

ON MINIMAL KERNELS AND LEVI CURRENTS ON WEAKLY COMPLETE COMPLEX MANIFOLDS

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ABSTRACT. A complex manifold X is *weakly complete* if it admits a continuous plurisubharmonic exhaustion function ϕ . The minimal kernels $\Sigma_X^k, k \in [0, \infty]$ (the loci where all $\mathcal{C}^k$ plurisubharmonic exhaustion functions fail to be strictly plurisubharmonic), introduced by Slodkowski-Tomassini, and the Levi currents, introduced by Sibony, are both concepts aimed at measuring how far X is from being Stein. We compare these notions, prove that all Levi currents are supported by all the Σ_X^k 's, and give sufficient conditions for points in Σ_X^k to be in the support of some Levi current.

When X is a surface and ϕ can be chosen analytic, building on previous work by the second author, Slodkowski, and Tomassini, we prove the existence of a Levi current precisely supported on Σ_X^∞ , and give a classification of Levi currents on X . In particular, unless X is a modification of a Stein space, every point in X is in the support of some Levi current.

1. INTRODUCTION

Given an abstract (and possibly very complicated) manifold, a natural question is whether it is possible to see it as a subset of a simpler space. In the real category, a fundamental theorem by Nash states that this is always possible, and in a very strong sense: every Riemannian manifold can be isometrically embedded in some $\mathbb{R}^N$. When moving to the complex category, we can then ask the following natural question: is it possible to embed any complex manifold in some $\mathbb{C}^N$, by means of a proper holomorphic map? We call *Stein* a manifold for which the above holds true. This time, the rigidity of holomorphic functions readily provides negative examples: for instance, the maximum principle implies that any holomorphic map on a compact complex manifold must be constant, and thus the manifold cannot be Stein. A central question is then to understand when a given complex manifold is Stein. More specifically, given a dimension n , one would like to understand the *obstructions* for an n -dimensional manifold to be Stein.

A major advance in this direction was provided by Grauert [4]: a complex manifold is Stein if and only if it admits a $\mathcal{C}^2$ strictly plurisubharmonic (psh for short) exhaustion function. The $\mathcal{C}^2$ assumption was relaxed to $\mathcal{C}^0$ by Narasimhan [14, 15]. In view of these results, it is natural to tackle the question by studying the positive cone $\text{Psh}_e^0(X)$ of all continuous psh exhaustion functions on X (or more generally the cone $\text{Psh}_e^k(X) := \text{Psh}_e^0(X) \cap \mathcal{C}^k$ for some $k \in [0, \infty]$), and in particular to find obstructions for them to be strictly psh. As a rough idea, such obstructions must correspond to the presence of some sets in X along which all continuous psh functions must necessarily be

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pluriharmonic. As a prototypical example, the blow-up of a point and its corresponding exceptional divisor give precisely this kind of obstruction.

A precise study of this kind of phenomena was started by Slodkowski and Tomassini in [27] in the setting of *weakly complete complex manifolds*, i.e., manifolds admitting a continuous psh exhaustion function. A crucial definition is the following: for $k \in [0, \infty]$ the *minimal kernel* of a manifold X (with respect to Psh_e^k) is

$$(1) \quad \Sigma_X^k := \{x \in X : \nexists u \in \text{Psh}_e^k(X) \text{ which is strictly psh at } x\}.$$

When $k \geq 2$, Σ_X^k coincides with the locus where $i\partial\bar{\partial}u$ is degenerate for all $u \in \text{Psh}_e^k(X)$. A key result of [27] is that, whenever $\text{Psh}_e^k(X)$ is not empty, there actually exists a function $\phi_0 \in \text{Psh}_e^k(X)$ (called *minimal*) which fails to be strictly psh *precisely* on the minimal kernel Σ_X^k . Moreover, the minimal kernels are *local maximum sets* (see Definition 3.1). Some finer properties are also established (some requiring at least the C^2 regularity, see for instance [27, Theorem 3.9]). Observe that the Σ_X^k 's are increasing in k , but it is not known whether equalities should occur in general, see for instance [26, Section 5.10]. It is interesting to note that a similar concept applied to *bounded* plurisubharmonic functions produces sets $\mathfrak{c}^k(X)$, called *cores*, that do indeed depend on the regularity, as shown in [5].

In [11, 12], the second author, Slodkowski, and Tomassini showed that, if X has complex dimension 2 and Psh_e^∞ contains at least one real analytic function, the minimal kernel is either a union of countably many compact (and negative) curves or equal to the whole manifold, by giving a full classification of the possible structures that such a manifold can present. An important point here is that, although in general the minimal kernel does not have a priori an analytic structure, however its intersection with any level of a psh exhaustion function does (at least in dimension 2); such analytic structure implies that, under said hypotheses, the kernel does not depend on the regularity we consider.

In [23], Sibony introduced the notion of *Levi current* (see Definition 2.1), which is related to the (non-)existence of strictly psh functions on a complex manifold and thus to the problem of determining whether a given manifold is Stein, see also [17, 21, 22, 24]. Extremal Levi-currents are supported on sets where all continuous psh functions are constant. In the case of infinitesimally homogeneous manifolds, a foliation is constructed and linked to the obstructions to Steinness.

Our goal here is to compare these two approaches, and in particular to use the notion of Levi current on X to study the analytic structure of the minimal kernels Σ_X^k . In order to do this, let us denote by Psh^k , for $0 \leq k \leq \infty$, the cone of C^k psh function on X and define, for $k \geq 1$, the *distribution* $\mathcal{E}^k$ in TX as

$$(2) \quad \mathcal{E}^k := \{(x, v) \in TX : (d\varphi)_x(v) = 0 \ \forall \varphi \in \text{Psh}^k(X)\}.$$

A *distribution* is a subset of TX whose intersection with $T_x X$ is a (real) vector subspace of the latter for every $x \in X$. In general, $\mathcal{E}^k$ will not be a subbundle of TX , as $\dim \mathcal{E}_x^k$ is not constant; however it is a closed subset of TX , hence the function $x \mapsto \dim \mathcal{E}_x^k$ is upper semicontinuous.

The following is our main result.

Theorem 1.1. *Let X be a weakly complete complex manifold and T a Levi current on X . Denote by Σ_X^k the minimal kernels of X and by F the union of the supports of all Levi currents on X . Then $F \subseteq \Sigma_X^k$ for all $k \geq 0$ and*

- (1) *if T has compact support K_T , then K_T is a local maximum set;*
- (2) *if K is a local maximum set, then there exists a Levi current supported on K . In particular, $K \cap F \neq \emptyset$.*

Moreover, if X is a surface, $\phi \in \mathcal{C}^k$, $k \geq 2$ a psh exhaustion function, and Y a regular connected component of a level set $\{\phi = c\}$,

- (3) *if $4 \leq k \leq \infty$ and $U \subseteq \Sigma_X^k$ is an open set in Y and there exists $x \in U$ such that $\dim \mathcal{E}_x^k = 2$, then X is a union of compact complex curves. In particular, $F = \Sigma_X^k = X$;*
- (4) *if $2 \leq k \leq \infty$ and $Y \subseteq \Sigma_X^k$, then there exists $c' < c$ such that the connected component of $\phi^{-1}([c', c])$ containing Y is contained in F . In particular, $Y \subseteq F$.*

Moreover, given any psh function $u \in \text{Psh}^0(X)$, any Levi current T can be naturally disintegrated as $T = \int T_c d\mu$, where μ is a positive measure on $\mathbb{R}$ and T_c is a Levi current supported on $\{u = c\}$, see Corollary 2.4. This in particular gives examples of Levi currents for which Item (1) applies. Notice that, whenever two level sets of $u, v \in \text{Psh}^0(X)$ do not coincide, this allows to further refine the description of the extremal Levi current. This motivates the definition (2) of the distributions $\mathcal{E}^k$.

It follows from [11, 27] that, when $k \geq 2$, for any level set Y of an exhaustion function $\phi \in \text{Psh}_e^0$, the set $\Sigma_X^k \cap Y$ is a local maximum set (or empty), see Lemma 3.4. Hence Item (2) applies for instance to such sets. Finally, the manifold $X = \mathbb{C} \times \mathbb{P}^1$ provides an example where the Items (3) and (4) apply.

Remark 1.2. It would be interesting to know if the equality holds in Item (2) (and in particular for the intersections between levels sets of a psh exhaustion function and the minimal kernel). Namely if, for any point in a local maximum set K , there exists a Levi current T such that $x \in \text{spt } T$, where $\text{spt } T$ denotes the support of T .

The paper is organized as follows. In Section 2 we recall the definition of Levi currents and the properties that we will need in the sequel. In Section 3 we prove Items (1) and (2) of Theorem 1.1. The first item is established for $\Sigma_X^k \cap \{\phi_0 = c_0\}$ (where c_0 is an attained value for a continuous minimal function ϕ_0 and $k \geq 2$) in [27, Theorem 3.6], and is actually a consequence of [22, Theorem 3.1], where it is proved through an integration by parts, see also [23, Proposition 4.2] for an analogous statement for F . We give here a different proof by means of a characterization of the local maximum property due to Slodkowski [25] which allows to bypass the use of Bremermann functions and Jensen measures as in [27]. In Section 4 we study the relation between the minimal kernels Σ_X^k and distributions in the tangent bundle TX given by directions satisfying some degeneracy condition. This leads to the proof of Item (3). The proof of Theorem 1.1 is completed in Section 5, where we establish Item (4). In Section 6 we consider the case where X is a surface and the exhaustion function in Theorem 1.1 can be chosen analytic. By exploiting the main result in [11], we deduce a classification of Levi currents in this case.

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2. LEVI CURRENTS ON COMPLEX MANIFOLDS

In this section we recall the definition of Levi current and give the properties that we need in the sequel. These results are essentially contained in [23, Section 4] and [24, Section 3], we sketch here the proofs for completeness. We let X be any complex manifold and we denote by $\text{Psh}^0(X)$ the space of continuous plurisubharmonic functions on X .

Definition 2.1 (Sibony [23]). A current T on X is called *Levi current* if

- (1) T is non zero;
- (2) T is of bidimension $(1, 1)$;
- (3) T is positive;
- (4) $i\partial\bar{\partial}T = 0$;
- (5) $T \wedge i\partial\bar{\partial}u = 0$ for all $u \in \text{Psh}^0(X)$.

A Levi current T is *extremal* if $T = T_1 = T_2$ whenever $T = (T_1 + T_2)/2$ for T_1, T_2 Levi currents.

Remark 2.2. A positive current T has locally finite mass, so we can think of it as the datum of a (positive) measure μ (its *mass measure*) and, for almost every point in the support of such measure, a m -vector field χ , where m is the dimension of the current, such that

$$T(\omega) = \int_X \langle \chi, \omega \rangle d\mu \quad \text{for every smooth form } \omega \text{ of degree } m.$$

See for instance [3, p. 310]; the mass measure is sometimes denoted by $\|T\|$, if no confusion arises with the total mass of the current.

Lemma 2.3. Take $u \in \text{Psh}^0(X)$ and let T be a Levi current. The currents

$$T \wedge \partial u, \quad T \wedge \bar{\partial} u, \quad \text{and} \quad T \wedge \partial u \wedge \bar{\partial} u$$

are all well defined and vanish identically on X .

Proof. The currents in the statement are well defined when u is smooth, and the arguments from [2, Section 2] and [23, Section 4] prove the good definition for $u \in \text{Psh}^0(X)$.

If $u \in \text{Psh}^0(X)$, then also $\exp(u) \in \text{Psh}^0(X)$. So, by Definition 2.1 of Levi current, $T \wedge i\partial\bar{\partial}\exp(u)$ vanishes identically. Hence, we have

$$0 = \exp(u)T \wedge i\partial\bar{\partial}u + i\exp(u)T \wedge \partial u \wedge \bar{\partial}u.$$

Given that $T \wedge i\partial\bar{\partial}u = 0$, we conclude that $T \wedge \partial u \wedge \bar{\partial}u = 0$. This gives the last identity. We prove now the first one, the proof for the second one is similar. Since T is positive, for any $(0,1)$ -form α by Cauchy-Schwarz's inequality we get

$$|\langle T, \partial u \wedge \alpha \rangle|^2 \leq c \langle T, \partial u \wedge \bar{\partial}u \rangle \cdot \langle T, \alpha \wedge \bar{\alpha} \rangle$$

where c is a constant independent of T, u , and α . Since the first factor in the RHS is zero by the first part of the proof, the assertion follows. $\square$

By a standard disintegration procedure, we obtain the following consequence.

Corollary 2.4. *Suppose T is a Levi current and $u \in \text{Psh}^0(X)$; then there exists a measure μ on $\mathbb{R}$ and a collection of currents T_c , $c \in \mathbb{R}$ such that*

- T_c is supported on $Y_c = \{x \in X : u(x) = c\}$ for all $c \in \mathbb{R}$;
- T_c is non zero for μ -almost every $c \in \mathbb{R}$;
- whenever $T_c \neq 0$, T_c is a Levi current;
- for every 2-form α on X we have

$$\langle T, \alpha \rangle = \int_{\mathbb{R}} \langle T_c, \alpha \rangle d\mu(c).$$

Moreover, if $u \in \text{Psh}^1(X)$ and c is a regular value for u , then $T_c = j_*S_c$, where j is the inclusion of Y_c in X and S_c a current on the real manifold Y_c .

Notice that T_c needs not be extremal, as it is easily seen considering $X = \mathbb{C} \times \mathbb{P}^1$.

Remark 2.5. Suppose now that X is weakly complete and let ϕ be a psh exhaustion function. By Corollary 2.4, every Levi current is obtained by averaging Levi currents which are supported on the level sets of ϕ ; as the latter is an exhaustion of X , its level sets are compact, so every Levi current on X is an integral average of compactly supported Levi currents, i.e., positive currents of bidimension $(1,1)$ which are $\partial\bar{\partial}$ -closed and compactly supported.

Corollary 2.6. *If T is a Levi current and $u \in \text{Psh}^1(X)$, then the 2-vector field associated to T belongs to the kernel of $\partial u \wedge \bar{\partial}u$ ($\|T\|$ -almost everywhere), whenever the latter is non-zero. Moreover, if there exists $v \in \text{Psh}^0(X)$ which is strictly plurisubharmonic at a point $x \in X$, then $x \notin \text{spt } T$ for any Levi current T .*

Proof. Recall that the vector field is associated to T as in Remark 2.2. The first statement is then equivalent to $T \wedge \partial u \wedge \bar{\partial}u = 0$, hence follows from Lemma 2.3. Suppose now that we have $v \in \text{Psh}^0(X)$ which is strictly psh at x and a Levi current T . First, by Richberg [18, Satz 4.3] we can assume that v is $\mathcal{C}^\infty$ and strictly psh near x . Then, if $\rho \in \mathcal{C}_0^\infty(X)$ is supported in a neighbourhood of x where v is strictly psh and $\|\rho\|_{\mathcal{C}^2}$ is small enough, then also $v + \rho$ is psh. In particular, we can choose ρ_i , $1 \leq i \leq 2n$ such that the $\ker(\partial(v + \rho_i) \wedge \bar{\partial}(v + \rho_i))_x$ are independent (over $\mathbb{R}$). This property holds true in a neighbourhood of x . Hence, as the vector field associated to T (on a full measure subset of the support T for the mass measure) should belong to all these subspaces, the only possibility is that $x \notin \text{spt } T$. This concludes the proof. $\square$

Lemma 2.7. *Let T be a Levi current such that $K = \text{spt } T$ is compact. If u is defined and plurisubharmonic in an open neighbourhood V of K and strictly plurisubharmonic at $x \in V$, then $x \notin \text{spt } T$.*

Proof. Let $V' \Subset V$ be an open neighbourhood of K containing x . Let $\chi \in \mathcal{C}_0^\infty(V)$ be such that $\chi|_{V'} \equiv 1$, then χu is defined on X and psh on V' . As $\text{spt } T \subseteq V'$, also $\text{spt}(T \wedge i\partial\bar{\partial}u) \subseteq V'$, so $T \wedge i\partial\bar{\partial}u$ is positive; moreover, as T is a Levi current, we have $i\partial\bar{\partial}T = 0$, hence

$$0 = \langle i\partial\bar{\partial}T, u \rangle = \langle T, i\partial\bar{\partial}u \rangle = \langle T \wedge i\partial\bar{\partial}u, 1 \rangle.$$

Therefore, $T \wedge i\partial\bar{\partial}u = 0$ as a (positive) measure.

Since u is strictly psh at x , this happens in a neighbourhood of x , so $T \wedge i\partial\bar{\partial}u$ is strictly positive in a neighbourhood of x unless T is zero there. This gives $x \notin \text{spt } T$ and concludes the proof. $\square$

Lemma 2.8. *Suppose that a current T satisfies requests 1 – 4 of Definition 2.1 and T has compact support. Then T is a Levi current.*

Proof. Given that T is compactly supported, so are uT , $T \wedge \partial u$, $T \wedge \bar{\partial}u$, and $T \wedge i\partial\bar{\partial}u$ for all $u \in \text{Psh}^0(X)$. Moreover, as T is positive and u is psh, $T \wedge i\partial\bar{\partial}u$ is non-negative measure on X ; therefore, it is zero if and only if $\langle T \wedge i\partial\bar{\partial}u, 1 \rangle = 0$.

Notice that, by Stokes' theorem, we have $\langle i\partial\bar{\partial}(uT), 1 \rangle = 0$, hence

$$0 = -\langle i\partial\bar{\partial}u \wedge T, 1 \rangle + \langle ui\partial\bar{\partial}T, 1 \rangle + \langle i\partial(\bar{\partial}u \wedge T), 1 \rangle - \langle i\bar{\partial}(\partial u \wedge T), 1 \rangle.$$

We have $i\partial\bar{\partial}T = 0$ by hypothesis, while $\langle i\partial(\bar{\partial}u \wedge T), 1 \rangle$ and $\langle i\bar{\partial}(\partial u \wedge T), 1 \rangle$ vanish by another application of Stokes' theorem. Therefore $T \wedge i\partial\bar{\partial}u = 0$, that is, T is a Levi current. $\square$

3. LOCAL MAXIMUM SETS

We establish here Items (1) and (2) of Theorem 1.1. We recall the following definition, see also [27, Section 2] and [19].

Definition 3.1. Let X be a complex manifold and $K \subset X$ be locally closed. We say that K is a *local maximum set* if every $x \in K$ has a neighbourhood U with the following property: for every compact set $K' \subset U$ and every function ψ which is psh in a neighbourhood of K' , we have

$$\max_{K \cap K'} \psi = \max_{K \cap bK'} \psi.$$

Proposition 3.2. *Suppose that X is weakly complete. If T is a Levi current with compact support, then $\text{spt } T$ is a local maximum set.*

Proof. Suppose that $K := \text{spt } T$ is not a local maximum set. By [25, Proposition 2.3] there exist $x \in K$, a neighbourhood B of x , with local coordinates z with origin in x , such that $B \equiv \{z \in \mathbb{C}^n : \|z\| < 1\}$, and $\psi : B \rightarrow \mathbb{R}$ strictly psh with $\psi(x) = 0$ and $\psi(y) \leq -\epsilon\|z(y)\|^2$ for all $y \in K \cap B$. Up to replacing ψ by an element of a continuous approximating sequence, we can directly assume that ψ is continuous. By taking a possibly smaller ball at x , we can also assume that $-\epsilon\|z(y)\|^2 - \epsilon/8 \leq \psi(y)$ on $K \cap B$. Set

$$A := \{y \in B : \|z(y)\|^2 > 3/4\} \quad \text{and} \quad V = \{y \in B : |\psi(y) + \epsilon\|z(y)\|^2| < \epsilon/4\}.$$

By the continuity of ψ and the bounds above, V is an open subset of B containing $K \cap B$. Consider $u \in \text{Psh}^0(X)$ such that $u(x) = -\epsilon/4$ and $\sup_B |u| < \epsilon/2$ (this function exists because of the assumption on X). Since $K \cap B \subset V$, there also exists $\chi \in \mathcal{C}_0^\infty(X)$ such that $\chi|_K \equiv 1$ and $\text{spt } \chi \cap B \subset V$. Define the function $v: X \rightarrow \mathbb{R}$ as

$$v = \begin{cases} \chi \max\{u, \psi\} & \text{on } B, \\ \chi u & \text{on } X \setminus B. \end{cases}$$

We claim that $v = \chi u$ on A . Observe that we only need to prove the equality for points in the support of χ . For every $p \in \text{spt } \chi \cap A$, we have that $p \in V$, and so $\psi(p) < -3\epsilon/4 + \epsilon/4 = -\epsilon/2$. Hence, $\psi(p) < u(p)$ and $v(p) = (\chi u)(p)$.

By construction, v coincides with ψ in a neighbourhood of x . It follows that v is psh in a neighbourhood of K and strictly psh in x . Therefore, we have $x \notin \text{spt } T$ by Lemma 2.7. This gives a contradiction with the choice of $x \in K$ and completes the proof. $\square$

Proposition 3.3. *Let $K \subset X$ be a local maximum compact set. There exists a Levi current T such that $\text{spt } T \subseteq K$.*

Proof. By [24, Theorem 3.1] (see also [23, Section 4]) and Lemma 2.8, if there are no Levi currents supported on K , there exists a smooth strictly psh function u on some open neighbourhood U of K . By slightly perturbing u , for every $x_0 \in K$ we can construct $2n$ continuous strictly psh functions on some neighbourhood $K \subset U' \subseteq U$ such that $du_1, \dots, du_{2n}$ are linearly independent at x_0 . This implies that, in a neighbourhood of x_0 , we have

$$\{u_1 = u_1(x_0)\} \cap \dots \cap \{u_{2n} = u_{2n}(x_0)\} = \{x_0\}.$$

By [26, Corollary 1.11] and [25, Theorem 4.2], for every family of continuous psh functions on U' there exists a local maximum set $K' \subseteq K$ with the property that all functions of the family are constant on K' . Choosing a point $x_0 \in K'$, the previous paragraph gives that x_0 is isolated in K' . This is a contradiction, and the proof is complete. $\square$

We conclude the section with the following result, that we will need to prove Item (3) of Theorem 1.1.

Lemma 3.4. *Let X be a weakly complete complex surface and Y a regular level for a C^1 psh exhaustion function ϕ . Then, for all $k \geq 2$, $\Sigma_X^k \cap Y$ is a local maximum set and, for all local maximum sets $K \subseteq \Sigma_X^k$, K is foliated by holomorphic discs, i.e., it is locally a union of disjoint holomorphic discs.*

Proof. The intersection $\Sigma_X^k \cap Y$ is a local maximum set by [11, Theorem 3.2]. As observed in [11, Proposition 3.5], the proof of the lemma is then essentially given in [27, Lemma 4.1], see in particular the Assertion in the proof of that lemma. $\square$

We point out that [27, Lemma 4.1] relies on a result by Shcherbina [20], which holds true only in dimension 2; this is the reason for restricting ourselves to the case of surfaces. It would be interesting to prove (or disprove) a similar statement in higher dimension.

4. KERNELS AND TANGENT DIRECTIONS

In this section we let X be a weakly complete complex manifold of dimension n and assume that $\text{Psh}^k(X)$ contains at least one exhaustion function $\phi : X \rightarrow \mathbb{R}$ for some $2 \leq k \leq \infty$. Recall that the minimal kernel Σ_X^k of X is defined as in (1) and the distribution $\mathcal{E}^k$ of TX as in (2). We consider further the distribution $\mathcal{S}^k$ of TX given by

$$(3) \quad \mathcal{S}^k := \{(x, \xi) \in TX : \xi \in T_x X, (i\partial\bar{\partial}u)_x(\xi, \xi) = 0 \forall u \in \text{Psh}^k(X)\}.$$

Similar objects have already appeared in relation to the study of the Levi problem, see for instance [7] in the case of homogeneous manifolds and [6, 27]. We also set

$$E_\ell^k := \{x \in X : \dim \mathcal{E}_x^k \geq \ell\} \quad \text{and} \quad S_1^k := \{x \in X : \dim \mathcal{S}_x^k \geq 1\}.$$

By definition, $E_\ell^k \subseteq E_{\ell-1}^k$ and E_ℓ^k is closed in $E_{\ell-1}^k$ for all $\ell \geq 1$. Observe moreover that $\mathcal{S}^k$ is a *complex* distribution.

Remark 4.1. Let T be a Levi current. Following Remark 2.2, we write T in terms of its mass measure and a vector-field, defined $\|T\|$ -almost everywhere. A Fubini integration shows that, for almost every point of the support of T with respect to the mass measure, the vector field associated to T at x belongs to the fibre $\mathcal{S}_x^k$ of $\mathcal{S}^k$ at x .

Proposition 4.2. *We have $\mathcal{S}^k \subseteq \mathcal{E}^k$, and $S_1^k = E_2^k = E_1^k = \Sigma_X^k$.*

Proof. It follows from the definition of $\mathcal{E}^k$ that $E_1^k = E_2^k$. Moreover, if $(x, v) \notin \mathcal{E}^k$, there exists $\psi \in \text{Psh}^k(X)$ such that $(d\psi)_x(v) \neq 0$; then

$$i\partial\bar{\partial}\exp(\psi) = \exp(\psi)i\partial\bar{\partial}\psi + i\exp(\psi)\partial\psi \wedge \bar{\partial}\psi \geq 0,$$

which implies that $(i\partial\bar{\partial}\exp(\psi))_x(v, v) \geq \exp(\psi(x))|\partial\psi_x(v)|^2 > 0$ and so $(x, v) \notin \mathcal{S}^k$. It follows that $\mathcal{S}^k \subseteq \mathcal{E}^k$.

We now prove that $E_1^k = \Sigma_X^k$. If $x \notin \Sigma_X^k$, then there exists $\psi \in \text{Psh}_e^k(X)$ which is strictly psh around x ; therefore, given any $\rho : X \rightarrow \mathbb{R}$ smooth with compact support near x , there exists $\epsilon > 0$ such that $\psi + \epsilon\rho$ is still psh. So, we can construct psh functions of class $\mathcal{C}^k$ whose differentials span the tangent space at x , which implies that these differentials do not have any nontrivial common kernel in $T_x X$. So $\mathcal{E}_x^k = \{0\}$, hence $E_1^k \subseteq \Sigma_X^k$.

On the other hand, if $\mathcal{E}_x^k = \{0\}$, we can find $2n$ linearly independent vectors $v_1, \dots, v_{2n} \in T_x X$ and $2n$ positive psh functions $\psi_1, \dots, \psi_{2n}$ of class $\mathcal{C}^k$ such that $d\psi_i(v_j) = \delta_{ij}$ for all $0 \leq i, j \leq 2n$. Let us define the function $\psi := \sum_{j=1}^{2n} \psi_j^2$. In order to conclude, it is enough to prove that ψ is strictly psh at x . Indeed, adding to the exhaustion function ϕ a suitable multiple of ψ , this implies that $x \notin \Sigma_X^k$. This gives $E_1^k \supseteq \Sigma_X^k$, hence $E_1^k = \Sigma_X^k$ because of the previous part.

In order to prove that ψ is strictly psh at x , we prove that the bilinear form given by $i\partial\bar{\partial}\psi$ is positive definite. By definition, we have

$$i\partial\bar{\partial}\psi = \sum_{j=1}^{2n} \psi_j i\partial\bar{\partial}\psi_j + \sum_{j=1}^{2n} i\partial\psi_j \wedge \bar{\partial}\psi_j.$$

The first sum in the right hand side of the above expression is non negative since the ψ_j 's are positive and psh. Setting $\Omega := \sum_{j=1}^{2n} i\partial\psi_j \wedge \bar{\partial}\psi_j$, it is then enough to check

that

$$\begin{cases} \Omega(v_j, v_j) > 0 & \text{for all } 1 \leq j \leq 2n; \\ \Omega(v_i, v_j) = 0 & \text{for all } 1 \leq i \neq j \leq 2n. \end{cases}$$

The fact that ψ_j is real implies that for all j

$$\partial\psi_j \wedge \bar{\partial}\psi_j = \partial\psi_j \wedge \bar{\partial}\bar{\psi}_j = \partial\psi_j \wedge \overline{\partial\psi_j}.$$

By the definition of $\{\psi_j\}$ and $\{v_j\}$, this implies that $(\partial\psi_j \wedge \bar{\partial}\psi_j)(v_j, v_j) > 0$ for all $1 \leq j \leq 2n$ and $(\partial\psi_j \wedge \bar{\partial}\psi_j)(v_i, v_l) = 0$ as soon as one among i and l is different from j . This implies the claim on Ω , and concludes the proof of the equality $E_1^k = \Sigma_X^k$.

In order to conclude, we need to prove that $S_1^k \supseteq \Sigma_X^k$. Take $x \in \Sigma_X^k$ and suppose by contradiction that, for every $v \in T_x X$ there is $\varphi_v : X \rightarrow \mathbb{R}$ which is $\mathcal{C}^k$, psh, and such that $(dd^c \varphi_v)_x(v, v) > 0$. Then, as above, we can construct a $\mathcal{C}^k$ function ψ which is strictly psh at x . This gives the desired contradiction and completes the proof. $\square$

The following result gives Item (3) of our main Theorem 1.1.

Proposition 4.3. *Let X be a weakly complete complex surface, ϕ a $\mathcal{C}^k$, $4 \leq k \leq \infty$, exhaustion psh function and Y a regular connected component of a level set $\{\phi = c\}$ of ϕ . Suppose that $U \subseteq Y$ is an open set in Y and $U \subseteq \Sigma_X^k$. If there exists $x \in U$ such that $\dim \mathcal{E}_x^k = 2$, then X is a union of compact complex curves. In particular, $\Sigma_X^k = F = X = E_2^k$, and E_3^k is contained in a (possibly empty) analytic subset of the singular levels for ϕ .*

We will need the following theorem by Nishino, see [16, Proposition 9 and Théorème II] and [13, Section 2.2.1].

Theorem 4.4 (Nishino). *Let X be a weakly complete surface that contains an uncountable family $\mathcal{F}$ of disjoint connected compact complex curves. Then there exist a Riemann surface R and a holomorphic map $h : X \rightarrow R$ with compact fibers.*

Proof of Proposition 4.3. By Theorem 4.4, to prove the first assertion it is enough to show that X contains uncountably many disjoint compact complex curves.

Since $x \in U$ is such that $\dim \mathcal{E}_x^k = 2$, a complement of $\mathcal{E}_x^k$ in $T_x X$ has real dimension 2, so there exists $\psi \in \text{Psh}_e^k(X)$ such that $(d\phi)_x$ and $(d\psi)_x$ are linearly independent; hence, the map $\psi|_Y : Y \rightarrow \mathbb{R}$ is not constant. Since $k \geq 4$, by Sard's theorem we can find regular values b for ψ arbitrarily close to $b_0 := \psi(x)$, therefore the sets $C_b = \{y \in Y : \psi(y) = b\}$ intersect the open set $U \subseteq \Sigma_X^k$.

For any $y \in C_b \cap \Sigma_X^k$, by Proposition 4.2 we have $T_y C_b = \mathcal{E}_y^k = \mathcal{S}_y^k$. Therefore, $C_b \cap \Sigma_X^k$ is a complex curve, being a real, smooth 2-dimensional manifold with complex tangent space. On the other hand, the set $C_b \setminus \Sigma_X^k$ is open in C_b . Let $z \in C_b$ be a boundary point (with respect to C_b); as $z \in \Sigma_X^k$, by Lemma 3.4 there is a holomorphic disc $f : \mathbb{D} \rightarrow X$ such that $f(\mathbb{D}) \subset Y$ and $f(0) = z$. If $\zeta \in \mathbb{D}$ is close enough to 0, then, setting $w = f(\zeta)$, we have $w \in \Sigma_X^k$, and $(d\phi)_w$ and $(d\psi)_w$ independent. This gives $w \in E_2^k \setminus E_3^k$, which in turn implies that $\mathcal{E}_w^k = T_w C_{\psi(w)}$. Therefore $f(\mathbb{D})$ coincides locally with a leaf $C_{b'}$. Hence C_b is contained in Σ_X^k , so it is a compact complex curve.

As b was taken arbitrarily among the regular values close enough to b_0 , we find uncountably many disjoint (since they correspond to distinct values) compact complex curves in X , as desired.

In order to conclude, we need to prove the final assertion on E_3^k . We proved above that there exists a holomorphic map $h: X \rightarrow R$ with compact fibres, where R is a Riemann surface; in fact, we can suppose that (h, R) is the Remmert reduction of X . It is then enough to prove that $E_3^k \subseteq \{h' = 0\}$: every psh function of X is of the form $\psi \circ h$ where $\psi: R \rightarrow \mathbb{R}$ is a psh function on R , so, the singular levels of h are contained in the singular levels of ϕ .

Let $x \in X$ be such that $h'(x) \neq 0$. Consider a strictly psh exhaustion function ψ for R (which we can assume to be C^∞ near x by [18]) and the family of functions $\mathcal{F} := \{\psi + \epsilon\rho\}$, where ρ is a smooth function compactly supported near $h(x)$. For every such ρ , $\psi + \epsilon\rho$ is still strictly psh for ϵ sufficiently small. Thus, we can obtain a set of generators for the tangent space given by differentials at $h(x)$ of psh functions in $\mathcal{F}$. Pre-composing the corresponding functions with h , we obtain that the space of differentials at x of psh functions on X has dimension at least 2. Hence, $x \notin E_3^k$, and the proof is complete. $\square$

Remark 4.5. Suppose that X is a surface and Y a regular level for an exhaustion function $\phi \in \text{Psh}_e^0(X)$. Let $K \subseteq Y \cap \Sigma_X$ be a local maximum set. By Lemma 3.4, Y and K are foliated by holomorphic discs. For every such disk, its tangent bundle is exactly the restriction of $\mathcal{S}$. By [1, Theorem 1.4], there exists a $\partial\bar{\partial}$ -closed positive current of bidimension $(1,1)$, directed by $\mathcal{S}$, supported in K . By Lemma 2.8, such current is a Levi current. This gives a different proof of Item (2) when $\dim X = 2$.

5. END OF THE PROOF OF THEOREM 1.1

It follows from Corollary 2.6 (or Lemma 2.7) that $\text{spt } T \subseteq \Sigma_X^k$ for every Levi current T and all $k \geq 0$. Thus, we have $F \subseteq \Sigma_X^k$ for all $k \geq 0$. Moreover, Items (1), (2), and (3) follow from Propositions 3.2, 3.3, and 4.3, respectively.

Let now Y be a regular connected component of a level set for an exhaustion psh function $\phi \in \text{Psh}_e^k(X)$ for some $k \geq 2$. The remaining item follows from the next proposition.

Proposition 5.1. *If $k \geq 2$ and $Y \subseteq \Sigma_X^k$, there exists $c' < c$ such that the connected component of $\phi^{-1}([c', c])$ containing Y is contained in F .*

Proof. We assume for simplicity that the level $\{\phi = c\}$ is regular and connected, the argument is similar otherwise. Since $k \geq 2$, by [27, Theorem 3.9] there is $c' < c$ such that, setting

$$K = \{x \in X : c' \leq \phi(x) \leq c\},$$

the form $(dd^c\phi)^2$ vanishes on the interior of K , hence on K . So, we have $K \subseteq \Sigma_X^k$.

Consider the current T given by

$$T := i\partial\phi \wedge \bar{\partial}\phi.$$

It is clear that T is a current of bidimension $(1,1)$, positive and directed by the complex subspace of the tangent of the levels of ϕ . Moreover, $i\partial\bar{\partial}T$ is induced by the form

$$i\partial\bar{\partial}(i\partial\phi \wedge \bar{\partial}\phi) = -(\partial\bar{\partial}\phi)^2.$$

So, $i\partial\bar{\partial}T$ vanishes where ϕ is not strictly psh, hence on Σ_X^k . Let B be the interior of K , then the restriction of T to B is a current of bidimension $(1,1)$, positive, $\partial\bar{\partial}$ -closed (in

B); moreover, given $u \in \text{Psh}^0(X)$, we have that $T \wedge \partial\bar{\partial}u = 0$ on Σ_X^k , so the restriction of T to B is a Levi current.

By construction and Lemma 2.3 we have $T \wedge \partial\phi = 0$, so we can disintegrate T along the levels of ϕ , see Corollary 2.4: there exist currents T_s with $s \in (c', c)$, such that, for α a 2-form with $\text{spt } \alpha \subset B$,

$$\langle T, \alpha \rangle = \int_{c'}^c \langle T_s, \alpha \rangle d\mu(s) \quad \text{for some measure } \mu \text{ on } (c', c).$$

Since $\phi \in \mathcal{C}^2$, the measure μ is absolutely continuous with respect to the Lebesgue measure on (c', c) .

As T is $\partial\bar{\partial}$ -closed in B , so is μ -almost every T_s in B ; therefore, for a dense open set of $s \in (c', c)$, T_s is a positive, $\partial\bar{\partial}$ -closed current of bidimension $(1, 1)$ and

$$\text{spt } T_s = \{x \in X : \phi(x) = s\}.$$

The set in the RHS is compact since ϕ is an exhaustion function. By Lemma 2.8, T_s is a Levi current.

In conclusion, the level set $\{x \in X : \phi(x) = s\}$ is contained in F for almost all $s \in (c', c)$, so $\phi^{-1}([c', c]) \subseteq F$, as F is closed (see [23, Proposition 4.2]). In particular, $Y \subseteq F$. $\square$

The proof of Theorem 1.1 is complete.

6. REAL ANALYTIC EXHAUSTION FUNCTION

A classification of those weakly complete complex surfaces X admitting an analytic exhaustion function is given in [11]. As a direct consequence, we can get an analogous complete classification of the possible Levi currents in this setting.

First notice that each exceptional divisor V in X corresponds to an extremal Levi current given by the current of integration $[V]$. Without loss of generality, to simplify our next statement, we can thus assume that X has no such divisors on the regular levels of α . The statement for a general X is then a direct consequence.

Theorem 6.1. *Let X be a weakly complete complex surface admitting an analytic exhaustion function α . Assume that X has no exceptional divisors on the regular levels of α . Then one of the following possibilities hold:*

- (1) X is Stein (and so, admits no Levi currents);
- (2) $F = \Sigma_X^\infty = X = \cup V_i$, where all the V_i are (disjoint) connected compact curves, and all extremal Levi currents are of the form $\lambda[V_i']$ for some positive λ , with V_i' an irreducible component of some V_i ;
- (3) $F = \Sigma_X^\infty = X$, every regular level Y_c of α is foliated by curves U_i , and the support of any extremal Levi currents on Y_c is equal to (a connected component of) Y_c .

Observe also that, although a priori we would only have $\Sigma_X^k \subseteq \Sigma_X^\infty$ for all $k \geq 0$, the above geometric description implies that $\Sigma_X^k = \Sigma_X^\infty$ for all $k \geq 0$.

Proof. It follows from [11, Theorem 1.1] that one of the following possibilities holds:

- (1) X is a Stein manifold;
- (2) X is proper over a (possibly singular) complex curve;

- (3) the connected components of the regular levels of α are foliated with dense complex curves.

In the first and second cases, the assertion follows directly. In the third case, a Levi current can be constructed, for instance, by means of [1, Theorem 1.4]. By proposition 3.2, the support of any Levi current is a local maximum set. By [9, Lemma 3.3], a local maximum set contained in a Levi-flat hypersurface must be a union of leaves of the Levi foliation.

Hence, in the third case, any Levi current on a regular level set of the exhaustion function is supported on the whole level set, as all the leaves of the Levi foliation are dense. This in particular applies to extremal Levi currents. The proof is complete. $\square$

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