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Subgroups of Bestvina-Brady groups

S. Blumer

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria

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ABSTRACT

In 1987, Droms [15] proved that all the subgroups of a right-angled Artin group (RAAG) defined by a finite simple graph Γ are themselves RAAGs if, and only if, Γ has no induced squares nor lines of length 3. The present work provides a similar result for specific normal subgroups of RAAGs, called Bestvina-Brady groups: We characterize those finite graphs in which every subgroup of such a group is itself a RAAG. In turn, we confirm several Galois theoretic conjectures for the pro-p analogues of these groups, and study their associated graded Lie algebras.

0. Introduction

Let Γ be a simple graph, i.e., a pair (V, E), where V is a set and E is a set of 2-element subsets of V. The associated **right-angled Artin group** G_Γ (RAAG, for short) is the group generated by the vertices of the graph, with a defining relation $vw = wv$ for each pair of vertices v, w joined by an edge in Γ:

$$\left\langle v : \ v \in V \,\middle|\, [v, w] : \ \{v, w\} \in E \right\rangle \tag{0.1}$$

A group G is said to be a RAAG if it is isomorphic to G_Γ for some graph Γ. In that case, Γ is determined by G (see Droms [14]), and is called the defining graph of G.

RAAGs were first introduced in the 1970s by A. Baudisch [3], and further developed in the 1980s by C. Droms [13–15] under the name of *graph groups*. Since then, they have been extensively studied as they provide a rich class of groups whose algebraic properties can be derived from those of the defining graph. The topic has also attracted much interest in geometric group theory, since every RAAG is the fundamental group of a non-positively curved cube complex known as the Salvetti complex [30], which can be constructed

E-mail address: simone.blumer@univie.ac.at.

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by taking a torus for each generator and gluing these tori together according to the commutation relations encoded in the defining graph.

Right-angled Artin groups sit in between the two extremal cases of free groups and free abelian groups, and they can thus be seen as interpolating those classes. It is then natural to ask which properties RAAGs share with free/free abelian groups. For instance, it is well known that subgroups of free and free abelian groups are of the same type. Nevertheless, the subgroups of a RAAG might not be RAAGs themselves: depending on the defining graph, there may exist finitely generated subgroups that do not even admit a finite number of defining relation, that is to say, the RAAG is not coherent (see Proposition 3.19).

This problem was studied by Droms [13], who proved a finitely generated RAAG is coherent if, and only if, the defining graph is **chordal**, i.e., it does not contain any induced cycle of length ≥ 4.

On the other hand, Droms [15] also gave a complete classification of finite graphs Γ such that all subgroups of G_Γ are RAAGs.

Theorem 0.1 *(Droms). Let Γ be a finite graph. Then, every subgroup of G_Γ is a RAAG if, and only if, no induced subgraph of Γ is either a square or a line-graph of length 3.*

Graphs without induced squares nor line-graphs of length 3 appear in various contexts and with several names, as trivially perfect graphs [18], graphs of elementary type [7], or quasi-threshold graphs [37]. In honour to Droms, in the context of RAAGs, these graphs are usually called **Droms graphs**, and we will adopt the same denomination. Clearly, Droms graphs are chordal.

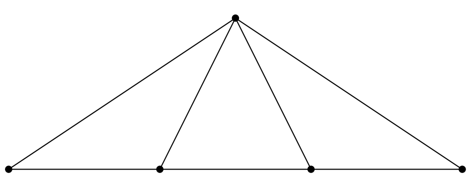
It follows from the proof of Theorem 0.1 that, for a finite graph Γ to be a Droms graph, it is also sufficient that all the *finitely generated* subgroups of G_Γ are RAAGs, i.e., G_Γ is locally RAAG.

Among the subgroups of RAAGs, some distinguished normal subgroups have been studied since their discovery due to Bestvina and Brady in [4], as they provide examples of finitely generated, yet non-finitely presented groups, that are of type FP. Given a graph Γ, one can define the corresponding **Bestvina-Brady group** as the kernel of the "length character" $\chi_\Gamma : G_\Gamma \to \mathbb{Z}$, obtained by extending $v \in V \mapsto 1 \in \mathbb{Z}$.

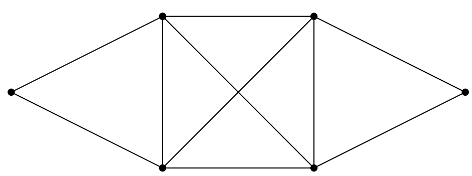
If Γ is a Droms graph, then clearly B_Γ is itself a RAAG, but the problem of detecting Bestvina-Brady groups that are RAAGs remains open in general. Recently, Ruffoni and Chang [9] solved this problem for 2-dimensional flag complexes, proving that, in case Γ does not contain complete graphs on 4 vertices, then B_Γ is a finitely generated RAAG if, and only if, Γ admits a tree 2-spanner. They also prove that the existence of a tree 2-spanner is sufficient, though not necessary, for the associated Bestvina-Brady group to be a RAAG.

Moved by Droms' Theorem 0.1, we are interested in understanding when the group B_Γ is locally RAAG: clearly, if G_Γ is locally RAAG, then so is B_Γ, but one can easily construct counterexamples to the converse statement (see Corollary 3.2). In particular, our main result is the following (see Theorem 2.8 for a more complete statement).

Main Theorem. *Let Γ be a finite connected simple graph. Then, every subgroup of the Bestvina-Brady group B_Γ is a RAAG if, and only if, Γ is* chordal *and no induced subgraph of Γ has either of the two forms:*



∇



$\bar{H}$

The graph ∇ is called the **gem graph**, and $\bar{H}$ is a graph where two gems are overlapped; we will call it an **overlapping-gems graph**.

The pro-p completions of right-angled Artin groups (and their generalizations) have been studied in [33] (resp. [7]) in the attempt of confirming several Galois theoretic conjectures on the realizability of a pro-p group as an absolute Galois group. In particular,

Theorem 0.2 *(Snopče, Zalesskii [33]). A finite graph Γ is a Droms graph if, and only if, the pro-p RAAG $G_{\Gamma,p}$ is isomorphic to the maximal pro-p quotient of the absolute Galois group of a field containing a primitive p^{th} root of 1.*

In Theorem 2.8 we deduce an analogue of that result for the pro-p version of the Bestvina-Brady groups.

Following [6], [19], and [36], we also study the Lie algebra counterpart ($\mathfrak{g}_\Gamma$, and $\mathfrak{b}_\Gamma$) of RAAGs and Bestvina-Brady groups. The advantage of working with positively graded Lie algebras relies on the fact that cohomology computations are easier than in the group-case, as one can make use of the theory of quadratic algebras ([6,29]). For instance, by means of a spectral sequence due to J.P. May [24], the cohomology ring of Bestvina-Brady groups defined on acyclic flag complexes is computed.

Proposition 0.3. *If Γ is a finite graph with acyclic flag complex over a field k, then*

$$H^\bullet(B_\Gamma, k) \simeq H^\bullet(G_\Gamma, k)/(\chi_\Gamma)$$

where (χ_Γ) denotes the ideal of $H^\bullet(G_\Gamma, k)$ generated by the class of the 1-cocycle $\chi_\Gamma : G_\Gamma \to k$.

In [32], Servatius, Droms and Servatius produced embeddings of non-abelian surface groups into (the derived subgroup of) RAAGs defined by graphs with induced cycles of length at least 5. By means of a Lie algebraic argument we give a new proof of the known fact that no non-abelian surface group appears as a Bestvina-Brady group. A similar proof applies by pure group theoretic means. We also provide a similar result for non-abelian Poincaré duality Lie algebras.

At the end of the paper we show that the Bestvina-Brady group of a graph is coherent precisely when so is the right-angled Artin group, that is, when the graph is chordal.

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1. Right-angled Artin & Bestvina-Brady groups and Lie algebras

Since we are interested both in right-angled Artin groups and their Lie algebraic counterparts, we will work in a more general setting. Henceforth, let $\mathcal{A}$ be either the category of positively-graded Lie algebras over a field, or the category of groups.

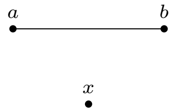
1.1. RAAGs

If Γ is a simple graph, we denote by $\mathfrak{G}_\Gamma$ the **RAAG object** given by the presentation (0.1) in the category $\mathcal{A}$, where the generators have degree 1 in the Lie algebra case. We will identify the canonical generators of $\mathfrak{G}_\Gamma$ with the vertices of Γ.

Many algebraic properties of $\mathfrak{G}_\Gamma$ (e.g., the form of the trivial coefficient cohomology, the coherence property, and the decomposability into free/direct product) do not depend on the chosen category, but only on the underlying graph.

Nevertheless, the naïve translation of Theorem 0.1 into the realm of Lie algebras is not true, as the following example shows.

Example 1.1 *(Kochloukova, Martínez Pérez [19]).* Consider the graph Γ with geometric realization



Then, Γ is a Droms graph and hence every (finitely generated) subgroup of G_Γ is itself a RAAG. The RAAG Lie algebra $\mathfrak{g}_\Gamma$ over an arbitrary field k can be given a presentation $\langle a, b, x \mid [a, b] \rangle$. On the other hand, the subalgebra $\mathfrak{m}$ generated by the elements $a, b, z = [x, a]$ and $t = [b, x]$ can be minimally presented as

$$\langle a, b, z = [x, a], t = [b, x] \mid [a, b], [z, b] + [t, a] \rangle ,$$

and hence it is not a RAAG Lie algebra (see [19, Sec. 6]). Notice that $\mathfrak{m}$ is not contained in $\mathfrak{g}'_\Gamma$, which is free by Proposition 3.19. ■

However, by [6], Theorem 0.1 remains true if we restrict to the *standard* subalgebras of the RAAG Lie algebra defined by a Droms graph: a finite simple graph Γ is Droms if, and only if, all standard subalgebras of $\mathfrak{g}_\Gamma$ are RAAG Lie algebras. Recall that a graded Lie algebra $\mathfrak{g} = \bigoplus_{i=1}^\infty \mathfrak{g}_i$ is said to be **standard** if it is finitely generated by its elements of degree 1, i.e., $\dim \mathfrak{g}_1 < \infty$ and $[\mathfrak{g}, \mathfrak{g}] = \bigoplus_{i\geq 2} \mathfrak{g}_i$. Such a Lie algebra is **Bloch-Kato** (BK, for short) if every standard subalgebra of $\mathfrak{g}$ is quadratic, i.e., it admits a complete set of homogeneous relations of degree 2. Equivalently, a standard Lie k-algebra $\mathfrak{g}$ is BK if its cohomology ring $H^\bullet(\mathfrak{g}, k)$ is universally Koszul (see [5] for a proof, and Conca [10] for the definition). Similarly, a pro-p group G is BK if the Galois cohomology ring $H^\bullet(H, \mathbb{F}_p)$ is a quadratic algebra for every closed subgroup H of G. For an exposition on Koszul (Lie) algebras, we refer the reader to Weigel [36], and for BK Lie algebras to the author's preprint [6].

Theorem 1.2. *Let Γ be a finite simple graph. Then, Γ is a Droms graph if, and only if, anyone of the following statements holds:*

1. *All subgroups of the RAAG G_Γ are RAAGs (possibly defined by infinite graphs) (Droms [15]),*
2. *All standard subalgebras of the RAAG Lie k-algebra $\mathfrak{g}_\Gamma$ are RAAG Lie algebras ([6, Prop. 3.17]),*
3. *The RAAG Lie algebra $\mathfrak{g}_\Gamma$ is BK ([5]),*
4. *The cohomology k-algebra $H^\bullet(\mathfrak{G}_\Gamma, k)$ (resp. the $\mathbb{F}_p$-algebra $H^\bullet(G_{\Gamma,p}, \mathbb{F}_p)$) is universally Koszul ([5, Ex. 4.4], and also Cassella, Quadrelli [8], and Snopce, Zalesskii [33] for the pro-p completion),*
5. *The pro-p RAAG $G_{\Gamma,p}$ is BK ([33]),*
6. *Every closed subgroup of $G_{\Gamma,p}$ is a pro-p RAAG ([33]),*
7. *The pro-p group $G_{\Gamma,p}$ is isomorphic to the maximal pro-p quotient $G_{\mathbb{K}}(p)$ of the absolute Galois groups of a field $\mathbb{K}$ containing a primitive p^{th} root of 1 ([33]).*

An object of $\mathcal{A}$ is called a **Droms object** if it is isomorphic to $\mathfrak{G}_\Gamma$ for a Droms graph Γ, i.e., it is a RAAG object whose defining graph is a Droms graph.

1.2. Bestvina-Brady objects

In this work, k will denote a ring whose nature depends on the chosen category.

(1) For Lie algebras, k is an arbitrary field on which the Lie algebra is defined,
(2) For abstract groups, k is either a field or the ring $\mathbb{Z}$ of integers.

There exists a natural morphism, the **length character** $\chi_\Gamma : \mathfrak{G}_\Gamma \to k$, sending each canonical generator (i.e., the vertices of Γ) to $1 \in k$. The kernel $\mathfrak{B}_\Gamma$ of χ_Γ is the (right-angled) **Bestvina-Brady object**. Here, $k = \mathbb{Z}$, in the group case. If Λ is induced in a graph Γ, since the character χ_Λ is the restriction of χ_Γ to the subobject $\mathfrak{G}_\Lambda$ of $\mathfrak{G}_\Gamma$, the object $\mathfrak{B}_\Lambda$ naturally embeds into $\mathfrak{B}_\Gamma$, and $\mathfrak{B}_\Lambda = \mathfrak{G}_\Lambda \cap \mathfrak{B}_\Gamma$.

We will denote groups by the letters G, B, H, ..., and graded Lie algebras by $\mathfrak{g}, \mathfrak{b}, \mathfrak{m}$, ... For instance, the right-angled Artin (pro-p) group is denoted by G_Γ (resp. $G_{\Gamma,p}$) and its Bestvina-Brady group by B_Γ. We denote by $B_{\Gamma,p}$ the kernel of the map into the p-adic integers $\chi_{\Gamma,p} : G_{\Gamma,p} \to \mathbb{Z}_p$. Similarly, $\mathfrak{g}_\Gamma$ and $\mathfrak{b}_\Gamma$ will denote respectively the RAAG and the Bestvina-Brady Lie algebras of Γ.

If p is an odd prime, then, by Bartholdi et al. [2], the p-restrictification (in the sense of [6]) of the RAAG Lie $\mathbb{F}_p$-algebra $\mathfrak{g}_\Gamma$ is denoted by $\mathfrak{g}_{\Gamma,p}$. The cohomology theory of $\mathfrak{g}_{\Gamma,p}$ is the same as that of $\mathfrak{g}_\Gamma$, and we will only focus on the non-restricted case. Similarly, the p-restrictification of $\mathfrak{b}_\Gamma$ is denoted by $\mathfrak{b}_{\Gamma,p}$, and it is the kernel of the natural map $\chi_{\Gamma,p} : \mathfrak{g}_{\Gamma,p} \to \mathbb{F}_p[T]$, defined by extending $v \in V(\Gamma) \mapsto T$ to a restricted Lie algebra homomorphism.

In order to avoid the different behaviours of groups and Lie algebras as in Example 1.1, we give the following definition of a *local* property. If $\mathcal{P}$ is a group theoretic (resp. Lie algebra theoretic) property, we say that a group G (resp. a graded Lie algebra $\mathfrak{g}$) is **locally** $\mathcal{P}$ if every finitely generated subgroup of G (resp. every *standard* subalgebra of $\mathfrak{g}$) satisfies the property $\mathcal{P}$. For instance, by Theorem 1.2, Droms objects are locally Droms.

The two works [4] and [20] prove that the topology of the flag complex of a graph determines cohomological finiteness properties of the corresponding Bestvina-Brady objects. Recall that the flag complex Δ_Γ of a graph Γ is the maximal simplicial complex whose 1-skeleton is Γ itself.

Theorem 1.3 *(Bestvina and Brady; Kochloukova and Martínez Pérez). Let Γ be a finite graph. If n is a natural number, then the following statements are equivalent:*

(1) *The flag complex Δ_Γ on Γ is $(n-1)$-acyclic over k, i.e., the reduced simplicial homology groups $\tilde{H}_i(\Delta_\Gamma, k)$ vanish over k for $i \le n-1$;*
(2) *The object $\mathfrak{B}_\Gamma$ is of type FP_n over k.*

If Γ is a connected graph, then Dicks and Leary [12] proved that the edges of any (directed) spanning tree of Γ form a generating system for $\mathfrak{B}_\Gamma$. More precisely, if T is a subtree of Γ containing all of its vertices, and E^+ is an chosen orientation of the edges of T — that is, if $\{v, w\}$ is an edge of T, then exactly one of the pairs (v, w) and (w, v) lies in E^+—, then the group B_Γ is generated by the elements vw^{-1} and the Lie algebra $\mathfrak{b}_\Gamma$ is generated by $v - w$, for any directed edge $(v, w) \in E^+$. The latter comes from the fact that $\mathfrak{b}_\Gamma$ is a finitely generated cocyclic ideal of the Koszul Lie algebra $\mathfrak{g}_\Gamma$, and hence it is standard (see [6, Prop. 2.8]). Similarly, the pro-p group $B_{\Gamma,p}$ is topologically generated by the same elements as B_Γ. Indeed, the explicit proof of Dicks and Leary for the presentation of Bestvina-Brady groups applies to their pro-p analogue as well. If Δ_Γ is simply connected, then the presentation of $B_{\Gamma,p}$ coincides with that of the pro-p completion of B_Γ — which is, in fact, the same as that of B_Γ but in the category of pro-p groups.

Moreover, Γ is connected if, and only if, $\mathfrak{b}_\Gamma$ is a standard subalgebra of $\mathfrak{g}_\Gamma$, and, more generally, by [6, Cor. 2.7], the Lie k-algebra $\mathfrak{b}_\Gamma$ is Koszul if, and only if, Δ_Γ is acyclic over k.

Corollary 1.4. *Let Γ be a finite simple graph. Then, Δ_Γ is acyclic over a field k if, and only if, the Bestvina-Brady Lie k-algebra $\mathfrak{b}_\Gamma$ is Koszul. In that case, the cohomology ring*

$$H^\bullet(\mathfrak{b}_\Gamma, k) \simeq H^\bullet(\mathfrak{g}_\Gamma, k)/(\chi_\Gamma \cdot H^1(\mathfrak{g}_\Gamma, k))$$

is a Koszul algebra, where $\chi_\Gamma : \mathfrak{g}_\Gamma \to k$ is the length character seen as a 1-cocycle.

Since the quadratic dual of the quadratic cover of $\mathfrak{b}_\Gamma$ is isomorphic to $H^\bullet(\mathfrak{g}_\Gamma, k)/\chi_\Gamma \cdot H^\bullet(\mathfrak{g}_\Gamma, k) =: A_\Gamma$ (see [6]), it follows from [29, Ch. 2, Cor. 3.3.] that A_Γ is Koszul precisely when so is $\mathfrak{b}_\Gamma$. In particular, since $H^\bullet(\mathfrak{g}_\Gamma, k)$ is the exterior Stanley-Reisner ring of the opposite graph of Γ, we get

Corollary 1.5. *Let Γ be a finite connected graph. The quotient A_Γ of the exterior k-algebra generated by the vertices of a finite simple graph Γ by the ideal generated by the elements $\sum_{v\in V(\Gamma)} v$ and $x \wedge y$, for $\{x, y\} \notin E(\Gamma)$, is Koszul if, and only if, the flag complex Δ_Γ is acyclic over k.*

In their seminal work [4], Bestvina and Brady also characterized the finite presentability of B_Γ in terms of the homotopy type of the flag complex of the underlying graph.

Theorem 1.6. *Let Γ be a finite graph. Then, B_Γ is finitely presented if, and only if, the flag complex of Γ is simply connected.*

A combination of Theorems 1.3 and 1.6 implies the existence of finitely generated groups which are of type FP but not finitely presented.

Notice that the same result does not hold for Bestvina-Brady Lie algebras, as, for graded Lie algebras over a field, being of type FP_2 and being finitely presented are equivalent conditions (see [36]). In fact, by Theorem 1.3, $\mathfrak{b}_\Gamma$ is finitely presented if, and only if, the first homology group over k of the flag complex of Γ vanishes, the latter being a weaker condition than simple connectedness.

1.3. Lower central series

For a (residually nilpotent) group G, denote by $\gamma_n(G)$ the lower central series of G, that is $\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [G, \gamma_n(G)]$ for $n > 0$. The associated graded object

$$\operatorname{gr}^\gamma G = \bigoplus_{n\geq 1} \gamma_n(G)/\gamma_{n+1}(G)$$

is a Lie ring, where the Lie brackets are induced by the commutator map $(g, h) \mapsto [g, h] = ghg^{-1}h^{-1}$. Similarly, if G is a pro-p group, then $\operatorname{gr}^\zeta G$ is computed in terms of the Jennings-Zassenhaus series, and is a p-restricted Lie algebra over $\mathbb{F}_p$. Since the filtration is clear from the context, we will drop the superscript and denote both these graded objects by $\operatorname{gr} G$.

As an example, if Γ is a finite graph, then $\operatorname{gr} G_\Gamma$ is isomorphic with the RAAG Lie ring $\mathfrak{g}_\Gamma(\mathbb{Z})$ (see [16]), and $\operatorname{gr} G_{\Gamma,p} \simeq \mathfrak{g}_{\Gamma,p}$ (see [2]). In the same way, Bestvina-Brady groups and Lie algebras are related by the following result, which is an easy consequence of [27, Thm. 5.6].

Lemma 1.7. *Let Γ be a finite simple graph. If Γ is connected, then $\operatorname{gr} B_\Gamma$ is naturally isomorphic with the Lie ring $\mathfrak{b}_\Gamma(\mathbb{Z}) = \ker(\mathfrak{g}_\Gamma(\mathbb{Z}) \to \mathbb{Z})$. Similarly, $\operatorname{gr} B_{\Gamma,p} \simeq \mathfrak{b}_{\Gamma,p}$.*

It follows that, for a connected graph Γ and a field k, the k-cohomology ring of the (pro-p) group B_Γ can be computed in terms of the bigraded algebra $H^{\bullet,\bullet}(\mathfrak{b}_\Gamma, k)$ of the Bestvina-Brady Lie k-algebra via a

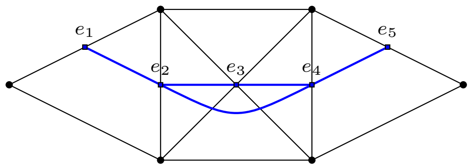


Fig. 1. The overlapping-gems graph $\bar{H}$ (in black) and the defining graph of $\mathfrak{B}_{\bar{H}}$ (in blue).

distinguished spectral sequence discovered by J.P. May (see [24]) with second page $E_2^{p,q} = H^{p+q,q}(\mathfrak{b}_\Gamma, k)$. For instance, if $\mathfrak{b}_\Gamma$ is a Koszul Lie algebra over a field k, then we get a k-algebra isomorphism $H^\bullet(B_\Gamma, k) \simeq H^\bullet(\mathfrak{b}_\Gamma, k)$, proving that B_Γ has Koszul cohomology over k. Hence, from Corollary 1.4 we deduce Proposition 0.3.

Examples 1.8. (1) Since the gem graph ∇ is a cone on the line graph L_3 of length 3, we get a direct product decomposition $\mathfrak{G}_\nabla = \mathfrak{G}_{L_3} \times k$. It follows that $\mathfrak{B}_\nabla \simeq \mathfrak{G}_{L_3}$, and hence $\mathfrak{B}_\nabla$ does not satisfy (1)–(4) of Theorem 1.2.

(2) Since the flag complex of the overlapping-gems graph $\bar{H}$ is contractible, by Theorems 1.3 and 1.6, $\mathfrak{B}_{\bar{H}}$ is finitely presented. We get the presentation (see [27, Cor. 2.3])

$$\mathfrak{B}_{\bar{H}} = \left\langle e_1, e_2, e_3, e_4, e_5 \,\middle|\, [e_1, e_2], [e_2, e_3], [e_2, e_4], [e_3, e_4], [e_4, e_5] \right\rangle$$

In particular, $\mathfrak{B}_{\bar{H}}$ is a RAAG, with underlying graph that is not Droms (Fig. 1). The same can be easily seen for the pro-p group $G_{\bar{H},p}$.

As communicated to the author by Ruffoni and Chang, $\bar{H}$ is the smallest graph without tree 2-spanners (see [9]) whose associated Bestvina-Brady group is isomorphic to a RAAG.

(3) Let Γ be the connected sum of two connected subgraphs Γ_1 and Γ_2 with a single common vertex v. This amounts to saying that $V(\Gamma) = V(\Gamma_1) \cup V(\Gamma_2)$, $V(\Gamma_1) \cap V(\Gamma_2) = \{v\}$, and $E(\Gamma)$ is the disjoint union of the $E(\Gamma_i)$'s. Then, the Dicks-Leary presentation [12] gives a free decomposition $\mathfrak{B}_\Gamma = \mathfrak{B}_{\Gamma_1} \amalg \mathfrak{B}_{\Gamma_2}$.

(4) Let $\Gamma = C_n$ be a n-cycle graph, with $n \geq 4$. Then, $\mathfrak{B}_\Gamma$ is a finitely generated subgroup of $\mathfrak{G}_\Gamma$ that admits no finite presentation by Theorem 1.3, as $\tilde{H}_1(C_n, k) \simeq k$. ■

2. Graph characterization

Many classes of graphs can be defined in terms of forbidden induced subgraphs. For instance, a finite simple graph Γ is chordal if it contains no induced n-cycle for $n \geq 4$, i.e., chordal graphs are $(C_n)_{n\geq 4}$-free. Similarly, Γ is a Droms graph if it does not contain any induced square nor line L_3 of length 3: it is (L_3, C_4)-free. Sometimes, a family C of finite graphs with forbidden induced subgraphs is closed with respect to some non-trivial operations (τ_i). This means that C has a subset Σ of *building blocks* that generates C by iterating those operations. If $\Sigma \neq C$, then we say that the family C has a **defining construction** in terms of the operations (τ_i) and building blocks Σ. For example, Droms graphs can be obtained by applying cones and disjoint unions on single vertices [15]. Another class of graphs admitting a defining construction is that of ptolemaic graphs.

2.1. Ptolemaic graphs

A graph Γ is said to be **ptolemaic** if it is connected, chordal and it does not contain any induced gem ∇. Ptolemaic graphs form an interesting and well studied class of graph for they are distance-hereditary, as it was proved in [1]. The same work characterizes ptolemaic graphs in terms of a defining construction that we recall here.

The operations are given by adding either a leaf or *any kind* of twin, with the exception that a false-twin can only be attached to a vertex that has complete neighbourhood.

(1) A **leaf** is a vertex that is adjacent to a single other vertex of the graph.
(2) A (true) **twin** is a vertex that has the same neighbourhood as that of another vertex.
(3) A **false-twin** is a vertex that shares the neighbourhood with another vertex to which it is not adjacent.

Lemma 2.1 *(Bandelt, Mulder [1]). Ptolemaic graphs have a defining construction given by the above operations and single vertices as building blocks, i.e., every finite ptolemaic graph is obtained from the singleton graph K_1 by iterating the operations of adding leaves or twins (with exceptions).*

For instance, block-graphs are ptolemaic graphs; recall that a block-graph is a graph obtained by connecting complete graphs at single common vertices.

We now introduce a generalization of block-graphs, where the building blocks are the connected Droms graphs instead of cliques.

Definition 2.2. A **tree of Droms graphs** is a graph obtained by performing connected sums at vertices of connected Droms graphs. In the notation as above, $\Sigma = \{\text{connected Droms graphs}\}$ and τ is the connected sum operation of graphs with base-point.

In particular, a tree of Droms graphs has a central element iff it is a Droms graph. It is easily seen that trees of Droms graphs always admit tree 2-spanners ([9]).

We recall the following definitions.

Definition 2.3. Let Γ be a connected graph. A vertex v is a **cut-vertex** for Γ if the induced subgraph Γ_v of Γ obtained by removing v is disconnected. A **block of** Γ **defined by** v is an induced subgraph spanned by v and the vertices of a single connected components of Γ_v.

The blocks of a graph are its biconnected components.

If v is a cut-vertex of a graph Γ with blocks $\Gamma_1, \Gamma_2, \dots$, then Γ is the connected sum of the blocks over the vertex v itself, i.e., $\Gamma = \Gamma_1 \underset{v}{\vee} \Gamma_2 \underset{v}{\vee} \dots$

Fact 2.4. Let Γ' be a graph with a cut-vertex v. If v is not central in Γ', then there is a block Γ_1 defined by v such that v is neither a cut-vertex nor a central vertex of Γ_1.

Proof. If $(\Gamma_i)_{i\in I}$ are the blocks of Γ' defined by v, then, by definition, v is not a cut-vertex for Γ_i, $i \in I$. If v were central in Γ_i, for all $i \in I$, then v would be central in Γ'. □

Lemma 2.5. *Let Γ be a tree of Droms graphs. Then, $\mathfrak{B}_\Gamma$ is a Droms object.*

Proof. We argue by induction on the number of vertices of Γ.

If Γ is a (connected) Droms graph, then $\mathfrak{G}_\Gamma$ is a Droms object by Theorem 1.2. Since Γ is connected, either $\mathfrak{B}_\Gamma$ is a standard subalgebra of $\mathfrak{g}_\Gamma$, or it is a finitely generated subgroup of G_Γ. Hence, we deduce, by Theorem 1.2, that $\mathfrak{B}_\Gamma$ is a Droms object.

If Γ is not a Droms graph, then, there are two trees of Droms graphs Γ_1 and Γ_2 with a common vertex v such that Γ is the connected sum of the Γ_i's along v. By induction, the $\mathfrak{B}_{\Gamma_i}$'s are Droms objects, and hence so is their free product $\mathfrak{B}_\Gamma = \mathfrak{B}_{\Gamma_1} \amalg \mathfrak{B}_{\Gamma_2}$ (see Example 1.8(3)). □

2.2. The defining construction of trees of Droms graphs

We start by showing the following key result.

Lemma 2.6. *Let* Γ *be a connected chordal* $(\bar{H}, \Delta)$*-free graph. Then,* Γ *has either a cut-vertex or a central vertex.*

Proof. We argue by induction on the number of vertices of Γ. If Γ has at most two vertices, then any of them is trivially a central vertex. Suppose that Γ has at least 2 vertices.

Since Γ is a chordal graph with no induced gem, it is a Ptolemaic graph, and hence we can use the defining construction of Γ by means of leaves and twins.

Case 1. Γ has a leaf.

If $\{v, w\}$ is a leaf with w of valency 1, then v is a cut-vertex for Γ.

Case 2. Γ has two (true) twins v and v'.

Let Γ' be the induced subgraph spanned by all the vertices $\neq v'$. By induction, Γ' has either a central vertex or a cut-vertex.

If Γ' has a central vertex z, then z is also a central vertex for Γ, because either $z = v$ or z is adjacent to v, and hence, in both cases, v' is adjacent to z.

If Γ' has no central vertices, then it must have a cut-vertex w. Two sub-cases might occur.

(1) *v is not a cut-vertex for* Γ'. In this case, w is a cut-vertex for Γ as well.
(2) *$v = w$ is a cut-vertex for* Γ'. Since Γ' has no central vertex, by Fact 2.4, there is a block Γ_1 of Γ such that v is neither a central nor a cut-vertex for Γ_1. Let Γ_2 be the union of the blocks different from Γ_1. By induction, Γ_1 contains either a central vertex or a cut-vertex.
   (a) If Γ_1 has no central vertices, then, by induction, it has a cut-vertex w'. In particular, $v \neq w'$ and w' is a cut-vertex for Γ.
   (b) Suppose that Γ_1 has a central vertex w_1, and let w_2 be a vertex in Γ_2 adjacent to v.
   If v is a leaf in Γ_1, then w_1 is a cut-vertex for Γ_1, and hence for Γ as in case (2)(a). Now suppose v is not a leaf of Γ_1, and let $X \neq \emptyset$ be the set of vertices of Γ_1 that are not adjacent to v; in particular, all $x \in X$ lie at distance 2 from v. If there existed a vertex $w_1' \neq w_1$ that is adjacent to both v and a vertex $x \in X$, then Γ would have an induced overlapping gems graph $\bar{H}$ spanned by $\{v, v', w_1, w_1', x, w_2\}$. It follows that all the vertices $w_1' \neq w_1$ of Γ_1 that are adjacent to v are not adjacent to any vertex of X, and hence w_1 is a cut-vertex for Γ.

It remains to consider the last case.

Case 3. Γ has two false twins v and v' (and the neighbourhood of v is a complete graph).

Let Γ' be the induced subgraph spanned by all the vertices $\neq v'$. By induction, Γ' has either a central vertex or a cut-vertex.

(1) Let z be a central vertex of Γ'. If $v = z$, then Γ' is a complete graph, and any vertex $w \notin \{v, v'\}$ is central in Γ.
   If $v \neq z$, then z is adjacent to v' as well, and hence it is central in Γ.

(2) If Γ' has no central vertex, then it must have a cut-vertex w. Since the neighbourhood of v is a complete graph, v cannot be a cut-vertex for Γ', i.e., $v \neq w$. It follows that w is a cut-vertex of Γ, and the proof is complete. □

We can now characterize the trees of Droms graphs in terms of forbidden induced subgraphs.

Theorem 2.7. *Let Γ be a finite simple graph. Then the following statements are equivalent:*

(1) *Γ is a connected chordal graph that does not contain any induced gem ∇ nor overlapping-gems $\bar{H}$,*
(2) *Γ is a tree of Droms graphs.*

Proof. If Γ is a tree of Droms graphs, then $\mathfrak{B}_\Gamma$ is a Droms object by Lemma 2.5. In particular, Γ is a ptolemaic graph with no induced $\bar{H}$ by Examples 1.8(1)–(2).

For the converse, suppose Γ is a connected, chordal, $(\bar{H}, \Delta)$-free graph and argue by induction on the number of vertices. The 1-vertex case is obvious. By Lemma 2.6, Γ has either a central vertex or a cut-vertex. If Γ has a cut vertex v, then there are two connected subgraphs Γ_1 and Γ_2 with common vertex v such that $\Gamma = \Gamma_1 \underset{v}{\vee} \Gamma_2$. By induction, both Γ_1 and Γ_2 are trees of Droms graphs, and hence so is Γ.

If Γ has no cut-vertex, let z be a central vertex. Consider the induced subgraph Γ' of Γ spanned by all the vertices $\neq z$. If Γ' is not a Droms graph, then it must contain an induced path of length 3, and hence its cone Γ contains an induced gem $\Delta = \nabla(L_3)$, contradicting the hypotheses. It follows that Γ' is a Droms graph, and hence so is its cone Γ. □

We deduce Theorem A, analogous to Theorem 1.2.

Theorem 2.8. *Let Γ be a finite connected simple graph. Then, Γ is a tree of Droms graph if, and only if, anyone of the following statements holds:*

(1) *All subgroups of B_Γ are RAAGs (possibly defined by infinite graphs),*
(2) *All standard subalgebras of $\mathfrak{b}_\Gamma$ are RAAG Lie algebras,*
(3) *The Lie algebra $\mathfrak{b}_\Gamma$ is BK,*
(4) *The Bestvina-Brady object $\mathfrak{B}_\Gamma$ is Droms,*
(5) *The cohomology algebra $H^\bullet(\mathfrak{b}_\Gamma, k)$ of the Bestvina-Brady Lie k-algebra is universally Koszul,*
(6) *The pro-p group $B_{\Gamma,p}$ is BK,*
(7) *Every closed subgroup of $B_{\Gamma,p}$ is a pro-p RAAG,*
(8) *There exists a field $\mathbb{K}$ containing a primitive p^{th} root of 1 such that $B_{\Gamma,p} \simeq G_\mathbb{K}(p)$.*

Proof. First notice that in light of Theorem 1.2, statements (1), (2), (4) and (7) are all equivalent to each other. Moreover, anyone of them implies (3), (5), (6) and (8).

If Γ is a tree of Droms graphs, then $\mathfrak{B}_\Gamma$ is a Droms object by Lemma 2.5, whence we get (1)–(8).

For the converse, suppose that Γ is a connected graph but it is not a tree of Droms graphs. By Theorem 2.7, Γ contains an induced subgraph Λ isomorphic to either ∇, $\bar{H}$ or the n-cycle C_n, $n \geq 4$. Now, by Examples 1.8, $\mathfrak{B}_\Lambda$ is either a RAAG object that is not Droms (if $\Lambda = \nabla$ or $\bar{H}$), or a finitely generated but not finitely presented object (if $\Lambda = C_n$, $n \geq 4$). In both cases, $\mathfrak{B}_\Gamma$ (resp. $B_{\Gamma,p}$) contains a sub-object (resp. a closed subgroup) whose cohomology ring over k (resp. $k = \mathbb{F}_p$) is not quadratic. It follows that (3) is false, and hence so are (1), (2), (4) and (5). Similarly, the pro-p analogues (6) and (7) cannot occur, and, by the Norm-residue isomorphism theorem (aka the Bloch-Kato conjecture [35]), neither can (8). □

Remark 2.9. When Γ is not a Droms graph, then B_Γ is not necessarily a RAAG, and its cohomology might be universally Koszul even when $H^\bullet(\mathfrak{b}_\Gamma, k)$ is not. The author is not aware of any example of a residually nilpotent group G such that $H^\bullet(G, k)$ is universally Koszul (k a field), and $H^\bullet(\operatorname{gr} G \otimes k, k)$ is not.

However, by (4), if Γ is a tree of Droms graphs and k is a field, then (5) also holds for the Bestvina-Brady (pro-p) group, that is, the universal Koszulity property holds for $H^\bullet(B_\Gamma, k)$ (resp. $H^\bullet(B_{\Gamma,p}, \mathbb{F}_p)$). In particular, since (8) is equivalent to (7), if $B_{\Gamma,p} \simeq G_{\mathbb{K}}(p)$, then the elementary type conjecture [17] and the universal Koszulity conjecture [25] are both confirmed within the class of Bestvina-Brady pro-p groups.

3. Examples

3.1. Free objects

In the context of (pro-p) groups, it is proved that B_T is a free (pro-p) group of finite rank when T is a finite tree ([33]). We now provide a proof for the Lie theoretic translation of that result.

Proposition 3.1. *Let Γ be a finite graph. Then, $\mathfrak{b}_\Gamma$ is a free Lie algebra of rank $|V(\Gamma)| - 1$ if, and only if, Γ is a tree.*

Proof. Let Γ be a finite connected graph with n vertices and m edges. It is a tree precisely when $n = m + 1$.

If Γ is a tree, then $\mathfrak{b}_\Gamma$ is finitely presented by Theorem 1.3. Since $\mathfrak{g}_\Gamma = \mathfrak{b}_\Gamma \rtimes k$, it follows that $b_2(\mathfrak{b}_\Gamma) = b_2(\mathfrak{g}_\Gamma) - b_1(\mathfrak{b}_\Gamma) = m - (n-1) = 0$, and hence $\mathfrak{b}_\Gamma$ is free by [5, Thm. 5.2].

Conversely, if $\mathfrak{b}_\Gamma$ is free of rank $n-1$, then it is of type FP and $m = b_2(\mathfrak{g}_\Gamma) = b_2(\mathfrak{b}_\Gamma) + b_1(\mathfrak{b}_\Gamma) = 0 + n - 1$, i.e., Γ is a tree. □

In turn, we deduce the following well-known result.

Corollary 3.2. *A finite graph Γ is a tree if, and only if, B_Γ is a free group of finite rank.*

Proof. If B_Γ is a free group of finite rank, then Γ is connected by Theorem 1.3, and $\mathfrak{b}_\Gamma \simeq \operatorname{gr} B_\Gamma$ is free by [31, Thm. 6.1] (see also [23]). As a consequence of Proposition 3.1, we deduce that Γ is a tree.

Conversely, if Γ is a tree, then the Bestvina-Brady Lie k-algebra $\mathfrak{b}_\Gamma$ is a free Lie algebra and $\operatorname{gr} B_\Gamma \otimes k \simeq \mathfrak{b}_\Gamma$. It follows from the May spectral sequence and the universal coefficient theorem that $\operatorname{cd} B_\Gamma \leq 1$, and hence B_Γ is a free group by the Stallings-Swan theorem. □

Remark 3.3. Proposition 3.1 and its corollary can also be proved by means of the Dicks-Leary presentation. Indeed, $\mathfrak{B}_\Gamma$ is finitely generated precisely when Γ is connected, and, in that case, it has no non-trivial relation if, and only if, Γ does not contain any induced cycle, i.e., when Γ is a tree.

3.2. Surface groups and Poincaré duality Lie algebras

RAAGs can contain surface groups of high genus. For instance, Servatius, Droms and Servatius proved in [32] that if an n-cycle is induced in a graph Γ, then the derived subgroup G'_Γ contains the fundamental group of a closed oriented surface of genus $g = 1 + (n-4)2^{n-3}$.

We now prove, by means of Theorem 2.8, that no oriented surface group of genus ≥ 2 is a Bestvina-Brady group. Notice that one can give a direct (easier) proof of the result by means of the Dicks-Leary presentation [12].

Corollary 3.4. *Suppose that the Bestvina-Brady Lie algebra $\mathfrak{b}_\Gamma$ is a quadratic 1-relator Lie algebra over a field k. Then, $\mathfrak{b}_\Gamma$ decomposes as the free product of an abelian Lie algebra of dimension 2 and a free Lie algebra.*

In particular, the fundamental group of a closed hyperbolic surface (i.e., of genus ≥ 2) is not isomorphic to a Bestvina-Brady group.

Proof. By [5], $\mathfrak{b} = \mathfrak{b}_\Gamma$ is BK, and hence Γ is a tree of Droms graphs by Theorem 2.8.

Now, [6, Lem. 2.2] provides the Betti numbers $b_i(\_) := \dim H^i(\_, k)$ of $\mathfrak{b}$ in terms of those of $\mathfrak{g}_\Gamma$:

$$1 \overset{\text{1-rel.}}{=} b_2(\mathfrak{b}) = b_2(\mathfrak{g}_\Gamma) - b_1(\mathfrak{g}_\Gamma) + 1.$$

Recall that the number of vertices (resp. of edges) of Γ equals the Betti number $b_1(\mathfrak{g}_\Gamma)$ (resp., $b_2(\mathfrak{g}_\Gamma)$), so that $|V| = |E|$.

By definition, a (non-induced) tree $T = (V(T), E(T))$ of $\Gamma = (V, E)$ is a spanning tree if $V = V(T)$. As Γ is connected, it admits a spanning tree T; One has $|E(T)| - |V(T)| + 1 = 0$, and hence $1 = |V| - |E(T)| = |E| - |E(T)|$, i.e., Γ is obtained from T by adding a single edge to a pair of its vertices, producing an induced cycle C. Since Γ is a tree of Droms graphs, C has length 3 and $\mathfrak{b}$ is the free product of $\mathfrak{b}_C \simeq k^2$ and a free Lie algebra.

Now, let G be the fundamental group of an orientable closed surface of genus n. Then, G can be presented as

$$G = \left\langle x_1, y_1, \ldots, x_n, y_n \,\middle|\, [x_1, y_1][x_2, y_2] \cdots [x_n, y_n] \right\rangle$$

The associated Lie algebra was computed by Labute [22]:

$$\operatorname{gr} G = \left\langle x_1, y_1, \ldots, x_n, y_n \,\middle|\, [x_1, y_1] + [x_2, y_2] + \cdots + [x_n, y_n] \right\rangle$$

If $G = B_\Gamma$ for some graph Γ, then, for any field k, $\operatorname{gr} G \otimes k$ is isomorphic with the Bestvina-Brady Lie k-algebra $\mathfrak{b}_\Gamma$. By the first part of the proof, $n = 1$. □

The same proof shows that a Demuškin group [11] (see also [25] and [21]) occurs as a Bestvina-Brady pro-p group only if it is 2-generated.

Recall that a Lie k-algebra $\mathfrak{g}$ is a **Poincaré duality Lie algebra** in dimension $n \in \mathbb{N}$ if it has cohomological dimension n, $H^n(\mathfrak{g}, k)$ is 1-dimensional, and the cup-product of the trivial coefficients cohomology ring defines non-degenerate pairings $H^i(\mathfrak{g}, k) \otimes H^{n-i}(\mathfrak{g}, k) \to H^n(\mathfrak{g}, k)$. For instance, all the finite dimensional Lie algebras of dimension n are Poincaré duality of dimension n, as well as any quadratic, freely indecomposable, 1-relator Lie algebra (with $n = 2$), i.e., the surface Lie algebra $\mathscr{G}_{2d}$.

Proposition 3.5. *Let Γ be a finite simple graph such that $\mathfrak{b}_\Gamma$ is a Poincaré duality Lie algebra in dimension n. Then, $\mathfrak{b}_\Gamma$ is abelian.*

Proof. Suppose $n \geq 1$. Since the cohomological dimension of $\mathfrak{b}_\Gamma$ is n, the graph Γ has a $(n+1)$-clique Δ.

Let v be any vertex of Γ, and let Γ_v be the induced subgraph spanned by the vertices $\neq v$. Then, $\mathfrak{b}_{\Gamma_v}$ is a proper subalgebra of $\mathfrak{b}_\Gamma$, and hence it has cohomological dimension $\leq n-1$ by [5, p. 792], proving that $\mathfrak{g}_{\Gamma_v}$ has cohomological dimension $\leq n$.

In particular, Γ_v contains no $(n+1)$-clique, which means that $v \in \Delta$, and hence $\Gamma = \Delta$. □

The same argument applies to any cocyclic ideal of $\mathfrak{g}_\Gamma$ in the place of $\mathfrak{b}_\Gamma$.

Since all RAAG objects are Bestvina-Brady (e.g., $\mathfrak{G}_\Gamma = \mathfrak{B}_{\nabla(\Gamma)}$), if Γ is a tree of Droms graphs, then all (standard) finitely generated subobjects of $\mathfrak{B}_\Gamma$ are Bestvina-Brady objects. The converse seems to be false.

Corollary 3.6. *Let Γ be a connected graph such that all the finitely generated subgroups of B_Γ are Bestvina-Brady groups. Then Γ is $(\bar{H}, \nabla, C_{n+5})$-free, $n \geq 0$.*

Proof. Let Λ be an induced subgraph of Γ.

If Λ is either a gem or an overlapping-gems graph, then B_Γ contains a copy of G_{L_3}. Let H be a 3-generated subgroup of G_Γ that is not a RAAG; such subgroup exists by the proof of Droms' theorem [15] (see also [33]).

Suppose by contradiction that H is isomorphic to a Bestvina-Brady group $B_{\Lambda'}$. Since G_{L_3} is coherent, then the flag complex $\Delta_{\Lambda'}$ is simply connected; moreover, Λ' has 4 vertices. Nevertheless, it is easy to see by inspection of all such graphs that the associated Bestvina-Brady groups are RAAGs, contradicting the assumption on H.

It remains to consider the case when Λ is a cycle of length ≥ 5. Now, the derived subgroup $G'_\Lambda \subset B_\Lambda$ contains the fundamental group of a closed hyperbolic surface, which is not a Bestvina-Brady group by Corollary 3.4. □

Example 3.7. Consider the square graph $\Gamma = C_4$ with ordered vertices (x_1, y_1, x_2, y_2).

If $\mathfrak{m}$ is a standard proper subalgebra of $\mathfrak{b}_\Gamma$, then it is generated by at most 2 elements, and hence $\mathfrak{m}$ is either free or abelian (see [6, Prop. 2.20]). This proves that all the standard subalgebra of $\mathfrak{b}_\Gamma$ are Bestvina-Brady Lie algebras; in turn, given the strong similarity between the group case and the Lie algebra case, this also suggests that all finitely generated subgroups of B_{C_4} might be Bestvina-Brady groups.

We now compute a presentation for $\mathfrak{b}_\Gamma$. Notice that $\mathfrak{b}_\Gamma$ is a homomorphic image of the Lie algebra $\mathfrak{h} = \langle x', y', e' \,|\, [x', y'] \rangle$ via the map f defined by $x' \mapsto x := x_2 - x_1$, $y' \mapsto y := y_2 - y_1$, and $e' \mapsto e := y_2 - x_2$. The dimension of the degree 3 component of $\mathfrak{g}_\Gamma$ is 4, and $\dim \mathfrak{h}_3 = 5$ (see e.g. [36]). Since $[x, [e, y]] = 0$ in $\mathfrak{b}_\Gamma$, the induced map $\bar{f} : \mathfrak{h}/([x', [e', y']]) \to \mathfrak{b}_\Gamma$ is an isomorphism in degrees ≤ 3.

Now, $\mathfrak{g}_\Gamma$ is Koszul and $\mathfrak{b}_\Gamma$ is a cocyclic ideal, so the non-diagonal pieces in the same homological degree of the cohomology of $\mathfrak{b}_\Gamma$ have equal dimensions, i.e., $H^{i,i+1}(\mathfrak{b}_\Gamma, k) \simeq H^{i,i+2}(\mathfrak{b}_\Gamma, k) \simeq \dots$.

It follows that the Bestvina-Brady Lie algebra $\mathfrak{b}_\Gamma$ admits the (infinite) minimal presentation

$$\langle x, y, e \,|\, [x, \mathrm{ad}(e)^n y] : \; n \geq 0 \rangle$$

where $\mathrm{ad}(e) : y \mapsto [e, y]$. ■

Question 3.8. Does B_{C_4} contain a subgroup that is not the Bestvina-Brady group of a graph?

The existence of non-abelian surface Lie subalgebras of $\mathfrak{g}_{C_n}$, $n \geq 5$, cannot be directly deduced from the case of groups.

Example 3.9. Consider the RAAG Lie algebra $\mathfrak{g} = \mathfrak{g}_{C_5}$ with presentation

$$\langle x_0, \dots, x_4 \,|\, [x_i, x_{i+1}] : \; i = 0, \dots, 4 \rangle$$

where the indices are mod 5. The subalgebra $\mathfrak{m}$ generated by the elements $p_i = [x_i, [x_{i+2}, x_{i+4}]]$, $q_i = [x_{i+1}, x_{i+3}]$ $(i = 0, \dots, 4)$ can be given a grading that makes it a standard Lie algebra, with $|p_i| = |q_i| = 1$. With these generators, the only non-trivial quadratic relation of $\mathfrak{m}$ is

$$\sum_{i=0}^{4} [p_i, q_i] = 0.$$

Is $\mathfrak{m}$ isomorphic with the surface Lie algebra $\mathscr{G}_{2\cdot 5}$? ■

Lemma 3.10. *Let $\mathfrak{g}$ be a Koszul Lie algebra, and let $\mathfrak{m}$ be a maximal standard subalgebra of $\mathfrak{g}$. Then,*

$$\dim H^{i,i+1}(\mathfrak{m}) \leq \dim H^{i,i}(\mathfrak{m}).$$

Proof. Consider a decomposition of $\mathfrak{g}$ as the HNN-extension $\mathrm{HNN}_\phi(\mathfrak{m}, t)$, where $\phi : \mathfrak{a} \to \mathfrak{m}$ is a derivation of degree 1. The Chiswell exact sequence (see [6]) reads

$$0 = H^{i,i+1}(\mathfrak{g}) \to H^{i,i+1}(\mathfrak{m}) \to H^{i,i}(\mathfrak{a}) \to H^{i+1}(\mathfrak{g}) \to H^{i+1,i+1}(\mathfrak{m}) \to H^{i+1,i}(\mathfrak{a}) = 0$$

It then follows for the bigraded Betti numbers $b_{ij}(\_) := \dim H^{ij}(\_, k)$

$$b_{i,i+1}(\mathfrak{m}) = b_{ii}(\mathfrak{a}) - b_{i+1}(\mathfrak{g}) + b_{i+1,i+1}(\mathfrak{m}).$$

For a Lie algebra $\mathfrak{f}$, denote by $\mathrm{q}\mathfrak{f}$ the quadratic cover of $\mathfrak{f}$, and by $\mathrm{q}\mathfrak{f}^!$ the Koszul dual of $\mathrm{q}\mathfrak{f}$. Since the dual maps $\mathrm{q}\mathfrak{m}^! \to \mathrm{q}\mathfrak{a}^!$ and $\mathfrak{g}^! \to \mathrm{q}\mathfrak{m}^!$ are surjective, and $(\mathrm{q}\mathfrak{m}^!)_n = H^{n,n}(\mathfrak{m})$, etc., we deduce that $b_{i,i+1}(\mathfrak{m}) \leq b_{i,i}(\mathfrak{m}) - (b_{i+1}(\mathfrak{g}) - b_{i+1,i+1}(\mathfrak{m})) \leq b_{i,i}(\mathfrak{m})$. □

In particular, if $\mathfrak{m}$ is a cocyclic ideal of $\mathfrak{g}$, we deduce from [6, Cor. 2.7] that $\dim H^{i,j}(\mathfrak{m}) \leq \dim H^{i,i}(\mathfrak{m})$ for all $j \geq i$.

We now give a direct proof of the latter result for $i = 2$, that is, we find constraints on the number of minimal relations of higher degree.

Proposition 3.11. *Let $\mathfrak{g}$ be a Koszul Lie algebra with a cocyclic ideal $\mathfrak{b}$. If $\mathfrak{b}$ is a standard Lie algebra and $\dim H^{2,2}(\mathfrak{b}) = m$, then either $\mathfrak{b}$ is quadratic, or $1 \leq \dim H^{2,3}(\mathfrak{b}) = \dim H^{2,4}(\mathfrak{b}) = \cdots \leq m$. In other words, for all $j > 2$, the minimal number of relations of degree j is bounded above by the number of the quadratic ones.*

Proof. By [6, Cor. 2.7], $\dim H^{i,j}(\mathfrak{b}) = \dim H^{i,i+1}(\mathfrak{b})$ for all $j > i$, and hence we deduce the result applying Lemma 3.10. Although the proof is complete, we can explicitly construct the higher degree relations in terms of those of degree 2.

If $x_1 \in \mathfrak{g}_1 \setminus \mathfrak{b}$, then $\mathrm{ad}(x_1)|_\mathfrak{b} : \mathfrak{b} \to \mathfrak{b}$ is a degree-1 derivation. Let $\mathfrak{f}$ be a free cover of $\mathfrak{b}$ and let $r_1, \ldots, r_m \in \mathfrak{f}_2$ be minimal relators of $\mathfrak{b}$, so that $\mathfrak{f}/(r_1, \ldots, r_m)$ is the quadratic cover of $\mathfrak{b}$. The map $\mathrm{ad}(x_1)|_\mathfrak{b}$ can be lifted to a degree-1 derivation ϕ of $\mathfrak{f}$.

Put $r_i^{(2)} := r_i$,

$$r_i^{(q+1)} := \phi(r_i^{(q)}) \in \mathfrak{f}_{q+1}, \quad \text{for } q \geq 2,$$

and let I be the ideal generated by the elements $r_i^{(q)}$, where $i = 1, \ldots, m$ and $q \geq 2$. Since $\mathrm{ad}(x_1)|_\mathfrak{b}$ is a derivation of $\mathfrak{b}$, the image of the $r_i^{(q)}$ into $\mathfrak{b}$ vanishes, and hence there is a natural surjection $\pi_\mathfrak{b} : \mathfrak{f}/I \to \mathfrak{b}$. We claim that $\pi_\mathfrak{b}$ is an isomorphism, and hence $\dim H^{2,3}(\mathfrak{b}) = \dim H^{2,3}(\mathfrak{f}/I) \leq m$.

By construction, ϕ defines a derivation $\mathfrak{f}/I \to \mathfrak{f}/I$ of degree 1, which allows us to consider the associated semidirect product $\mathfrak{n} = \mathfrak{f}/I \rtimes_\phi V$ where V is a 1-dimensional Lie algebra (i.e., $\mathfrak{n}$ is an HNN-extension $\mathrm{HNN}_\phi(\mathfrak{f}/I, t)$). This is clearly a quadratic Lie algebra containing $\mathfrak{f}/I$, and $\pi_\mathfrak{b} \circ \phi = \mathrm{ad}(x_1)|_\mathfrak{b} \circ \pi_\mathfrak{b}$.

Therefore, the universal property of HNN-extensions implies that the map $\pi_\mathfrak{b}$ induces a (surjective) homomorphism $\pi : \mathfrak{n} \to \mathfrak{g}$.

Since the map π between quadratic Lie algebras is an isomorphism in degrees 1 and 2, it is an isomorphism in all the degrees, and hence $\pi_\mathfrak{b} = \pi|_{\mathfrak{f}/I}$ is also injective.

Finally notice that, if $\phi(I_2)$ is contained in the ideal generated by I_2, then $\mathfrak{f}/I \simeq \mathfrak{b}$ is quadratic. □

Remark 3.12.

(1) If the quadratic cover of $\mathfrak{b}$ is free (i.e., $m = 0$), then $\mathfrak{b}$ is necessarily free.
(2) It follows from the proof of [6, Cor. 2.7] that it is enough to assume that $\mathfrak{g}$ is 3-Koszul, meaning that $H^{i,j}(\mathfrak{g}) = 0$ when $i < j$, $i \leq 3$. Indeed, in that case, one still has $H^{2,3}(\mathfrak{b}) \simeq H^{2,4}(\mathfrak{b}) \simeq \dots$
(3) The result also provides a procedure to compute (minimal) presentations of some cocyclic ideals of Koszul Lie algebras. We now extend the computations of Example 3.7 to longer cycles.

Example 3.13. Let C_{n+1} be the cycle of length $n+1 \geq 4$. Consider the RAAG Lie algebra $\mathfrak{g} = \mathfrak{g}_{C_{n+1}}$ generated by the vertices x_i $(0 \leq i \leq n)$ subject to the relations $[x_i, x_{i+1}]$ $(0 \leq i \leq n)$, where the indices are mod $n+1$. We compute the defining relations of $\mathfrak{b} = \mathfrak{b}_{C_{n+1}}$. Notice that $\mathfrak{b}$ is finitely generated, and hence standard.

One has $b_2(\mathfrak{g}) = n+1$, $b_{2,2}(\mathfrak{b}) = b_2(\mathfrak{g}) - b_1(\mathfrak{b}) = 1$. Since $\mathfrak{b}$ is not finitely presented, it is not quadratic, and Proposition 3.11 implies that $\dim H^{2,j}(\mathfrak{b}) = 1$ for all $j \geq 3$.

Set $e_i = x_i - x_{i+1}$, for $i = 0, \dots, n$, so that $\mathfrak{b}$ is generated by the elements $e_1, \dots, e_n$. There holds

$$\sum_{1 \leq i < j \leq n} [e_i, e_j] = 0,$$

and this is the unique relation (up to scalar multiplication) of degree 2 of $\mathfrak{b}$. Since $\mathrm{ad}(x_1)$ is a derivation of $\mathfrak{b}$, we deduce a degree 3 relation

$$\sum_{1 \leq i < j \leq n} ([[x_1, e_i], e_j] + [e_i, [x_1, e_j]]) = 0$$

and degree-$(d+2)$ relations

$$\sum_{k=0}^{d} \binom{d}{k} [\mathrm{ad}(x_1)^k e_i, \mathrm{ad}(x_1)^{d-k} e_j] = 0$$

Nevertheless, these are not written in the generators of $\mathfrak{b}$; we consider the relation of degree 3 (i.e., $d = 1$), for which we can find a closed form.

For $i = 1, \dots, n$, set

$$s_i = e_1 + \dots + e_{i-1} \text{ and } t_i = e_{i+1} + \dots + e_n$$

(one has $s_1 = t_n = 0$). Since $[x_1, e_i] = [s_i, e_i]$,

$$\begin{aligned}
&\sum_{1 \leq i < j \leq n} ([[x_1, e_i], e_j] + [e_i, [x_1, e_j]]) = \sum_{1 \leq i < j \leq n} ([[s_i, e_i], e_j] + [e_i, [s_j, e_j]]) = \\
&= \sum_{i=1}^{n-1} [[s_i, e_i], e_{i+1} + \dots + e_n] + \sum_{j=2}^{n} [e_1 + \dots + e_{j-1}, [s_j, e_j]] = \\
&= \sum_{i=1}^{n-1} [[s_i, e_i], t_i] + \sum_{j=2}^{n} [s_j, [s_j, e_j]] = \sum_{i=2}^{n} [[s_i, e_i], t_i] + \sum_{j=2}^{n} [s_j, [s_j, e_j]] = \quad (\text{as } t_n = s_1 = 0) \\
&= \sum_{i=2}^{n} [[s_i, e_i], t_i - s_i]
\end{aligned}$$

We have thus found a minimal degree-3 relation $r_3 = \sum_{i=2}^{n} [[s_i, e_i], t_i - s_i]$, meaning that $H_{2,3}(\mathfrak{b}) = k \cdot [r_3]$.

Explicitly, it can be written as

$$\sum_{i=2}^{n}[[e_1+\cdots+e_{i-1},e_i],e_n+\cdots+e_{i+1}-e_{i-1}-\cdots-e_1]$$

For $n=3$, we recover the degree 3 relation of Example 3.7. ■

3.3. Bloch-Kato version of the b_2-conjecture

The following question was raised by Weigel in [36]:

Question 3.14. Let A be a Koszul algebra of finite cohomological dimension d over a field k. Is it true that

$$\dim(\operatorname{Ext}_A^{2,2}(k,k))\leq\frac{d-1}{2d}\dim(\operatorname{Ext}_A^{1,1}(k,k))^2?$$

This has positive answer in case the *eigenvalues* of A are all real numbers (see [6]); in particular, this is true when $d=2$.

Aiming to answer this question in case A is the universal enveloping algebra of a Lie algebra, the author introduced in [6] an invariant of a finitely presented graded Lie algebra $\mathfrak{g}$ in terms of its low-degree Betti numbers as

$$\omega(\mathfrak{g}):=(\operatorname{cd}(\mathfrak{g})-1)b_1(\mathfrak{g})^2-2\operatorname{cd}(\mathfrak{g})b_2(\mathfrak{g}).$$

It was proved by Weigel [36] following Turán [34] that, if $\mathfrak{g}$ is a RAAG Lie algebra, then $\omega(\mathfrak{g})\geq 0$, giving a positive answer to Question 3.14 within that class of Lie algebras.

Now let $\mathfrak{b}$ be a cocyclic ideal of type FP_2 of a Koszul Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ has cohomological dimension $d=n+1$, then a similar computation to that of [6, Lem. 3.15] shows that

$$(n+1)\omega(\mathfrak{b})=n\omega(\mathfrak{g})-(b_1(\mathfrak{g})-n-1)^2.$$

In particular, the invariant ω of $\mathfrak{g}$ is related with that of any finitely presented cocyclic ideals of $\mathfrak{g}$.

Proposition 3.15. *Let Γ be a graph of clique number $n+1$ admitting a tree 2-spanner (see [9]), e.g., Γ is a tree of Droms graphs. Then,*

$$n\omega(\mathfrak{g}_\Gamma)\geq(b_1(\mathfrak{g}_\Gamma)-n-1)^2$$

Hence, if Γ has v vertices and e edges, then

$$n(v^2-2e-2)\geq(v-1)^2.$$

Proof. Since $\mathfrak{b}_\Gamma$ is a RAAG Lie algebra, the inequality $\omega(\mathfrak{b}_\Gamma)\geq 0$ follows from Turàn's theorem (see also [6, Sec. 3.3]). □

The following is a graph-theoretic reformulation of Question 2 of [36] for Bestvina-Brady Lie algebras which are Koszul.

Question 3.16. Let Γ be a finite simple graph with acyclic (over an arbitrary field) flag complex of dimension n. Is it true that

$$n(v^2 - 2e - 1) \geq (v-1)^2 \tag{3.1}$$

where v and e are the number of vertices and edges of Γ, respectively?

Equivalently, is it true that finite acyclic flag complexes have dimension at least $\frac{(v-1)^2}{v^2-2e-1}$?

Remark 3.17. The inequality has been confirmed for $v \leq 8$ by means of an easy — yet far from being optimal — code in SageMath.

```
for vert in [1..8]:
  for G in graphs(vert):
    if G.is_connected():
      D = G.clique_complex()
      if D.is_acyclic():
        edges = G.num_edges()
        n = D.dimension()
        om = n * (vert**2 - 2 * edges - 1) - (vert - 1)**2
        if om < 0:
          print(vert, edges)
          G.show()
```

For 2-dimensional acyclic flag complexes, we deduce an interesting upper bound for the number of edges.

Proposition 3.18. *Let k be a field and let Δ be a k-acyclic, 2-dimensional flag complex with v vertices and e edges. Then the inequality (3.1) holds:*

$$(v+1)^2 \geq 4(e+1).$$

Proof. Let Γ be the 1-skeleton of Δ. Since $\mathfrak{b}_\Gamma$ is Koszul and has cohomological dimension at most 2, the result follows from the fact that Question 3.14 has a positive answer for $\mathfrak{b}_\Gamma$ ([36]). □

For arbitrary graphs, the best estimate for the dimension of a flag complex Δ was given by Myers and Liu [26]

$$\dim \Delta \geq \frac{2e}{v^2 - 2e}$$

and is implied by (3.1).

3.4. Coherence of Bestvina-Brady objects

By [19] and [13], a graph Γ is chordal iff the object $\mathfrak{G}_\Gamma$ is coherent (i.e., *all* the finitely generated subobjects admit a finite presentation); moreover, in the Lie algebra case, this is equivalent to the fact that $\mathfrak{g}_\Gamma$ is locally of type FP over k. Hence the k-cohomology of $\mathfrak{B}_\Gamma$ is a Koszul algebra when Γ is connected and chordal. In turn, by the universal coefficient theorem, this gives an algebraic proof that the flag complex of a connected chordal graph is acyclic (see also [28] for a purely combinatorial one).

The coherence property of Bestvina-Brady groups is equivalent to that of its ambient RAAG.

Proposition 3.19. *Let Γ be a finite simple graph. Then, the following statements are equivalent:*

(1) Γ *is a chordal graph,*
(2) *The Bestvina-Brady Lie algebra* $\mathfrak{b}_\Gamma$ *is locally of type FP,*
(3) *The Bestvina-Brady object* $\mathfrak{B}_\Gamma$ *is coherent,*
(4) *The derived sub-object* $\mathfrak{B}_\Gamma' = [\mathfrak{B}_\Gamma, \mathfrak{B}_\Gamma]$ *is free.*

Proof. If Γ is chordal, then the RAAG object $\mathfrak{G}_\Gamma$ is coherent, with free derived sub-object, $\mathfrak{g}_\Gamma$ is locally of type FP (see [13,19,32]), and hence the same holds for $\mathfrak{B}_\Gamma$. This proves that (1) implies (2)–(4).

Suppose now that Γ contains an induced n-cycle C for $n \geq 4$. Then, the sub-object $\mathfrak{B}_C$ of $\mathfrak{B}_\Gamma$ is finitely generated but not of type FP_2 by Theorem 1.3, proving that $\mathfrak{B}_\Gamma$ is neither coherent nor locally of type FP. Moreover, since $\mathfrak{B}'_C = \mathfrak{G}'_C$ is a non-free sub-object of $\mathfrak{B}'_\Gamma$, the latter cannot be free. □

Declaration of competing interest

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