4), by (3.6), (3.7), (3.5) and (3.11), passing to the limit as $h\to\infty$ with $\delta=\delta_h$, we get that for all $r\in(0,r_0)$ and $n\ge\bar n$

$$\begin{aligned}
r\int_{\partial B_r} (A\nabla U_n\cdot\nabla U_n)\,ds - 2r\int_{\partial B_r} \frac{1}{\mu}(A\nabla U_n\cdot\nu)^2\,ds \ge{}& \int_{B_r} (\mathrm{div}\beta)A\nabla U_n\cdot\nabla U_n\,dx\\
&- 2\int_{B_r} D\beta(A\nabla U_n)\cdot\nabla U_n\,dx + \int_{B_r} (dA\nabla U_n\nabla U_n)\cdot\beta\,dx\\
&+ r\int_{\partial B_r} \tilde f U_n^2\,ds - \int_{B_r} (\tilde f\,\mathrm{div}\beta + \nabla\tilde f\cdot\beta)U_n^2\,dx,
\end{aligned} \tag{3.12}$$

under assumption (2.9).

Step two. In this step we will take the limit as $n\to\infty$ in (3.9) and (3.12). We start by noting that, since A is symmetric, we have the following identity

$$A\nabla U_n\cdot\nabla U_n - A\nabla U\cdot\nabla U = A\nabla U_n\cdot\nabla(U_n-U) + A\nabla U\cdot\nabla(U_n-U).$$

By this, (2.24), and (2.15), applying Hölder's inequality, using that $U_n \to U$ in $H^1(B_r)$ thanks to Proposition 2.3, and consequently that $\{\nabla U_n\}$ is bounded in $L^2(B_r)$, we thus have that

$$\begin{aligned}
&\int_{B_r} |\mathrm{div}\beta||A\nabla U_n \cdot \nabla U_n - A\nabla U \cdot \nabla U|\, dx \\
&\leq \int_{B_r} |\mathrm{div}\beta||A\nabla U_n||\nabla(U_n - U)|\, dx + \int_{B_r} |\mathrm{div}\beta||A\nabla U||\nabla(U_n - U)|\, dx \\
&\leq 2(N+2)\left(\int_{B_r} |\nabla U_n||\nabla(U_n - U)|\, dx + \int_{B_r} |\nabla U||\nabla(U_n - U)|\, dx\right) \\
&\leq 2(N+2)\left(\int_{B_r} |\nabla(U_n - U)|^2\, dx\right)^{1/2}\left[\left(\int_{B_r} |\nabla U_n|^2\, dz\right)^{1/2} + \left(\int_{B_r} |\nabla U|^2\, dz\right)^{1/2}\right] \to 0
\end{aligned} \tag{3.13}$$

as $n \to \infty$. Similarly to the above identity, we can write

$$D\beta(A\nabla U_n) \cdot \nabla U_n - D\beta(A\nabla U) \cdot \nabla U = D\beta(A\nabla(U_n - U)) \cdot \nabla U_n + D\beta(A\nabla U) \cdot \nabla(U_n - U),$$

obtained by adding $\pm D\beta(A\nabla U) \cdot \nabla U_n$ to the left-hand side. Using this, (2.23), (2.15), and arguing as when inferring (3.13), we can deduce that for some $\mathrm{const} > 0$ independent of r

$$\begin{aligned}
&\int_{B_r} |D\beta(A\nabla U_n) \cdot \nabla U_n - D\beta(A\nabla U) \cdot \nabla U|\, dx \\
&\leq \int_{B_r} \|D\beta\||A\nabla(U_n - U)||\nabla U_n|\, dx + \int_{B_r} \|D\beta\||A\nabla U||\nabla(U_n - U)|\, dx \\
&\leq \mathrm{const}\left(\int_{B_r} |\nabla(U_n - U)||\nabla U_n|\, dx + \int_{B_r} |\nabla U||\nabla(U_n - U)|\, dx\right) \to 0
\end{aligned} \tag{3.14}$$

as $n \to \infty$. In analogous way, we can write

$$dA\nabla U_n \nabla U_n - dA\nabla U \nabla U = dA\nabla U_n \nabla(U_n - U) + dA\nabla U \nabla(U_n - U),$$

obtained by adding $\pm dA\nabla U \nabla U_n$ to the left-hand side, and applying (2.25) and (2.26). Exploiting the last identity, (2.22), (2.27), and arguing as when inferring (3.13), we have that for some $\mathrm{const} > 0$ independent of r

$$\begin{aligned}
\int_{B_r} |dA\nabla U_n \nabla U_n \cdot \beta - dA\nabla U \nabla U \cdot \beta|\, dx &\leq \int_{B_r} |dA\nabla U_n \nabla U_n - dA\nabla U \nabla U||\beta|\, dx \\
&\leq \mathrm{const}\, r\left(\int_{B_r} |dA\nabla U_n \nabla(U_n - U)|\, dx + \int_{B_r} |dA\nabla U \nabla(U_n - U)|\, dx\right) \\
&\leq \mathrm{const}\, r\left(\int_{B_r} |\nabla U_n||\nabla(U_n - U)|\, dx + \int_{B_r} |\nabla U||\nabla(U_n - U)|\, dx\right) \to 0
\end{aligned} \tag{3.15}$$

as $n \to \infty$. To conclude the convergence as $n \to \infty$ of the right-hand side of (3.9), we observe that

$$(\beta \cdot \nabla U_n)\tilde{f} U_n - (\beta \cdot \nabla U)\tilde{f} U = (\beta \cdot \nabla U_n)\tilde{f}(U_n - U) + \tilde{f} U(\beta \cdot \nabla(U_n - U))$$

obtained by adding $\pm(\beta \cdot \nabla U_n)\tilde{f} U$ to the left-hand side; therefore, by this and (2.22), applying Hölder's inequality and (2.28), we have that for some $\mathrm{const} > 0$ independent of r

$$\begin{aligned}
&\int_{B_r} |(\beta \cdot \nabla U_n)\tilde{f} U_n - (\beta \cdot \nabla U)\tilde{f} U|\, dx \\
&\qquad \leq \int_{B_r} |(\beta \cdot \nabla U_n)\tilde{f}(U_n - U)|\, dx + \int_{B_r} |\tilde{f} U(\beta \cdot \nabla(U_n - U))|\, dx \\
&\quad \leq \mathrm{const}\, r^\delta \left(\int_{B_r} |\nabla U_n| \frac{|U_n - U|}{|x|}\, dx + \int_{B_r} \frac{|U(x)|}{|x|}|\nabla(U_n - U)|\, dx\right) \\
&\quad \leq \mathrm{const}\, r^\delta \left(\left[\int_{B_r} |\nabla U_n|^2\, dx\right]^{1/2}\left[\int_{B_r} |\nabla(U_n - U)|^2\, dx + \frac{N-1}{2r}\int_{\partial B_r} |U_n - U|^2\, ds\right]^{1/2}\right. \\
&\qquad \left. + \left[\int_{B_r} |\nabla(U_n - U)|^2\, dx\right]^{1/2}\left[\int_{B_r} |\nabla U|^2\, dx + \frac{N-1}{2r}\int_{\partial B_r} U^2\, ds\right]^{1/2}\right) \to 0
\end{aligned} \tag{3.16}$$

as $n \to \infty$, where in addition we have used the convergence $U_n \to U$ in $H^1(B_r)$ proved in Proposition 2.3, which in turn implies that $\{\nabla U_n\}$ is bounded in $L^2(B_r)$ and that $U_n \to U$ in $L^2(\partial B_r)$ by the continuity of the trace operator in (2.55). As for the left-hand side of (3.9), arguing precisely as when deriving (3.13), we have that

$$\int_{B_{r_0}} A\nabla U_n \cdot \nabla U_n\, dx \to \int_{B_{r_0}} A\nabla U \cdot \nabla U\, dx \quad \text{as } n \to \infty. \tag{3.17}$$

From this, using the coarea formula, we deduce that up to a subsequence still denoted with U_n

$$\int_{\partial B_r} A\nabla U_n \cdot \nabla U_n \, ds \to \int_{\partial B_r} A\nabla U \cdot \nabla U \, ds \quad \text{as } n \to \infty, \text{ for a.e. } r \in (0, r_0). \tag{3.18}$$

At last, using (2.20), (2.15), the Hölder inequality, the convergence $U_n \to U$ in $H^1(B_{r_0})$ and consequently the boundedness of $\{\nabla U_n\}$ in $L^2(B_{r_0})$, we have that

$$\begin{aligned}\int_{B_{r_0}} \frac{1}{\mu}|(A\nabla U_n \cdot \nu)^2 - (A\nabla U \cdot \nu)^2| \, ds &\leq 2\int_{B_{r_0}} |A\nabla(U_n - U)\cdot \nu||A\nabla(U_n + U)\cdot \nu| \, ds \\ &\leq 4\left(\int_{B_{r_0}} |\nabla(U_n - U)|^2 \, dx\right)^{1/2}\left(\int_{B_{r_0}} |\nabla(U_n + U)|^2 \, dx\right)^{1/2} \to 0\end{aligned}$$

as $n \to \infty$; hence up to a subsequence still denoted with U_n, it holds that

$$\int_{\partial B_r} \frac{1}{\mu}(A\nabla U_n \cdot \nu)^2 \, ds \to \int_{\partial B_r} \frac{1}{\mu}(A\nabla U \cdot \nu)^2 \, ds \quad \text{as } n \to \infty, \text{ for a.e. } r \in (0, r_0). \tag{3.19}$$

Putting together (3.13), (3.14), (3.15), (3.16), (3.18), (3.19), and passing to the limit as $n \to \infty$ in (3.9), we have shown the validity of (3.1) for a.e. $r \in (0, r_0)$, under assumption (2.8).

On the other hand, under assumption (2.9), using (2.24), and applying the Hölder inequality and (1.5), we have that for some $\mathrm{const} > 0$ independent of r

$$\begin{aligned}&\int_{B_r} |\tilde{f}\,\mathrm{div}\beta||U_n^2 - U^2| \, dx \leq (N+2)\int_{B_r} |\tilde{f}||U_n - U||U_n + U| \, dx \\ &\leq (N+2)\left(\int_{B_r} |\tilde{f}||U_n - U|^2 \, dx\right)^{1/2}\left(\int_{B_r} |\tilde{f}||U_n + U|^2 \, dx\right)^{1/2} \\ &\leq \mathrm{const}\, r^{\frac{2p-N-1}{p}}\left(\int_{B_r} |\nabla(U_n - U)|^2 \, dx + \frac{N-1}{2r}\int_{\partial B_r} |U_n - U|^2 \, ds\right)^{1/2} \\ &\quad \cdot \left(\int_{B_r} |\nabla(U_n + U)|^2 \, dx + \frac{N-1}{2r}\int_{\partial B_r} |U_n + U|^2 \, ds\right)^{1/2} \to 0\end{aligned} \tag{3.20}$$

as $n \to \infty$, where in addition we have used the convergence $U_n \to U$ in $H^1(B_r)$ proved in Proposition 2.3, which in turn, summed to the continuity of the trace operator in (2.55), implies that $U_n \to U$ in $L^2(\partial B_r)$ and $\{U_n\}$ is bounded in $L^2(\partial B_r)$. Moreover, still under assumption (2.9), taking advantage of (2.22) and applying again the Hölder inequality, we get that for some $\mathrm{const} > 0$ independent of r

$$\int_{B_r} |\nabla \tilde{f} \cdot \beta||U_n^2 - U^2| \, dx \leq \mathrm{const}\, r \int_{B_r} |\nabla \tilde{f}||U_n - U||U_n + U| \, dx \to 0 \tag{3.21}$$

as $n \to \infty$, reasoning as in (3.20). Eventually, under assumption (2.9), using that $\tilde{f} \in L^\infty_{\mathrm{loc}}(B_{r_0} \setminus \{0\})$, it also holds that for some $\mathrm{const} > 0$ independent of r

$$\int_{\partial B_r} |\tilde{f}||U_n^2 - U^2| \, ds \leq \mathrm{const} \int_{\partial B_r} |U_n - U||U_n + U| \to 0 \quad \text{as } n \to \infty, \tag{3.22}$$

applying in addition the Hölder inequality, exploiting the convergence $U_n \to U$ in $L^2(\partial B_r)$ and the boundedness of $\{U_n\}$ in $L^2(\partial B_r)$, both as a consequence of Proposition 2.3 and the continuity of the trace map in (2.55). Assembling (3.13), (3.14), (3.15), (3.18), (3.19), (3.20), (3.21) and (3.22), and passing to the limit as $n \to \infty$ in (3.12), we have shown the validity of (3.2) for a.e. $r \in (0, r_0)$, under assumption (2.9). ☐

Lemma 3.2. *Let f satisfy either (2.8) or (2.9). Let $U \in H^1_{\tilde{\Gamma}}(B_{\tilde{r}})$ be a weak solution to problem (2.5) and let $\nu = \nu(x) = x/|x|$ for every $x \in \partial B_r$. Then, for a.e. $r \in (0, r_0)$*

$$\int_{B_r} A\nabla U \cdot \nabla U \, dx - \int_{B_r} \tilde{f} U^2 \, dx = \int_{\partial B_r} (A\nabla U \cdot \nu) U \, ds. \tag{3.23}$$

Proof. We multiply Eq. (2.50) with U_n itself and then we integrate over $\mathcal{B}_{r,n} \setminus B_\delta$. Taking into account that $U_n = G_n = 0$ on $\gamma_{r,n}$, and consequently U_n can be extended to zero in $B_r \setminus \mathcal{B}_{r,n}$, an integration by parts leas to the following identity

$$\begin{aligned}&\int_{B_r \setminus B_\delta} A\nabla U_n \cdot \nabla U_n \, dx - \int_{B_r \setminus B_\delta} \tilde{f} U_n^2 \, dx \\ &\qquad = \int_{\partial B_r} (A\nabla U_n \cdot \nu) U_n \, ds - \int_{\partial B_\delta} (A\nabla U_n \cdot \nu) U_n \, ds.\end{aligned} \tag{3.24}$$

Using (2.15), we can easily deduce that $A\nabla U_n \cdot \nabla U_n \in L^1(B_r)$; thus from the absolute continuity of the Lebesgue integral, it follows that

$$\int_{B_r \setminus B_\delta} A\nabla U_n \cdot \nabla U_n \, dx \to \int_{B_r} A\nabla U_n \cdot \nabla U_n \, dx \quad \text{as } \delta \searrow 0. \tag{3.25}$$

Furthermore, we have that $\tilde{f}U_n^2 \in L^1(B_r)$ either by (2.28) under assumption (2.8), or by the Hölder inequality and (1.5) under assumption (2.9); so, again by the absolute continuity of the Lebesgue integral, we obtain that

$$\int_{B_r\setminus B_\delta} \tilde{f}U_n^2\,dx \to \int_{B_r} \tilde{f}U_n^2\,dx \quad \text{as } \delta \searrow 0. \tag{3.26}$$

As for the limit as $\delta \searrow 0$ of the last term in (3.24), it is sufficient noting that $(A\nabla U_n\cdot\nu)U_n \in L^1(B_r)$ thanks to (2.15); as a result, by the Lebesgue dominated convergence theorem, it holds that

$$\int_{\partial B_\delta} (A\nabla U_n\cdot\nu)U_n\,ds = \int_{B_r} \chi_{\partial B_\delta}(x)(A\nabla U_n\cdot\nu)U_n\,dx \to 0 \quad \text{as } \delta \searrow 0. \tag{3.27}$$

Summing (3.25), (3.26) and (3.27), and passing to the limit as $\delta \searrow 0$ in (3.24), we have that for all $r \in (0,r_0)$ and $n \geq \bar{n}$

$$\int_{B_r} A\nabla U_n\cdot\nabla U_n\,dx - \int_{B_r} \tilde{f}U_n^2\,dx = \int_{\partial B_r} (A\nabla U_n\cdot\nu)U_n\,ds. \tag{3.28}$$

To conclude with, we have to pass to the limit as $n\to\infty$ in the previous identity. In particular, thanks to the Hölder inequality we have that

$$\begin{aligned}\int_{B_r} |\tilde{f}U_n^2 - \tilde{f}U^2|\,dx &= \int_{B_r} |\tilde{f}||U_n - U||U_n + U|\,dx\\ &\leq \left(\int_{B_r} |\tilde{f}||U_n - U|^2\,dx\right)^{1/2}\left(\int_{B_r} |\tilde{f}||U_n + U|^2\,dx\right)^{1/2} \to 0 \quad \text{as } n\to\infty,\end{aligned} \tag{3.29}$$

if in addition either we use (2.28) if $\tilde{f}$ satisfies assumption (2.8), or we exploit once more the Hölder inequality and (1.5) if $\tilde{f}$ satisfies assumption (2.9). Moreover, using (2.15) and the Hölder inequality, we also infer that

$$\begin{aligned}&\int_{B_r} |(A\nabla U_n\cdot\nu)U_n - (A\nabla U\cdot\nu)U|\,dx\\ &\leq \int_{B_r} |(A\nabla(U_n - U)\cdot\nu)U_n|\,dx + \int_{B_r} |(A\nabla U\cdot\nu)(U_n - U)|\,dx\\ &\leq 2\left[\left(\int_{B_r} |U_n|^2\,dx\right)^{1/2}\left(\int_{B_r} |\nabla(U_n - U)|^2\,dx\right)^{1/2}\right.\\ &\quad\left. + \left(\int_{B_r} |\nabla U|^2\,dx\right)^{1/2}\left(\int_{B_r} |U_n - U|^2\,dx\right)^{1/2}\right] \to 0 \quad \text{as } n\to\infty.\end{aligned}$$

Therefore by the coarea formula, we can assert that up to a subsequence still denoted with U_n

$$\int_{\partial B_r} (A\nabla U_n\cdot\nu)U_n\,ds \to \int_{\partial B_r} (A\nabla U\cdot\nu)U\,ds \quad \text{as } n\to\infty, \text{ for a.e. } r\in(0,r_0). \tag{3.30}$$

Hence combining (3.17) with (3.29) and (3.30), and passing to the limit as $n \to \infty$ in (3.28), we finally get (3.23) for a.e. $r \in (0,r_0)$. □

4. The monotonicity formula

In this section we prove a monotonocity formula for the Almgren frequency function associated with solutions of problem (2.5). To such aim, we fix a non-trivial weak solution $U \in H^1_{\tilde{\Gamma}}(B_{\tilde{r}})$ to problem (2.5) and we define the energy function as

$$D : r \in (0,\tilde{r}) \mapsto D(r) := \frac{1}{r^{N-1}}\left(\int_{B_r} A\nabla U\cdot\nabla U\,dx - \int_{B_r} \tilde{f}U^2\,dx\right), \tag{4.1}$$

and the height function as

$$H : r \in (0,\tilde{r}) \mapsto H(r) := \frac{1}{r^N}\int_{\partial B_r} \mu U^2\,ds. \tag{4.2}$$

In the following lemma we show that the function $H(r)$ is strictly positive if r is sufficiently small.

Lemma 4.1. *Let $r_0 > 0$ be as in Lemma 2.1. Then the function H defined in (4.2) is strictly positive in $(0,r_0]$.*

Proof. We assume by contradiction that $H(r_1) = 0$ for some $r_1 \in (0,r_0]$. Hence, taking into account (2.20), we can deduce that the trace of U is identically zero on ∂B_{r_1}. So, taking $\varphi = U$ in (2.10), we have that

$$\int_{B_{r_1}} A\nabla U\cdot\nabla U\,dx = \int_{B_{r_1}} \tilde{f}U^2\,dx.$$

From this and Lemma 2.1, we can conclude that necessarily $U \equiv 0$ in B_{r_1}, which in turn implies that $U \equiv 0$ in $B_{\tilde{r}}$ as a consequence of unique continuation principles for second order elliptic equations with Lipschitz coefficients (see e.g. [1]). In particular this contradicts the fact that $U \not\equiv 0$ by assumption. □

We are now in the position to define Almgren's frequency function as follows

$$\mathcal{N} : r \in (0, r_0] \mapsto \mathcal{N}(r) := \frac{D(r)}{H(r)}. \tag{4.3}$$

We premise a simple result which will be crucial to estimate D'.

Lemma 4.2. *Every $U \in H^1(B_r)$ with $r \in (0, r_0]$ satisfies the following inequality*

$$\int_{B_r} |\nabla U|^2 \, dz \le 4r^{N-1} \left(D(r) + \frac{N-1}{4} H(r) \right). \tag{4.4}$$

Proof. Taking advantage of (2.29) and recalling the definitions of D and H given in (4.1) and (4.2) respectively, we have that

$$\int_{B_r} |\nabla U|^2 \, dz \le 4r^{N-1} \left(D(r) + Cr^{\varepsilon} H(r) \right),$$

for every $U \in H^1(B_r)$ with $r \in (0, r_0]$. Thus (4.4) is a trivial consequence of this and (2.31). □

In the following lemma we show that $\mathcal{N}$ is bounded from below.

Lemma 4.3. *We have that*

$$\mathcal{N}(r) > -\frac{N-1}{4} \quad \text{for every } r \in (0, r_0]. \tag{4.5}$$

Proof. By (2.29), (4.1) and (4.2), we have that

$$r^{N-1} D(r) \ge -Cr^{N-1+\varepsilon} H(r) \quad \text{for every } r \in (0, r_0].$$

Hence, recalling the definition of $\mathcal{N}$ given in (4.3), we deduce that

$$\mathcal{N}(r) \ge -Cr^{\varepsilon} \quad \text{for every } r \in (0, r_0], \tag{4.6}$$

which gives (4.5), if in addition we take into account (2.31). □

In the next lemma we exhibit an estimate from below for the derivative of D: to derive such a result, the Pohozaev-type inequality will turn out to be fundamental.

Lemma 4.4. *Let f satisfy either (2.8) or (2.9). It holds that*

$$D \in W^{1,1}_{\mathrm{loc}}(0, \tilde{r}) \tag{4.7}$$

and for a.e. $r \in (0, r_0)$

$$D'(r) \ge \frac{2}{r^{N-1}} \int_{\partial B_r} \frac{1}{\mu} (A\nabla U \cdot \nu)^2 \, ds + O(r^{\bar{\varepsilon}}) \left(D(r) + \frac{N-1}{2} H(r) \right) \quad \text{as } r \to 0^+, \tag{4.8}$$

where $\nu = \nu(x) := x/|x|$ for every $x \in \partial B_r$ and

$$\bar{\varepsilon} = \begin{cases} 0 & \text{if } -1 + \varepsilon \ge 0, \\ -1 + \varepsilon & \text{if } -1 + \varepsilon \le 0, \end{cases} \tag{4.9}$$

being $\varepsilon > 0$ as in (2.30).

Proof. (4.7) holds since D is the product of the $W^{1,\infty}_{\mathrm{loc}}(0, \tilde{r})$-function $r \mapsto r^{1-N}$ and

$$r \mapsto \int_{B_r} A\nabla U \cdot \nabla U \, dx - \int_{B_r} \tilde{f} U^2 \, dx$$

which is in $W^{1,1}(0, \tilde{r})$; this last fact can be deduced by using the coarea formula, the fact that $U \in H^1(B_{\tilde{r}})$, (2.15), and additionally either (2.28) under assumption (2.8), or the Hölder inequality and (1.5) under assumption (2.9). Moreover the coarea formula also allows us to infer that

$$D'(r) = \frac{1}{r^{N-1}} \left(\int_{\partial B_r} A\nabla U \cdot \nabla U \, ds - \int_{\partial B_r} \tilde{f} U^2 \, ds \right) + \frac{1-N}{r^N} \left(\int_{B_r} A\nabla U \cdot \nabla U \, dx - \int_{B_r} \tilde{f} U^2 \, dx \right)$$

in the sense of distributions and a.e. in $(0, \tilde{r})$. Then we use (3.1) and (3.2) under assumption (2.8) and (2.9) respectively to estimate from below the term $\int_{\partial B_r} A\nabla U \cdot \nabla U \, ds$, obtaining that for a.e. $r \in (0, r_0)$

$$\begin{aligned} D'(r) \ge{}& \frac{2}{r^{N-1}} \int_{\partial B_r} \frac{1}{\mu} (A\nabla U \cdot \nu)^2 \, ds + \frac{1}{r^N} \int_{B_r} (\operatorname{div}\beta) A\nabla U \cdot \nabla U \, dx - \frac{2}{r^N} \int_{B_r} D\beta (A\nabla U) \cdot \nabla U \, dx \\ &+ \frac{1}{r^N} \int_{B_r} (dA\nabla U\nabla U) \cdot \beta \, dx + \frac{1-N}{r^N} \int_{B_r} A\nabla U \cdot \nabla U \, dx - \frac{1}{r^{N-1}} \int_{\partial B_r} \tilde{f} U^2 \, ds \\ &+ \frac{2}{r^N} \int_{B_r} (\beta \cdot \nabla U) \tilde{f} U \, dx + \frac{N-1}{r^N} \int_{B_r} \tilde{f} U^2 \, dx, \end{aligned} \tag{4.10}$$

under assumption (2.8), and

$$\begin{aligned} D'(r) \geq & \frac{2}{r^{N-1}} \int_{\partial B_r} \frac{1}{\mu}(A\nabla U \cdot \nu)^2 \, ds + \frac{1}{r^N} \int_{B_r} (\mathrm{div}\beta) A\nabla U \cdot \nabla U \, dx \\ & - \frac{2}{r^N} \int_{B_r} D\beta(A\nabla U) \cdot \nabla U \, dx + \frac{1}{r^N} \int_{B_r} (dA\nabla U \nabla U) \cdot \beta \, dx \\ & + \frac{1-N}{r^N} \int_{B_r} A\nabla U \cdot \nabla U \, dx - \frac{1}{r^N} \int_{B_r} (\tilde{f}\mathrm{div}\beta + \nabla \tilde{f} \cdot \beta) U^2 \, dx + \frac{N-1}{r^N} \int_{B_r} \tilde{f} U^2 \, dx, \end{aligned} \tag{4.11}$$

under assumption (2.9). First of all we observe that

$$\begin{aligned} & \frac{1}{r^N} \int_{B_r} (\mathrm{div}\beta) A\nabla U \cdot \nabla U \, dx - \frac{2}{r^N} \int_{B_r} D\beta(A\nabla U) \cdot \nabla U \, dx + \frac{1}{r^N} \int_{B_r} (dA\nabla U \nabla U) \cdot \beta \, dx \\ & + \frac{1-N}{r^N} \int_{B_r} A\nabla U \cdot \nabla U \, dx = \frac{O(r)}{r^N} \int_{B_r} |\nabla U|^2 \, dx \\ & \qquad = O(1) \left(D(r) + \frac{N-1}{4} H(r) \right) \quad \text{as } r \to 0^+, \end{aligned} \tag{4.12}$$

as a direct consequence of (2.13), (2.21) and (2.27), and thanks to (4.4). Now we specifically focus on (4.10). For this, under assumption (2.8) we use (2.28), (2.20) and (4.4) to deduce that

$$\begin{aligned} \left| \int_{B_r} \tilde{f} U^2 \, dx \right| & \leq \mathrm{const}\, r^{\delta} \left[\left(\frac{2}{N-1} \right)^2 \int_{B_r} |\nabla U|^2 \, dx + \frac{4}{N-1} r^{-1} \int_{\partial B_r} \mu U^2 \, ds \right] \\ & \leq \mathrm{const}\, r^{\delta+N-1} \left(D(r) + \frac{N-1}{2} H(r) \right), \end{aligned} \tag{4.13}$$

for some $\mathrm{const} > 0$ independent of r; moreover by (2.20) we also have that

$$\left| \int_{\partial B_r} \tilde{f} U^2 \, ds \right| \leq \mathrm{const}\, r^{-2+\delta} \int_{\partial B_r} \mu U^2 \, ds \leq \mathrm{const}\, r^{-2+\delta+N} H(r), \tag{4.14}$$

for some $\mathrm{const} > 0$ independent of r; lastly, taking into consideration (2.22), applying the Hölder inequality to $|x|^{-1}|U|$ and $|\nabla U|$, and using (2.28), (2.20) and (4.4), we obtain that

$$\begin{aligned} \left| \int_{B_r} (\beta \cdot \nabla U) \tilde{f} U \, dx \right| & \leq \mathrm{const}\, r^{\delta} \int_{B_r} |x|^{-1} |U| |\nabla U| \, dx \\ & \leq \mathrm{const}\, r^{\delta} \sqrt{r^{N-1} \left(D(r) + \frac{N-1}{2} H(r) \right)} \cdot \sqrt{r^{N-1} \left(D(r) + \frac{N-1}{4} H(r) \right)} \\ & \leq \mathrm{const}\, r^{-1+\delta+N} \left(D(r) + \frac{N-1}{2} H(r) \right), \end{aligned} \tag{4.15}$$

for some $\mathrm{const} > 0$ independent of r. Putting together (4.10), (4.12), (4.13), (4.14) and (4.15), we can conclude that a.e. in $(0, r_0)$ under assumption (2.8)

$$\begin{aligned} D'(r) \geq & \frac{2}{r^{N-1}} \int_{\partial B_r} \frac{1}{\mu}(A\nabla U \cdot \nu)^2 \, ds + O(1) \left(D(r) + \frac{N-1}{4} H(r) \right) \\ & + O(r^{-1+\delta}) \left(D(r) + \frac{N-1}{2} H(r) \right) \quad \text{as } r \to 0^+. \end{aligned}$$

From this (4.8) immediately follows under assumption (2.8). In order to conclude the proof, we turn to concentrate on (4.11). In particular, under assumption (2.9), by the Hölder inequality, (1.5), (2.20) and ultimately (4.4), we have that

$$\begin{aligned} \left| \int_{B_r} \tilde{f} U^2 \, dx \right| & \leq \mathrm{const}\, r^{\frac{2p-N-1}{p}} \left(\int_{B_r} |\nabla U|^2 \, dx + \frac{N-1}{r} \int_{\partial B_r} \mu U^2 \, ds \right) \\ & \leq \mathrm{const}\, r^{\frac{2p-N-1}{p}+N-1} \left(D(r) + \frac{N-1}{2} H(r) \right), \end{aligned} \tag{4.16}$$

for some $\mathrm{const} > 0$ independent of r; furthermore, from (2.21), (2.22), the Hölder inequality, (1.5), (2.20) and (4.4), we arrive at

$$\begin{aligned} \left| \int_{B_r} (\tilde{f}\mathrm{div}\beta + \nabla \tilde{f} \cdot \beta) U^2 \, dx \right| & \leq \mathrm{const}\, r^{\frac{2p-N-1}{p}} \left(\int_{B_r} |\nabla U|^2 \, dx + \frac{N-1}{r} \int_{\partial B_r} \mu U^2 \, ds \right) \\ & \leq \mathrm{const}\, r^{\frac{2p-N-1}{p}+N-1} \left(D(r) + \frac{N-1}{2} H(r) \right), \end{aligned} \tag{4.17}$$

for some $\mathrm{const} > 0$ independent of r. Combining (4.11), (4.12), (4.16) and (4.17), we can deduce that a.e. in $(0, r_0)$ under assumption (2.9)

$$\begin{aligned} D'(r) \geq & \frac{2}{r^{N-1}} \int_{\partial B_r} \frac{1}{\mu}(A\nabla U \cdot \nu)^2 \, ds + O(1) \left(D(r) + \frac{N-1}{4} H(r) \right) \\ & + O(r^{-1+\frac{2p-N-1}{p}}) \left(D(r) + \frac{N-1}{2} H(r) \right) \quad \text{as } r \to 0^+, \end{aligned}$$

which implies (4.8) under assumption (2.9). □

In the following lemma we compute the derivative of H: differently to the result in [2, Lemma 4.1], a perturbing term which behaves as H itself appears.

Lemma 4.5. *We have that*

$$H \in W^{1,1}_{\mathrm{loc}}(0,\tilde r) \tag{4.18}$$

and

$$H'(r) = \frac{2}{r^N}\int_{\partial B_r} \mu U \frac{\partial U}{\partial \nu}\, ds + O(H(r)) \quad \text{as } r \to 0^+, \tag{4.19}$$

in a distributional sense and a.e. in $(0,\tilde r)$*, where* $\nu = \nu(x) := x/|x|$ *for every* $x \in \partial B_r$*.*

Proof. We have that $H \in L^1_{\mathrm{loc}}(0,\tilde r)$ since it turns out to be the product between the $L^\infty_{\mathrm{loc}}(0,\tilde r)$-function $r \mapsto r^{-N}$ and

$$r \mapsto \int_{\partial B_r} \mu U^2\, ds,$$

which belongs to $L^1(0,\tilde r)$. To show this, it is sufficient to use that $U \in H^1(B_{\tilde r})$ and that,

$$\mu(x) \le \text{const} \quad \text{for every } x \in B_{\tilde r}, \tag{4.20}$$

for some $\text{const} > 1$ independent on r, as a consequence of (2.18). As for the derivative of H, we can reason precisely as in the proof of [3, Lemma 3.1] (keeping in mind that here $s = 1/2$ and hence the weight t^{1-2s} does not occur) to deduce that the distributional derivative of H is given by

$$H'(r) = \frac{2}{r^N}\int_{\partial B_r} \mu U \frac{\partial U}{\partial \nu}\, ds + \frac{1}{r^N}\int_{\partial B_r} (\nabla\mu\cdot\nu)U^2\, ds. \tag{4.21}$$

Exploiting (2.19) and (4.20), we find out that the right-hand side of (4.21) belongs to $L^1_{\mathrm{loc}}(0,\tilde r)$. Thus (4.18) is proved and (4.21) holds a.e. in $(0,\tilde r)$ (not only in a distributional sense). Eventually, in order to get (4.19), it is enough to observe that

$$\frac{1}{r^N}\int_{\partial B_r} (\nabla\mu\cdot\nu)U^2\, ds = O(1)H(r) \quad \text{as } r \to 0,$$

by (2.19) and (2.20), and recalling the definition of H given in (4.2). □

In the next lemma we prove an equivalent formulation of the L^1_{loc}-derivative of H and an important identity relating the derivative of H with D.

Lemma 4.6. *It holds that a.e. in* $(0,\tilde r)$

$$H'(r) = \frac{2}{r^N}\int_{\partial B_r} (A\nabla U\cdot\nu)U\, ds + O(H(r)) \quad \text{as } r \to 0, \tag{4.22}$$

with $\nu = \nu(x) := x/|x|$ *for every* $x \in \partial B_r$*, and a.e. in* $(0, r_0)$

$$H'(r) = \frac{2}{r}D(r) + O(H(r)) \quad \text{as } r \to 0. \tag{4.23}$$

Proof. We immediately notice that once (4.22) is shown, then (4.23) can be easily derived by using (3.23) and recalling the definition of D given in (4.1). Therefore we have to prove (4.22): to this aim, we observe that a.e. in $(0,\tilde r)$

$$\begin{aligned}\frac{2}{r^N}\int_{\partial B_r} \mu\left(\nabla U\cdot\frac{x}{|x|}\right)U\, ds &= \frac{2}{r^N}\int_{\partial B_r} \mu\left(\nabla U\cdot\frac{\beta}{|x|}\right)U\, ds - \frac{2}{r^N}\int_{\partial B_r} \mu\left(\nabla U\cdot\frac{\beta-x}{|x|}\right)U\, ds\\ &= \frac{2}{r^N}\int_{\partial B_r}\left(A\nabla U\cdot\frac{x}{|x|}\right)U\, ds - \frac{1}{r^N}\int_{\partial B_r}\mu\left(\nabla(U^2)\cdot\frac{\beta-x}{|x|}\right) ds,\end{aligned} \tag{4.24}$$

where in the first row we have simply added $\pm\mu\left(\nabla U\cdot\frac{\beta}{|x|}\right)U$ to the integrand function and in the second row we have used (2.17) and the fact that A is symmetric. Now, by (2.16) and (2.17) we have that

$$\mu\left(\frac{\beta-x}{|x|}\right)\cdot x = 0;$$

then applying the divergence theorem before, using the coarea formula after, we deduce that a.e. in $(0,\tilde r)$

$$0 = \int_{\partial B_r} U^2 \operatorname{div}\left(\frac{\mu(\beta-x)}{|x|}\right) ds + \int_{\partial B_r}\mu\left(\nabla(U^2)\cdot\frac{\beta-x}{|x|}\right) ds. \tag{4.25}$$

Now we explicitly compute $\operatorname{div}\left(\frac{\mu(\beta-x)}{|x|}\right)$, having that

$$\operatorname{div}\left(\frac{\mu(\beta-x)}{|x|}\right) = \nabla\mu\cdot\frac{\beta-x}{|x|} + \frac{\mu\operatorname{div}(\beta-x)}{|x|} - \mu(\beta-x)\cdot\frac{x}{|x|^3},$$

which, together with (2.18), (2.19) and (2.21), yields

$$\operatorname{div}\left(\frac{\mu(\beta-x)}{|x|}\right)=O(1)\quad\text{as } |x|\to 0. \tag{4.26}$$

In conclusion, the thesis follows by combining (4.19), (4.24), (4.25), (4.26), recalling that $\nu = x/|x|$ and taking into account (2.20). □

Exploiting the previous results we are able to prove the desired estimate from below of the derivative of $\mathcal{N}$, which in turn will allow us to show as byproducts the boundedness from above of $\mathcal{N}$ and the existence of its limit as $r\to 0$.

Lemma 4.7. *It holds that*

$$\mathcal{N}\in W^{1,1}_{\mathrm{loc}}(0,r_0), \tag{4.27}$$

and a.e. in $(0,r_0)$

$$\mathcal{N}'(r)\geq O(r^{\bar\varepsilon})\left(\mathcal{N}(r)+\frac{N-1}{2}\right)\quad \text{as } r\to 0^+, \tag{4.28}$$

where $\bar\varepsilon\in(-1,0)$ *is defined in* (4.9).

Proof. (4.27) follows by assembling the regularity of D with the regularity and the positivity of H (see (4.7), (4.18) and Lemma 4.1 respectively). In order to derive (4.28) we explicitly compute the distributional derivative of $\mathcal{N}$: exploiting (4.8), (4.22) and (4.23), we have that a.e. in $(0,r_0)$

$$\begin{aligned}\mathcal{N}'(r)\geq{}&\frac{2r\left[\left(\int_{\partial B_r}\frac{(A\nabla U\cdot\nu)^2}{\mu}\,ds\right)\left(\int_{\partial B_r}\mu U^2\,ds\right)-\left(\int_{\partial B_r}U(A\nabla U\cdot\nu)\,ds\right)^2\right]}{\left(\int_{\partial B_r}\mu U^2\,ds\right)^2}\\&+O(r)+O(r^{\bar\varepsilon})\left(\mathcal{N}(r)+\frac{N-1}{2}\right)+\frac{O(r^2)}{r^N H(r)}\int_{\partial B_r}(A\nabla U\cdot\nu)U\,ds\quad\text{as } r\to 0.\end{aligned} \tag{4.29}$$

Using (4.22), (4.23), and recalling the definition of $\mathcal{N}$ given in (4.3), we infer that a.e. in $(0,r_0)$

$$\frac{1}{r^N H(r)}\int_{\partial B_r}(A\nabla U\cdot\nu)U\,ds=\frac{H'(r)}{2H(r)}+O(1)=\frac{\mathcal{N}(r)}{r}+O(1)\quad\text{as } r\to 0.$$

Plugging this into (4.29) and applying the Cauchy–Schwarz inequality to $\frac{A\nabla U\cdot\nu}{\sqrt{\mu}}$ and $\sqrt{\mu}U$ as vectors in $L^2(\partial B_r)$, we arrive at

$$\mathcal{N}'(r)\geq\left(O(r^{\bar\varepsilon})+O(r)\right)\left(\mathcal{N}(r)+\frac{N-1}{2}\right)\quad\text{as } r\to 0^+ \text{ and a.e. in } (0,r_0),$$

which, taking into account that $\bar\varepsilon\in(-1,0)$ by (4.9), implies (4.28). □

Lemma 4.8. *There exists a positive constant* $C_4>0$ *such that*

$$\mathcal{N}(r)\leq C_4\quad \textit{for every } r\in(0,r_0). \tag{4.30}$$

Proof. Making use of (4.28), we have that for a.e. $\rho\in(0,r_0)$

$$\mathcal{N}'(\rho)\geq -C_3\,\rho^{\bar\varepsilon}\left(\mathcal{N}(\rho)+\frac{N-1}{2}\right) \tag{4.31}$$

for some $C_3>0$ independent of r (up to take r_0 smaller from the beginning, without loss of generality). From this, observing that

$$\mathcal{N}'(\rho)=\left(\mathcal{N}(\cdot)+\frac{N-1}{2}\right)'(\rho),$$

taking into consideration (4.5) and integrating with respect to $\rho\in(r,r_0)$ for any $r\in(0,r_0)$, we achieve (4.30). □

Lemma 4.9. *The limit* $\lim_{r\to 0^+}\mathcal{N}(r)$ *does exist. Moreover it is finite and nonnegative.*

Proof. From (4.27) it follows that for every $r\in(0,r_0)$

$$\mathcal{N}(r_0)-\mathcal{N}(r)=\int_r^{r_0}\mathcal{N}'(\rho)\,d\rho. \tag{4.32}$$

Now we write $\mathcal{N}'$ as the sum of two terms, as follows

$$\mathcal{N}'(\rho)=\xi_1(\rho)+\xi_2(\rho)\quad\text{for a.e. } \rho\in(0,r_0), \tag{4.33}$$

where

$$\xi_1(\rho):=\mathcal{N}'(\rho)+C_3\,\rho^{\bar\varepsilon}\left(C_4+\frac{N-1}{2}\right)\geq 0, \tag{4.34}$$

as a consequence of (4.30) and (4.31), and

$$\xi_2(\rho) := -C_3\,\rho^{\bar{\varepsilon}}\left(C_4 + \frac{N-1}{2}\right) \in L^1(0, r_0), \tag{4.35}$$

since $\bar{\varepsilon} > -1$. Hence, if we insert (4.33) into (4.32), passing to the limit as $r \to 0^+$, we can conclude that $\lim_{r\to 0^+} \mathcal{N}(r)$ does exist, in light of (4.34) and (4.35) by applying the monotone convergence theorem and Lebesgue's dominated convergence theorem. In particular such a limit is finite in virtue of (4.5) and (4.30), and it is positive thanks to (4.6). □

We set

$$\ell := \lim_{r\to 0^+} \mathcal{N}(r). \tag{4.36}$$

and by Lemma 4.9 we have that $\ell \in \mathbb{R}_+$.

In the rest of this section we prove some results giving information on the growth of H.

Lemma 4.10. *There exists a positive constant $C_5 > 0$ such that*

$$H(r) \le C_5 r^{2\ell} \quad \textit{for all } r \in (0, r_0), \tag{4.37}$$

and for any $\sigma > 0$ there exists a positive constant $C_6 = C_6(\sigma) > 0$ such that

$$H(r) \ge C_6 r^{2\ell+\sigma} \quad \textit{for all } r \in (0, r_0). \tag{4.38}$$

Proof. We first show (4.37). For this, we first notice that by Lemma 4.9 for every $\rho \in (0, r_0)$

$$\mathcal{N}(\rho) - \ell = \int_0^\rho \mathcal{N}'(\tau)\,d\tau, \tag{4.39}$$

with $\ell \in \mathbb{R}_+$ as in (4.36). Moreover from (4.34) we can deduce that

$$\mathcal{N}'(\tau) \ge -C_3\,\tau^{\bar{\varepsilon}}\left(C_4 + \frac{N-1}{2}\right) \quad \text{for a.e. } \tau \in (0, r_0).$$

Merging this with (4.39), we obtain that for every $\rho \in (0, r_0)$

$$\mathcal{N}(\rho) - \ell \ge -\text{const}\,\rho^{\bar{\varepsilon}+1}, \tag{4.40}$$

for some $\text{const} > 0$. At this point, we exploit (4.23) and (4.40) to infer that for a.e. $\rho \in (0, r_0)$

$$\frac{H'(\rho)}{H(\rho)} \ge \frac{2\ell}{\rho} - \text{const}\,\rho^{\bar{\varepsilon}} \quad \text{as } r \to 0$$

(up to choose r_0 smaller from the beginning); then integrating the above inequality over (r, r_0) for all $r \in (0, r_0)$, we get (4.37). Now we prove (4.38). To such purpose, we use (4.23) and (4.36) to claim that for any $\sigma > 0$ there exists $r_\sigma > 0$ such that for a.e. $\rho \in (0, r_\sigma)$

$$\frac{H'(\rho)}{H(\rho)} \le \frac{2\ell + \sigma}{\rho}.$$

Integrating the above inequality over (r, r_σ) with $r \in (0, r_\sigma)$, we deduce that $H(r) \ge \text{const}\, r^{2\ell+\sigma}$ for every $r \in (0, r_\sigma)$ for some $\text{const} > 0$. The validity of this last inequality for every $r \in [r_\sigma, r_0)$ is a trivial consequence of (4.18). (4.38) is thereby proved. □

Lemma 4.11. *The function $H(r)/r^{2\ell}$ admits a finite limit as $r \to 0^+$.*

Proof. In view of (4.37), it remains to prove the existence of the limit. To this aim, we compute the derivative of $H(\rho)/\rho^{2\ell}$, obtaining that for a.e. $\rho \in (0, r_0)$

$$\frac{d}{d\rho}\left(\frac{H(\rho)}{\rho^{2\ell}}\right) = \frac{2H(\rho)}{\rho^{2\ell+1}}\left[\int_0^\rho \mathcal{N}'(\tau)\,d\tau + O(\rho)\right], \tag{4.41}$$

using (4.23) and (4.39). Taking advantage of (4.33), (4.34) and (4.35), the right-hand side of (4.41) becomes

$$\frac{2H(\rho)}{\rho^{2\ell+1}}\int_0^\rho \xi_1(\tau)\,d\tau - \frac{2H(\rho)}{\rho^{2\ell}}\left(\frac{C_3}{\bar{\varepsilon}+1}\left(C_4 + \frac{N-1}{2}\right)\rho^{\bar{\varepsilon}} + O(1)\right).$$

So, integrating (4.41) over (r, r_0) for any $r \in (0, r_0)$, we get

$$\frac{H(r_0)}{r_0^{2\ell}} - \frac{H(r)}{r^{2\ell}} = \int_r^{r_0}\left[\frac{2H(\rho)}{\rho^{2\ell+1}}\int_0^\rho \xi_1(\tau)\,d\tau\right]d\rho - \int_r^{r_0}\frac{2H(\rho)}{\rho^{2\ell}}\left(C_7\,\rho^{\bar{\varepsilon}} + O(1)\right)d\rho, \tag{4.42}$$

with $C_7 := \frac{C_3}{\bar{\varepsilon}+1}\left(C_4 + \frac{N-1}{2}\right) > 0$. We now focus on the right-hand side of (4.42): the limit as $r \to 0^+$ of the first term does exist as a consequence of the monotone convergence theorem, taking into account (4.34); the limit of the second term does exist as well and is finite, applying the Lebesgue dominated convergence theorem, thanks to (4.37) and the fact that $\bar{\varepsilon} > -1$. From these considerations and (4.42), we can deduce the existence of the limit of $H(r)/r^{2\ell}$ as $r \to 0^+$. The lemma is thereby proved. □

5. The blow-up argument

In this section we investigate the convergence properties and then we get information on the limit profile as $\lambda \to 0^+$ of the rescaled and renormalized family of functions $\{U^\lambda\}_{\lambda\in(0,r_0)}$ defined as follows: for any $\lambda \in (0, r_0)$

$$U^\lambda : x \in B_{\frac{r_0}{\lambda}} \mapsto U^\lambda(x) := \frac{U(\lambda x)}{\sqrt{H(\lambda)}}, \tag{5.1}$$

where $U \in H^1_{\tilde{\Gamma}}(B_{\tilde{r}})$ is a fixed non-trivial weak solution to (2.5). We notice that the family is well-defined thanks to Lemma 4.1. Moreover the word *renormalized* is justified by

$$\int_{\partial B_1} \mu(\lambda\cdot)|U^\lambda(\cdot)|^2 ds = 1. \tag{5.2}$$

Furthermore, since U weakly solves problem (2.5), by direct computations we have that U^λ is a weak solution to

$$\begin{cases} -\mathrm{div}(A(\lambda\cdot)\nabla U^\lambda(\cdot)) = \lambda^2 \tilde{f}(\lambda\cdot)U^\lambda(\cdot) & \text{in } B_{\frac{r_0}{\lambda}} \setminus \tilde{\Gamma}, \\ U^\lambda = 0 & \text{on } \tilde{\Gamma}, \end{cases} \tag{5.3}$$

for every fixed $\lambda \in (0, r_0)$. This has to be interpreted in the sense that U^λ belongs to the space

$$H^1_{\tilde{\Gamma}}(B_{\frac{r_0}{\lambda}}) := \overline{C^\infty(\overline{B_{\frac{r_0}{\lambda}}} \setminus \tilde{\Gamma})}^{\|\cdot\|_{H^1(B_{\frac{r_0}{\lambda}})}}, \tag{5.4}$$

namely the closure with respect to the H^1-norm of the space of all $C^\infty(\overline{B_{\frac{r_0}{\lambda}}})$-functions vanishing in a neighbourhood of $\tilde{\Gamma}$, and it holds that

$$\int_{B_{\frac{r_0}{\lambda}}} A(\lambda x)\nabla U^\lambda(x)\cdot\nabla v(x)\,dx = \lambda^2 \int_{B_{\frac{r_0}{\lambda}}} \tilde{f}(\lambda x)U^\lambda(x)v(x)\,dx \quad \text{for every } v \in C^\infty_c(B_{\frac{r_0}{\lambda}} \setminus \tilde{\Gamma}). \tag{5.5}$$

In the following lemma we prove that the family $\{U^\lambda\}_{\lambda\in(0,r_0)}$ is bounded in $H^1(B_1)$ which is the smallest possible space since as $\lambda \to 0^+$ the ball $B_{\frac{r_0}{\lambda}}$ becomes larger and larger.

Lemma 5.1. *There exists a positive constant $M > 0$ such that*

$$\|U^\lambda\|_{H^1(B_1)} \le M \quad \textit{for every } \lambda \in (0, r_0).$$

Proof. Thanks to (2.29) and (2.31), applying a suitable change of variable, we have that

$$\frac{1}{4}\lambda^{N-1}H(\lambda)\int_{B_1} |\nabla U^\lambda|^2\,dz \le \lambda^{N-1}\left(D(\lambda) + \frac{N-1}{4}H(\lambda)\right).$$

Hence dividing each member of the above inequality by $\lambda^{N-1}H(\lambda)$ and taking into account (4.30), we deduce that

$$\|\nabla U^\lambda\|^2_{L^2(B_1)} \le 4C_4 + N - 1. \tag{5.6}$$

Moreover from (2.28) we can infer that

$$\left(\frac{N-1}{2}\right)^2 \lambda^{N-1}H(\lambda)\int_{B_1} |U^\lambda|^2\,dx \le \lambda^{N-1}H(\lambda)\left(\int_{B_1} |\nabla U^\lambda|^2\,dx + N - 1\right),$$

exploiting (2.20) in order to make H appear on the right-hand side. Dividing each member of the last inequality by $\lambda^{N-1}H(\lambda)$ and using (5.6), we obtain that

$$\|U^\lambda\|^2_{L^2(B_1)} \le \frac{16C_4}{(N-1)^2} + \frac{8}{N-1}. \tag{5.7}$$

Combining (5.6) and (5.7), we arrive at the thesis. □

The following three lemmas are thought to derive the boundedness of the $L^2(\partial B_1)$-norm of a slightly modified version of ∇U^λ (see Lemma 5.5 below), which in turn will come into play to establish the convergence-type result of Proposition 5.10 below.

Lemma 5.2. *There exists a positive constant $M_1 > 0$ such that for every $\lambda \in (0, \frac{r_0}{2})$ and for every $R \in [1, 2]$*

$$M_1^{-1}H(\lambda) \le H(R\lambda) \le M_1 H(\lambda). \tag{5.8}$$

Proof. Using (4.23), (4.5) and (4.30), we can deduce that there exist two positive constants $c_1, c_2 > 0$ such that

$$-\frac{c_1}{r} \le \frac{H'(r)}{H(r)} \le \frac{c_2}{r}$$

for a.e. $r \in (0, r_0)$, up to select r_0 smaller from the beginning. Integrating over $(\lambda, R\lambda)$ with $R \in (1, 2]$ and $\lambda \in (0, \frac{r_0}{R})$, we obtain that

$$2^{-c_1} \leq \frac{H(R\lambda)}{H(\lambda)} \leq 2^{c_2}.$$

The above inequality still holds if $R = 1$. So (5.8) follows after observing that $(0, \frac{r_0}{2}) \subseteq (0, \frac{r_0}{R})$. □

Lemma 5.3. *Let $M_1 > 0$ be as in Eq. (5.8). Then for every $\lambda \in (0, \frac{r_0}{2})$ and for every $R \in [1, 2]$ it holds that*

$$\int_{B_R} |U^\lambda|^2 \, dx \leq M_1 2^{N+1} \int_{B_1} |U^{R\lambda}|^2 \, dx,$$

and

$$\int_{B_R} |\nabla U^\lambda|^2 \, dx \leq M_1 2^{N-1} \int_{B_1} |\nabla U^{R\lambda}|^2 \, dx.$$

Proof. We omit the proof since one can proceed exactly as in the proof of [2, Lemma 5.3], using Lemma 5.2 and applying suitable changes of variable. □

Lemma 5.4. *Let U^λ be as in (5.1) with $\lambda \in (0, r_0)$. Then there exist $M_2 > 0$ and $\bar{\lambda} > 0$ such that for every $\lambda \in (0, \bar{\lambda})$ there exists $R_\lambda \in [1, 2]$ such that*

$$\int_{\partial B_{R_\lambda}} |\nabla U^\lambda|^2 \, ds \leq M_2 \int_{B_{R_\lambda}} |\nabla U^\lambda|^2 \, dx.$$

Proof. For every fixed $\lambda \in (0, \frac{r_0}{2})$ the function

$$g_\lambda : r \mapsto g_\lambda(r) := \int_{B_r} |\nabla U^\lambda(x)|^2 \, dx \tag{5.9}$$

is absolutely continuous in $[0, 2]$ and thus

$$g'_\lambda(r) = \int_{\partial B_r} |\nabla U^\lambda(x)|^2 \, ds$$

in a distributional sense and a.e. in $(0, 2)$. Now we assume by contradiction that for every $M_2 > 0$ there exists a sequence $\lambda_n \to 0^+$ such that $g'_{\lambda_n}(r) > M_2 \, g_{\lambda_n}(r)$ a.e. in $(0, 2)$ for every $n \geq 1$; so, integrating with respect to r, we deduce that $g_{\lambda_n}(2) > M_2 \, g_{\lambda_n}(1)$ for every $n \geq 1$. From this we can infer that

$$\liminf_{\lambda \to 0^+} g_\lambda(1) \leq e^{-M_2} \limsup_{\lambda \to 0^+} g_\lambda(2), \tag{5.10}$$

where the lim sup in the right-hand side is less than $+\infty$ as a consequence of the boundedness of $\{U^\lambda\}_{\lambda \in (0, r_0/2)}$ in $H^1(B_2)$ (this comes from Lemmas 5.1 and 5.3). Thus, passing to the limit as $M_2 \to \infty$ in (5.10) and recalling the definition of g_λ given in (5.9), we get that

$$\liminf_{\lambda \to 0^+} \int_{B_1} |\nabla U^\lambda(x)|^2 \, dx = 0. \tag{5.11}$$

Now we claim that there exists a sequence $\lambda_n \to 0^+$ such that $U^{\lambda_n} \rightharpoonup W$ in $H^1(B_1)$ for some $W \in H^1(B_1)$ and in addition

$$\lim_{n \to \infty} \int_{B_1} |\nabla U^{\lambda_n}(x)|^2 \, dx = 0.$$

This can be deduced by combining (5.11) and Lemma 5.1. In particular, this and the weak lower semicontinuity of L^2-norm imply that

$$\int_{B_1} |\nabla W(x)|^2 \, dx = 0.$$

Hence we have that W is equal to a non-zero constant since

$$\int_{\partial B_1} |W|^2 \, ds = 1, \tag{5.12}$$

which is a consequence of (5.2) and the compactness of the trace operator (2.55). We are almost done: indeed we notice that since U^{λ_n} is in the space defined in (5.4) then U^{λ_n} belongs to

$$\{v \in H^1(B_1) : v = 0 \text{ on } \tilde{\Gamma} \text{ in a trace sense}\}, \tag{5.13}$$

which is weakly closed in $H^1(B_1)$; thus necessarily $W \equiv 0$ in B_1, producing a contradiction with (5.12). □

Lemma 5.5. *There exists a positive constant $M_3 > 0$ such that for every $\lambda \in (0, \min\{\bar{\lambda}, \frac{r_0}{2}\})$*

$$\int_{\partial B_1} |\nabla U^{\lambda R_\lambda}|^2 \, ds \le M_3,$$

being $R_\lambda \in [1, 2]$ as in Lemma 5.4.

Proof. We pass over the proof since one can reason precisely as in the proof of [2, Lemma 5.5] taking advantage of Lemmas 5.2–5.4 and applying suitable changes of variable. □

For future reference, it is useful to write the following lemma which is a straightforward consequence of Lemma 5.5.

Lemma 5.6. *If $\lambda_n \to 0^+$, then there exist a subsequence $\{\lambda_{n_k}\}_{k\ge1}$ and a $L^2(\partial B_1)$- function h such that*

$$A(\lambda_{n_k} R_{\lambda_{n_k}} \cdot)\nabla U^{\lambda_{n_k} R_{\lambda_{n_k}}} \cdot \nu \rightharpoonup h \quad \text{in } L^2(\partial B_1) \text{ as } k \to \infty,$$

with $R_{\lambda_{n_k}} \in [1, 2]$ as in Lemma 5.4.

Proof. Let $\lambda_n \to 0^+$. From Lemma 5.5 we can deduce that there exist a subsequence $\{\lambda_{n_k}\}_{k\ge1}$ and a $L^2(\partial B_1)$- function h such that for any $\psi \in L^2(\partial B_1)$

$$\int_{\partial B_1} \frac{\partial U^{\lambda_{n_k} R_{\lambda_{n_k}}}}{\partial \nu} \psi \, ds \to \int_{\partial B_1} h\psi \, ds \quad \text{as } k \to \infty,$$

being $R_{\lambda_{n_k}} \in [1, 2]$ chosen as in Lemma 5.4. Summing this fact with (2.13), we have that

$$\begin{aligned} &\int_{\partial B_1} (A(\lambda_{n_k} R_{\lambda_{n_k}} \cdot)\nabla U^{\lambda_{n_k} R_{\lambda_{n_k}}} \cdot \nu)\psi \, ds \\ &= \int_{\partial B_1} \frac{\partial U^{\lambda_{n_k} R_{\lambda_{n_k}}}}{\partial \nu} \psi \, ds + O(\lambda_{n_k} R_{\lambda_{n_k}}) \int_{\partial B_1} (\nabla U^{\lambda_{n_k} R_{\lambda_{n_k}}} \cdot \nu)\psi \, ds \\ &\to \int_{\partial B_1} h\psi \, ds \quad \text{as } k \to \infty, \end{aligned}$$

where in addition we have used that

$$\left| \int_{\partial B_1} (\nabla U^{\lambda_{n_k} R_{\lambda_{n_k}}} \cdot \nu)\psi \, ds \right| \le M_3 \left(\int_{\partial B_1} |\psi|^2 \, ds \right)^{\frac{1}{2}},$$

by the Hölder inequality and thanks to Lemma 5.5. The proof is thereby complete. □

Lemma 5.7. *Let ℓ be as in (4.36). For any sequence $\lambda_n \to 0^+$, given $R_{\lambda_n} \in [1, 2]$ as in Lemma 5.4, there exists a subsequence $\{\lambda_{n_k}\}_{k\ge1}$ and an homogeneous function $\overline{U} \in H^1(B_1)$ of degree ℓ, i.e.*

$$\overline{U}(x) = |x|^\ell \, \overline{U}\left(\frac{x}{|x|}\right) \quad \text{for all } x \in B_1, \tag{5.14}$$

such that

$$U^{\lambda_{n_k} R_{\lambda_{n_k}}} \to \overline{U} \quad \text{in } H^1(B_1) \text{ as } k \to \infty. \tag{5.15}$$

Moreover $\Psi := \overline{U}|_{\mathbb{S}^N}$ is an L^2-normalized eigenfunction of problem (1.8) associated with $\ell(\ell + N - 1)$.

Remark 5.8. For future goals, we stress that from Lemma 5.7 the number $\ell(\ell + N - 1)$ turns out to be an eigenvalue of problem (1.8) and hence there exists $k_0 \ge 1$ such that $\ell(\ell + N - 1) = \mu_{k_0}$.

Proof of Lemma 5.7. Let $\lambda_n \to 0^+$. Taking $R_{\lambda_n} \in [1, 2]$ as in Lemma 5.4, we have that $\{U^{\lambda_n R_{\lambda_n}}\}_{n\ge1}$ is uniformly bounded in $H^1(B_1)$ with respect to n, as a consequence of Lemma 5.1. Therefore there exist a subsequence $\{\lambda_{n_k}\}_{k\ge1}$ and $\overline{U} \in H^1(B_1)$ such that

$$U^k \rightharpoonup \overline{U} \text{ in } H^1(B_1) \text{ as } k \to \infty, \tag{5.16}$$

having set $U^k := U^{\lambda_{n_k} R_{\lambda_{n_k}}}$ for every $k \ge 1$ in order to lighten the notation. In particular, it holds that

$$\overline{U} \not\equiv 0; \tag{5.17}$$

indeed, combining (5.16) and the compactness of the trace operator defined in (2.55), we have that

$$U^k \to \overline{U} \text{ in } L^2(\partial B_1), \tag{5.18}$$

which in turn, together with (2.18) and (5.2), allows us to deduce that

$$\int_{\partial B_1} |\overline{U}|^2 \, ds = 1. \tag{5.19}$$

Now we aim at finding the boundary value problem solved by $\overline{U}$. For this, we first claim that $\overline{U}$ satisfies

$$\int_{B_1} \nabla \overline{U} \cdot \nabla \varphi \, dx = 0 \quad \text{for every } \varphi \in C_c^\infty(B_1 \setminus \tilde{\Gamma}). \tag{5.20}$$

To prove this, we fix any function $\varphi \in C_c^\infty(B_1 \setminus \tilde{\Gamma})$. Since

$$B_1 \subset B_{r_0/(\lambda_{n_k} R_{\lambda_{n_k}})} \text{ for sufficiently large } k \geq 1, \tag{5.21}$$

in particular we have that $\varphi \in C_c^\infty(B_{r_0/(\lambda_{n_k} R_{\lambda_{n_k}})} \setminus \tilde{\Gamma})$ for sufficiently large $k \geq 1$. Thus, taking in (5.5) $\lambda = \lambda_{n_k} R_{\lambda_{n_k}}$ for sufficiently large k, we have that

$$\int_{B_1} A(\lambda_{n_k} R_{\lambda_{n_k}} x) \nabla U^k(x) \cdot \nabla \varphi(x) \, dx = (\lambda_{n_k} R_{\lambda_{n_k}})^2 \int_{B_1} \tilde{f}(\lambda_{n_k} R_{\lambda_{n_k}} x) U^k(x) \varphi(x) \, dx \tag{5.22}$$

for sufficiently large k. Reasoning exactly as in the proof of Lemma 5.6 and exploiting (5.16), we obtain that

$$\int_{B_1} A(\lambda_{n_k} R_{\lambda_{n_k}} x) \nabla U^k(x) \cdot \nabla \varphi(x) \, dx \to \int_{B_1} \nabla \overline{U}(x) \cdot \nabla \varphi(x) \, dx \quad \text{as } k \to \infty. \tag{5.23}$$

Additionally we observe that under assumption (2.8)

$$\begin{aligned}
&(\lambda_{n_k} R_{\lambda_{n_k}})^2 \left| \int_{B_1} \tilde{f}(\lambda_{n_k} R_{\lambda_{n_k}} x) U^k(x) \varphi(x) \, dx \right| \\
&\leq (\lambda_{n_k} R_{\lambda_{n_k}})^\delta \left(\int_{B_1} \frac{|U^k(x)|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \cdot \left(\int_{B_1} \frac{|\varphi(x)|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \\
&\leq \frac{4(\lambda_{n_k} R_{\lambda_{n_k}})^\delta}{(N-1)^2} \left(\int_{B_1} |\nabla U^k(x)|^2 \, dx + (N-1) \int_{\partial B_1} \mu(\lambda_{n_k} R_{\lambda_{n_k}} \cdot) |U^k|^2 \, ds \right)^{\frac{1}{2}} \cdot \left(\int_{B_1} |\nabla \varphi(x)|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq \frac{4(\lambda_{n_k} R_{\lambda_{n_k}})^\delta}{(N-1)^2} \left(\int_{B_1} |\nabla U^k(x)|^2 \, dx + N - 1 \right)^{\frac{1}{2}} \left(\int_{B_1} |\nabla \varphi(x)|^2 \, dx \right)^{\frac{1}{2}} = o(1) \text{ as } k \to \infty,
\end{aligned}$$

where we have applied in order the Hölder inequality, (2.28), (2.20), (5.2) and the result in Lemma 5.1. Instead, under assumption (2.9), we have that for some $\text{const} > 0$

$$\begin{aligned}
&(\lambda_{n_k} R_{\lambda_{n_k}})^2 \left| \int_{B_1} \tilde{f}(\lambda_{n_k} R_{\lambda_{n_k}} x) U^k(x) \varphi(x) \, dx \right| \\
&\leq \text{const} \, (\lambda_{n_k} R_{\lambda_{n_k}})^{\frac{4p-N-1}{p}} \left(\int_{B_1} |\nabla U^k|^2 \, dx + (N-1) \int_{\partial B_1} \mu(\lambda_{n_k} R_{\lambda_{n_k}} \cdot) |U^k|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{B_1} |\nabla \varphi|^2 \, dx \right)^{\frac{1}{2}} \\
&\leq \text{const} \, (\lambda_{n_k} R_{\lambda_{n_k}})^{\frac{4p-N-1}{p}} \left(\int_{B_1} |\nabla U^k|^2 \, dx + N - 1 \right)^{\frac{1}{2}} \left(\int_{B_1} |\nabla \varphi(x)|^2 \, dx \right)^{\frac{1}{2}} = o(1) \text{ as } k \to \infty,
\end{aligned}$$

using again the Hölder inequality, (1.5), (2.20), (5.2) and the result in Lemma 5.1. So in both cases we have that

$$(\lambda_{n_k} R_{\lambda_{n_k}})^2 \int_{B_1} \tilde{f}(\lambda_{n_k} R_{\lambda_{n_k}} x) U^k(x) \varphi(x) \, dx \to 0 \quad \text{as } k \to \infty. \tag{5.24}$$

Thus summing (5.23) and (5.24), passing to the limit as $k \to \infty$ in (5.22), we get (5.20). Moreover we remark that, since U^k belongs to the space defined in (5.4) for every k, in virtue of (5.21), for sufficiently large k it holds that U^k belongs to the space (5.13), which is weakly closed in $H^1(B_1)$; so, thanks to (5.16), we also have that $\overline{U}$ belongs to the above space. Putting together this information with (5.20), we can deduce that $\overline{U}$ is a weak solution to

$$\begin{cases} -\Delta \overline{U} = 0 & \text{in } B_1 \setminus \tilde{\Gamma}, \\ \overline{U} = 0 & \text{on } \tilde{\Gamma}. \end{cases} \tag{5.25}$$

Our next goal is to prove that (5.16) actually holds in a strong sense, namely we want to prove that

$$U^k \to \overline{U} \text{ in } H^1(B_1) \text{ as } k \to \infty. \tag{5.26}$$

To this aim, we test Eq. (5.3) with any $\varphi \in C_c^\infty(\overline{B_1} \setminus \tilde{\Gamma})$ (since it belongs to $C_c^\infty(\overline{B_{r_0/(\lambda_{n_k} R_{\lambda_{n_k}})}} \setminus \tilde{\Gamma})$ for sufficiently large k), obtaining that for sufficiently large k

$$\begin{aligned}
\int_{B_1} A(\lambda_{n_k} R_{\lambda_{n_k}} x) \nabla U^k(x) \cdot \nabla \varphi(x) \, dx = &(\lambda_{n_k} R_{\lambda_{n_k}})^2 \int_{B_1} \tilde{f}(\lambda_{n_k} R_{\lambda_{n_k}} x) U^k(x) \varphi(x) \, dx \\
&+ \int_{\partial B_1} (A(\lambda_{n_k} R_{\lambda_{n_k}} x) \nabla U^k(x) \cdot \nu(x)) \varphi(x) \, dx.
\end{aligned} \tag{5.27}$$

In light of (5.23), (5.24) and Lemma 5.6, up to a further subsequence still denoted with λ_{n_k}, taking the limit in (5.27) as $k \to \infty$, we get that for every $\varphi \in C_c^\infty(\overline{B_1} \setminus \tilde{\Gamma})$

$$\int_{B_1} \nabla \overline{U} \cdot \nabla \varphi \, dx = \int_{\partial B_1} h \varphi \, ds.$$

The above identity still holds by density if we choose as φ the function $\overline{U}$ itself since $\overline{U}$ belongs to the space (5.13) as observed above, and thus we can conclude that

$$\int_{B_1} |\nabla \overline{U}|^2 \, dx = \int_{\partial B_1} h \overline{U} \, ds. \tag{5.28}$$

Furthermore we are allowed to compute (5.27) at $\varphi = U^k$ by a density argument since U^k belongs to the space (5.13) for sufficiently large k, thus having that

$$\begin{aligned} \lim_{k\to\infty} \int_{B_1} A(\lambda_{n_k} R_{\lambda_{n_k}} x) \nabla U^k(x) \cdot \nabla U^k(x) \, dx &= \lim_{k\to\infty} \int_{\partial B_1} (A(\lambda_{n_k} R_{\lambda_{n_k}} x) \nabla U^k(x) \cdot \nu(x)) U^k(x) \, ds \\ &= \int_{\partial B_1} h \overline{U} \, ds = \int_{B_1} |\nabla \overline{U}|^2 \, dx, \end{aligned} \tag{5.29}$$

if in addition we use (5.24) (which can proven in the same way as before even choosing $\varphi = U^k$), Lemma 5.6, (5.18) and ultimately (5.28). From this, proceeding as in the proof of Lemma 5.6, we obtain that

$$\lim_{k\to\infty} |\nabla U^k|^2 \, dx = \int_{B_1} |\nabla \overline{U}|^2 \, dx,$$

which, combined with (5.16), leads to have that $\nabla U^k \to \nabla \overline{U}$ in $L^2(B_1)$. This, together with (2.28) and (5.18), implies that $U^k \to \overline{U}$ in $L^2(B_1)$. The proof of (5.26) is thereby complete.

Let us now prove that $\overline{U}$ is homogeneous of degree ℓ. We then investigate the associated Almgren function, that, based on (5.25), is defined as follows (provided that the denominator is non-null):

$$\mathcal{N}_{\overline{U}} : t \in (0,1] \mapsto \mathcal{N}_{\overline{U}}(t) := \frac{D_{\overline{U}}(t)}{H_{\overline{U}}(t)}, \tag{5.30}$$

where

$$D_{\overline{U}}(t) := t^{1-N} \int_{B_t} |\nabla \overline{U}|^2 \, dx,$$

and

$$H_{\overline{U}}(t) := t^{-N} \int_{\partial B_t} |\overline{U}|^2 \, ds. \tag{5.31}$$

A similar discussion as that one used to prove Lemma 4.1 (taking $\tilde{f} \equiv 0$) ensures that $H_{\overline{U}} > 0$, taking advantage of (5.17).

Let us now introduce the Almgren function associated with U^k for every $k \geq 1$. For this we define, for any fixed $k \geq 1$, the following two functions

$$D_k : t \in (0,1] \mapsto$$
$$D_k(t) := t^{1-N} \left(\int_{B_t} A(\lambda_{n_k} R_{\lambda_{n_k}} \cdot) \nabla U^k \cdot \nabla U^k \, dx - (\lambda_{n_k} R_{\lambda_{n_k}})^2 \int_{B_t} \tilde{f}(\lambda_{n_k} R_{\lambda_{n_k}} \cdot) |U^k|^2 \, dx \right),$$

and

$$H_k : t \in (0,1] \mapsto H_k(t) := t^{-N} \int_{\partial B_t} \mu(\lambda_{n_k} R_{\lambda_{n_k}} \cdot) |U^k|^2 \, ds.$$

We notice that

$$H_k(t) = \frac{H(\lambda_{n_k} R_{\lambda_{n_k}} t)}{H(\lambda_{n_k} R_{\lambda_{n_k}})}, \tag{5.32}$$

and thus $H_k > 0$ by Lemma 4.1 (we stress that the fact that $\lambda_{n_k} \to 0^+$ guarantees that $\lambda_{n_k} R_{\lambda_{n_k}} \in (0, r_0]$). Hence the Almgren function given by

$$\mathcal{N}_k : t \in (0,1] \mapsto \mathcal{N}_k(t) := \frac{D_k(t)}{H_k(t)}$$

is well-defined and satisfies

$$\mathcal{N}_k(t) = \mathcal{N}(\lambda_{n_k} R_{\lambda_{n_k}} t), \tag{5.33}$$

thanks to (5.32) and since it also holds that

$$D_k(t) = \frac{D(\lambda_{n_k} R_{\lambda_{n_k}} t)}{H(\lambda_{n_k} R_{\lambda_{n_k}})}.$$

So from (5.29), (5.24) (which is still valid with U^k in place of φ), (5.18), (2.18), (5.33) and (4.36), we can deduce that for every $t \in (0,1]$

$$\mathcal{N}_{\overline{U}}(t) = \lim_{k\to\infty} \mathcal{N}_k(t) = \lim_{k\to\infty} \mathcal{N}(\lambda_{n_k} R_{\lambda_{n_k}} t) = \ell. \tag{5.34}$$

Thus $\mathcal{N}'_{\overline{U}}(t) = 0$ for a.e. $t \in (0,1)$ and consequently, considering that $\overline{U}$ is a weak solution to (5.25) and arguing as to prove Lemma 4.7, we have that for a.e. $t \in (0,1)$

$$0 \le \frac{2t\left[\left(\int_{\partial B_t}\left|\frac{\partial \overline{U}}{\partial \nu}\right|^2 ds\right)\left(\int_{\partial B_t} |\overline{U}|^2\, ds\right) - \left(\int_{\partial B_t} \overline{U}\frac{\partial \overline{U}}{\partial \nu}\, ds\right)^2\right]}{\left(\int_{\partial B_t} |\overline{U}|^2\, ds\right)^2} \le \mathcal{N}'_{\overline{U}}(t) = 0,$$

by Cauchy–Schwarz's inequality. This in particular implies that $\overline{U}$ and $\partial\overline{U}/\partial\nu$ have the same direction in $L^2(\partial B_t)$ for a.e. $t \in (0,1)$ and hence, writing any $x \in \partial B_t$ as $x = t\vartheta$ with $\vartheta = \frac{x}{|x|} \in \mathbb{S}^N$, we have that

$$\frac{\partial \overline{U}}{\partial t}(t\vartheta) = \frac{\partial \overline{U}}{\partial \nu}(t\vartheta) = \eta(t)\overline{U}(t\vartheta) \quad \text{for a.e. } t \in (0,1) \text{ and for every } \vartheta \in \mathbb{S}^N, \tag{5.35}$$

for some function $\eta = \eta(t)$ defined a.e. in $(0,1)$. Multiplying the above identity by $\overline{U}(t\vartheta)$ itself and then integrating with respect to ϑ over $\mathbb{S}^N$, by (5.31), [2, Lemma 4.1] applied to $H_{\overline{U}}(t)$, (5.30) and at last (5.34), we can deduce that

$$\eta(t) = \frac{1}{2}\frac{H'_{\overline{U}}(t)}{H_{\overline{U}}(t)} = \frac{\mathcal{N}_{\overline{U}}(t)}{t} = \frac{\ell}{t},$$

which in turn allows us to establish that η is summable staying far from 0. Plugging this into (5.35) and integrating with respect to t over $(r,1)$ for any fixed $r \in (0,1)$, we obtain that

$$\overline{U}(r\vartheta) = r^{\ell}\overline{U}(\vartheta) = r^{\ell}\Psi(\vartheta) \quad \text{for every } r \in (0,1) \text{ and } \vartheta \in \mathbb{S}^N, \tag{5.36}$$

where $\Psi := \overline{U}|_{\mathbb{S}^N}$. By (5.19) we have that

$$\int_{\mathbb{S}^N} |\Psi|^2\, ds = 1,$$

so that Ψ is non-trivial on $\mathbb{S}^N$. Furthermore, using (5.36), since $\overline{U}$ belongs to (5.13) then $\Psi \in \mathcal{H}_\Theta$ (defined in (1.9)); from (5.25), we can find that Ψ satisfies

$$\Delta_{\mathbb{S}^N}\Psi(\vartheta) + \ell(N+\ell-1)\Psi(\vartheta) = 0$$

in a weak sense, that is (1.10) holds. So by definition Ψ turns out to be an eigenfunction of problem (1.8) associated with $\ell(N+\ell-1)$. □

Lemma 5.9. *Let ℓ be as in (4.36) and $k_0 \ge 1$ be as in Remark 5.8. Then*

$$\ell = \frac{k_0}{2}. \tag{5.37}$$

Proof. Let $k_0 \ge 1$ be as in Remark 5.8. In order to prove (5.37), we claim that

$$\ell = -\frac{N-1}{2} + \sqrt{\left(\frac{N-1}{2}\right)^2 + \mu_{k_0}}.$$

Once this is shown, (5.37) follows from the explicit formula (1.11) for the eigenvalues of problem (1.8). In order to prove the claim, we take into account Remark 5.8 to conclude that ℓ solves

$$y^2 + y(N-1) - \mu_{k_0} = 0,$$

which in general admits two solutions $y_\pm$ given by

$$y_\pm = -\frac{N-1}{2} \pm \sqrt{\left(\frac{N-1}{2}\right)^2 + \mu_{k_0}}.$$

We are allowed to exclude that $\ell = y_-$ since in such a case we would have that

$$\overline{U}(x) = |x|^{y_-}\Psi\left(\frac{x}{|x|}\right) \notin L^{2^*}(B_1),$$

which contradicts (5.13) due to the validity of (1.5). The claim is thereby proved and hence the proof of lemma is complete. □

Proposition 5.10. *Let $k_0 \ge 1$ be as in Lemma 5.9. For any sequence $\lambda_n \to 0^+$ there exist a subsequence $\{\lambda_{n_k}\}_{k\ge1}$ and an L^2-normalized eigenfunction Ψ of problem (1.8) associated with the eigenvalue μ_{k_0} such that*

$$U^{\lambda_{n_k}}(x) \to |x|^{\frac{k_0}{2}}\Psi\left(\frac{x}{|x|}\right) \quad \text{in } H^1(B_1) \text{ as } k \to \infty.$$

Proof. Let $\lambda_n \to 0^+$. By Lemma 5.1 we have that $\{U^{\lambda_n}\}_{n\geq 1}$ is uniformly bounded in $H^1(B_1)$ with respect to n and so there exist a subsequence $\{\lambda_{n_k}\}_{k\geq 1}$ and an $H^1(B_1)$-function $\underline{U}$ such that

$$U^{\lambda_{n_k}} \rightharpoonup \underline{U} \quad \text{in } H^1(B_1) \text{ as } k \to \infty. \tag{5.38}$$

Now we observe that, if we take $R_{\lambda_{n_k}} \in [1,2]$ as in Lemma 5.4, there exist a subsequence (still denoted with λ_{n_k}) and $\overline{R} \in [1,2]$ such that $R_{\lambda_{n_k}} \to \overline{R}$. Moreover we notice that, in virtue of (5.15), up to a further subsequence (still denoted with λ_{n_k}) the sequences $U^{\lambda_{n_k} R_{\lambda_{n_k}}}$ and $\nabla U^{\lambda_{n_k} R_{\lambda_{n_k}}}$ are uniformly dominated by two functions in $L^2(B_1)$ and it also holds that $U^{\lambda_{n_k} R_{\lambda_{n_k}}} \to \overline{U}$ and $|\nabla U^{\lambda_{n_k} R_{\lambda_{n_k}}}|^2 \to |\nabla \overline{U}|^2$ a.e. in B_1, being $\overline{U}$ the limit profile in Lemma 5.7. In addition, in light of Lemma 5.2, we have that up to a further subsequence (still denoted with λ_{n_k})

$$\left\{ \frac{H(\lambda_{n_k} R_{\lambda_{n_k}})}{H(\lambda_{n_k})} \right\}_{k\geq 1}$$

admits finite limit as $k \to \infty$ that will be denoted by η. Now we consider any $v \in C^\infty(\overline{B_1})$. All the previous considerations and the Lebesgue dominated convergence theorem imply that

$$\begin{aligned}
\lim_{k\to\infty} \int_{B_1} U^{\lambda_{n_k}}(x) v(x)\, dx &= \lim_{k\to\infty} R_{\lambda_{n_k}}^{N+1} \sqrt{\frac{H(\lambda_{n_k} R_{\lambda_{n_k}})}{H(\lambda_{n_k})}} \int_{B_{1/R_{\lambda_{n_k}}}} U^{\lambda_{n_k} R_{\lambda_{n_k}}}(x) v(R_{\lambda_{n_k}} x)\, dx \\
&= \overline{R}^{N+1} \sqrt{\eta} \int_{B_{1/\overline{R}}} \overline{U}(x) v(\overline{R} x)\, dx \\
&= \sqrt{\eta} \int_{B_1} \overline{U}_{\overline{R}}(x) v(x)\, dx,
\end{aligned}$$

where $\overline{U}_{\overline{R}}(x) := \overline{U}(x/\overline{R})$ for every $x \in B_1$. By a density argument we thus have that $U^{\lambda_{n_k}} \rightharpoonup \sqrt{\eta}\,\overline{U}_{\overline{R}}$ in $L^2(B_1)$. Combining this with (5.38), we obtain that $\underline{U} = \sqrt{\eta}\,\overline{U}_{\overline{R}}$ and therefore $U^{\lambda_{n_k}} \rightharpoonup \sqrt{\eta}\,\overline{U}_{\overline{R}}$ in $H^1(B_1)$. Actually this last convergence holds in a strong sense: indeed, we have that

$$\begin{aligned}
\lim_{k\to\infty} \int_{B_1} |\nabla U^{\lambda_{n_k}}(x)|^2\, dx &= \lim_{k\to\infty} R_{\lambda_{n_k}}^{N-1} \frac{H(\lambda_{n_k} R_{\lambda_{n_k}})}{H(\lambda_{n_k})} \int_{B_{1/R_{\lambda_{n_k}}}} |\nabla U^{\lambda_{n_k} R_{\lambda_{n_k}}}(x)|^2\, dx \\
&= \overline{R}^{N-1} \eta \int_{B_{1/\overline{R}}} |\nabla \overline{U}(x)|^2\, dx = \eta \int_{B_1} |\nabla \overline{U}_{\overline{R}}(x)|^2\, dx,
\end{aligned}$$

which in turn implies that $\nabla U^{\lambda_{n_k}} \to \sqrt{\eta}\nabla(\overline{U}_{\overline{R}})$ in $L^2(B_1)$; now summing this with the compactness of the trace operator (2.55), from (2.28) we can infer that

$$U^{\lambda_{n_k}} \to \sqrt{\eta}\,\overline{U}_{\overline{R}} \quad \text{in } H^1(B_1) \text{ as } k \to \infty. \tag{5.39}$$

In order to prove that $\sqrt{\eta}\,\overline{U}_{\overline{R}} = \overline{U}$, we exploit (5.14) to deduce that

$$\overline{U}_{\overline{R}} = \frac{1}{\overline{R}^{\ell}} \overline{U}. \tag{5.40}$$

Hence we finish if we prove that $\frac{\sqrt{\eta}}{\overline{R}^{\ell}} = 1$. To this purpose, we observe that

$$\int_{\partial B_1} \mu(\lambda_{n_k}) |U^{\lambda_{n_k}}|^2\, ds = \int_{\partial B_1} |U^{\lambda_{n_k}}|^2\, ds + O(\lambda_{n_k}) \int_{\partial B_1} |U^{\lambda_{n_k}}|^2\, ds \to \int_{\partial B_1} \eta |\overline{U}_{\overline{R}}|^2\, ds \tag{5.41}$$

as $k \to \infty$, thanks to (2.18), (5.39) combined with the compactness of the trace operator (2.55), which in turn together with Lemma 5.1 leads to the uniform boundedness of $\{U^{\lambda_{n_k}}\}_{k\geq 1}$ in $L^2(\partial B_1)$. Summing (5.2), (5.41), (5.40) and (5.19), we get

$$1 = \int_{\partial B_1} \eta |\overline{U}_{\overline{R}}|^2\, ds = \eta \int_{\partial B_1} \frac{1}{\overline{R}^{\ell}} |\overline{U}|^2\, ds = \frac{\eta}{\overline{R}^{\ell}},$$

as desired. So, pushing this into (5.39), taking into account (5.40), we obtain that $U^{\lambda_{n_k}} \to \overline{U}$ in $H^1(B_1)$ as $k \to \infty$. The proof is thereby complete if we put together this with Lemma 5.7, Remark 5.8 and at last Lemma 5.9. □

Our next aim is to prove that the limit of $H(r)/r^{k_0}$ as $r \to 0^+$, which we already know to be finite by Lemmas 4.11 and 5.9, is strictly positive. For this (as made in [2]) an asymptotic expansion of the Fourier coefficients associated with U is needed: in order to fully state the result, for every $k \geq 1$ let $m_k \geq 1$ be the multiplicity of the eigenvalue μ_k and let $\{Y_{k,m}\}_{m\in\{1,2,\dots,m_k\}}$ be a $L^2(\mathbb{S}^N)$-orthonormal basis of the eigenspace associated with μ_k; then we define for every $k \geq 1$, $m \in \{1,2,\dots,m_k\}$ and $\lambda \in (0, r_0)$

$$\varphi_{k,m}(\lambda) = \int_{\mathbb{S}^N} U(\lambda\vartheta) Y_{k,m}(\vartheta)\, ds \tag{5.42}$$

and

$$Y_{k,m}(\lambda) = -\int_{B_\lambda} (A - \mathrm{Id}_{N+1})\nabla U(x) \cdot \frac{1}{|x|}\nabla_{\mathbb{S}^N} Y_{k,m}\left(\frac{x}{|x|}\right) dx + \int_{B_\lambda} \tilde{f}(x)U(x)Y_{k,m}\left(\frac{x}{|x|}\right) dx$$
$$+ \int_{\partial B_\lambda} (A - \mathrm{Id}_{N+1})\nabla U(x) \cdot \frac{x}{|x|} Y_{k,m}\left(\frac{x}{|x|}\right) ds.$$

We simply rewrite the expansion in [2, Lemma 6.5] adapted to our setting, namely with $\tilde{f}$ satisfying either (2.8) or (2.9), without providing the proof since it is precisely the same (indeed in [2] at this point of discussion the authors apply a diffeomorphism that gives rise to a new formulation of their problem with the same features of our problem (2.5)).

Lemma 5.11. *Let $k_0 \geq 1$ be as in Lemma 5.9 and let m_{k_0} the multiplicity of the eigenvalue μ_{k_0}. For every $m \in \{1, 2, \ldots, m_{k_0}\}$ and $R \in (0, r_0]$*

$$\varphi_{k_0,m}(\lambda) = \lambda^{\frac{k_0}{2}} \left(\frac{\varphi_{k_0,m}(R)}{R^{\frac{k_0}{2}}} + \frac{2N + k_0 - 2}{2(N + k_0 - 1)} \int_\lambda^R t^{-N-\frac{k_0}{2}} Y_{k_0,m}(t)\, dt \right.$$
$$\left. + \frac{k_0 R^{-N+1-k_0}}{2(N + k_0 - 1)} \int_0^R t^{\frac{k_0}{2}-1} Y_{k_0,m}(t)\, dt \right) + O(\lambda^{\frac{k_0}{2}+\bar{\varepsilon}+1}) \quad \text{as } \lambda \to 0^+, \tag{5.43}$$

being $\bar{\varepsilon} \in (-1, 0)$ defined in (4.9).

Lemma 5.12. *Let $k_0 \geq 1$ be as in Lemma 5.9. Then*

$$\lim_{r\to 0^+} \frac{H(r)}{r^{k_0}} > 0. \tag{5.44}$$

Proof. Expanding the function $\vartheta \mapsto U(\lambda\vartheta) \in L^2(\mathbb{S}^N)$ for every fixed $\lambda \in (0, r_0)$ in Fourier series with respect to the $L^2(\mathbb{S}^N)$-orthonormal basis $\{Y_{k,m} : k \geq 1, m \in \{1, 2, \ldots, m_k\}\}$, we have that

$$U(\lambda\vartheta) = \sum_{k\geq 1} \sum_{m=1}^{m_k} \varphi_{k,m}(\lambda) Y_{k,m}(\vartheta).$$

From this and (4.2), applying a suitable change of variable, using (2.18) and the Parseval identity, we get

$$H(\lambda) = (1 + O(\lambda)) \sum_{k\geq 1} \sum_{m=1}^{m_k} |\varphi_{k,m}(\lambda)|^2 \quad \text{for every fixed } \lambda \in (0, r_0). \tag{5.45}$$

Now we assume by contradiction that the limit in (5.44) is zero; so from (5.45) it follows that

$$\lim_{\lambda\to 0^+} \frac{\varphi_{k_0,m}(\lambda)}{\lambda^{\frac{k_0}{2}}} = 0 \quad \text{for every } m \in \{1, 2, \ldots, m_{k_0}\}.$$

This, together with (5.43), implies that for every $m \in \{1, 2, \ldots, m_{k_0}\}$ and $R \in (0, r_0]$

$$\frac{\varphi_{k_0,m}(R)}{R^{\frac{k_0}{2}}} + \frac{2N + k_0 - 2}{2(N + k_0 - 1)} \int_0^R t^{-N-\frac{k_0}{2}} Y_{k_0,m}(t)\, dt + \frac{k_0 R^{-N+1-k_0}}{2(N + k_0 - 1)} \int_0^R t^{\frac{k_0}{2}-1} Y_{k_0,m}(t)\, dt = 0,$$

which in turn, substituted into (5.43), gives us that for every $m \in \{1, 2, \ldots, m_{k_0}\}$

$$\varphi_{k_0,m}(\lambda) = O(\lambda^{\frac{k_0}{2}+\bar{\varepsilon}+1}) \quad \text{as } \lambda \to 0^+, \tag{5.46}$$

if in addition we take into consideration that for every $m \in \{1, 2, \ldots, m_{k_0}\}$

$$\int_0^\lambda t^{-N-\frac{k_0}{2}} Y_{k_0,m}(t)\, dt = O(\lambda^{\bar{\varepsilon}+1}) \quad \text{as } \lambda \to 0^+$$

(for this we suggest the reader to consult the proof of [2, Lemma 6.5]). Therefore, exploiting (5.42) and (5.46), we can deduce that for every $m \in \{1, 2, \ldots, m_{k_0}\}$

$$\sqrt{H(\lambda)}(U^\lambda, Y_{k_0,m})_{L^2(\mathbb{S}^N)} = O(\lambda^{\frac{k_0}{2}+\bar{\varepsilon}+1}) \quad \text{as } \lambda \to 0^+. \tag{5.47}$$

Now by (4.38) we observe that $\sqrt{H(\lambda)} \geq \text{const}\, \lambda^{\frac{k_0+\bar{\varepsilon}+1}{2}}$ for some const > 0 depending only on $\bar{\varepsilon}$ and for every $\lambda \in (0, r_0)$; so using this, from (5.47) we obtain that

$$(U^\lambda, Y)_{L^2(\mathbb{S}^N)} = O(\lambda^{\frac{\bar{\varepsilon}+1}{2}}) \quad \text{as } \lambda \to 0^+ \tag{5.48}$$

for every $Y \in \mathrm{span}\{Y_{k_0,m} : m \in \{1, 2, \ldots, m_{k_0}\}\}$. On the other hand, by Proposition 5.10 and the continuity of the trace map in (2.55), for any sequence $\lambda_n \to 0^+$ there exist a subsequence (which we will still denote with λ_n) and an L^2-normalized function $\Psi \in \mathrm{span}\{Y_{k_0,m} : m \in \{1, 2, \ldots, m_{k_0}\}\}$ such that

$$U^{\lambda_n} \to \Psi \quad \text{in } L^2(\mathbb{S}^N) \text{ as } n \to \infty.$$

We end up by observing that this leads to

$$(U^{\lambda_n}, \Psi)_{L^2(\mathbb{S}^N)} \to \|\Psi\|^2_{L^2(\mathbb{S}^N)} = 1 \quad \text{as } n \to \infty,$$

which is in contradiction with (5.48). □

We are now in the condition of proving the asymptotic behaviour around 0 of solutions to (2.5) and accordingly for solutions to (1.3).

Theorem 5.13. *Let $k_0 \geq 1$ be as in Lemma 5.9. If m_{k_0} is the multiplicity of the eigenvalue μ_{k_0} and $\{Y_{k_0,m}\}_{m\in\{1,2,\dots,m_{k_0}\}}$ is a $L^2(\mathbb{S}^N)$-orthonormal basis of the eigenspace associated with μ_{k_0}, then for every $m \in \{1,2,\dots,m_{k_0}\}$*

$$\frac{U(\lambda x)}{\lambda^{\frac{k_0}{2}}} \to |x|^{\frac{k_0}{2}} \sum_{m=1}^{m_{k_0}} \beta_m Y_{k_0,m}\left(\frac{x}{|x|}\right) \quad \text{in } H^1(B_1) \text{ as } \lambda \to 0^+, \tag{5.49}$$

where $(\beta_1, \beta_2, \dots, \beta_{m_{k_0}}) \in \mathbb{R}^{m_{k_0}} \setminus \{0\}$ and for every $m \in \{1,2,\dots,m_{k_0}\}$

$$\begin{aligned} \beta_m =& \frac{\varphi_{k_0,m}(R)}{R^{\frac{k_0}{2}}} + \frac{2N+k_0-2}{2(N+k_0-1)} \int_0^R t^{-N-\frac{k_0}{2}} Y_{k_0,m}(t)\,dt \\ &+ \frac{k_0 R^{-N+1-k_0}}{2(N+k_0-1)} \int_0^R t^{\frac{k_0}{2}-1} Y_{k_0,m}(t)\,dt \quad \text{for all } R \in (0, r_0]. \end{aligned} \tag{5.50}$$

Proof. By Proposition 5.10 it holds that for every sequence $\lambda_n \to 0^+$ there exist a subsequence $\{\lambda_{n_k}\}$ and a non-null vector $(\beta_1, \beta_2, \dots, \beta_{m_{k_0}}) \in \mathbb{R}^{m_{k_0}}$ such that

$$\frac{U(\lambda_{n_k} x)}{\lambda_{n_k}^{k_0/2}} \to |x|^{\frac{k_0}{2}} \sum_{m=1}^{m_{k_0}} \beta_m Y_{k_0,m}\left(\frac{x}{|x|}\right) \quad \text{in } H^1(B_1) \text{ as } k \to \infty, \tag{5.51}$$

if we also use Lemmas 4.11 and 5.12. We remark that if we show (5.50) then (5.49) automatically follows from Urysohn's subsequence principle. Hence we proceed by combining (5.42), (5.51) with the continuity of the trace map in (2.55), obtaining that for every $m \in \{1,2,\dots,m_{k_0}\}$

$$\lim_{k\to\infty} \frac{\varphi_{k_0,m}(\lambda_{n_k})}{\lambda_{n_k}^{k_0/2}} = \sum_{j=1}^{m_{k_0}} \beta_j \int_{\mathbb{S}^N} Y_{k_0,j}(\vartheta) Y_{k_0,m}(\vartheta)\,ds = \beta_m, \tag{5.52}$$

being $\{Y_{k_0,m}\}_{m\in\{1,2,\dots,m_{k_0}\}}$ orthonormal. So assembling (5.43) and (5.52), we arrive at (5.50). This completes the proof. □

Proof of Theorem 1.1. If $u \in H^1(B_{\tilde r})$ is a non-trivial weak solution to (1.3), letting F as in Section 2.1, $U = u \circ F \in H^1(B_{\tilde r})$ is a non-trivial weak solution to (2.5). Theorem 5.13 ensures that there exist $k_0 \geq 1$ and an eigenfunction Ψ of problem (1.8) associated with μ_{k_0} such that

$$\frac{U(\lambda x)}{\lambda^{\frac{k_0}{2}}} \to |x|^{\frac{k_0}{2}} \Psi\left(\frac{x}{|x|}\right) \quad \text{in } H^1(B_1) \text{ as } \lambda \to 0^+. \tag{5.53}$$

Setting $U_\lambda(\cdot) := U(\lambda \cdot)$, it holds that

$$u(\lambda x) = U_\lambda\left(\frac{F^{-1}(\lambda x)}{\lambda}\right) \quad \text{and} \quad \nabla(u(\lambda x)) = \nabla U_\lambda\left(\frac{F^{-1}(\lambda x)}{\lambda}\right) DF^{-1}(\lambda x) \tag{5.54}$$

for every $x \in B_1$ and $\lambda < \tilde r$. Moreover, from (2.2) and (2.3), we can deduce that as $\lambda \to 0^+$

$$\left|\frac{F^{-1}(\lambda x)}{\lambda} - x\right| \to 0 \quad \text{and} \quad \|DF^{-1}(\lambda x) - \mathrm{Id}_{N+1}\| \to 0 \quad \text{uniformly in } B_1.$$

So the thesis follows from this, (5.53) and (5.54). □

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Data availability

No data was used for the research described in the article.

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