

THE ω -BOREL INVARIANT FOR REPRESENTATIONS INTO $SL(n, \mathbb{C}_\omega)$

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ABSTRACT. Let Γ be the fundamental group of a complete hyperbolic 3-manifold M with toric cusps. By following [BBI] we define the ω -Borel invariant $\beta_n^\omega(\rho_\omega)$ associated to a representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$, where \mathbb{C}_ω is a field introduced by [Par12] which can be constructed as a quotient of a suitable subset of $\mathbb{C}^{\mathbb{N}}$ with the data of a non-principal ultrafilter ω on \mathbb{N} and a real divergent sequence λ_l such that $\lambda_l \geq 1$.

Since a sequence of ω -bounded representations ρ_l into $SL(n, \mathbb{C})$ determines a representation ρ_ω into $SL(n, \mathbb{C}_\omega)$, for $n = 2$ we study the relation between the invariant $\beta_2^\omega(\rho_\omega)$ and the sequence of Borel invariants $\beta_2(\rho_l)$. We conclude by showing that if a sequence of representations $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ induces a representation $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$ which determines a reducible action on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ with non-trivial length function, then it holds $\beta_2^\omega(\rho_\omega) = 0$.

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1. INTRODUCTION

Given a finitely generated group Γ , the character variety $X(\Gamma, SL(n, \mathbb{C}))$ is an algebraic variety obtained as *GIT*-quotient of the representation variety $R(\Gamma, SL(n, \mathbb{C}))$ by the conjugation action of $SL(n, \mathbb{C})$. When Γ is the fundamental group of a complete hyperbolic 3-dimensional manifold M with toric cusps, it is possible to attach to every equivalence class of representations a suitable invariant called Borel invariant. Indeed, in [BBI] the authors prove that the Borel class $\beta(n)$, already introduced and studied in [Gon93], is a generator for the cohomology group $H_{cb}^3(PSL(n, \mathbb{C}))$. Thus, given a representation $\rho : \Gamma \rightarrow PSL(n, \mathbb{C})$, we can construct a class into $H_b^3(\Gamma)$ by pulling back $\beta(n)$ along ρ_b^* and then evaluate this new class on a fundamental class $[N, \partial N] \in H^3(N, \partial N)$. Here N is a compact core of M . When $n = 2$ this invariant is exactly

the volume of the representation defined as the integral of the pullback of the standard volume form $\omega_{\mathbb{H}^3}$ along any pseudo-developing map D , as written both in [Dun99] and in [Fra04] (see for instance [Kim16] for a proof of the equivalence). The Borel invariant of a representation $\rho : \Gamma \rightarrow SL(n, \mathbb{C})$ will be the Borel invariant of the induced representation into $PSL(n, \mathbb{C})$. Moreover, since this invariant remains unchanged under conjugation, we have a well-defined function on the character variety $X(\Gamma, SL(n, \mathbb{C}))$, called Borel function, which is continuous with respect to the topology of the pointwise convergence.

Inspired by the work of Thurston about the compactification of the Teichmüller space for a closed surface of genus g exposed in [Th88] and generalizing the constructions for algebraic curves appeared in [CS83], in [MS84] J. Morgan and P. Shalen proposed a new way to compactify a generic algebraic variety V given a generating set \mathcal{F} for the algebra of regular functions $\mathbb{C}[V]$. This particular method applied to the character variety $X(\Gamma, SL(2, \mathbb{C}))$ allows to interpret the ideal points of the compactification as projective length functions of isometric Γ -actions on real trees which are constructed as Bass–Serre trees associated to $SL(2, \mathbb{K}_v)$, where \mathbb{K}_v is a suitable valued field (see [Ser80]). A more geometric approach based on Gromov–Hausdorff convergence was suggested by both [Bes88] and [Pau88]. Lately [Par12] extended this interpretation to the more general case of $X(\Gamma, SL(n, \mathbb{C}))$ by viewing an ideal point as a projective vectorial length function relative to an isometric action, this time on a Euclidean building of type A_{n-1} . The method suggested by [Par12] to obtain the Euclidean building and its isometric Γ -action is based on asymptotic cones and it reminds the ones already exposed both in [Bes88] and in [Pau88].

In the attempt to link all these ideas, one could naturally ask if it is possible to extend continuously the Borel function to the ideal points of the compactification of $X(\Gamma, SL(n, \mathbb{C}))$. Going further, one could be interested in studying the possible values attained at ideal points and trying to formulate a rigidity result, which would generalize [BBI, Theorem 1].

The aim of this paper is to make a small step towards this direction by defining a numerical invariant, the ω -Borel invariant, associated to a representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$, where \mathbb{C}_ω is a field obtained as a quotient of a suitable subset of $\mathbb{C}^{\mathbb{N}}$ by an equivalence relation which depends on a non-principal ultrafilter ω on \mathbb{N} and a real divergent sequence λ_l with $\lambda_l \geq 1$. The motivation of this definition relies on the interpretation of the limit action of Γ on the Euclidean building of type A_{n-1} as a representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$, as proved in [Par12, Theorem 5.2].

The first section is dedicated to preliminary definitions, in particular we recall the definition of the field \mathbb{C}_ω and the notion of bounded cohomology of locally compact groups. In the second section we give

the definition of the ω -Borel cohomology class $\beta^\omega(n)$ which will be an element of $H_b^3(SL^\delta(n, \mathbb{C}_\omega))$. In the last section we define the ω -Borel invariant $\beta_n^\omega(\rho_\omega)$ for a representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$ and we describe some of its properties. In particular we focus our attention on the case $n = 2$. We show that if a sequence of representations $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ induces a representation $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$ which determines a reducible action on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ with non-trivial length function, then it holds $\beta_2^\omega(\rho_\omega) = 0$.

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2. PRELIMINARY DEFINITIONS

2.1. The field \mathbb{C}_ω . For more details regarding the definitions and the results contained in this section we refer to [Par12, Section 3.3]. We start by recalling the notion of ultrafilter and some fundamental properties that we are going to exploit lately.

Definition 2.1. An *ultrafilter* ω on a set X is a family of subsets of X which satisfies:

- The empty set is not contained in ω , that is $\emptyset \notin \omega$.
- If $A \subset B$ and $A \in \omega$, then $B \in \omega$.
- Given a collection A_1, \dots, A_n such that $A_i \in \omega$ for every $i = 1, \dots, n$, then $A_1 \cap \dots \cap A_n \in \omega$.
- Given A_1, \dots, A_n such that $A_1 \sqcup \dots \sqcup A_n = X$, there exists exactly one $i_0 \in \{1, \dots, n\}$ so that $A_{i_0} \in \omega$.

An ultrafilter is *principal* and centered at $x \in X$ if for every set $A \in \omega$ it holds $x \in A$. Otherwise we say that the ultrafilter is *non-principal*.

The importance of ultrafilters relies on their power to force convergence of sequences of points in a topological space X by selecting a suitable limit point. For the sake of clarity we first need to introduce the following

Definition 2.2. Let X be a topological space and let $(x_k)_{k \in \mathbb{N}}$ be a sequence of points in X . Fix an ultrafilter ω on the set of natural numbers \mathbb{N} . We say that the sequence ω -converges to x_0 if for every open neighborhood U of x_0 we have $\{k \in \mathbb{N} : x_k \in U\} \in \omega$.

A priori a sequence may admit no limit or several limits if the topology of the space X does not have good properties. To guarantee the existence and the uniqueness of the limit we need a compact Hausdorff space. Indeed, it holds

Proposition 2.3. *Let X be a topological space which is compact and Hausdorff. Then, for any ultrafilter ω on \mathbb{N} and any sequence $(x_k)_{k \in \mathbb{N}}$ of points in X , there exists a unique point $x_0 \in X$ such that*

$$\omega\text{-}\lim_{k \rightarrow \infty} x_k = x_0.$$

Another remarkable property of ultrafilters is the compatibility with continuous functions between topological spaces.

Proposition 2.4. *Let $f : X \rightarrow Y$ be a continuous function between two compact Hausdorff spaces. Let ω be an ultrafilter on \mathbb{N} . For any sequence $(x_k)_{k \in \mathbb{N}}$ of points in X we have*

$$\omega\text{-}\lim_{k \rightarrow \infty} f(x_k) = f(\omega\text{-}\lim_{k \rightarrow \infty} x_k).$$

We are now ready to describe the construction of the field \mathbb{C}_ω . Let ω be a non-principal ultrafilter on \mathbb{N} and let $(\lambda_k)_{k \in \mathbb{N}}$ be a real sequence that diverges to infinity and such that $\lambda_k \geq 1$ for every k . We define

$$\mathbb{C}_\omega = \{(a_k) \in \mathbb{C}^{\mathbb{N}} \mid \exists C > 0, \forall k \ |a_k|^{\frac{1}{\lambda_k}} < C\} / \sim_\omega$$

where $(a_k)_{k \in \mathbb{N}} \sim_\omega (b_k)_{k \in \mathbb{N}}$ if and only if $\omega\text{-}\lim_{k \rightarrow \infty} |a_k - b_k|^{\frac{1}{\lambda_k}} = 0$. It is easy to verify that the operations of pointwise sum and pointwise multiplication defined over $\mathbb{C}^{\mathbb{N}}$ are compatible with the equivalence relation \sim_ω . Thus they define two operations of sum and multiplication over \mathbb{C}_ω , which make \mathbb{C}_ω a field. There is a natural field embedding of \mathbb{C} into \mathbb{C}_ω given by the constant sequences.

If we denote by a_ω the equivalence class $[(a_k)]$ of the sequence $(a_k)_{k \in \mathbb{N}}$, the function

$$|a_\omega|^\omega := \omega\text{-}\lim_{k \rightarrow \infty} |a_k|^{\frac{1}{\lambda_k}}$$

is an ultrametric absolute value on \mathbb{C}_ω , that is it satisfies

$$|a_\omega + b_\omega|^\omega \leq \max\{|a_\omega|^\omega, |b_\omega|^\omega\}$$

for every pair $a_\omega, b_\omega \in \mathbb{C}_\omega$. It is worth noticing the elements of \mathbb{C} , seen as the subfield of constant sequences, have all norm equal to 1.

Definition 2.5. The ultrametric field $(\mathbb{C}_\omega, |\cdot|^\omega)$ is called the *asymptotic cone* of $(\mathbb{C}, |\cdot|)$ with respect to the scaling sequence $(\lambda_k)_{k \in \mathbb{N}}$ and the ultrafilter ω .

If we consider the distance induced by the absolute value $|\cdot|^\omega$ and we endow \mathbb{C}_ω with the metric topology, we obtain a topological field which is complete (see [Par12, Remark 3.10]), but it is not locally compact.

Proposition 2.6. *The field \mathbb{C}_ω is not locally compact with respect to the metric topology induced by the absolute value $|\cdot|^\omega$.*

Proof. Since \mathbb{C}_ω is a normed space, local compactness can be checked by verifying the compactness of the unit closed ball. Hence, it suffices to show that the closed ball

$$\overline{B}_1(0) := \{a_\omega \in \mathbb{C}_\omega \mid |a_\omega|^\omega \leq 1\}$$

is not compact. We are going to show that it is not sequentially compact. Consider the sequence $(n)_{n \in \mathbb{N}}$ where each element n has to be thought of as an element of \mathbb{C}_ω thanks to the standard embedding given by constant sequences. Given two different elements n and m it is clear that their distance in \mathbb{C}_ω is always equal to 1, indeed

$$|n - m|^\omega = \omega\text{-}\lim_{k \rightarrow \infty} |n - m|^{\frac{1}{\lambda_k}} = 1.$$

Hence it cannot exist a subsequence of $(n)_{n \in \mathbb{N}}$ which converges, as desired. \square

The construction exposed above can be repeated, rather than for a field, for every m -dimensional normed vector space $(V, \|\cdot\|)$ over \mathbb{C} . More precisely, we define

$$V_\omega := \{(v_k) \in V^{\mathbb{N}} \mid \exists C > 0, \forall k \ \|v_k\|^{\frac{1}{\lambda_k}} < C\} / \sim_\omega,$$

where $(v_k)_{k \in \mathbb{N}}$ and $(u_k)_{k \in \mathbb{N}}$ are equivalent if and only if $\omega\text{-}\lim_{k \rightarrow \infty} \|u_k - v_k\|^{\frac{1}{\lambda_k}} = 0$. Let v_ω be the equivalence class determined by $(v_k)_{k \in \mathbb{N}}$. It is possible to endow V_ω with a structure of m -dimensional \mathbb{C}_ω -vector space by considering the operations induced by pointwise sum and by pointwise scalar multiplication. As before, we have a well-defined norm $\|\cdot\|^\omega$ given by

$$\|v_\omega\|^\omega := \omega\text{-}\lim_{k \rightarrow \infty} \|v_k\|^{\frac{1}{\lambda_k}}.$$

Definition 2.7. The \mathbb{C}_ω -vector space $(V_\omega, \|\cdot\|^\omega)$ is the *asymptotic cone* of the vector space $(V, \|\cdot\|)$ with respect to the scaling sequence $(\lambda_k)_{k \in \mathbb{N}}$ and the ultrafilter ω .

We now focus our attention on the set of complex square matrices of order n , namely $M(n, \mathbb{C})$. If we choose as norm over $M(n, \mathbb{C})$ the standard matrix norm, we can apply the construction above to the normed vector space $(M(n, \mathbb{C}), \|\cdot\|)$. In this particular case we are able to enrich the structure of $M(n, \mathbb{C})_\omega$ by considering a multiplication. Indeed, the classic multiplication rows-by-columns is compatible with \sim_ω and hence it defines a structure of \mathbb{C}_ω -algebra on $M(n, \mathbb{C})_\omega$.

Definition 2.8. The normed algebra $(M(n, \mathbb{C})_\omega, \|\cdot\|^\omega)$ is called the *asymptotic cone* of the algebra $(M(n, \mathbb{C}), \|\cdot\|)$ with respect to the scaling sequence $(\lambda_k)_{k \in \mathbb{N}}$ and the ultrafilter ω .

Definition 2.9. A sequence $(g_k) \in GL(n, \mathbb{C})^{\mathbb{N}}$ is ω -bounded if

$$\exists C > 0 : \forall k \ \|g_k\|^{\frac{1}{\lambda_k}}, \|g_k^{-1}\|^{\frac{1}{\lambda_k}} < C.$$

The previous condition implies that the sequence $(g_k)_{k \in \mathbb{N}}$ defines an element of $M(n, \mathbb{C})_\omega$ which admits a multiplicative inverse. We denote by $GL(n, \mathbb{C})_\omega$ the set of all the invertible elements of $M(n, \mathbb{C})_\omega$. This is a group with respect to the multiplication rows-by-columns. We denote by $SL(n, \mathbb{C})_\omega$ the subgroup

$$SL(n, \mathbb{C})_\omega := \{g_\omega \in GL(n, \mathbb{C})_\omega \mid \exists (g_k)_{k \in \mathbb{N}} \in g_\omega : \forall k \det(g_k) = 1\}.$$

Since we can also consider the normed algebra $(M(n, \mathbb{C}_\omega), \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ is the standard supremum norm with respect to $|\cdot|^\omega$, it is natural to ask whether this algebra is isomorphic to $M(n, \mathbb{C})_\omega$ as normed algebra. The answer is given by [Par12, Corollary 3.18], which states that there is a natural isomorphism as normed \mathbb{C}_ω -algebras between $M(n, \mathbb{C})_\omega$ and $M(n, \mathbb{C}_\omega)$. Moreover this isomorphism induces an isomorphism of groups between $SL(n, \mathbb{C})_\omega$ and $SL(n, \mathbb{C}_\omega)$.

We conclude this section by introducing the space $\mathbb{P}^1(\mathbb{C})_\omega$. In order to do this, we first need to recall the construction of the asymptotic cone of \mathbb{H}^3 .

Definition 2.10. Let $(x_k)_{k \in \mathbb{N}}$ be a sequence of basepoints in \mathbb{H}^3 . Consider the space

$$C_\omega(\mathbb{H}^3, d/\lambda_k, x_k) := \{(y_k) \in (\mathbb{H}^3)^\mathbb{N} \mid \exists C > 0, \forall k \ d(x_k, y_k) < C\lambda_k\} / \sim_\omega$$

where $(y_k)_{k \in \mathbb{N}} \sim_\omega (y'_k)_{k \in \mathbb{N}}$ if and only if $\omega\text{-}\lim_{k \rightarrow \infty} d(y_k, y'_k)/\lambda_k = 0$. Denote by y_ω the equivalence class of the sequence $(y_k)_{k \in \mathbb{N}}$. If we define

$$d_\omega(y_\omega, y'_\omega) = \omega\text{-}\lim_{k \rightarrow \infty} d(y_k, y'_k)/\lambda_k$$

we get a metric and the metric space $(C_\omega(\mathbb{H}^3, d/\lambda_k, x_k), d_\omega)$ is the *asymptotic cone* with respect to the ultrafilter ω , the scaling sequence $(\lambda_k)_{k \in \mathbb{N}}$ and the sequence of basepoints $(x_k)_{k \in \mathbb{N}}$.

Assume to fix the origin O of the Poincaré model of \mathbb{H}^3 as the constant sequence of basepoints for the asymptotic cone construction. It should be clear that there exists a natural surjection

$$\pi : \mathbb{P}^1(\mathbb{C})^\mathbb{N} \rightarrow \partial_\infty C_\omega(\mathbb{H}^3, d/\lambda_k, O)$$

defined as it follows. Thinking of $\mathbb{P}^1(\mathbb{C})$ as the boundary at infinity of \mathbb{H}^3 , a sequence of points $(\xi_k) \in \mathbb{P}^1(\mathbb{C})^\mathbb{N}$ determines in a unique way a sequence of geodesic rays $(c_k)_{k \in \mathbb{N}}$ starting from O and ending at $(\xi_k)_{k \in \mathbb{N}}$. These rays allows us to define a geodesic ray $c_\omega : [0, \infty) \rightarrow C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ given by $c_\omega(t) := [c_k(\lambda_k t)]$. Hence, we can define $\pi((\xi_k)_{k \in \mathbb{N}}) := c_\omega(\infty)$. The space $\mathbb{P}^1(\mathbb{C})_\omega$ will be the quotient of $\mathbb{P}^1(\mathbb{C})^\mathbb{N}$ by the equivalence relation induced by the surjection π . In this way $\mathbb{P}^1(\mathbb{C})_\omega$ is clearly identified with boundary at infinity of $C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ and hence inherits in a natural way an action of $SL(2, \mathbb{C})_\omega$ given by $[h_k] \cdot [\xi_k] := [h_k \cdot \xi_k]$. This action is well defined because the action of

$SL(2, \mathbb{C})_\omega$ on $C_\omega(\mathbb{H}^3, d/\lambda_k, O)$ is well defined (see [Par12, Proposition 3.20]). Moreover, since the Bass–Serre tree $\Delta^{BS}(SL(2, \mathbb{C}_\omega))$ associated to $SL(2, \mathbb{C}_\omega)$ is naturally isometric to $C_\omega(\mathbb{H}^3, d/\lambda_k, O)$, as shown in [Par12, Proposition 3.21], the space $\mathbb{P}^1(\mathbb{C})_\omega$ can be identified also with $\mathbb{P}^1(\mathbb{C}_\omega)$ and this identification is compatible with the actions of $SL(2, \mathbb{C})_\omega$ and $SL(2, \mathbb{C}_\omega)$, respectively.

2.2. Bounded cohomology of locally compact groups. From now until the end of this section we denote by G a locally compact group. We endow \mathbb{R} with the structure of a trivial normed G -module, where the considered norm is the standard Euclidean one. The space of bounded continuous functions is

$C_{cb}^n(G, \mathbb{R}) := C_{cb}(G^{n+1}, \mathbb{R}) = \{f : G^{n+1} \rightarrow \mathbb{R} \mid f \text{ is continuous and } \|f\|_\infty < \infty\}$
 where the supremum norm is defined as

$$\|f\|_\infty := \sup_{g_0, \dots, g_n \in G} |f(g_0, \dots, g_n)|$$

and $C_{cb}^n(G, \mathbb{R})$ is endowed with the following G -module structure

$$(g.f)(g_0, \dots, g_n) := f(g^{-1}g_0, \dots, g^{-1}g_n)$$

for every element $g \in G$ and every function $f \in C_{cb}^n(G, \mathbb{R})$ (here the notation $g.f$ stands for the action of the element g on f). We denote by δ_n the homogeneous boundary operator of degree n , namely

$$\begin{aligned} \delta_n : C_{cb}^n(G, \mathbb{R}) &\rightarrow C_{cb}^{n+1}(G, \mathbb{R}), \\ \delta_n f(g_0, \dots, g_{n+1}) &= \sum_{i=0}^{n+1} (-1)^i f(g_0, \dots, \hat{g}_i, \dots, g_{n+1}), \end{aligned}$$

where the notation \hat{g}_i indicates that the element g_i has been omitted.

There is a natural embedding of \mathbb{R} into $C_{cb}^0(G, \mathbb{R})$ given by the constant functions on G . This allows us to consider the following chain complex of G -modules

$$0 \longrightarrow \mathbb{R} \longrightarrow C_{cb}^0(G, \mathbb{R}) \xrightarrow{\delta_0} C_{cb}^1(G, \mathbb{R}) \xrightarrow{\delta_1} \dots$$

and thanks to the compatibility of δ_n with respect to the G -action, we can consider the submodules of G -invariant vectors

$$0 \longrightarrow C_{cb}^0(G, \mathbb{R})^G \xrightarrow{\delta_0} C_{cb}^1(G, \mathbb{R})^G \xrightarrow{\delta_1} C_{cb}^2(G, \mathbb{R})^G \xrightarrow{\delta_2} \dots$$

Like in any other chain complex, we define the set of the n^{th} -bounded continuous cocycles as

$$Z_{cb}^n(G, \mathbb{R})^G := \ker(\delta_n : C_{cb}^n(G, \mathbb{R})^G \rightarrow C_{cb}^{n+1}(G, \mathbb{R})^G)$$

and the set of the n^{th} -bounded continuous coboundaries

$$B_{cb}^n(G, \mathbb{R})^G := \text{im} \left(\delta_{n-1} : C_{cb}^{n-1}(G, \mathbb{R})^G \rightarrow C_{cb}^n(G, \mathbb{R})^G \right), \text{ and } B_{cb}^0(G, \mathbb{R}) := 0.$$

Definition 2.11. The *continuous bounded cohomology* in degree n of G with real coefficients is the space

$$H_{cb}^n(G) := H_{cb}^n(G, \mathbb{R}) = \frac{Z_{cb}^n(G, \mathbb{R})^G}{B_{cb}^n(G, \mathbb{R})^G},$$

with the quotient seminorm

$$\|[f]\|_\infty := \inf \|f\|_\infty,$$

where the infimum is taken over all the possible representatives of $[f]$.

It is possible to gain information about the bounded cohomology of G also by studying suitable spaces on which G acts. More precisely, let X be a measurable space on which G acts measurably, that is the action map $\theta : G \times X \rightarrow X$ is measurable (G is equipped with the σ -algebra of the Haar measurable sets). We set

$$\mathcal{B}^\infty(X^n, \mathbb{R}) := \{f : X^n \rightarrow \mathbb{R} \mid f \text{ is measurable and } \sup_{x \in X^n} |f(x)| < \infty\},$$

and we endow it with the structure of Banach G -module given by

$$(g.f)(x_1, \dots, x_n) := f(g^{-1}.x_1, \dots, g^{-1}.x_n),$$

for every $g \in G$ and every $f \in \mathcal{B}^\infty(X^n, \mathbb{R})$. If $\delta_n : \mathcal{B}^\infty(X^n, \mathbb{R}) \rightarrow \mathcal{B}^\infty(X^{n+1}, \mathbb{R})$ is the standard homogeneous coboundary operator, for $n \geq 1$ and $\delta_0 : \mathbb{R} \rightarrow \mathcal{B}^\infty(X, \mathbb{R})$ is the inclusion given by constant functions, we get a cochain complex $(\mathcal{B}^\infty(X^\bullet, \mathbb{R}), \delta_\bullet)$. We denote by $\mathcal{B}_{\text{alt}}^\infty(X^{n+1}, \mathbb{R})$ the Banach G -submodule of alternating cochains, that is the set of elements satisfying

$$f(x_{\sigma(0)}, \dots, x_{\sigma(n)}) = \text{sgn}(\sigma)f(x_0, \dots, x_n),$$

for every permutation $\sigma \in S_{n+1}$.

Definition 2.12. Let E be a Banach G -module. The *continuous submodule* of E is defined by

$$\mathcal{C}E := \{v \in E \mid \lim_{g \rightarrow e} \|g.v - v\| = 0\}.$$

A *resolution* of E is an exact complex $(E^\bullet, \partial_\bullet)$ of Banach G -modules such that $E^0 = E$ and $E^n = 0$ for every $n \leq -1$.

$$0 \longrightarrow E \xrightarrow{\partial_0} E^1 \xrightarrow{\partial_1} E^2 \xrightarrow{\partial_2} \dots$$

We say that $(E^\bullet, \partial_\bullet)$ is a *strong resolution* if the continuous subcomplex $(\mathcal{C}E^\bullet, \partial_\bullet)$ admits a contracting homotopy, that is a sequence of maps $h_n : \mathcal{C}E^{n+1} \rightarrow \mathcal{C}E^n$ such that $\|h_n\| \leq 1$ and $h_{n+1} \circ \partial_n + \partial_n \circ h_{n-1} = \text{id}_{E^n}$ for all $n \in \mathbb{N}$.

In [BI02, Proposition 2.1] the authors prove that the complex of bounded measurable functions $(\mathcal{B}^\infty(X^\bullet, \mathbb{R}), \delta_\bullet)$ is a strong resolution of \mathbb{R} . Since the homology of any strong resolution of the trivial Banach G -module \mathbb{R} maps in a natural way to the continuous bounded cohomology of G by [BM02, Proposition 1.5.2.], there exists a canonical map

$$\mathfrak{c}^\bullet : H^\bullet(B^\infty(X^{\bullet+1}, \mathbb{R})^G) \rightarrow H_{cb}^\bullet(G).$$

More precisely, every bounded measurable G -invariant cocycle $f : X^{n+1} \rightarrow \mathbb{R}$ determines canonically a class $\mathfrak{c}^n[f] \in H_{cb}^n(G)$. The same result holds for the subcomplex $(\mathcal{B}_{\text{alt}}^\infty(X^\bullet, \mathbb{R}), \delta_\bullet)$ of alternating cochains.

3. THE ω -BOREL COCYCLE

3.1. The cocycle Vol^ω . From now until the end of the paper we will consider the spaces $\mathbb{P}^1(\mathbb{C})_\omega$ and $\mathbb{P}^1(\mathbb{C}_\omega)$ identified, hence we will refer to any of these two as they were the same space. The same will be done also for the groups $SL(n, \mathbb{C})_\omega$ and $SL(n, \mathbb{C}_\omega)$. Moreover, to avoid a heavy notation we are going to refer to any sequence $(x_l)_{l \in \mathbb{N}}$ by dropping the parenthesis every time that we are considering the sequence itself instead of any of its single term.

In this section we are going to construct a generalization of the hyperbolic volume function which will live on $\mathbb{P}^1(\mathbb{C}_\omega)^4$. This generalization will reveal the fundamental tool to define the ω -Borel cocycle.

Before starting, we want to underline a delicate point. Since we want to exploit the properties of the standard Borel cocycle, one could try to define the new function Vol^ω simply by taking the ω -limit of the volumes, that is $\text{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) = \omega\text{-lim}_{l \rightarrow \infty} \text{Vol}(x_l^0, \dots, x_l^3)$, where x_l^i is any representative of x_ω^i . Unfortunately this definition is not correct. Indeed, if we suppose to have 3 points that coincide, say $x_\omega^0 = x_\omega^1 = x_\omega^2$, different choices of representatives lead to different values of the ω -limit of their volumes. Hence, we need to be careful.

Let $\mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$ be the space of 4-tuples of distinct points on $\mathbb{P}^1(\mathbb{C}_\omega)$. As in the standard case, there is a natural cross ratio function

$$cr_\omega : \mathbb{P}^1(\mathbb{C}_\omega)^{(4)} \rightarrow \mathbb{C}_\omega \setminus \{0, 1\}, \quad cr_\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) = \frac{(x_\omega^0 - x_\omega^2)(x_\omega^1 - x_\omega^3)}{(x_\omega^0 - x_\omega^3)(x_\omega^1 - x_\omega^2)},$$

which is well defined by its purely algebraic nature. Every x_ω^i may be considered in \mathbb{C}_ω or equal to ∞ . If we define the Bloch–Wigner function by

$$D_2 : \mathbb{C} \rightarrow \mathbb{R}, \quad D_2(z) := \Im(\text{Li}_2(z)) + \arg(1-z) \log |z|,$$

where $\text{Li}_2(z)$ is the dilogarithm function, by still denoting D_2 its continuous extension on $\mathbb{P}^1(\mathbb{C})$, we can formulate the following

Definition 3.1. The ω -Bloch–Wigner function is given by

$$D_2^\omega : \mathbb{C}_\omega \cup \{\infty\} \rightarrow \mathbb{R}, \quad D_2^\omega(x_\omega) := \omega\text{-}\lim_{l \rightarrow \infty} D_2(x_l) \text{ for } x_\omega \in \mathbb{C}_\omega \text{ and } D_2^\omega(\infty) := 0.$$

where x_l is any representative of the equivalence class x_ω .

Lemma 3.2. *If x_l and y_l are two sequences representing the same element in \mathbb{C}_ω , then*

$$\omega\text{-}\lim_{l \rightarrow \infty} D_2(x_l) = \omega\text{-}\lim_{l \rightarrow \infty} D_2(y_l).$$

Proof. Since $\mathbb{P}^1(\mathbb{C})$ is compact and $\omega\text{-}\lim_{l \rightarrow \infty} |x_l - y_l|^{\frac{1}{|x_l|}} = 0$, both sequences x_l and y_l will converge to the same limit in $\mathbb{C} \cup \{\infty\}$. Denote by ξ this point. As a consequence of Proposition 2.4 and by the continuity of D_2 we have

$$\omega\text{-}\lim_{l \rightarrow \infty} D_2(x_l) = D_2(\omega\text{-}\lim_{l \rightarrow \infty} x_l) = D_2(\xi) = D_2(\omega\text{-}\lim_{l \rightarrow \infty} y_l) = \omega\text{-}\lim_{l \rightarrow \infty} D_2(y_l),$$

as claimed. \square

The previous lemma guarantees that the definition of the ω -Bloch–Wigner function is correct since it does not depend on the choice of the representative of the class x_ω .

Definition 3.3. The ω -volume function for a 4-tuple of points $(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^4$ is defined as

$$\text{Vol}^\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) = \begin{cases} D_2^\omega(cr_\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3)) & \text{if } (x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^{(4)}, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.4. We are going to denote by Vol the composition $D_2 \circ cr$, where D_2 is the standard Bloch–Wigner function and cr is the cross ratio on $\mathbb{P}^1(\mathbb{C})$. Fix a 4-tuple $(x_\omega^0, \dots, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^4$ of distinct points. Thanks to the natural identification between $\mathbb{P}^1(\mathbb{C}_\omega)$ and $\mathbb{P}^1(\mathbb{C})_\omega$, we can think of each x_ω^i as the class of a sequence x_l^i of points in $\mathbb{P}^1(\mathbb{C})$. Now, it is easy to see that

$$cr_\omega(x_\omega^0, \dots, x_\omega^3) = [cr(x_l^0, \dots, x_l^3)]$$

in \mathbb{C}_ω (if the x_ω^i are all distinct, also the terms of the sequences x_l^i are distinct ω -almost every $l \in \mathbb{N}$). By exploiting the previous identity, we can rewrite the definition of Vol^ω as follows

$$\begin{aligned} \text{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) &= D_2^\omega(\text{cr}_\omega(x_\omega^0, \dots, x_\omega^3)) = \omega\text{-}\lim_{l \rightarrow \infty} D_2(\text{cr}(x_l^0, \dots, x_l^3)) \\ &= \omega\text{-}\lim_{l \rightarrow \infty} \text{Vol}(x_l^0, \dots, x_l^3), \end{aligned}$$

and this is completely independent of the choice of representatives x_l^0, \dots, x_l^3 . Hence Vol^ω coincides with the ω -limit of the standard volumes $\text{Vol}(x_l^0, \dots, x_l^3)$ on a 4-tuple $(x_\omega^0, \dots, x_\omega^3) \in \mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$, where x_l^i is any representative for x_ω^i . Even though we have already underlined that this is not true on the whole space $\mathbb{P}^1(\mathbb{C}_\omega)^4$, we can always choose suitable representatives for x_ω^i such that

$$\text{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) = \omega\text{-}\lim_{l \rightarrow \infty} \text{Vol}(x_l^0, \dots, x_l^3).$$

Proposition 3.5. *The function Vol^ω is a bounded, alternating, $GL(2, \mathbb{C}_\omega)$ -invariant cocycle.*

Proof. Most of the properties we stated follow directly from the properties of the standard volume function Vol . We are going to show $GL(2, \mathbb{C}_\omega)$ -invariance, for instance. From now until the end of the proof we are going to pick suitable representative sequences for points in $\mathbb{P}^1(\mathbb{C}_\omega)$ such that

$$\text{Vol}^\omega(x_\omega^0, \dots, x_\omega^3) = \omega\text{-}\lim_{l \rightarrow \infty} \text{Vol}(x_l^0, \dots, x_l^3).$$

Let $g_\omega \in GL(2, \mathbb{C}_\omega)$. We want to show that $g_\omega \cdot \text{Vol}^\omega = \text{Vol}^\omega$.

$$\begin{aligned} g_\omega \cdot \text{Vol}^\omega(x_\omega^0, x_\omega^1, x_\omega^2, x_\omega^3) &= \text{Vol}^\omega(g_\omega^{-1} \cdot x_\omega^0, \dots, g_\omega^{-1} \cdot x_\omega^3) \\ &= \omega\text{-}\lim_{l \rightarrow \infty} \text{Vol}(g_l^{-1} \cdot x_l^0, \dots, g_l^{-1} \cdot x_l^3) \end{aligned}$$

and thanks to the equivariance of the classic volume function we get

$$\omega\text{-}\lim_{l \rightarrow \infty} \text{Vol}(g_l^{-1} \cdot x_l^0, \dots, g_l^{-1} \cdot x_l^3) = \omega\text{-}\lim_{l \rightarrow \infty} \text{Vol}(x_l^0, \dots, x_l^3) = \text{Vol}^\omega(x_\omega^0, \dots, x_\omega^3),$$

as required. The strategy to prove the alternating property and the cocycle property of Vol^ω is the same as above and we omit it.

Finally, the boundedness is obvious since the ω -Bloch–Wigner is nothing more than the ω -limit of a sequence of real values all bounded by ν_3 on $\mathbb{P}^1(\mathbb{C}_\omega)^{(4)}$ and it coincides with 0 on the complementary. Here ν_3 is the volume of a regular ideal hyperbolic tetrahedron in \mathbb{H}^3 . \square

3.2. The cocycle B_n^ω . In order to define the ω -Borel invariant for a representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$, we first need to define the ω -Borel cocycle. We are going to follow the same construction exposed in [BBI, Section 3]. Let $\mathfrak{S}_k^\omega(m)$ be the following space

$$\mathfrak{S}_k^\omega(m) := \{(x_\omega^0, \dots, x_\omega^k) \in (\mathbb{C}_\omega^m)^{k+1} \mid \langle x_\omega^0, \dots, x_\omega^k \rangle = \mathbb{C}_\omega^m\} / GL(m, \mathbb{C}_\omega)$$

where $GL(m, \mathbb{C}_\omega)$ acts on $(k+1)$ -tuples of vectors by the diagonal action and $\langle x_\omega^0, \dots, x_\omega^k \rangle$ is the \mathbb{C}_ω -linear space generated by $x_\omega^0, \dots, x_\omega^k$. It is obvious that if $k < m-1$ the space defined above is empty. For every m -dimensional vector space V over \mathbb{C}_ω and any $(k+1)$ -tuple of spanning vectors $(x_\omega^0, \dots, x_\omega^k) \in V^{k+1}$, we choose an isomorphism $V \rightarrow \mathbb{C}_\omega^m$. Since any two different choices of isomorphisms are related by an element $g_\omega \in GL(m, \mathbb{C}_\omega)$, we get a well defined element of $\mathfrak{S}_k^\omega(m)$ which will be denoted by $[V; (x_\omega^0, \dots, x_\omega^k)]$. For

$$\mathfrak{S}_k^\omega := \bigsqcup_{m \geq 0} \mathfrak{S}_k^\omega(m) = \mathfrak{S}_k^\omega(0) \sqcup \dots \sqcup \mathfrak{S}_k^\omega(k+1)$$

we have two different face maps $\varepsilon_i^{(k)}, \eta_i^{(k)} : \mathfrak{S}_k^\omega \rightarrow \mathfrak{S}_{k-1}^\omega$ given by

$$\begin{aligned} \varepsilon_i^{(k)}[\mathbb{C}_\omega^m; (x_\omega^0, \dots, x_\omega^k)] &:= [\langle x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k \rangle; (x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k)], \\ \eta_i^{(k)}[\mathbb{C}_\omega^m; (x_\omega^0, \dots, x_\omega^k)] &:= [\mathbb{C}_\omega^m / \langle x_\omega^i \rangle; (x_\omega^0, \dots, \hat{x}_\omega^i, \dots, x_\omega^k)]. \end{aligned}$$

Since these maps satisfy the same relations as in [BBI], that is for all $0 \leq i < j \leq k$

$$\begin{aligned} \varepsilon_j^{(k-1)} \varepsilon_i^{(k)} &= \varepsilon_i^{(k-1)} \varepsilon_{j+1}^{(k)}, \\ \eta_j^{(k-1)} \eta_i^{(k)} &= \eta_i^{(k-1)} \eta_{j+1}^{(k)}, \\ \eta_j^{(k-1)} \varepsilon_i^{(k)} &= \varepsilon_i^{(k-1)} \eta_{j+1}^{(k)}, \end{aligned}$$

we can define a boundary operator

$$D_k : \mathbb{Z}[\mathfrak{S}_k^\omega] \rightarrow \mathbb{Z}[\mathfrak{S}_{k-1}^\omega], \quad D_k(\sigma) := \sum_{i=0}^k (-1)^i (\varepsilon_i^{(k)}(\sigma) - \eta_i^{(k)}(\sigma)),$$

where $\mathbb{Z}[\mathfrak{S}_k^\omega]$ is the free abelian group generated by \mathfrak{S}_k^ω and it is equal to 0 for $k \leq -1$. We still denote by $\varepsilon_i^{(k)}$ and $\eta_i^{(k)}$ the linear extensions of face maps to $\mathbb{Z}[\mathfrak{S}_k^\omega]$. In this way we have constructed a chain complex $(\mathbb{Z}[\mathfrak{S}_\bullet^\omega], D_\bullet)$. With the purpose of dualizing this complex, we recall that we have a natural action of the symmetric group S_{k+1} on \mathfrak{S}_k^ω , hence we can define

$$\mathbb{R}_{\text{alt}}(\mathfrak{S}_k^\omega) := \{f : \mathfrak{S}_k^\omega \rightarrow \mathbb{R} \mid f \text{ is alternating with respect to the } S_{k+1}\text{-action}\}$$

and we can define D_k^* as the dual of $D_k \otimes id_{\mathbb{R}}$. The construction above produces a cochain complex $(\mathbb{R}_{\text{alt}}(\mathfrak{S}_\bullet^\omega), D_\bullet^*)$.

We are going now to define a cocycle living in $\mathbb{R}_{\text{alt}}(\mathfrak{S}_3^\omega)$ which will be used to construct the ω -Borel cocycle. Since the ω -volume function

Vol^ω introduced in the previous section can be thought of as defined on $(\mathbb{C}_\omega^2 \setminus \{0\})^4$, it is extendable to

$$\text{Vol}^\omega : \mathfrak{S}_3^\omega \rightarrow \mathbb{R}$$

where we set $\text{Vol}^\omega|_{\mathfrak{S}_3^\omega(m)}$ to be identically zero if $m \neq 2$ and

$$\text{Vol}^\omega[\mathbb{C}_\omega^2; (v_\omega^0, \dots, v_\omega^3)] := \begin{cases} \text{Vol}^\omega(v_\omega^0, \dots, v_\omega^3) & \text{if each } v_\omega^i \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

By the compatibility of the ω -limit with respect to finite sums, it should be clear that

Proposition 3.6. *The function $\text{Vol}^\omega \in \mathbb{R}_{\text{alt}}(\mathfrak{S}_3^\omega)$ is a cocycle, that is $D_4^*(\text{Vol}^\omega) = 0$.*

Since the proof of this proposition is the same as [BBI, Lemma 8, Lemma 9] we omit it. In order to define the ω -Borel cocycle we are going to introduce the spaces of affine flags in \mathbb{C}_ω^n . A complete flag F_ω in \mathbb{C}_ω^n is a sequence of linear subspaces

$$F_\omega^0 \subset F_\omega^1 \subset \dots \subset F_\omega^n$$

such that every F_ω^i has dimension i as \mathbb{C}_ω -vector space. An affine flag (F_ω, v_ω) is a complete flag F_ω together with an n -tuple of vectors $v_\omega = (v_\omega^1, \dots, v_\omega^n) \in (\mathbb{C}_\omega^n)^n$ such that

$$F_\omega^i = \mathbb{C}_\omega v_\omega^i + F_\omega^{i-1}, \quad i \geq 1.$$

It is clear that the group $GL(n, \mathbb{C}_\omega)$ acts naturally on the space of flags $\mathcal{F}(n, \mathbb{C}_\omega)$ and on the space of affine flags $\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)$ of \mathbb{C}_ω^n . Let $\mathbb{Z}[\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1}]$ be the abelian group generated by $\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1}$ and let ∂_k be the standard boundary map induced by the face maps $\varepsilon_i^{(k)} : \mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1} \rightarrow \mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^k$ consisting in dropping the i^{th} -component for $1 \leq k \leq n-1$. Moreover set $\partial_0 : \mathbb{Z}[\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)] \rightarrow 0$. We are ready now to define

$$T_k : (\mathbb{Z}[\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^k], \partial_k) \rightarrow (\mathbb{Z}[\mathfrak{S}_k^\omega], D_k)$$

which will enable us to construct a morphism between the dual of the complexes above (more precisely on their alternating versions). Given a multi-index $\mathbf{J} \in \{0, 1, \dots, n-1\}^{k+1}$, we start by defining

$$\tau_{\mathbf{J}} : \mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1} \rightarrow \mathfrak{S}_k^\omega$$

as the function

$$\tau_{\mathbf{J}}((F_{0,\omega}, v_{0,\omega}), \dots, (F_{k,\omega}, v_{k,\omega})) := \left[\frac{\langle F_{0,\omega}^{j_0+1}, \dots, F_{k,\omega}^{j_k+1} \rangle}{\langle F_{0,\omega}^{j_0}, \dots, F_{k,\omega}^{j_k} \rangle}; (v_{0,\omega}^{j_0+1}, \dots, v_{k,\omega}^{j_k+1}) \right]$$

and finally

$$T_k((F_{0,\omega}, v_{0,\omega}), \dots, (F_{k,\omega}, v_{k,\omega})) := \sum_{\mathbf{J} \in \{0, \dots, n-1\}^{k+1}} \tau_{\mathbf{J}}((F_{0,\omega}, v_{0,\omega}), \dots, (F_{k,\omega}, v_{k,\omega})).$$

If we now recall that there exists a natural action of S_{k+1} on $\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1}$ and dualize the complex considered so far, we get the cocomplex of alternating cochains $(\mathbb{R}_{\text{alt}}(\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1}), \partial_k^*)$ (here ∂_k^* is the dual of $\partial_k \otimes id_{\mathbb{R}}$). By denoting T_k^* the dual map of $T_k \otimes id_{\mathbb{R}}$, the same proof of [BBI, Lemma 11] guarantees that T_k^* is a morphism a complexes taking values in $(\mathbb{R}_{\text{alt}}(\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1}))^{GL(n, \mathbb{C}_\omega)}$.

Definition 3.7. We define the ω -Borel function of degree n as

$$\begin{aligned} B_n^\omega((F_{0,\omega}, v_{0,\omega}), \dots, (F_{3,\omega}, v_{3,\omega})) &:= T_3^*(\text{Vol}^\omega) = \\ &= \sum_{\mathbf{J} \in \{0, \dots, n-1\}^4} \text{Vol}^\omega \left[\frac{\langle F_{0,\omega}^{j_0+1}, \dots, F_{3,\omega}^{j_3+1} \rangle}{\langle F_{0,\omega}^{j_0}, \dots, F_{3,\omega}^{j_3} \rangle}; (v_{0,\omega}^{j_0+1}, \dots, v_{3,\omega}^{j_3+1}) \right]. \end{aligned}$$

Using the same approach of [BBI] it is straightforward to prove that

Proposition 3.8. *The function B_n^ω is a bounded, alternating, strict $GL(n, \mathbb{C}_\omega)$ -invariant cocycle on the space $\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^4$ of 4-tuples of affine flags which naturally descends to the space $\mathcal{F}(n, \mathbb{C}_\omega)^4$ of 4-tuples of flags. Moreover, for every 4-tuple of flags $(F_{0,\omega}, \dots, F_{3,\omega}) \in \mathcal{F}(n, \mathbb{C}_\omega)^4$ we have the following bound*

$$|B_n^\omega(F_{0,\omega}, \dots, F_{3,\omega})| \leq \frac{n(n^2 - 1)}{6} \nu_3.$$

We want now to use [BI02, Proposition 2.1] in order to obtain the desired cohomology class. Before doing this we need to underline a delicate point in the discussion. By Proposition 2.6 the field \mathbb{C}_ω is not locally compact with respect to the topology induced by the ultrametric absolute value. In particular the group $SL(n, \mathbb{C}_\omega)$ cannot be locally compact with respect to the topology inherited by $M(n, \mathbb{C}_\omega)$ seen as $\mathbb{C}_\omega^{n^2}$. Hence it is meaningless to refer to the Haar measure or to the Haar σ -algebra for $SL(n, \mathbb{C}_\omega)$. In order to overcome these difficulties, we are going to consider $SL^\delta(n, \mathbb{C}_\omega)$, that is the group $SL(n, \mathbb{C}_\omega)$ endowed with the discrete topology. The same for $GL^\delta(n, \mathbb{C}_\omega)$. Moreover, in order to apply correctly [BI02, Proposition 2.1], we are going to consider the discrete σ -algebra on both \mathfrak{S}_k^ω and $\mathcal{F}(n, \mathbb{C}_\omega)$.

Recall that $\mathfrak{S}_k^\omega(n)$ is a space on which the symmetric group S_{k+1} acts naturally. Let $\mathcal{B}_{\text{alt}}^\infty(\mathfrak{S}_k^\omega)$ be the Banach space of bounded alternating Borel functions on \mathfrak{S}_k^ω . The restriction of D_k^* gives us back a complex of Banach spaces $(\mathcal{B}_{\text{alt}}^\infty(\mathfrak{S}_\bullet^\omega), D_\bullet^*)$.

By restricting the map T_k^* to the subcomplexes of bounded Borel functions and by applying [BI02, Proposition 2.1] to $(\mathcal{B}_{\text{alt}}^\infty(\mathcal{F}(n, \mathbb{C}_\omega)^{\bullet+1}), \partial_\bullet)$, we get a map

$$S_\omega^k(n) : H^k(\mathcal{B}_{\text{alt}}^\infty(\mathfrak{S}_\bullet^\omega)) \rightarrow H_b^k(GL^\delta(n, \mathbb{C}_\omega)).$$

Definition 3.9. With the notation above, we define the ω -Borel cohomology class of degree n as

$$\beta^\omega(n) := S_\omega^3(n)(\text{Vol}^\omega) = \mathfrak{c}^3[B_n^\omega],$$

where $\mathfrak{c}^3 : H^3(\mathcal{B}_{\text{alt}}^\infty(\mathcal{F}(n, \mathbb{C}_\omega)^{\bullet+1})^{GL(n, \mathbb{C}_\omega)}) \rightarrow H_b^3(GL^\delta(n, \mathbb{C}_\omega))$ is the canonical map of [BI02, Proposition 2.1].

Remark 3.10. We have the following commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{C}_\omega^\times & \longrightarrow & GL(n, \mathbb{C}_\omega) & \longrightarrow & PGL(n, \mathbb{C}_\omega) \longrightarrow 1 \\ & & \uparrow & & \uparrow & & \downarrow \cong \\ 1 & \longrightarrow & \mu_n & \longrightarrow & SL(n, \mathbb{C}_\omega) & \longrightarrow & PSL(n, \mathbb{C}_\omega) \longrightarrow 1 \end{array}$$

where \mathbb{C}_ω^\times is the group of invertible elements of \mathbb{C}_ω and μ_n is the group of the n -th roots of unity. Since these groups are both amenable, by functoriality of bounded cohomology it is possible to conclude that $H_b^3(GL^\delta(n, \mathbb{C}_\omega)) \cong H_b^3(SL^\delta(n, \mathbb{C}_\omega))$. In particular, we are going to think of the class $\beta^\omega(n)$ as an element of both $H_b^3(GL^\delta(n, \mathbb{C}_\omega))$ and $H_b^3(SL^\delta(n, \mathbb{C}_\omega))$.

4. THE ω -BOREL INVARIANT FOR A REPRESENTATION ρ_ω

Let Γ be the fundamental group of a complete hyperbolic 3-manifold M with toric cusps. This means that we can decompose the manifold M as $M = N \cup \bigcup_{i=1}^h C_i$, where N is any compact core of M and for every $i = 1, \dots, h$ the component C_i is a cuspidal neighborhood diffeomorphic to $T_i \times (0, \infty)$, where T_i is a torus whose fundamental group corresponds to a suitable abelian parabolic subgroup of $PSL(2, \mathbb{C})$. Our aim is to define a numerical invariant associated to any representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$. Let $i : (M, \emptyset) \rightarrow (M, M \setminus N)$ be the natural inclusion map. Since the fundamental group of the boundary ∂N is abelian, hence amenable, it can be proved that the maps $i_b^* : H_b^k(M, M \setminus N) \rightarrow H_b^k(M)$ induced at the level of bounded cohomology groups are isometric isomorphisms for $k \geq 2$ (see [BBF14]). Moreover, it holds $H_b^k(M, M \setminus N) \cong H_b^k(N, \partial N)$ by homotopy invariance of bounded cohomology. If we denote by c the canonical comparison map $c : H_b^k(N, \partial N) \rightarrow H^k(N, \partial N)$, we can consider the composition

$$H_b^3(SL^\delta(n, \mathbb{C}_\omega)) \xrightarrow{(\rho_\omega)_b^*} H_b^3(\Gamma) \cong H_b^3(M) \xrightarrow{(i_b^*)^{-1}} H_b^3(N, \partial N) \xrightarrow{c} H^3(N, \partial N),$$

where the isomorphism that appears in this composition holds since M is aspherical. By choosing a fundamental class $[N, \partial N]$ for $H_3(N, \partial N)$ we are ready to give the following

Definition 4.1. The ω -Borel invariant associated to a representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$ is given by

$$\beta_n^\omega(\rho_\omega) := \langle (c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*) \beta^\omega(n), [N, \partial N] \rangle,$$

where the brackets $\langle \cdot, \cdot \rangle$ indicate the Kronecker pairing.

Remark 4.2. The previous definition is independent of the choice of the compact core N . Moreover, it can be easily extended to any lattice of $PSL(2, \mathbb{C})$.

We are going to generalize some of the classic results valid for the standard Borel invariant. The proofs are identical to the ones exposed in [BBI]. Before starting, we recall the existence of natural transfer maps

$$H_b^\bullet(\Gamma) \xrightarrow{\text{trans}_\Gamma} H_{cb}^\bullet(PSL(2, \mathbb{C})) \quad H^\bullet(N, \partial N) \xrightarrow{\tau_{DR}} H_c^\bullet(PSL(2, \mathbb{C})),$$

where $H_c^\bullet(PSL(2, \mathbb{C}))$ denotes the continuous cohomology groups of $PSL(2, \mathbb{C})$. We remind the reader that the continuous cohomology groups of a locally compact group G are constructed as the continuous bounded cohomology groups just by dropping the requirement of boundedness of cochains.

The transfer maps are defined as it follows. Let V_k be the set $C_b((\mathbb{H}^3)^{k+1}, \mathbb{R})$ of real bounded continuous functions on $(k+1)$ -tuples of points of \mathbb{H}^3 . With the standard homogeneous boundary operators and the structure of Banach $PSL(2, \mathbb{C})$ -module given by

$$(g.f)(x^0, \dots, x^n) := f(g^{-1}x^0, \dots, g^{-1}x^n), \quad \|f\|_\infty = \sup_{x^0, \dots, x^n \in \mathbb{H}^3} |f(x^0, \dots, x^n)|$$

for every $f \in C_b((\mathbb{H}^3)^{n+1}, \mathbb{R})$ and $g \in PSL(2, \mathbb{C})$, we get a complex $V_\bullet = C_b((\mathbb{H}^3)^{\bullet+1}, \mathbb{R})$ of Banach $PSL(2, \mathbb{C})$ -modules that allows us to compute the continuous bounded cohomology of $PSL(2, \mathbb{C})$. More precisely, it holds

$$H^k(V_\bullet^{PSL(2, \mathbb{C})}) \cong H_{cb}^k(PSL(2, \mathbb{C}))$$

for every $k \geq 0$. Moreover, by substituting $PSL(2, \mathbb{C})$ with Γ , we have in an analogous way that

$$H^k(V_\bullet^\Gamma) \cong H_b^k(\Gamma)$$

for every $k \geq 0$. The previous considerations allow us to define the map

$$\begin{aligned} \text{trans}_\Gamma : V_k^\Gamma &\rightarrow V_k^{PSL(2, \mathbb{C})}, \\ \text{trans}_\Gamma(c)(x_0, \dots, x_n) &:= \int_{\Gamma \backslash PSL(2, \mathbb{C})} c(\bar{g}x_0, \dots, \bar{g}x_n) d\mu(\bar{g}), \end{aligned}$$

where c is any Γ -invariant element of V_k and μ is any invariant probability measure on $\Gamma \backslash PSL(2, \mathbb{C})$. Here \bar{g} stands for the equivalence class of g into $\Gamma \backslash PSL(2, \mathbb{C})$.

Since $\text{trans}_\Gamma(c)$ is $PSL(2, \mathbb{C})$ -equivariant and trans_Γ commutes with the coboundary operator, we get a well-defined map

$$\text{trans}_\Gamma : H_b^\bullet(\Gamma) \rightarrow H_{cb}^\bullet(PSL(2, \mathbb{C})).$$

We now pass to the description of the map τ_{DR} . If $\pi : \mathbb{H}^3 \rightarrow M = \Gamma \backslash \mathbb{H}^3$ is the natural covering projection, we set $U := \pi^{-1}(M \setminus N)$. Recall that the relative cohomology group $H^k(N, \partial N)$ is isomorphic to the cohomology group $H^k(\Omega^\bullet(\mathbb{H}^3, U)^\Gamma)$ of the Γ -invariant differential forms on \mathbb{H}^3 which vanishes on U . Since, by Van Est isomorphism we have that $H_c^k(PSL(2, \mathbb{C}), \mathbb{R}) \cong \Omega^k(\mathbb{H}^3)^{PSL(2, \mathbb{C})}$, we define

$$\tau_{DR} : \Omega^k(\mathbb{H}^3, U)^\Gamma \rightarrow \Omega^k(\mathbb{H}^3)^{PSL(2, \mathbb{C})}, \quad \tau_{DR}(\alpha) := \int_{\Gamma \backslash PSL(2, \mathbb{C})} \bar{g}^* \alpha d\mu(\bar{g}),$$

where μ and \bar{g} are the same as before. The map τ_{DR} commutes with the coboundary operators inducing a map

$$\tau_{DR} : H^k(N, \partial N) \cong H^k(\Omega^\bullet(\mathbb{H}^3, U)^\Gamma) \rightarrow H^k(\Omega^\bullet(\mathbb{H}^3)^{PSL(2, \mathbb{C})}) \cong H_c^k(PSL(2, \mathbb{C})).$$

For a more detailed description of the above maps we suggest to the reader to check [BBI13, Section 3.2].

Proposition 4.3. *For $k \geq 2$ the diagram*

$$\begin{array}{ccc} H^k(\mathcal{B}_{\text{alt}}^\infty(\mathfrak{S}_\bullet^\omega)) & \xrightarrow{S_\omega^{k(n+1)}} & H_b^k(GL^\delta(n+1, \mathbb{C}_\omega)) \\ & \searrow S_\omega^k(n) & \downarrow \\ & & H_b^k(GL^\delta(n, \mathbb{C}_\omega)) \end{array}$$

commutes. The vertical arrow is induced by the left corner injection $GL(n, \mathbb{C}_\omega) \rightarrow GL(n+1, \mathbb{C}_\omega)$. In particular we have that $\beta^\omega(n+1)$ restricts to $\beta^\omega(n)$.

Proof. Let $i_n : \mathbb{C}_\omega^n \rightarrow \mathbb{C}_\omega^{n+1}$ be the injection $i_n(x_\omega^1, \dots, x_\omega^n) := (x_\omega^1, \dots, x_\omega^n, 0)$. By an abuse of notation we define

$$i_n : \mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega) \rightarrow \mathcal{F}_{\text{aff}}(n+1, \mathbb{C}_\omega)$$

as $i_n((F_\omega, v_\omega)) = (\tilde{F}_\omega, \tilde{v}_\omega)$ where for $0 \leq j \leq n$ we have $\tilde{F}_\omega^j = i_n(F_\omega^j)$, $\tilde{v}_\omega^j = i_n(v_\omega^j)$ and $\tilde{v}_\omega^{n+1} = e_{n+1}$. If we set $\mathbf{J} \in \{0, \dots, n\}^{k+1}$ and $I = \{i : 0 \leq i \leq k \text{ such that } j_i = n\}$, it is easy to verify that if $I = \emptyset$ this implies $\mathbf{J} \in \{0, \dots, n-1\}^{k+1}$ and

$$\tau_{\mathbf{J}}(i_n(F_{0,\omega}, v_{0,\omega}), \dots, i_n(F_{k,\omega}, v_{k,\omega})) = \tau_{\mathbf{J}}((F_{0,\omega}, v_{0,\omega}), \dots, (F_{k,\omega}, v_{k,\omega}))$$

while if $I \neq \emptyset$, then

$$\tau_{\mathbf{J}}(i_n(F_{0,\omega}, v_{0,\omega}), \dots, i_n(F_{k,\omega}, v_{k,\omega})) = [\mathbb{C}_\omega; (\delta_0^I, \dots, \delta_k^I)],$$

where $\delta_i^I = [e_{n+1}]$ if $i \in I$ and 0 otherwise. The previous considerations imply that i_n induces a commutative diagram of complexes

$$\begin{array}{ccc} \mathcal{B}_{\text{alt}}^\infty(\mathfrak{S}_k^\omega) & \xrightarrow{T_k^*} & \mathcal{B}_{\text{alt}}^\infty(\mathcal{F}_{\text{aff}}(n+1, \mathbb{C}_\omega)^{k+1}) \\ & \searrow T_k^* & \downarrow i_n^* \\ & & \mathcal{B}_{\text{alt}}^\infty(\mathcal{F}_{\text{aff}}(n, \mathbb{C}_\omega)^{k+1}) \end{array}$$

and since the map i_n^* implements the restriction in bounded cohomology, the commutativity of the diagram which appears in the statement follows. In particular, by focusing our attention on the case of $k = 3$ we get

$$i_n^*(B_{n+1}^\omega) = i_n^* \circ T_3^*(\text{Vol}^\omega) = T_3^*(\text{Vol}^\omega) = B_n^\omega$$

as claimed. \square

Proposition 4.4. *For any representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$ the composition*

$$H_b^3(SL^\delta(n, \mathbb{C}_\omega)) \longrightarrow H_b^3(\Gamma) \xrightarrow{\text{trans}_\Gamma} H_{cb}^3(PSL(2, \mathbb{C}))$$

maps $\beta^\omega(n)$ to $\frac{\beta_n^\omega(\rho_\omega)}{\text{Vol}(M)}\beta(2)$. In particular, it holds the following bound

$$|\beta_n^\omega(\rho_\omega)| \leq \frac{n(n^2 - 1)}{6} \text{Vol}(M),$$

as in the classic case.

Proof. Recall that we have the following commutative diagram

$$\begin{array}{ccc} H_b^3(SL^\delta(n, \mathbb{C}_\omega)) & & \\ \downarrow (\rho_\omega)_b^* & & \\ H_b^3(\Gamma) & \xrightarrow{\text{trans}_\Gamma} & H_{cb}^3(PSL(2, \mathbb{C})) \\ \downarrow \cong & & \downarrow c \\ H_b^3(N, \partial N) & & H_c^3(PSL(2, \mathbb{C})) \\ \downarrow c & \xrightarrow{\tau_{DR}} & \downarrow c \\ H^3(N, \partial N) & & H_c^3(PSL(2, \mathbb{C})). \end{array}$$

Since $H_{cb}^3(PSL(2, \mathbb{C})) \cong \mathbb{R}$, there exists a suitable $\lambda \in \mathbb{R}$ such that

$$\text{trans}_\Gamma \circ (\rho_\omega)_b^*(\beta^\omega(n)) = \lambda\beta(2).$$

Hence by composing both sides with the comparison map c , we obtain

$$c \circ \text{trans}_\Gamma \circ (\rho_\omega)_b^*(\beta^\omega(n)) = c(\lambda\beta(2)) = \lambda(c\beta(2)) = \lambda\beta(2).$$

If we pick up $\omega_{N, \partial N} \in H^3(N, \partial N)$ in such a way that its evaluation on the fundamental class $[N, \partial N]$ gives us back $\text{Vol}(M)$, we have that $\tau_{DR}(\omega_{N, \partial N}) = \beta(2)$. In particular

$$\tau_{DR}(c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*(\beta^\omega(n))) = \lambda\tau_{DR}(\omega_{N, \partial N})$$

and by injectivity of the map τ_{DR} in top degree we get

$$(c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*)(\beta^\omega(n)) = \lambda\omega_{N, \partial N}.$$

If we evaluate both sides on the fundamental class, we obtain

$$\beta_n^\omega(\rho_\omega) = \langle (c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*)(\beta^\omega(n)), [N, \partial N] \rangle = \langle \lambda\omega_{N, \partial N}, [N, \partial N] \rangle = \lambda\text{Vol}(M).$$

At the same time it holds

$$|\lambda| = \frac{\|\text{trans}_\Gamma \circ (\rho_\omega)_b^*\beta^\omega(n)\|}{\|\beta(2)\|} \leq \frac{n(n^2 - 1)}{6},$$

from which it follows

$$|\beta_n^\omega(\rho_\omega)| \leq \frac{n(n^2 - 1)}{6}\text{Vol}(M),$$

as claimed. \square

Recall that there is a natural inclusion of fields of \mathbb{C} into \mathbb{C}_ω given by constant sequences. In particular we have natural embeddings of \mathbb{C}^m into \mathbb{C}_ω^m and of $SL(n, \mathbb{C})$ into $SL(n, \mathbb{C}_\omega)$. Since every representation $\rho : \Gamma \rightarrow SL(n, \mathbb{C})$ determines a representation $\hat{\rho}$ into $SL(n, \mathbb{C}_\omega)$ by composing it with the previous embedding, it is quite natural to ask which is the relation between $\beta_n^\omega(\hat{\rho})$ and $\beta_n(\rho)$. We have the following

Proposition 4.5. *Let $\rho : \Gamma \rightarrow SL(n, \mathbb{C})$ be a representation. If we denote by $\hat{\rho} : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$ the representation obtained by composing ρ with the natural embedding of $SL(n, \mathbb{C})$ into $SL(n, \mathbb{C}_\omega)$, we have*

$$\beta_n^\omega(\hat{\rho}) = \beta_n(\rho).$$

Proof. We are going to prove that the cohomology class $\beta^\omega(n)$ restricts naturally to the class $\beta(n)$. Let $j : SL(n, \mathbb{C}) \rightarrow SL(n, \mathbb{C}_\omega)$ be the natural embedding. By endowing both spaces with the discrete topology, we have a continuous morphism of groups that induces a map

$$j_b^* : H_b^3(SL^\delta(n, \mathbb{C}_\omega)) \rightarrow H_b^3(SL^\delta(n, \mathbb{C})).$$

We want to prove that $j_b^*(\beta^\omega(n)) = \beta(n)$. From this it will follow

$$\begin{aligned}\beta_n^\omega(\hat{\rho}) &= \langle (c \circ (i_b^*)^{-1} \circ \hat{\rho}_b^*)\beta^\omega(n), [N, \partial N] \rangle = \langle (c \circ (i_b^*)^{-1} \circ (j \circ \rho)_b^*)\beta^\omega(n), [N, \partial N] \rangle \\ &= \langle (c \circ (i_b^*)^{-1} \circ \rho_b^* \circ j_b^*)\beta^\omega(n), [N, \partial N] \rangle = \langle (c \circ (i_b^*)^{-1} \circ \rho_b^*)\beta(n), [N, \partial N] \rangle = \beta_n(\rho).\end{aligned}$$

Similarly to what we have done for the field \mathbb{C}_ω , we define the configuration space

$$\mathfrak{S}_k(m) := \{(x^0, \dots, x^k) \in (\mathbb{C}^m)^{k+1} \mid \langle x^0, \dots, x^k \rangle = \mathbb{C}^m\} / GL(m, \mathbb{C}).$$

for every $k \geq m - 1$. This family of spaces is exactly the family introduced by [BBI]. There exists a natural family of maps given by

$$\hat{j}_k(m) : \mathfrak{S}_k(m) \rightarrow \mathfrak{S}_k^\omega(m), \quad \hat{j}_k(m)[\mathbb{C}^m; (v^0, \dots, v^k)] := [\mathbb{C}_\omega^m; (v^0, \dots, v^k)],$$

where each vector v^i which appears on the right-hand side of the equation is thought of as an element of \mathbb{C}_ω^m . This function is well-defined because v^0, \dots, v^k are generators also for \mathbb{C}_ω^m as a \mathbb{C}_ω -vector space and the identifications induced via conjugation by $GL(m, \mathbb{C})$ are respected. By denoting

$$\hat{j}_k := \hat{j}_k(0) \sqcup \hat{j}_k(1) \sqcup \dots \sqcup \hat{j}_k(k+1),$$

we get the following commutative diagram

$$\begin{array}{ccc} H^3(\mathcal{B}_{\text{alt}}^\infty(\mathfrak{S}_\bullet^\omega)) & \xrightarrow{S_\omega^3(n)} & H_b^3(SL^\delta(n, \mathbb{C}_\omega)) \\ H^3(\hat{j}_\bullet^*) \downarrow & & \downarrow j_b^* \\ H^3(\mathcal{B}_{\text{alt}}^\infty(\mathfrak{S}_\bullet)) & \xrightarrow{S^3(n)} & H_b^3(SL^\delta(n, \mathbb{C})), \end{array}$$

where \hat{j}_\bullet^* are the maps induced by \hat{j}_\bullet on the Borel cochains. We will prove that $\text{Vol} = \text{Vol}^\omega \circ \hat{j}_3$, that is $H^3(\hat{j}_\bullet^*)[\text{Vol}^\omega] = [\text{Vol}]$. Let $m \in \{0, \dots, 4\}$. It is clear that $\text{Vol} = \text{Vol}^\omega \circ \hat{j}_3(m)$ for $m \neq 2$ because both sides are equal to zero. Let now consider $[\mathbb{C}^2; (v^0, \dots, v^3)] \in \mathfrak{S}_3(2)$. If any of these vectors is 0 both functions evaluated on the 4-tuple give us back 0. Hence, we can suppose that each v^i is different from 0. If the vectors v^0, \dots, v^3 are in general position into \mathbb{C}^2 , they still remain in general position into \mathbb{C}_ω^2 . Thus

$$\begin{aligned}\text{Vol}^\omega \circ \hat{j}_3(2)[\mathbb{C}^2; (v^0, \dots, v^3)] &= \text{Vol}^\omega[\mathbb{C}_\omega^2; (v^0, \dots, v^3)] = \omega\text{-}\lim_{l \rightarrow \infty} \text{Vol}(v^0, \dots, v^3) \\ &= \text{Vol}(v^0, \dots, v^3) = \text{Vol}[\mathbb{C}^2; (v^0, \dots, v^3)].\end{aligned}$$

In the same way if (v^0, \dots, v^3) are not in general position into \mathbb{C}^2 , they will not be in general position into \mathbb{C}_ω^2 either, so both $\text{Vol}^\omega \circ \hat{j}_3(2)$ and Vol will evaluate to be zero, as desired. \square

We want now to express $\beta_n^\omega(\rho_\omega)$ in terms of boundary maps. Recall that the complement of N is M is given by a finite union $\bigcup_{i=1}^h C_i$ of cuspidal neighborhoods. For every $i = 1, \dots, h$ the fundamental group $\pi_1(C_i) = H_i$ is an abelian parabolic subgroup of $PSL(2, \mathbb{C})$, hence it has a unique fixed point ξ_i in $\mathbb{P}^1(\mathbb{C})$. We define the set

$$\mathcal{C}(\Gamma) := \bigcup_{i=1}^h \Gamma \cdot \xi_i.$$

Definition 4.6. If $\Gamma = \pi_1(M)$ as above, given a representation $\rho_\omega : \Gamma \rightarrow SL(n, \mathbb{C}_\omega)$, a *decoration* for ρ_ω is a map

$$\varphi_\omega : \mathcal{C}(\Gamma) \rightarrow \mathcal{F}(n, \mathbb{C}_\omega)$$

that is equivariant with respect to ρ_ω .

Recall now that the cocycle B_n^ω is a strict cocycle, as in the standard case. Hence the class $(c \circ (i_b^*)^{-1} \circ (\rho_\omega)_b^*) \beta^\omega(n)$ can be represented in $H_b^3(\Gamma)$ by $\varphi_\omega^*(B_n^\omega)$, where φ_ω is a decoration for ρ_ω (we refer to [BI02, Corollary 2.7] for this result about the pullback of strict cocycles along boundary maps). In order to realize the corresponding cocycle in $H_b^3(N, \partial N)$, we identify the universal cover \tilde{N} of N with \mathbb{H}^3 minus a set of Γ -equivariant horoballs, each one centered at an element $\xi \in \mathcal{C}(\Gamma)$. We define a map $p : \tilde{N} \rightarrow \mathcal{C}(\Gamma)$ in two steps. We first send each horospherical section to the corresponding element. Then, for the interior of \tilde{N} , we map a fundamental domain to a chosen $\xi_0 \in \mathcal{C}(\Gamma)$ and we extend equivariantly. In this way, any bounded Γ -invariant cocycle $c : \mathcal{C}(\Gamma) \rightarrow \mathbb{R}$ determines a relative cocycle on $(N, \partial N)$ as it follows

$$\{\sigma : \Delta^3 \rightarrow \tilde{N}\} \mapsto c(p(\sigma(e_0)), \dots, p(\sigma(e_3))).$$

If τ is a relative triangulation of $(N, \partial N)$ and $\tilde{\tau}$ is the lifted triangulation of a fundamental domain in $(\tilde{N}, \partial \tilde{N})$, the ω -Borel invariant $\beta_n^\omega(\rho_\omega)$ can be computed by the following formula

$$\beta_n^\omega(\rho_\omega) = \sum_{\tilde{\sigma} \in \tilde{\tau}} B_n^\omega(\varphi_\omega(p(\tilde{\sigma}(e_0))), \varphi_\omega(p(\tilde{\sigma}(e_1))), \varphi_\omega(p(\tilde{\sigma}(e_2))), \varphi_\omega(p(\tilde{\sigma}(e_3))))$$

where $\tilde{\sigma}$ is a lifted copy of the simplex $\sigma \in \tau$.

5. THE CASE $n = 2$ AND PROPERTIES OF THE INVARIANT $\beta_2^\omega(\rho_\omega)$

In this section we are going to focus our attention on the case of representations into $SL(2, \mathbb{C}_\omega)$. Suppose to have a sequence of representations $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ that determines a representation $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$. A sequence of decorations φ_l for ρ_l produces in a natural way a decoration φ_ω . Indeed it suffices to compose the

standard projection $\pi : \mathbb{P}^1(\mathbb{C})^{\mathbb{N}} \rightarrow \mathbb{P}^1(\mathbb{C})_\omega \cong \mathbb{P}^1(\mathbb{C}_\omega)$ with the product map $\prod \varphi_l : \mathbb{P}^1(\mathbb{C}) \rightarrow \mathbb{P}^1(\mathbb{C})^{\mathbb{N}}$. We say that a decoration is *non-degenerate* if for every $\xi_0, \dots, \xi_3 \in \mathcal{C}(\Gamma)$ we have that the 4-tuple $(\varphi_\omega(\xi_0), \dots, \varphi_\omega(\xi_3))$ contains at least 3 distinct points. If the decoration φ_ω is non-degenerate we have

$$\begin{aligned} \beta_2^\omega(\rho_\omega) &= \sum_{\tilde{\sigma} \in \tilde{\tau}} B_2^\omega(\varphi_\omega(p(\tilde{\sigma}(e_0))), \varphi_\omega(p(\tilde{\sigma}(e_1))), \varphi_\omega(p(\tilde{\sigma}(e_2))), \varphi_\omega(p(\tilde{\sigma}(e_3)))) \\ &= \omega\text{-}\lim_{l \rightarrow \infty} \sum_{\tilde{\sigma} \in \tilde{\tau}} B_2(\varphi_l(p(\tilde{\sigma}(e_0))), \varphi_l(p(\tilde{\sigma}(e_1))), \varphi_l(p(\tilde{\sigma}(e_2))), \varphi_l(p(\tilde{\sigma}(e_3)))) \\ &= \omega\text{-}\lim_{l \rightarrow \infty} \beta_2(\rho_l), \end{aligned}$$

where the last equality is obtained by applying Corollary 2.7 of [BI02]. The third equality exploits the non-degeneracy of the decoration φ_ω . Hence we get

Proposition 5.1. *Let $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ be a sequence of representations with decorations φ_l . Let $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$ be the representation associated to the sequence ρ_l . If the decoration φ_ω produced by the sequence φ_l is non-degenerate, we have*

$$\beta_2^\omega(\rho_\omega) = \omega\text{-}\lim_{l \rightarrow \infty} \beta_2(\rho_l).$$

Corollary 5.2. *Let $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ be a sequence of representations with decorations φ_l . Let $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$ be the representation associated to the sequence ρ_l . Suppose $\beta_2^\omega(\rho_\omega) = \text{Vol}(M)$. If the decoration φ_ω produced by the sequence φ_l is non-degenerate, there must exist a sequence $g_l \in SL(2, \mathbb{C})$ and a representation $\rho_\infty : \Gamma \rightarrow SL(2, \mathbb{C})$ such that*

$$\omega\text{-}\lim_{l \rightarrow \infty} g_l \rho_l(\gamma) g_l^{-1} = \rho_\infty(\gamma).$$

Proof. Thanks to the assumption of non-degeneracy, by applying Proposition 5.1 we deduce that $\omega\text{-}\lim_{l \rightarrow \infty} \beta_2(\rho_l) = \text{Vol}(M)$. The statement now follows directly by [FS18, Theorem 1.1]. \square

Remark 5.3. The representation ρ_∞ which appears in the previous corollary as limit of the sequence ρ_l has to be a lift of the standard lattice embedding $i : \Gamma \rightarrow PSL(2, \mathbb{C})$.

Assume that a sequence of representations $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ diverges to a ideal point of the character variety $X(\Gamma, SL(2, \mathbb{C}))$ and let $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$ be the representation associated to the sequence. Recall that the identification between $SL(2, \mathbb{C}_\omega)$ and $SL(2, \mathbb{C})_\omega$ implies that the representation ρ_ω produces in a natural way an isometric action of Γ on the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$. We are going to restrict

our attention to reducible actions with non-trivial length function. We first recall the following

Definition 5.4. Let \mathcal{T} be a real tree on which Γ acts via isometries. We say that the action is *reducible* if one of the following holds:

- The action of Γ admits a global fixed point.
- There exists an end $\varepsilon \in \partial_\infty \mathcal{T}$ fixed by Γ .
- There exists a Γ -invariant line $L \subset \mathcal{T}$.

Proposition 5.5. *Let $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ be a sequence of representations and suppose it determines a representation $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$ such that the isometric action induced by ρ_ω on $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ has non-trivial length function. If the action is reducible then $\beta_2^\omega(\rho_\omega) = 0$.*

Proof. Since the length function associated to the action induced by ρ_ω is non-trivial then the action does not admit a global fixed point. Moreover, since the action is reducible, it must admit either a fixed end or an invariant line. Suppose that there exists an end fixed by Γ . By [Par12, Proposition 3.20] the asymptotic cone $C_\omega(\mathbb{H}^3, d/\lambda_l, O)$ is naturally identified with the Bass–Serre tree $\Delta^{BS}(SL(2, \mathbb{C}_\omega))$ associated to $SL(2, \mathbb{C}_\omega)$. Hence, there must exist an end of $\Delta^{BS}(SL(2, \mathbb{C}_\omega))$ fixed by the representation ρ_ω . Thus the image $\rho_\omega(\Gamma)$ is a subgroup of a suitable Borel subgroup N_ω of $SL(2, \mathbb{C}_\omega)$ and hence it is solvable, so amenable by [Zim84, Corollary 4.1.7]. This implies that the map $(\rho_\omega)_b^* = 0$ from which we conclude $\beta_2^\omega(\rho_\omega) = 0$.

Suppose now that the action of Γ admits an invariant line. This time the image $\rho_\omega(\Gamma)$ will be isomorphic to a subgroup of $\text{Isom}(\mathbb{R})$. Being $\text{Isom}(\mathbb{R})$ the semidirect group of the two amenable groups $\mathbb{Z}/2\mathbb{Z}$ and \mathbb{R} , it will be amenable by [Zim84, Proposition 4.1.6]. As before we will have $(\rho_\omega)_b^* = 0$, hence $\beta_2^\omega(\rho_\omega) = 0$. \square

Remark 5.6. Another way to prove Proposition 5.5 is by using decorations. Indeed, if the action determined by ρ_ω admits a fixed end $\varepsilon_\omega \in \partial_\infty \Delta^{BS}(SL(2, \mathbb{C}_\omega))$ and since the boundary at infinity can be identified with $\mathbb{P}^1(\mathbb{C}_\omega)$, then the map $\varphi_\omega(\xi) = \varepsilon_\omega$ for $\xi \in \mathcal{C}(\Gamma)$ is a decoration and trivially it results $\beta_2^\omega(\rho_\omega) = 0$.

In the same way if the action admits an invariant line L_ω , we denote by ε_ω^1 and ε_ω^2 the ends of the line L_ω . For every $\xi \in \mathcal{C}(\Gamma)$ we can choose either ε_ω^1 or ε_ω^2 as the image of ξ for the decoration φ_ω . This implies that every possible choice produces a decoration for ρ_ω such that it results $\beta_2^\omega(\rho_\omega) = 0$.

Let $S = \{\gamma_1, \dots, \gamma_s\}$ be a generating set for the group Γ . Recall that if a sequence of representations $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ diverges in the character variety $X(\Gamma, SL(2, \mathbb{C}))$ to an ideal point of the Morgan–Shalen compactification, then the real sequence

$$\lambda_l := \inf_{x \in \mathbb{H}^3} \sqrt{\sum_{i=1}^s d(\rho_l(\gamma_i)x, x)}$$

is positive and divergent. As written in [Par12, Theorem 5.2], for any non-principal ultrafilter ω on \mathbb{N} , by fixing $(\lambda_l)_{l \in \mathbb{N}}$ as scaling sequence, we can construct in a natural way a representation $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$ via the representations ρ_l .

Corollary 5.7. *Let $\rho_l : \Gamma \rightarrow SL(2, \mathbb{C})$ be a sequence of representations diverging to an ideal point of the Morgan–Shalen compactification of the character variety $X(\Gamma, SL(2, \mathbb{C}))$. Let $\rho_\omega : \Gamma \rightarrow SL(2, \mathbb{C}_\omega)$ be the natural representation determined by the sequence $(\rho_l)_{l \in \mathbb{N}}$. If the representation is reducible, then $\beta_2^\omega(\rho_\omega) = 0$.*

Proof. It follows directly from Proposition 5.5 by observing that the ρ_ω has non-trivial length function since it is associated to diverging sequence of representations. \square

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