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Extending intuitionistic operations, orderings, and entropy measures on generalized fuzzy orthopartitions

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ABSTRACT

Generalized fuzzy orthopartitions extend the traditional concept of partitions to include both fuzziness and uncertainty. A generalized fuzzy orthopartition is a collection of intuitionistic fuzzy sets representing equivalence classes and satisfying a specific pair of axioms, which capture the idea that the classes must be disjoint and cover the initial universe. The aim of this article is twofold. Firstly, we aggregate and order generalized fuzzy orthopartitions by extending intuitionistic operations and relations. Secondly, we introduce and study entropy measures on generalized fuzzy orthopartitions by employing entropies on intuitionistic fuzzy sets already existing in the literature.

1. Introduction

Generalized fuzzy orthopartitions extend the traditional concept of partitions to include both fuzziness and uncertainty [9]. The equivalence classes of generalized fuzzy orthopartitions are formally represented by intuitionistic fuzzy sets (IFS), which are mathematical objects addressing some of the limitations of classical fuzzy sets. It is well known that in fuzzy set theory, an element belongs to a set with a certain membership degree between 0 and 1 [40]. In IFS theory, an element is characterized by two values so that their sum is at most 1: the membership and non-membership degrees to which it belongs to a set [2,5]. Thus, in a generalized fuzzy orthopartition, we have incomplete and vague information about the membership classes of the elements. Namely, let $\mu_i(u)$ and $\nu_i(u)$ be the membership and non-membership degrees of an element u to a class C_i where $\mu_i(u) + \nu_i(u) \leq 1$, we only know that u belongs to C_i with a degree in the interval $[\mu_i(u), 1 - \nu_i(u)]$, which is not precisely determined. The intuitionistic fuzzy sets of a generalized fuzzy orthopartition must verify two axioms: the first one captures the idea that the equivalence classes of the partition are disjoint and the second one is a covering requirement. Generalized fuzzy orthopartitions are more general than other models existing in the literature. Indeed, a generalized fuzzy orthopartition is an *orthopartition* based on classical sets [14] whenever the membership and non-membership degrees to which elements belong to the classes are Boolean, namely, they are 0 or 1 (in this case, it can be seen as a special Boolean possibility distribution, as shown in [8]); it is a *fuzzy orthopartition* [7,16], when it satisfies additional axioms. We principally focus on the natural and intuitive relationship between generalized fuzzy orthopartitions and *Ruspini (fuzzy) partitions*. In a Ruspini partition, equivalence classes are represented by fuzzy sets and the total membership degree of each element (distributed among all equivalence classes) must be 1 [29]. Due to its definition, a generalized fuzzy orthopartition can be transformed into a Ruspini partition, once the membership degree of each element u to each class C_i is a uniquely determined value in the interval

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$[\mu_i(u), 1 - \nu_i(u)]$. More precisely, since the intuitionistic fuzzy sets $(\mu_1, \nu_1), \dots, (\mu_n, \nu_n)$ of a generalized fuzzy orthopartition satisfy a particular pair of properties, it is always possible to determine at least a collection of fuzzy sets $\pi_1, \dots, \pi_n$ forming a Ruspini partition such that $\mu_i(u) \leq \pi_i(u) \leq 1 - \nu_i(u)$ for each element u of the universe and for each $i \in \{1, \dots, n\}$ [7]. On the other hand, this is not true for any collection of intuitionistic fuzzy sets. Thus, we can easily realise that a generalized fuzzy orthopartition coincides with a Ruspini partition, once enough knowledge is provided so that all the degrees of memberships are precisely known; namely, for each element u and each class C_i, we have a sufficient quantity of information to describe the membership degree of u to C_i with a specific value in $[\mu_i(u), 1 - \nu_i(u)]$, instead of the whole interval. Briefly speaking, generalized fuzzy orthopartitions can be viewed as Ruspini partitions with uncertainty. According to this consideration, each generalized fuzzy orthopartition is intuitively associated with a class of Ruspini partitions.

Moreover, generalized fuzzy orthopartitions have been connected in [6] with *credal partitions* that are important structures describing generalized partitions in evidence theory, and in particular in evidential clustering [30,32].

The first goal of this article is to aggregate generalized fuzzy orthopartitions by extending some basic intuitionistic operations introduced in [2,4,19,27]. Many scholars have paid attention to the aggregation techniques for intuitionistic fuzzy information, due to the capacity of intuitionistic fuzzy sets to model several application scenarios, especially in the context of decision-making (some examples are given in [21,37,39,41,42]).

Let & be an intuitionistic fuzzy set operation, it is mandatory to combine two generalized fuzzy orthopartitions with the same size $O = \{A_1, \dots, A_n\}$ and $O' = \{A'_1, \dots, A'_n\}$ by applying & to each pair of components of O and O' having the same index: $O \&_O O' = \{A_1 \& A'_1, \dots, A_n \& A'_n\}$. We discover that $O \&_O O'$ is not always a generalized fuzzy orthopartition, although $A_1 \& A'_i, \dots, A_n \& A'_n$ are intuitionistic fuzzy sets. We also determine some conditions on O and O' under which $O \&_O O'$ satisfies all the axioms of a generalized fuzzy orthopartition. Generally, such conditions concern the membership and non-membership degrees of the elements that have to be upper-bounded by a threshold depending on the cardinality of O and O'. The list of all operations proposed in this article with the corresponding conditions is exhibited by Table 6. Two of these operations $\cap_O$ and $*_O$ have been already defined on fuzzy orthopartitions in [7]. We also show that special classes of generalized fuzzy orthopartitions equipped with a pair of operations $\cap_O$ and $\cup_O$ (extending the minimum and maximum on fuzzy sets) form bounded lattices. Then, we introduce the orderings $\preceq_O$, $\preceq^1_O$ and $\preceq^*_O$ on generalized fuzzy orthopartitions by using the intuitionistic relations $\preceq$, $\preceq^1$, and $\preceq^*$ defined in [2,5,11].

As a further purpose of the article, we present and study some entropy measures on generalized fuzzy orthopartitions by focusing on entropies on intuitionistic fuzzy sets already existing in the literature [23,33]. Over the years, entropy measures with different scopes, have been defined and studied by taking into account fuzzy sets [20], intuitionistic fuzzy sets, and other representation models of uncertainty and imprecision, for instance in the context of belief functions [25,26]. The importance of entropy's role in artificial intelligence is widely discussed in [10]. The entropy measures introduced in this article quantify the *fuzziness* of generalized fuzzy orthopartitions: they describe the closeness (or the distance) of a considered generalized fuzzy orthopartition to a standard partition or a Ruspini partition. Let us recall that different measures called *lower and upper entropies* have already been defined on fuzzy orthopartitions in [7] by generalizing the logical entropy of standard partitions presented by Ellerman [22]. As shown hereafter by Remark 8 (page 7), when adapting the concepts of lower and upper entropies to intuitionistic fuzzy sets, a novel intuitionistic entropy measure can be defined.

Since this work focuses on operations, orderings, and entropy measures on generalized fuzzy orthopartitions, it can be considered as a follow-up to the previous publication [7], where the operations $\cap_O$ and $*_O$, the orderings $\preceq_O$ and $\preceq^*_O$, and the lower and upper entropies have been defined and studied on fuzzy orthopartitions.

The article is organized as follows. Section 2 recalls basic notions about intuitionistic fuzzy sets. Section 3 reviews concepts related to generalized fuzzy orthopartitions, which have been defined in [7] and [9]. Section 4 presents novel operations on generalized fuzzy orthopartitions. Section 5 introduces and studies new entropy measures on generalized fuzzy orthopartitions. Section 6 discusses the potential developments of this work. Lastly, an appendix is introduced at the end of the article to show some examples related to Section 4.

Let us underline that all definitions and results of this article are given by assuming that the initial universe is finite. In the following, let $U = \{u_1, \dots, u_{|U|}\}$ be such a finite universe.

2. Intuitionistic fuzzy sets

This section is devoted to intuitionistic fuzzy sets. In particular, some intuitionistic operators and entropy measures are recalled.

Definition 1. An intuitionistic fuzzy set (IFS) A on U is a pair of functions $\mu_A : U \to [0,1]$ and $\nu_A : U \to [0,1]$ such that

$$\mu_A(u) + \nu_A(u) \leq 1, \text{ for any } u \in U. \tag{1}$$

Let $u \in U$, $\mu_A(u)$ and $\nu_A(u)$ are called the *membership* and *non-membership degrees* of u to the IFS A, respectively. Analogously, μ_A and ν_A are called *membership* and *non-membership* functions.

Definition 2. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set on U. The *hesitation margin* of A is a function $h_A : U \to [0,1]$ such that

$$h_A(u) = 1 - (\mu_A(u) + \nu_A(u)), \text{ for each } u \in U.$$

Let $u \in U$, the value $h_A(u)$ expresses the degree of *indeterminacy* (or *uncertainty*) of the membership of u to the IFS A.

Remark 1. Intuitionistic fuzzy sets can be understood as fuzzy sets with uncertainty:

(a) An IFS (μ_A, ν_A) represents a fuzzy set $\alpha_A : U \to [0,1]$ so that $\alpha_A(u)$ belongs to the interval $[\mu_A(u), 1-\nu_A(u)]$ (equivalently, $[\mu_A(u), \mu_A(u) + h_A(u)]$), hence $\alpha_A(u)$ is partially known. Recall that (μ_A, ν_A) is equivalent to an *interval-valued fuzzy set*, which is a function assigning the closed sub-interval $[\mu_A(u), 1-\nu_A(u)]$ of [0,1] to each $u \in U$ [3,18].
(b) Of course, when $h_A(u) = 0$ for each $u \in U$, (μ_A, ν_A) is equivalent to the fuzzy set μ_A.

However, intuitionistic fuzzy sets could assume a different semantics. For example, in the intuitionistic fuzzy Analytic Hierarchy Process (AHP) [38], the triple $(\mu_i(u), \nu_i(u), h_i(u))$ describes the preference degree of one criterion/alternative over another. Also, in [13], intuitionistic fuzzy sets express linguistics terms.

Now, let us show the list of orderings and operations on IFSs that are used in this article. In the sequel, we consider the order on functions on $U \to [0,1]$ defined as: for all $f, g : U \to [0,1]$, $f \le g$ if and only if $f(u) \le g(u)$ for each $u \in U$.

Definition 3. [2,5,11] Let (μ_A, ν_A) and (μ_B, ν_B) be intuitionistic fuzzy sets on U. Then,

(i) $(\mu_A, \nu_A) \preceq (\mu_B, \nu_B)$ if and only if $\mu_A \le \mu_B$ and $\nu_A \ge \nu_B$;
(ii) $(\mu_A, \nu_A) \preceq^* (\mu_B, \nu_B)$ if and only if $\mu_A \le \mu_B$ and $\nu_A \le \nu_B$;
(iii) $(\mu_A, \nu_A) \preceq^1 (\mu_B, \nu_B)$ if and only if "$\mu_A \le \mu_B$ and $\nu_A \ge \nu_B$" or "$\mu_A \ge \mu_B$ and $\nu_A \le \nu_B$".

Also, let $\& : [0,1] \times [0,1] \to [0,1]$, we write $f \& g$ to indicate the function from [0,1] to [0,1] so that $(f \& g)(u) = f(u) \& g(u)$ for each $u \in U$.

Definition 4. [2,5] Let (μ_A, ν_A) and (μ_B, ν_B) be intuitionistic fuzzy sets on U. Then, the following operations can be defined:

(i) $(\mu_A, \nu_A) \cap (\mu_B, \nu_B) = (\min\{\mu_A, \mu_B\}, \max\{\nu_A, \nu_B\})$;
(ii) $(\mu_A, \nu_A) \cup (\mu_B, \nu_B) = (\max\{\mu_A, \mu_B\}, \min\{\nu_A, \nu_B\})$;
(iii) $(\mu_A, \nu_A) + (\mu_B, \nu_B) = (\mu_A + \mu_B - \mu_A\mu_B, \nu_A\nu_B)$;
(iv) $(\mu_A, \nu_A) \odot (\mu_B, \nu_B) = (\mu_A\mu_B, \nu_A + \nu_B - \nu_A\nu_B)$;
(v) $(\mu_A, \nu_A) * (\mu_B, \nu_B) = \left(\dfrac{\mu_A + \mu_B}{2}, \dfrac{\nu_A + \nu_B}{2}\right)$;
(vi) $(\mu_A, \nu_A) \times (\mu_B, \nu_B) = \left(\dfrac{\mu_A + \mu_B}{2(\mu_A\mu_B + 1)}, \dfrac{\nu_A + \nu_B}{2(\nu_A\nu_B + 1)}\right)$;
(vii) $(\mu_A, \nu_A) - (\mu_B, \nu_B) = (\mu_A, \nu_A) \cap \neg(\mu_B, \nu_B)$, namely

$$(\mu_A, \nu_A) - (\mu_B, \nu_B) = (\min\{\mu_A, \nu_B\}, \max\{\nu_A, \mu_B\});$$

(viii) $\neg(\mu_A, \nu_A) = (\nu_A, \mu_A)$;
(ix) $\Box(\mu_A, \nu_A) = (\mu_A, 1 - \mu_A)$;
(x) $\Diamond(\mu_A, \nu_A) = (1 - \nu_A, \nu_A)$;
(xi) $C(\mu_A, \nu_A) = (p, q)$, where $p : U \to [0,1]$ and $q : U \to [0,1]$ are the constant functions such that

$$p(\bar{u}) = \max_{u \in U} \mu_A(u) \text{ and } q(\bar{u}) = \min_{u \in U} \nu_A(u), \text{ for each } \bar{u} \in U;$$

(xii) $\mathcal{I}(\mu_A, \nu_A) = (r, s)$, where $r : U \to [0,1]$ and $s : U \to [0,1]$ are the constant functions such that

$$r(\bar{u}) = \min_{u \in U} \mu_A(u) \text{ and } s(\bar{u}) = \max_{u \in U} \nu_A(u), \text{ for each } \bar{u} \in U.$$

Remark 2. Let us point out that

(a) $\preceq$ is the ordered relation associated with $\cap$:
let (μ_A, ν_A) and (μ_B, ν_B) be IFSs on U,

$$(\mu_A, \nu_A) \preceq (\mu_B, \nu_B) \text{ if and only if } (\mu_A, \nu_A) \cap (\mu_B, \nu_B) = (\mu_A, \nu_A); \tag{2}$$

(b) $\cap$, $\cup$, and $\neg$ respectively generalize the operations of intersection, union, and complement on fuzzy sets;
(c) $\preceq$ is a restriction of $\preceq^1$;
(d) if $(\mu_A, \nu_A) \preceq^1 (\mu_B, \nu_B)$, we can say that (μ_A, ν_A) is *less fuzzy* than (μ_B, ν_B). In other words, (μ_B, ν_B) is more closer than (μ_A, ν_A) to be a fuzzy set, by considering that $h_B(u) \le h_A(u)$ for each $u \in U$.

Hereafter, we denote Int the collection of all intuitionistic fuzzy sets on U.

Theorem 1. *[2,27] The algebraic structure $(Int, \cap, \cup)$ is a* lattice.

Furthermore, $(Int, \cap, \cup, (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{0}))$ is a *bounded lattice*, where $\mathbf{0}$ and $\mathbf{1}$ are the constant functions assigning 0 and 1 to each $u \in U$, respectively. This means that

$$(\mathbf{0}, \mathbf{1}) \preceq (\mu_A, \nu_A) \text{ and} (\mu_A, \nu_A) \preceq (\mathbf{1}, \mathbf{0}), \text{ for each } (\mu_A, \nu_A) \in Int.$$

In line with the lattice theory notation, the IFSs $(\mathbf{0}, \mathbf{1})$ and $(\mathbf{1}, \mathbf{0})$ are called *bottom* and *top* of the lattice. Also, the structure $(Int, \preceq)$, where $\preceq$ is defined by (2), is the lattice-ordered set corresponding to $(Int, \cap, \cup)$.

Intuitionistic entropy measures Entropy measures have been defined for intuitionistic fuzzy sets to measure their fuzziness. Namely, they quantify the information missing or existing to classify the elements of the given universe with certainty.

Some entropy measures are defined in [12,23,33]. Here we present the definitions of the main ones, the entropy η_1 proposed by [33] based on the distance between the membership and the non-membership of each element, and the entropy η_2 proposed by [12] based on their hesitancy margin.[1]

Definition 5. Let (μ_A, ν_A) be an intuitionistic fuzzy set on U. Then,

(i) $\eta_1(\mu_A, \nu_A) = 1 - \frac{1}{2|U|} \sum_{u \in U} |\mu_A(u) - \nu_A(u)|$;

(ii) $\eta_2(\mu_A, \nu_A) = \sum_{u \in U} h_A(u)$.

The previous entropy measures satisfy monotonicity properties as shown hereafter in Theorem 2.

We first need the following definition introduced in [10], where an intuitionistic fuzzy set is transformed into a new one, starting from a pair of elements of the initial universe, by means of the union and the intersection of intuitionistic fuzzy sets.

Definition 6. Let (μ, ν) be an intuitionistic fuzzy set on U, we consider the intuitionistic fuzzy set $(\mu_{(u_1,u_2)}, \nu_{(u_1,u_2)})$ defined on $U \setminus \{u_1\}$ as follows:

$$\mu_{(u_1,u_2)}(u) = \begin{cases} \mu(u) & \text{if } u \neq u_2; \\ \max\{\mu(u_1), \mu(u_2)\} & \text{otherwise;} \end{cases}$$

$$\nu_{(u_1,u_2)}(u) = \begin{cases} \nu(u) & \text{if } u \neq u_2; \\ \min\{\nu(u_1), \nu(u_2)\} & \text{otherwise.} \end{cases}$$

Let us focus on the concept described by Definition 6. Notice that the IFS $(\mu_{(u_1,u_2)}, \nu_{(u_1,u_2)})$ can be understood as (μ, ν) on a universe smaller than U, which is $U \setminus \{u_1\}$. However, the information related to (μ, ν) on u_1 is not lost because it is contained in the value assumed by $(\mu_{(u_1,u_2)}, \nu_{(u_1,u_2)})$ on u_2; indeed, it is obtained by aggregating $\mu(u_1)$ and $\nu(u_1)$ with $\mu(u_2)$ and $\nu(u_2)$, respectively.

Example 1. Consider the IFS (μ, ν) on $\{u_1, u_2, u_3\}$, where $\mu(u_1) = 0.3$, $\nu(u_1) = 0.5$, $\mu(u_2) = 0.2$, $\nu(u_2) = 0.1$, $\mu(u_3) = 0.1$, and $\nu(u_3) = 0.1$. According to Definition 6, $(\mu_{(u_1,u_2)}, \nu_{(u_1,u_2)})$ is an IFS on $\{u_2, u_3\}$ so that $\mu_{(u_1,u_2)}(u_2) = \max\{\mu(u_1), \mu(u_2)\} = \max\{0.3, 0.2\} = 0.3$, $\nu_{(u_1,u_2)}(u_2) = \min\{\nu(u_1), \nu(u_2)\} = \min\{0.5, 0.1\} = 0.1$, $\mu_{(u_1,u_2)}(u_3) = \mu(u_3) = 0.1$, and $\nu_{(u_1,u_2)}(u_3) = \nu(u_3) = 0.1$.

The following theorem connects η_1 and η_2 to the orderings on IFSs. We omit its proof because it is straightforward.

Theorem 2. *Let $(\mu_A, \nu_A), (\mu_B, \nu_B)$, and (μ, ν) be intuitionistic fuzzy sets on U.*

(a) *If $(\mu_A, \nu_A) \preceq^1 (\mu_B, \nu_B)$, then $\eta_1(\mu_A, \nu_A) \leq \eta_1(\mu_B, \nu_B)$;*
(b) *if $(\mu_A, \nu_A) \preceq^* (\mu_B, \nu_B)$, then $\eta_2(\mu_A, \nu_A) \geq \eta_2(\mu_B, \nu_B)$;*
(c) *$\eta_2(\mu_{(u_1,u_2)}, \nu_{(u_1,u_2)}) \leq \eta_2(\mu, \nu)$, where $(\mu_{(u_1,u_2)}, \nu_{(u_1,u_2)})$ is the intuitionistic fuzzy set on $U \setminus \{u_1\}$ given by Definition 6.*

Properties (a) and (b) shown that η_1 satisfies the *O-monotonicity* w.r.t. $\preceq^1$, and that η_2 satisfies the *O-monotonicity* w.r.t. $\preceq^*$. This kind of monotonicity shows a relation between an order on the elements and an order induces by the entropy [10]. Property (c) is related to another kind of monotonicity, the *R-monotonicity* that shows a relation between the reduction of the size of a partition and its entropy [10].

According to (a), the fuzzier the set, the bigger its entropy η_1. Reciprocally due to (b), the fuzzier the set, the smaller its entropy η_2. Lastly, by (c), when reducing the initial universe, the entropy η_2 of IFSs decreases.

[1] In order to simplify the notation, we wrote $\eta((\mu_A, \nu_A)) = \eta(\mu_A, \nu_A)$ when $\eta \in \{\eta_1, \eta_2\}$, throughout the article.

3. Generalized fuzzy orthopartitions

This section presents basic notions concerning generalized fuzzy orthopartition which have been introduced in [7] and [9]. From now on, we set $I = \{1, \ldots, n\}$, where $n \in \mathbb{N}$. The integer n denotes the cardinality of the generalized partitions considered in this paper; hence, we are assuming that they are formed of at least two equivalence classes.

Generalized fuzzy orthopartitions are meant to extend the concept of Ruspini partitions given in the following definition [29].

Definition 7. [29] A Ruspini partition $\pi = \{\pi_i \mid i \in I\}$ of U is a family of fuzzy sets such that

$$\sum_{i \in I} \pi_i(u) = 1, \text{ for each } u \in U.$$

Definition 7 describes a partition of n blocks, where an element $u \in U$ belongs to the class i with the degree $\pi_i(u) \in [0, 1]$.

A generalized fuzzy orthopartition is a more general model representing a partition of n blocks, where the membership and non-membership degrees to which elements belong to the equivalence classes are specified. Thus, each equivalence class of a generalized fuzzy orthopartition is mathematically described by an intuitionistic fuzzy set.

Definition 8. [9] Let $O = \{(\mu_i, \nu_i) \mid i \in I\}$ be a family of intuitionistic fuzzy sets of U. Then, O is a *generalized fuzzy orthopartition* of U if and only if the following properties hold for each $u \in U$:

(i) $\sum_{i \in I} \mu_i(u) \leq 1$;
(ii) $\sum_{i \in I} (\mu_i(u) + h_i(u)) \geq 1$.

Remark 3. A generalized fuzzy orthopartition can be viewed as a partition, where equivalence classes are characterized by vagueness and uncertainty. Let $O = \{(\mu_i, \nu_i) \mid i \in I\}$ be a generalized fuzzy orthopartition, what information do we possess about the relationship between the elements of the universe and the classes? Let $u \in U$, the relationship between u and the class i is captured by (μ_i, ν_i). Then, by Remark 1, we know that u belongs to the class i with a degree of the interval $[\mu_i(u), \mu_i(u) + h_i(u)]$, which is not precisely determined.

The meaning of the axioms of Definition 8 is strictly connected to the idea that generalized fuzzy orthopartitions are intended to represent Ruspini partitions with uncertainty. Let us give more details. Supposing that the uncertainty is solved in O, we denote the specific membership degree of u to the class i with $\pi_i(u)$, which belongs to $[\mu_i(u), \mu_i(u) + h_i(u)]$. When Axioms (i) and (ii) being not satisfying by u,

$$\pi_1(u) + \ldots + \pi_n(u) > 1 \text{ and } \pi_1(u) + \ldots + \pi_n(u) < 1,$$

respectively. Therefore, according to the fuzzy clustering interpretation,

1. if $\pi_1(u) + \ldots + \pi_n(u) > 1$, the blocks are not disjoint;
2. if $\pi_1(u) + \ldots + \pi_n(u) < 1$, then the blocks do not completely cover the universe.

Consequently, Axiom (i) captures the idea that the intuitionistic fuzzy sets in O must represent mutually disjoint blocks of U and Axiom (ii) is a covering requirement.

Remark 4. By Definition 2, $\mu_i(u) + h_i(u) = 1 - \nu_i(u)$ for each $i \in I$. Then, Axiom (ii) of Definition 8 is equivalent to $\sum_{i \in I} (1 - \nu_i(u)) \geq 1$, which can be algebraically rewritten as $\sum_{i \in I} \nu_i(u) \leq n - 1$.

Remark 5. A generalized fuzzy orthopartition $O = \{(\mu_i, \nu_i) \mid i \in I\}$ is called a standard partition if and only if

$$\text{“}\mu_i(u), \nu_i(u) \in \{0, 1\} \text{ and } \mu_i(u) + \nu_i(u) = 1\text{”, for each } i \in I \text{ and } u \in U.$$

For example, the generalized fuzzy orthopartition $O = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ defined by Table 1 coincides with the standard partition made of two equivalence classes, it is $\{\{u_1\}, \{u_2\}\}$.

It is easy to see that a generalized fuzzy orthopartition $O = \{(\mu_i, \nu_i) \mid i \in I\}$ coincides with a Ruspini partition, when

$$h_i(u) = 0 \text{ for each } u \in U \text{ and } i \in I. \tag{3}$$

More precisely, when all the degrees of uncertainty are 0, O contains the same information as the Ruspini partition $\{\mu_i \mid i \in I\}$. Indeed, according to Remark 3, given $u \in U$ and $i \in I$, we can say that the truth degree to which u belongs to the class i is $\mu_i(u)$.

Table 1
Definition of the elements of O.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	1	0	0	1
u_2	0	1	1	0

Table 2
Definition of the elements of O.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0.2	0.3	0	0.5
u_2	0.1	0.4	0.6	0

Table 3
Definition of the elements of $\{\pi_1, \pi_2\}$ and $\{\pi'_1, \pi'_2\}$.

u	$\pi_1(u)$	$\pi_2(u)$
u_1	0.6	0.4
u_2	0.2	0.8

u	$\pi'_1(u)$	$\pi'_2(u)$
u_1	0.5	0.5
u_2	0.3	0.7

Additionally, once its uncertainty is solved, any generalized fuzzy orthopartition could coincide with a Ruspini partition. This is the case when it is possible to specify for each $i \in I$ and $u \in U$, the exact degree $\pi_i(u)$ from $[\mu_i(u), \mu_i(u) + h_i(u)]$ to which u belongs to the class i such that $\pi_1(u) + \ldots + \pi_n(u) = 1$. Certainly, when the values $\pi_1(u), \ldots, \pi_n(u)$ become precisely known, it is not always true that $\pi_1(u) + \ldots + \pi_n(u) = 1$. Consequently, it is not a given that a generalized fuzzy orthopartition generates a Ruspini partition. However, as explained in our previous works (see [7] and [9]), Axioms (i) and (ii) of Definition 8 guarantee that it is always possible assigning at least a Ruspini partition to O; namely for each element u of the universe, we can find a combination of values in $[\mu_1(u), \mu_1(u) + h_1(u)], \ldots, [\mu_n(u), \mu_n(u) + h_n(u)]$ so that the corresponding function $\pi_1, \ldots, \pi_n$ from U to [0,1] form a Ruspini partition.

Ruspini partitions associated with a generalized fuzzy orthopartition are formally defined as follows.

Definition 9. A Ruspini partition $\pi = \{\pi_i \mid i \in I\}$ of U is *compatible* with a generalized fuzzy orthopartition $O = \{(\mu_i, \nu_i) \mid i \in I\}$ of U if and only if

$$\mu_i(u) \le \pi_i(u) \le \mu_i(u) + h_i(u), \text{ for each } u \in U \text{ and for each } i \in I.$$

In the following, Π_O denotes the collection of all Ruspini partitions compatible with O. Observe that a Ruspini partition $\pi = \{\pi_1, \ldots, \pi_n\}$ generates a generalized fuzzy orthopartition $O = \{(\mu_1, \nu_1), \ldots, (\mu_n, \nu_n)\}$ for any choice of $\mu_i(u)$ and $\nu_i(u)$ so that $\mu_i(u) \le \pi_i(u) \le \mu_i(u) + h_i(u)$. This is another way to characterize the partitions in Π_O: $\pi \in \Pi_O$ if and only if π can generate O.

Example 2. Let $O = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ be a generalized fuzzy orthopartition of the universe $\{u_1, u_2\}$, where μ_1, ν_1, μ_2, and ν_2 are defined in Table 2.

O could coincide with any Ruspini partition of two blocks so that

- u_1 belongs to the first block with a degree between $\mu_1(u_1) = 0.2$ and $1 - \nu_1(u_1) = 0.7$;
- u_1 belongs to the second block with a degree between $\mu_2(u_1) = 0$ and $1 - \nu_2(u_1) = 0.5$;
- u_2 belongs to the first block with a degree between $\mu_1(u_2) = 0.1$ and $1 - \nu_1(u_2) = 0.6$;
- u_2 belongs to the second block with a degree between $\mu_2(u_2) = 0.6$ and $1 - \nu_2(u_2) = 1$.

Consequently, O corresponds to an entire class of Ruspini partitions of $\{u_1, u_2\}$. Two examples are $\{\pi_1, \pi_2\}$ and $\{\pi'_1, \pi'_2\}$ defined in Table 3.

However, as mentioned before, casually choosing values in $[\mu_1(u_1), \mu_1(u_1) + h_1(u_1)]$, $[\mu_2(u_1), \mu_2(u_1) + h_2(u_1)]$, $[\mu_1(u_2), \mu_1(u_2) + h_1(u_2)]$, and $[\mu_2(u_2), \mu_2(u_2) + h_2(u_2)]$, we can obtain two functions $\tilde{\pi}_1$ and $\tilde{\pi}_2$ that does not form a Ruspini partition. For example, if $\tilde{\pi}_1(u_1) = 0.2$, $\tilde{\pi}_2(u_1) = 0$, $\tilde{\pi}_1(u_2) = 0.6$, and $\tilde{\pi}_2(u_2) = 0.6$, then $\tilde{\pi}_1(u_1) + \tilde{\pi}_2(u_1) < 1$ and $\tilde{\pi}_1(u_2) + \tilde{\pi}_2(u_2) > 1$.

Remark 6. It can be observed that when $O = \{(\mu, \nu)\}$ (namely, $n = 1$), Axiom (i) of Definition 8 is always verified by O due to (1). Also, Axiom (ii) is verified by O if and only if $\nu(u) = 0$ for each $u \in U$. Consequently, the unique Ruspini partition $\pi = \{\alpha\}$ compatible with O is compose of the fuzzy set $\alpha : U \to [0, 1]$ so that $\alpha(u) = 1$ for each $u \in U$. Then, α corresponds to the trivial partition where all elements of the universe belong to a unique block.

Now, we recall the concepts of lower and upper entropy introduced in [7] capturing the capacity of a generalized fuzzy orthopartition to distinguish the elements of the initial universe by means of its IFSs. These entropies are based on the definition of entropy for Ruspini partitions also introduced in [7]:

Definition 10. Let $\pi = \{\pi_i \mid i \in I\}$ be a Ruspini partition of U, the *entropy* of π is given by

$$\mathcal{H}(\pi) = \frac{\sum\limits_{(u,u')\in U^2} \max\{|\pi_i(u) - \pi_i(u')| \text{ such that } i \in I\}}{|U^2|},$$

where U^2 is the Cartesian product $\{(u,u') \mid u,u' \in U\}$.

The measure given in Definition 10 was also defined in the framework of rough set theory [31].

Definition 11. Let $O = \{(\mu_i, \nu_i) \mid i \in I\}$ be a generalized fuzzy orthopartition of U, the *lower and upper entropies* are respectively defined as

$$\mathcal{H}_*(O) = \min\{\mathcal{H}(\pi) \mid \pi \in \Pi_O\} \text{ and } \mathcal{H}^*(O) = \max\{\mathcal{H}(\pi) \mid \pi \in \Pi_O\}.$$

The entropies introduced above are extensions of the logical entropy for standard partitions presented by Ellerman in [22].

In the sequel, given $n \in \mathbb{N}$, we denote by OR the collection of all sequences of n IFSs forming a generalized fuzzy orthopartition of U. Moreover, the IFSs of $O_k \in OR$ are denoted as $(\mu_1^k, \nu_1^k), \dots, (\mu_n^k, \nu_n^k)$. Until now, the following operations and orderings have been defined on OR [7]:

Definition 12. Let $O_1, O_2 \in OR$, then

(i) $O_1 \cap_O O_2 = \{(\mu_1^1, \nu_1^1) \cap (\mu_1^2, \nu_1^2), \dots, (\mu_n^1, \nu_n^1) \cap (\mu_n^2, \nu_n^2)\}$;
(ii) $O_1 *_O O_2 = \{(\mu_1^1, \nu_1^1) * (\mu_1^2, \nu_1^2), \dots, (\mu_n^1, \nu_n^1) * (\mu_n^2, \nu_n^2)\}$;

where $\cap$ and $*$ are given by Definition 4 (axioms (i) and (v)).

Definition 13 *(Orders on generalized fuzzy orthopartitions).* Let $O_1, O_2 \in OR$, then

(i) $O_1 \preceq_O O_2$ if and only if $(\mu_i^1 \nu_i^1) \preceq (\mu_i^2, \nu_i^2)$ for each $i \in I$;
(ii) $O_1 \preceq_O^* O_2$ if and only if $(\mu_i^1, \nu_i^1) \preceq^* (\mu_i^2, \nu_i^2)$ for each $i \in I$;

where $\preceq$ and $\preceq^*$ are given by Definition 3 (axioms (i) and (ii)).

The lower and upper entropies satisfy the O-monotonicity w.r.t. $\preceq_O^*$ (see [7] for more details):

Theorem 3. *Let $O_1, O_2 \in OR$. If $O_1 \preceq_O^* O_2$, then*

$$\mathcal{H}_*(O_1) \le \mathcal{H}_*(O_2) \textit{ and } \mathcal{H}^*(O_1) \ge \mathcal{H}^*(O_2).$$

Remark 7. We can also consider as an entropy the length of the closed interval $\mathcal{E}_O = [\mathcal{H}_*(O), \mathcal{H}^*(O)]$ of [0,1], which is denoted with $|\mathcal{E}_O|$ (as usual, the length of the interval $[a,b]$ is given by $b-a$). In particular, $|\mathcal{E}_O|$ measures the fuzziness of O, namely it quantifies how close O is to being a Ruspini partition. Indeed, by Theorem 3, if $O_1 \preceq_O^* O_2$, then $|\mathcal{E}_{O_1}| \ge |\mathcal{E}_{O_2}|$. This means that O_2 is less fuzzy than O_1; in other words, O_2 is closer than O_1 to being a Ruspini partition. As a consequence, the smaller the interval $\mathcal{E}_O$, the closer O is to being a Ruspini partition.

Consistently with this consideration, we can observe that if O is a Ruspini partition (i.e. $h_i(u) = 0$ for each $u \in U$ and $i \in I$), then $\mathcal{H}_*(O) = \mathcal{H}^*(O)$ and the length of $\mathcal{E}_O$ is 0.

Remark 8. We can obviously apply Definitions 9 and 11 to generalized fuzzy orthopartitions that are singletons. More generally, a novel measure of entropy for intuitionistic fuzzy sets arises by translating the concepts explained above to intuitionistic fuzzy sets.[2]

Let (μ, ν) be an IFS of U, according to Definition 9, $\Pi_{(\mu,\nu)}$ is a collection of fuzzy sets defined as follows:

$$\alpha \in \Pi_{(\mu,\nu)} \text{ if and only if } \mu(u) \le \alpha(u) \le 1 - \nu(u) \text{ for each } u \in U.$$

[2] Recall that by Remark 8, $O = \{(\mu, \nu)\}$ is a generalized fuzzy orthopartition if and only if $\nu(u) = 0$ for each $u \in U$.

Therefore, $\Pi_{(\mu,\nu)}$ is the class of all fuzzy sets that (μ,ν) could coincide with, once the uncertainty of (μ,ν) is solved. Finally, according to Definition 11 and Remark 7, $|\mathcal{E}_{(\mu,\nu)}|$ quantifies the fuzziness of the intuitionistic fuzzy set (μ,ν). Then, $(\mu,\nu) \preceq^* (\mu',\nu')$, when (μ',ν') is closer than (μ,ν) to being a fuzzy set.

4. Aggregating fuzzy orthopartitions

In this section, we introduce and study novel operations on the generalized fuzzy orthopartitions of OR, starting from the operations of Definition 4 (Subsection 4.1). Then, we individuate sub-classes of OR that are closed under the considered operations (Subsection 4.2).

4.1. Operations on generalized fuzzy orthopartitions

Let O_1 and O_2 be generalized fuzzy orthopartitions, a simple way to define a new operation between O_1 and O_2 is to aggregate their IFSs (component by component) by using one of the operations of Definition 4. Analogously, a unary operator can be proposed on generalized fuzzy orthopartitions by applying one of the unary operators of Definition 4 to each of their IFSs.[3]

Definition 14. Let $O_1, O_2 \in OR$, let $\& \in \{\cap, \cup, +, \odot, *, \times, -\}$, and let $\triangle \in \{\neg, \Box, \Diamond, \mathcal{C}, \mathcal{I}\}$. Then, associated binary operator $\&_O$ and unary operator $\triangle_O$ for orthopartitions are defined as:

(i) $O_1 \&_O O_2 = \{(\mu_1^1, \nu_1^1)\&(\mu_1^2, \nu_1^2), \dots, (\mu_n^1, \nu_n^1)\&(\mu_n^2, \nu_n^2)\}$;
(ii) $\triangle_O O_1 = \{\triangle(\mu_1^1, \nu_1^1), \dots, \triangle(\mu_n^1, \nu_n^1)\}$.

As illustrated in the appendix by some examples, not all the operations given by Definition 14 are OR-closed. Additionally, Theorem 4 shows that $*_O$, $\Box_O$, and $\times_O$ are the only ones satisfying the closeness under OR.

Theorem 4. *OR is closed under $*_O$, $\Box_O$, and $\times_O$.*

Proof. **($*_O$).** The proof for $*_O$ is similar to that of the analogous theorem presented in [7].
($\Box_O$). Let $O \in OR$. Let us recall that $\Box_O O = \{(\mu_1, 1-\mu_1), \dots, (\mu_n, 1-\ \mu_n)\}$ by Definition 4 (ix). Let $u \in U$, since O is a generalized fuzzy orthopartition, we get $\sum_{i\in I} \mu_i(u) \le 1$ from Definition 8 (i). Then, Axiom (i) of Definition 8 trivially holds for $\Box_O O$ too.

Moreover, it is consequently true that $n - \sum_{i\in I} \mu_i(u) \le n-1$, which means that $\sum_{i\in I}^{n}(1-\mu_i(u)) \le n-1$. So, by Remark 4, Axiom (ii) of Definition 8 holds for $\Box_O O$.
($\times_O$). Let $O_1, O_2 \in OR$, we want to prove that Axioms (i) and (ii) of Definition 8 are satisfied by $O_1 \times_O O_2$. So, let us recall that the ith intuitionistic fuzzy set of $O_1 \times_O O_2$ is

$$\left(\frac{\mu_i^1 + \mu_i^2}{2(\mu_i^1\mu_i^2 + 1)}, \frac{\nu_i^1 + \nu_i^2}{2(\nu_i^1\nu_i^2 + 1)}\right)$$

from Definition 4 (vi).
(Axiom (i)). Let $u \in U$, since O_1 and O_2 are generalized fuzzy orthopartitions, $\sum_{i\in I} \mu_i^1(u) \le 1$ and $\sum_{i\in I} \mu_i^2(u) \le 1$ from Definition 8 (i). Then, $\sum_{i\in I}(\mu_i^1(u) + \mu_i^2(u)) = \sum_{i\in I} \mu_i^1(u) + \sum_{i\in I} \mu_i^2(u) \le 2$. Diving each member of the previous inequality by 2, we get $\sum_{i\in I} \frac{\mu_i^1(u) + \mu_i^2(u)}{2} \le 1$. Of course, this implies that $\sum_{i\in I} \frac{\mu_i^1(u) + \mu_i^2(u)}{2(\mu_i^1(u)\mu_i^2(u) + 1)} \le 1$.
(Axiom (ii)). Let $u \in U$, since O_1 and O_2 are generalized fuzzy orthopartitions, $\sum_{i\in I} \nu_i^1(u) \le n-1$ and $\sum_{i\in I} \nu_i^2(u) \le n-1$ from Definition 8 (ii) and Remark 4. Using a sequence of implications similar to those of the previous point, we can easily discover that

$$\sum_{i\in I} \frac{\nu_i^1(u) + \nu_i^2(u)}{2(\nu_i^1(u)\nu_i^2(u) + 1)} \le n-1. \quad \Box$$

[3] The operations $\cap_O$ and $*_O$ are already given by Definition 12. Moreover, they are studied taking into account the definition of fuzzy orthopartitions introduced in [7].

4.2. Sub-classes of OR closed under the operations $\cap_O$, $\cup_O$, $+_O$, $\odot_O$, $-_O$, $\neg_O$, $\Diamond_O$, $\mathcal{I}_O$, and $\mathcal{C}_O$

Let us define two special collections of fuzzy orthopartitions of OR depending on a threshold α of [0,1]:

- $OR^{\alpha}_{\mu} = \{O \in OR \mid \mu(u) \leq \alpha$ for each $(\mu, \nu) \in O$ and $u \in U\}$;
- $OR^{\alpha}_{\nu} = \{O \in OR \mid \nu(u) \leq \alpha$ for each $(\mu, \nu) \in O$ and $u \in U\}$.

Therefore, OR^{α}_{μ} is made of all generalized fuzzy orthopartitions of OR such that the membership functions of their IFSs are bounded above by α. Analogously, in the case of OR^{α}_{ν}, all the non-membership functions are bounded above by α. We can also observe that both OR^{α}_{μ} and OR^{α}_{ν} coincide with OR, when $\alpha = 1$.

Example 3. Consider the generalized fuzzy orthopartition O_1 defined by Table A.7. Then, we can see that $O_1 \in OR^{0.5}_{\mu}$, $O_1 \in OR^{0.6}_{\nu}$, and $O_1 \in OR^{0.6}_{\mu} \cap OR^{0.6}_{\nu}$. But, $O_1 \notin OR^{0.5}_{\nu}$.

The next theorems individuate the conditions on α so that OR^{α}_{μ}, OR^{α}_{ν}, and $OR^{\alpha}_{\mu} \cap OR^{\alpha}_{\nu}$ are closed under some of the operations of Definition 14.

Theorem 5. *OR^{α}_{ν} is closed under $\cap_O$, when $\alpha \leq 1 - \frac{1}{n}$.*

Proof. Let $O_1, O_2 \in OR^{\alpha}_{\nu}$, we want to prove that Axioms (i) and (ii) of Definition 8 are satisfied by $O_1 \cap_O O_2$.
Recall that let $i \in I$, $(\mu^1_i, \nu^1_i) \cap (\mu^2_i, \nu^2_i) = (\min\{\mu^1_i, \mu^2_i\}, \max\{\nu^1_i, \nu^2_i\})$ from Definition 4 (i).

(Axiom (i)). Let $u \in U$, we intend to prove that $\sum_{i\in I} \min\{\mu^1_i(u), \mu^2_i(u)\} \leq 1$. Considering that O_1 is a generalized fuzzy orthopartition, $\sum_{i\in I} \mu^1_i(u) \leq 1$. Moreover, it is clear that $\min\{\mu^1_i(u), \mu^2_i(u)\} \leq \mu^1_i(u)$ for each $i \in I$. Then, $\sum_{i\in I} \min\{\mu^1_i(u), \mu^2_i(u)\} \leq \sum_{i\in I} \mu^1_i(u)$. Finally, $\sum_{i\in I} \min\{\mu^1_i(u), \mu^2_i(u)\} \leq 1$.

(Axiom (ii)). Let $u \in U$, by Remark 4, we need to verify that the inequality $\sum_{i\in I} \max\{\nu^1_i(u), \nu^2_i(u)\} \leq n - 1$ holds. First of all, we know that $O_1, O_2 \in OR^{\alpha}_{\nu}$, then $\nu^1_i(u), \nu^2_i(u) \leq \alpha$ for each $i \in I$. Thus, $\max\{\nu^1_i(u), \nu^2_i(u)\} \leq \alpha$ for each $i \in I$. As a consequence, $\sum_{i\in I} \max\{\nu^1_i(u), \nu^2_i(u)\} \leq n\alpha$. By hypothesis, $\alpha \leq 1 - \frac{1}{n}$. Hence, $n\alpha \leq n - 1$.
Ultimately, $\sum_{i\in I} \max\{\nu^1_i(u), \nu^2_i(u)\} \leq n - 1$. By Remark 4, this means that $O_1 \cap_O O_2$ satisfies Axiom (ii) of Definition 8.

Moreover, for each $u \in U$, $\nu^1_i(u), \nu^2_i(u) \leq \alpha$ by hypothesis. Thus, for each $u \in U$, $\max\{\nu^1_i(u), \nu^2_i(u)\} \leq \alpha$. Lastly, $O_1 \cap_O O_2 \in OR^{\alpha}_{\nu}$. □

Theorem 6. *OR^{α}_{μ} is closed under $\cup_O$, when $\alpha \leq \frac{1}{n}$.*

Proof. Let $O_1, O_2 \in OR^{\alpha}_{\mu}$, we want to prove that Axioms (i) and (ii) of Definition 8 are satisfied by $O_1 \cup_O O_2$.
Recall that let $i \in I$, $(\mu^1_i, \nu^1_i) \cup (\mu^2_i, \nu^2_i) = (\max\{\mu^1_i, \mu^2_i\}, \min\{\nu^1_i, \nu^2_i\})$ from Definition 4 (ii).

(Axiom (i)). Let $u \in U$, we intend to prove that $\sum_{i\in I} \max\{\mu^1_i(u), \mu^2_i(u)\} \leq 1$. First of all, we know that O_1 and O_2 are generalized fuzzy orthopartitions of OR^{α}_{μ} from the hypothesis. Then, $\mu^1_i(u), \mu^2_i(u) \leq \alpha$ for each $i \in I$. Thus, $\max\{\mu_i(u)^1, \mu^2_i(u)\} \leq \alpha$ for each $i \in I$. As a consequence, $\sum_{i\in I} \max\{\mu^1_i(u), \mu^2_i(u)\} \leq n\alpha$. Also, $n\alpha$ is less than or equal to 1, considering that $\alpha \leq \frac{1}{n}$ by hypothesis.

(Axiom (ii)). Let $u \in U$, we intend to prove that

$$\sum_{i\in I} (1 - \min\{\nu^1_i(u), \nu^2_i(u)\}) \geq 1.$$

Considering that O_1 is a generalized fuzzy orthopartition,

$$\sum_{i\in I} (1 - \nu^1_i(u)) \geq 1. \tag{4}$$

Furthermore, it is clear that $\min\{\nu^1_i(u), \nu^2_i(u)\} \leq \nu^1_i(u)$ for each $i \in I$. Then, $1 - \min\{\nu^1_i(u), \nu^2_i(u)\} \geq 1 - \nu^1_i(u)$ for each $i \in I$. Hence, $\sum_{i\in I} (1 - \min\{\nu^1_i(u), \nu^2_i(u)\}) \geq \sum_{i\in I} (1 - \nu^1_i(u))$. Finally, from (4) and the last inequality, we can conclude that the thesis holds.

Moreover, for each $u \in U$, $\mu_i^1(u), \mu_i^2(u) \leq \alpha$ by hypothesis. Thus, for each $u \in U$, $\max\{\mu_i^1(u), \mu_i^2(u)\} \leq \alpha$. Lastly, $O_1 \cup_O O_2 \in OR_\mu^\alpha$. □

Theorem 7. *$O_1 +_O O_2 \in OR$ for each $O_1, O_2 \in OR_\mu^\alpha$, when $\alpha \leq \frac{1}{2n}$.*

Proof. Let $O_1, O_2 \in OR_\mu^\alpha$, we want to prove that Axioms (i) and (ii) of Definition 8 are satisfied by $O_1 +_O O_2$. Recall that the ith intuitionistic fuzzy set of $O_1 +_O O_2$ is

$$(\mu_i^1 + \mu_i^2 - \mu_i^1\mu_i^2, \nu_i^1\nu_i^2)$$

from Definition 4 (iii).

(Axiom (i)). Let $u \in U$. Since $O_1, O_2 \in OR_\mu^\alpha$, we get $\sum_{i\in I} \mu_i^1(u) \leq n\alpha$ and $\sum_{i\in I} \mu_i^2(u) \leq n\alpha$ (the reasoning is the same given in the previous theorems). Then, $\sum_{i\in I}(\mu_i^1(u) + \mu_i^2(u)) \leq n\alpha + n\alpha = 2n\alpha$. By hypothesis, $\alpha \leq \frac{1}{2n}$. Hence, $2n\alpha \leq 1$. Consequently, $\sum_{i\in I}(\mu_i^1(u) + \mu_i^2(u)) \leq 1$. Thus, we can conclude that $\sum_{i\in I}(\mu_i^1(u) + \mu_i^2(u)) - \sum_{i\in I}\mu_i^1(u)\mu_i^2(u) \leq 1$. Finally, it is easy to understand that $\sum_{i\in I} \mu_i^1(u) + \mu_i^2(u) - \mu_i^1(u)\mu_i^2(u) \leq 1$.

(Axiom (ii)). Let $u \in U$. Since O_1 is a generalized fuzzy orthopartition of U, we get $\sum_{i\in I}(1 - \nu_i^1(u)) \geq 1$ (see Definition 8 (Axiom (ii)) and Remark 4). Moreover, considering that $\nu_i^1(u), \nu_i^2(u) \in [0,1]$, it is true that $\nu_i^1(u)\nu_i^2(u) \leq \nu_i^1(u)$ for each $i \in I$. Hence, $1 - \nu_i^1(u) \leq 1 - (\nu_i^1(u)\nu_i^2(u))$ for each $i \in I$. As a consequence, $\sum_{i\in I}(1 - \nu_i^1(u)\nu_i^2(u)) \geq \sum_{i\in I}(1 - \nu_i^1(u))$. Ultimately, we can conclude that $\sum_{i\in I}(1 - (\nu_i^1(u)\nu_i^2(u))) \geq 1$. □

Remark 9. Let $O_1, O_2 \in OR_\mu^\alpha$, where $\alpha \leq \frac{1}{2n}$. Despite $O_1 +_O O_2$ being a generalized fuzzy orthopartition, it does belong to OR_μ^α. Let us show an example. Suppose that $O_1 = \{(\mu_1^1, \nu_1^1), (\mu_2^1, \nu_2^1)\}$ and $O_2 = \{(\mu_1^2, \nu_1^2), (\mu_2^2, \nu_2^2)\}$ belong to OR_μ^α, where $\alpha = 0.2$. Also, assume that $\mu_1^1(u) = 0.2$ and $\mu_1^2(u) = 0.2$, where $u \in U$. The condition of Theorem 7 is satisfied by α because $0.2 \leq 0.25$. Then, $O_1 +_O O_2$ is a generalized fuzzy orthopartition. However, $O_1 +_O O_2$ does not belong to OR_μ^α. Indeed, the first IFS of $O_1 +_O O_2$ is defined by $\mu_1^1 + \mu_1^2 - \mu_1^1\mu_1^2$. Consequently, $\mu_1^1(u)\mu_1^2(u) - \mu_1^1(u)\mu_1^2(u) = 0.2 + 0.2 - (0.04) = 0.36$, which is greater than $\alpha = 0.2$.

Theorem 8. *$O_1 \odot_O O_2 \in OR$ for each $O_1, O_2 \in OR_\nu^\alpha$, when $\alpha \leq \frac{1}{2}\left(1 - \frac{1}{n}\right)$.*

Proof. Let $O_1, O_2 \in OR_\nu^\alpha$, we want to prove that Axioms (i) and (ii) of Definition 8 are satisfied by $O_1 \odot_O O_2$. Recall that the ith intuitionistic fuzzy set of $O_1 \odot_O O_2$ is

$$(\mu_i^1\mu_i^2, \nu_i^1 + \nu_i^2 - \nu_i^1\nu_i^2)$$

from Definition 4 (iv).

(Axiom (i)). Let $u \in U$. Since O_1 is a generalized fuzzy orthopartition on U, we get $\sum_{i\in I} \mu_i^1(u) \leq 1$ from Definition 8 (i). Also, considering that $\mu_i^1(u), \mu_i^2(u) \in [0,1]$, $\mu_i^1(u)\mu_i^2(u) \leq \mu_i^1(u)$ for each $i \in I$. Therefore, $\sum_{i\in I} \mu_i^1(u)\mu_i^2(u) \leq \sum_{i\in I}\mu_i^1(u)$. Consequently, we get $\sum_{i\in I} \mu_i^1(u)\mu_i^2(u) \leq 1$.

(Axiom (ii)). Let $u \in U$. Since $O_1, O_2 \in OR_\nu^\alpha$, we get $\sum_{i\in I} \nu_i^1(u) \leq n\alpha$ and $\sum_{i\in I} \nu_i^2(u) \leq n\alpha$. Then, $\sum_{i\in I}(\nu_i^1(u) + \nu_i^2(u)) \leq n\alpha + n\alpha = 2n\alpha$. By hypothesis, $2n\alpha \leq n - 1$. Thus, $\sum_{i\in I} \nu_i^1(u) + \nu_i^2(u) \leq n - 1$. Finally, it is simple to deduce that $\sum_{i\in I}(\nu_i^1(u) + \nu_i^2(u) - \nu_i^1(u)\nu_i^2(u)) \leq n - 1$. □

Remark 10. Let $O_1, O_2 \in OR_\nu^\alpha$, where $\alpha \leq \frac{1}{2}\left(1 - \frac{1}{n}\right)$. Despite $O_1 \odot_O O_2$ being a generalized fuzzy orthopartition, it does always belong to OR_ν^α. This can be verified with an example similar to that shown by Remark 9.

Theorem 9. *$OR_\mu^\alpha \cap OR_\nu^\alpha$ is closed under $-_O$, when $\alpha \leq 1 - \frac{1}{n}$.*

Proof. Let $O_1, O_2 \in OR^{\alpha}_{\mu} \cap OR^{\alpha}_{\nu}$, we want to prove that Axioms (i) and (ii) of Definition 8 are satisfied by $O_1 -_O O_2$. Recall that the ith intuitionistic fuzzy sets of $O_1 -_O O_2$ is $(\min\{\mu^1_i, \nu^2_i\}, \max\{\nu^1_i, \mu^2_i\})$ from Definition 4 (vii).

(Axiom (i).) Let $u \in U$. Then, $\min\{\mu^1_i(u), \nu^2_i(u)\} \leq \mu_i(u)$ for each $i \in I$. Hence, $\sum_{i\in I} \min\{\mu^1_i(u), \nu^2_i(u)\} \leq \sum_{i\in I} \mu_i(u)$. Since O_1 is a generalized fuzzy orthopartition, we get $\sum_{i\in I} \mu_i(u) \leq 1$. Finally, $\sum_{i\in I} \min\{\mu^1_i(u), \nu^2_i(u)\} \leq 1$.

(Axiom (ii).) Let $u \in U$. Analogously to the previous point, we know that $\sum_{i\in I} \max\{\nu^1_i(u), \mu^2_i(u)\} \leq n\alpha$ because $\nu^1_i(u), \mu^2_i(u) \leq \alpha$ for each $i \in I$ (recall that $O_1, O_2 \in OR^{\alpha}_{\mu} \cap OR^{\alpha}_{\nu}$). Furthermore, considering that $n\alpha \leq n-1$ from hypothesis, $\sum_{i\in I} \max\{\nu^1_i(u), \mu^2_i(u)\} \leq n-1$.

Moreover, for each $u \in U$, $\mu^1_i(u), \mu^2_i(u), \nu^1_i(u), \nu^2_i(u) \leq \alpha$ by hypothesis. Thus, for each $u \in U$, $\min\{\mu^1_i(u), \nu^2_i(u)\} \leq \alpha$ and $\max\{\nu^1_i(u), \mu^2_i(u)\} \leq \alpha$. Lastly, we can conclude $O_1 -_O O_2 \in OR^{\alpha}_{\mu} \cap OR^{\alpha}_{\nu}$. □

Theorem 10. *$\neg_O O \in OR$ for each $O \in OR^{\alpha}_{\nu}$, when $\alpha \leq \frac{1}{n}$ and $n \geq 2$.*

Proof. Let $O \in OR^{\alpha}_{\nu}$, we intend to prove that $\neg_O O$ satisfies Axioms (i) and (ii) of Definition 8. Recall that the ith intuitionistic fuzzy set of $\neg_O O$ is (ν_i, μ_i) from Definition 4 (viii).

(Axiom (i)). Let $u \in U$. Since $O \in OR^{\alpha}_{\nu}$, $\sum_{i\in I} \nu_i(u) \leq n\alpha$. Therefore, $\sum_{i\in I} \nu_i(u) \leq 1$ because $n\alpha \leq 1$ from hypothesis.

(Axiom (ii)). Since O is a generalized fuzzy orthopartition on U, we get $\sum_{i\in I} \mu_i(u) \leq 1$ from Definition 8 (i). Trivially, $\sum_{i\in I} \mu_i(u) \leq n-1$, due to the assumption $n \geq 2$. □

Remark 11. Let $O \in OR^{\alpha}_{\nu}$, where $\alpha \leq \frac{1}{n}$ and $n \geq 2$. Then, $\neg_O O$ is a generalized fuzzy orthopartition, but it does not always belong to OR^{α}_{ν}.

Remark 12. Let $O \in OR$ such that $n = 1$, namely $O = \{(\mu, \nu)\}$. By Remark 6, $\nu(u) = 0$ for each $u \in U$. Also, we have $\neg_O O \in OR$, only when $\mu(u) = 0$ for each $u \in U$. Therefore, $\neg_O O$ is a generalized fuzzy orthopartition if and only if (μ, ν) is the IFS on U such that $\mu(u) = \nu(u) = 0$ for each $u \in U$.

Let us define the sub-class of OR:

$$\overline{OR} = \left\{ O = \{(\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\} \in OR \mid \sum_{i\in I} (\mu_i(u) + h_i(u)) = 1 \right\}, \tag{5}$$

where it makes sense to consider the operation $\Diamond_O$.

Theorem 11. *$\overline{OR}$ is closed under $\Diamond_O$.*

Proof. Let $O \in \overline{OR}$. Recall that the ith intuitionistic fuzzy set of $\Diamond_O O$ is $(1-\nu_i, \nu_i)$.

If $O \in \overline{OR}$, then $\sum_{i\in I} (\mu_i(u) + h_i(u)) = \sum_{i\in I} (1 - \nu_i(u)) = 1$ by (5). Then, Axioms (ii) is clearly satisfied by $\Diamond_O O$. The same holds for Axiom (i) by considering that the inequality $\sum_{i\in I} (\mu_i(u) + h_i(u)) = 1$ trivially implies the equality $\sum_{i\in I} \mu_i(u) \leq 1$.

Clearly, $\Diamond_O O \in \overline{OR}$, considering that the membership functions of O and $\Diamond_O$ are the same. □

Remark 13. Let $O = \{(\mu_1, \nu_1), \dots, (\mu_n, \nu_n)\} \in OR$. When $\Diamond_O O$ is a generalized fuzzy orthopartition, it coincides with the Ruspini partition $\{1-\nu_1, \dots, 1-\ \nu_n\}$ as explained in Section 3.

Let us show an example. Consider the generalized fuzzy orthopartition O of $\{u_1, u_2\}$ defined by Table 4. Then, $\Diamond_O O \in OR$ is defined by Table 5.

We can observe that $O \in \overline{OR}$ because $(\mu_1(u_1) + h_1(u_1)) + (\mu_2(u_1) + h_2(u_1)) = (0.5 + 0.1) + (0.2 + 0.2) = 1$ and $(\mu_1(u_2) + h_1(u_2)) + (\mu_2(u_2) + h_2(u_2)) = (0.5 + 0.2) + (0 + 0.3) = 1$. Hence, by Theorem 11, $\Diamond_O O$ is a generalized fuzzy orthopartition of $\{u_1, u_2\}$.

Moreover, $\Diamond_O O$ is equivalent to the Ruspini partition $\pi = \{\pi_1, \pi_2\}$ of $\{u_1, u_2\}$ such that $\pi_1(u_1) = 1 - \nu'_1(u_1) = 0.6$, $\pi_2(u_1) = 1 - \nu'_2(u_1) = 0.4$, $\pi_1(u_2) = 1 - \nu'_1(u_2) = 0.7$, and $\pi_2(u_2) = 1 - \nu'_2(u_2) = 0.3$.

According to the previous considerations, the operation $\Diamond_O$ transforms (when it is possible) a generalized fuzzy orthopartition into a Ruspini partition by taking into account exclusively the non-membership degrees of the elements.

Table 4
Definition of the elements of O.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0.5	0.4	0.2	0.6
u_2	0.5	0.3	0	0.7

Table 5
Definition of the elements of $\Diamond_O O$.

u	$\mu_1'(u)$	$\nu_1'(u)$	$\mu_2'(u)$	$\nu_2'(u)$
u_1	0.6	0.4	0.4	0.6
u_2	0.7	0.3	0.3	0.7

The following proposition shows that $\overline{OR}$ is the biggest set of fuzzy orthopartitions of OR closed under $\Diamond_O$.

Proposition 1. *Let $O \in OR$, then*

$\Diamond_O O \in OR$ if and only if $O \in \overline{OR}$.

Proof. ($\Leftarrow$). This implication corresponds to Theorem 11.

($\Rightarrow$). Let $O \notin \overline{OR}$, we intend to prove that $\Diamond_O O$ is not a generalized fuzzy orthopartition. By (5), since $O \notin \overline{OR}$, there exists $u \in U$ such that $\sum_{i\in I}(\mu_i(u) + h_i(u)) > 1$, namely

$$\sum_{i\in I}(1 - \nu_i(u)) > 1. \tag{6}$$

Recalling that $\Diamond_O O = \{(1-\nu_1, \nu_1), \dots, (1-\nu_n, \nu_n)\}$, it is clear that (6) implies that Axiom (i) of Definition 8 is not satisfied by $\Diamond_O O$. Then, the thesis clearly holds. $\square$

Observe that by Remark 13 and Proposition 1, $\overline{OR}$ can be viewed as the collection of all generalized fuzzy orthopartitions of OR, which can be identified with Ruspini partitions when we only consider the non-membership degrees of the elements.

Now, in order to consider the operations $\mathcal{C}_O$ and $\mathcal{I}_O$ on generalized fuzzy orthopartitions, we deal with the sub-classes OR^*_α and OR^{**}_α of OR defined as follows:

$$OR^*_\alpha = \{O \in OR \mid \forall i \in I \text{ “}\exists \bar{u} \in U \text{ with } \nu_i(\bar{u}) \leq \alpha\text{” and “}\forall u \in U\ \ \mu_i(u) \leq \alpha\text{”}\} \tag{7}$$

and

$$OR^{**}_\alpha = \{O \in OR \mid \forall i \in I \text{ “}\exists \bar{u} \in U \text{ with } \mu_i(\bar{u}) \leq \alpha\text{” and “}\forall u \in U\ \ \nu_i(u) \leq \alpha\text{”}\}. \tag{8}$$

Remark 14. It is clear that $OR^*_\alpha \subseteq OR^\alpha_\mu$ and $OR^{**}_\alpha \subseteq OR^\alpha_\nu$.

Example 4. Let O_1 and O be the generalized fuzzy orthopartitions given by Tables A.7 and A.12, respectively. Thus, we can notice that $O_1 \in O^*_{0.5}$. This is because

$$\mu^1_1(u_1) = 0.1, \mu^1_1(u_2) = 0.3, \mu^1_2(u_1) = 0.4, \text{ and } \mu^1_2(u_2) = 0.5, \tag{9}$$

which are all less than or equal to 0.5. Moreover, we get $\nu^1_1(u_1) = 0.3$ and $\nu^1_2(u_2) = 0.1$, which are less than 0.5. According to (9), $O_1 \in OR^{0.5}_\mu$.

Now, let us focus on O. Certainly, $O \in O^{**}_{0.5}$. In fact, we get

$$\nu_1(u_1) = 0.3, \nu_1(u_2) = 0.4, \nu_2(u_1) = 0.5, \text{ and } \nu_2(u_2) = 0, \tag{10}$$

which are less than 0.5. Furthermore, despite $\mu_2(u_2) = 0.6$, which is greater than 0.5, the remaining values $\mu_1(u_1), \mu_1(u_2), \mu_2(u_1)$ are less than 0.5. Additionally, according to (10), it is also true that $O \in OR^{0.5}_\nu$.

Remark 15. Given $O = \{(\mu, \nu)\} \in OR$, then $\mathcal{C}_O O$ is always a generalized fuzzy orthopartition. This is due to the following considerations.

By Definition 4 (xi), $\mathcal{C}_O O = \{(\mu', \nu')\}$, where

$$\mu'(\bar{u}) = \max_{u\in U} \mu(u) \quad \text{and} \quad \nu'(\bar{u}) = \min_{u\in U} \nu(u) \quad \text{for each } \bar{u} \in U. \tag{11}$$

By Remark 6, $\nu(u) = 0$ for each $u \in U$. Then, by (11), $\nu'(u) = 0$ for each $u \in U$. Using Remark 6 again, we are sure that $\mathcal{C}_O O \in OR$.

Theorem 12. *OR^*_α is closed under $\mathcal{C}_O$, when $\alpha \le \frac{1}{n}$.*

Proof. Let $O \in OR^*_\alpha$, we intend to prove that $\mathcal{C}_O O$ satisfies Axioms (i) and (ii) of Definition 8. Recall that the ith intuitionistic fuzzy set (p_i, q_i) of $\mathcal{C}_O O$ is (p, q), where the constant functions p and q are given by Definition 4 (xi).

(Axiom (i)). Let $\bar{u} \in U$, then $p_i(\bar{u}) = \max_{u \in U} \mu_i(u)$ for each $i \in I$. Since $O \in OR^*_\alpha$, $\mu_i(u) \le \alpha$ for each $u \in U$ and $i \in I$. So, we are sure that $p_i(\bar{u}) \le \alpha$ for each $i \in I$. Lastly, $\sum_{i \in I} p_i(\bar{u}) \le n\alpha$. Also, $\alpha \le \frac{1}{n}$ from hypothesis. Namely, $n\alpha \le 1$. Then, we can deduce that $\sum_{i \in I} p_i(\bar{u}) \le 1$.

(Axiom (ii)). Let $\bar{u} \in U$, then $q_i(\bar{u}) = \min_{u \in U} \nu_i(u)$ for each $i \in I$. Let us distinguish two cases: $n = 1$ and $n \ge 2$.
If $n = 1$, then $\sum_{i \in I} q_i(\bar{u}) = 0$. This is because $\nu_i(u) = 0$ for each $u \in U$ (see Remark 6). Hence, $\sum_{i \in I} q_i(\bar{u}) \le n - 1$ trivially holds.
Suppose that $n \ge 2$. Since $O \in OR^*_\alpha$, there exists $u \in U$ such that $\nu_i(u) \le \alpha$. Therefore, we get $q_i(\bar{u}) \le \alpha$ for each $i \in I$, which implies that $\sum_{i \in I} q_i(\bar{u}) \le n\alpha$. Then, considering that $\alpha \le \frac{1}{n}$ from hypothesis, $\sum_{i \in I} q_i(\bar{u}) \le 1$. Also, $n \ge 2$ implies that $1 \le n - 1$. Thus, we can conclude with no effort that $\sum_{i \in I} q_i(\bar{u}) \le n - 1$.

Finally, it is easy to understand that $\mathcal{C}_O O \in OR^*_\alpha$ because $p_i(u), q_i(u) \le \alpha$ for each $i \in I$ and $u \in U$. □

Remark 16. Similarly to Remark 15, we can observe that $\mathcal{I}_O O$ is always a generalized fuzzy orthopartition, when $O = \{(\mu, \nu)\} \in OR$.

Theorem 13. *OR^{**}_α is closed under $\mathcal{I}_O$, when $\alpha \le \frac{1}{n}$.*

Proof. Let $O \in OR^{**}_\alpha$, we intend to prove that $\mathcal{I}_O O$ satisfies Axioms (i) and (ii) of Definition 8. Recall that the ith intuitionistic fuzzy set (r_i, s_i) of $\mathcal{I}_O O$ is (r, s), where the constant functions r and s are given by Definition 4 (xii).

(Axiom (i)). Let $\bar{u} \in U$, then $r_i(\bar{u}) = \min_{u \in U} \mu_i(u)$ for each $i \in I$. Since $O \in OR^{**}_\alpha$, for each $i \in I$, there exists $u \in U$ such that $\mu_i(u) \le \alpha$. Then, $r_i(\bar{u}) \le \alpha$ for each $i \in I$. Of course, this leads that $\sum_{i \in I} r_i(\bar{u}) \le n\alpha$. Therefore, $\sum_{i \in I} r_i(\bar{u}) \le 1$, considering that $n\alpha \le 1$ from hypothesis.

(Axiom (ii)). Let $\bar{u} \in U$, then $s_i(\bar{u}) = \max_{u \in U} \nu_i(u)$ for each $i \in I$.
As in the proof of the previous theorem, let us distinguish two cases: $n = 1$ and $n \ge 2$.
If $n = 1$, then $\sum_{i \in I} s_i(\bar{u}) = 0$. This is because $\nu_i(u) = 0$ for each $u \in U$ (see Remark 6). Hence, $\sum_{i \in I} s_i(\bar{u}) \le n - 1$ trivially holds.
Suppose that $n \ge 2$. Since $O \in OR^{**}_\alpha$, $\nu_i(u) \le \alpha$ for each $i \in I$ and $u \in U$. Therefore, we get $s_i(\bar{u}) \le \alpha$ for each $i \in I$, which implies that $\sum_{i \in I} s_i(\bar{u}) \le n\alpha$. By hypothesis, $n\alpha \le 1$. Also, $n \ge 2$ implies that $1 \le n - 1$. Then, we can simply deduce that $\sum_{i \in I} s_i(\bar{u}) \le n - 1$.

Finally, it is easy to understand that $\mathcal{I}_O O \in OR^{**}_\alpha$ because $s_i(u), r_i(u) \le \alpha$ for each $i \in I$ and $u \in U$. □

Table 6 lists all the operations investigated in this section together with the corresponding sub-classes of OR and conditions where and under which they are defined.

Remark 17. The operations listed in Table 6 should be used to aggregate generalized fuzzy orthopartitions to solve concrete problems. Certainly, the choice of the most suitable operation depends on the real problem we want to solve. It is clear that the limitation of such operations is that they can be only considered on a part of all generalized fuzzy orthopartitions. As a consequence, we can also think about choosing the operation requiring the most general condition, in order to deal with a vast number of cases. For example, we could prefer the operation $*_O$, instead of $\cap_O$.
Moreover, in order to be free to choose any operation

$$\& \in \{\cap_O, \cup_O, +_O, \odot_O, *_O, \times_O, -_O, \neg_O, \Box_O, \Diamond_O, \mathcal{C}_O, \mathcal{I}_O\},$$

we could think to relax the definition of generalized fuzzy orthopartitions (Definition 8) considering the conditions

$$\sum_{i \in I} \mu_i(u) \le 1 + \varepsilon_\& \text{ and } \sum_{i \in I} \mu_i(u) + h_i(u) \ge 1 - \varepsilon^*_\& \text{ with } \varepsilon, \varepsilon^* \in [0,1],$$

Table 6
Operations on generalized fuzzy orthopartitions of Section 4. The first column lists the operations. The second column refers to the sub-classes of OR, where the operations are defined. The third column shows the conditions under which the operations can be considered. The last column lists the number of the theorems corresponding to the operations.

Operation	$X \subseteq OR$	Condition	Theorem
$\cap_O$	OR^α_ν	$\alpha \leq 1 - \frac{1}{n}$	5
$\cup_O$	OR^α_μ	$\alpha \leq \frac{1}{n}$	6
$+_O$	OR^α_μ	$\alpha \leq \frac{1}{2n}$	7
$\odot_O$	OR^α_ν	$\alpha \leq \frac{1}{2}\left(1 - \frac{1}{n}\right)$	8
$*_O$	OR	None	4
$\times_O$	OR	None	4
$-_O$	$OR^\alpha_\mu \cap OR^\alpha_\nu$	$\alpha \leq 1 - \frac{1}{n}$	9
$\neg_O$	OR^α_ν	$\alpha \leq \frac{1}{n}$ and $n \geq 2$	10
$\Box_O$	OR	None	4
$\Diamond_O$	$\overline{OR}$	None	11
$\mathcal{C}_O$	OR^*_α	$\alpha \leq \frac{1}{n}$	12
$\mathcal{I}_O$	OR^{**}_α	$\alpha \leq \frac{1}{n}$	13

instead of the standard ones. Such new conditions lead to the definition of a collection $OR_\&$ of objects larger than OR. Then, the thresholds could be chosen by finding a compromise between the generality of the models in $OR_\&$ (how much they are different from generalized fuzzy orthopartitions given by Definition 8) and the condition so that $OR_\&$ is closed under &.

Generalized fuzzy orthopartitions as bounded lattices We can prove that $OR^\alpha_\mu \cap OR^\alpha_\nu$ equipped with the operations $\cap_O$ and $\cup_O$ is an example of a bounded lattice, under the condition $\alpha \leq \frac{1}{n}$.

The top and the bottom are respectively denoted with the symbols $O_{(0,\alpha)}$ and $O_{(\beta,0)}$ with $\alpha, \beta \in [0,1]$ and they are defined as follows:

- $(\mu, \nu) \in O_{(0,\alpha)}$ if and only if $\mu(u) = 0$; $\nu(u) = \alpha$ for each $u \in U$ and
- $(\mu, \nu) \in O_{(\beta,0)}$ if and only if $\mu(u) = \beta$ and $\nu(u) = 0$ for each $u \in U$.

In other words, $O_{(0,\alpha)}$ is made of intuitionistic fuzzy sets that are all equal one each other and where the first component is the constant function assigning 0 to each $u \in U$ and the second component is the constant function assigning α to each $u \in U$. In a dual way, $O_{(\beta,0)}$ is made of intuitionistic fuzzy sets that are all equal one each other and where the first component is the constant function assigning β to each $u \in U$ and the second component is the constant function assigning 0 to each $u \in U$.

Remark 18. Of course, both $O_{(\alpha,0)}$ and $O_{(0,\beta)}$ belong to $OR^\alpha_\mu \cap OR^\beta_\nu$.

Theorem 14. *Let $\alpha, \beta \in [0,1]$ such that $\alpha, \beta \leq \frac{1}{n}$, then*

$$(OR^\alpha_\mu \cap OR^\beta_n, \cap_O, \cup_O, O_{(0,\beta)}, O_{(\alpha,0)}) \tag{12}$$

is a bounded lattice.

Proof. It can be easily shown that $(OR^\alpha_\mu \cap OR^\beta_\nu, \cap_O, \cup_O)$ is a lattice.

- Firstly, let us show that $OR^\alpha_\mu \cap OR^\beta_\nu$ is closed under $\cap_O$ and $\cup_O$.
  Then, we consider the generalized fuzzy orthopartitions O_1 and O_2 of $OR^\alpha_\mu \cap OR^\beta_\nu$.
  Since $O_1, O_2 \in OR^\alpha_\mu$ and $\alpha \leq \frac{1}{n}$, $O_1 \cup_O O_2$ is a generalized fuzzy orthopartition from Theorem 6.
  Let us prove that $O_1 \cap_O O_2$ is a generalized fuzzy orthopartition too.
  We distinguish two cases: $n = 1$ and $n \geq 2$.
  Let $n = 1$, then $O_1 = \{(\mu^1, \nu^1)\}$ and $O_2 = \{(\mu^2, \nu^2)\}$.
  Clearly, Axioms (i) and (ii) of Definition 8 are satisfied by $O_1 \cap_O O_2$: let $u \in U$,
  - $\min\{\mu^1(u), \mu^2(u)\} \leq \alpha$, which is less than or equal to 1 from the hypothesis;

– $\max\{\nu^1(u), \nu^2(u)\} = 0$, which is equal to $n-1$ (recall that $\nu^1(u) = \nu^2(u) = 0$ for each $u \in U$).
Then, $O_1 \cap_O O_2 \in OR$.
Let $n \geq 2$. Trivially, we have $\frac{1}{n} \leq \left(1 - \frac{1}{n}\right)$. Therefore, since $O_1, O_2 \in OR_\nu^\beta$ and $\beta \leq 1 - \frac{1}{n}$, $O_1 \cap_O O_2$ is a generalized fuzzy orthopartition from Theorem 5.
Now, we can easily verify that $O_1 \cap_O O_2$ and $O_1 \cup_O O_2$ belong to OR_μ^α and OR_ν^β. The ith intuitionistic fuzzy sets of $O_1 \cap_O O_2$ and $O_1 \cup_O O_2$ are respectively

$$(\min\{\mu_i^1, \mu_i^2\}, \max\{\nu_i^1, \nu_i^2\}) \text{ and } (\max\{\mu_i^1, \mu_i^2\}, \min\{\nu_i^1, \nu_i^2\}).$$

Since $O_1, O_2 \in OR_\mu^\alpha \cap OR_\nu^\beta$, it must be true that $\mu_i^1(u), \mu_i^2(u) \leq \alpha$ and $\nu_i^1(u), \nu_i^2(u) \leq \beta$ for each $u \in U$. Hence, we get

$$\min\{\mu_i^1(u), \mu_i^2(u)\}, \max\{\mu_i^1(u), \mu_i^2(u)\} \leq \alpha \text{ and } \min\{\nu_i^1(u), \nu_i^2(u)\},$$
$$\max\{\nu_i^1(u), \nu_i^2(u)\} \leq \beta, \text{ for each } u \in U.$$

As a consequence, we can deduce that $O_1 \cap_O O_2, O_1 \cup_O O_2 \in OR_\mu^\alpha \cap OR_\nu^\beta$.

- Secondly, the set of all intuitionistic fuzzy sets equipped with the operations $\cap$ and $\cup$ is a lattice (see Section 2), then the same properties clearly hold on their extensions $\cap_O$ and $\cup_O$. Furthermore, considering that $\min\{\mu_i^1(u), \mu_i^2(u)\}, \min\{\mu_i^1(u), \mu_i^2(u)\} \leq \alpha$, $O_1 \cup_O O_2$ belongs to OR_μ^α too.

Let $O \in O_\mu^\alpha \cap O_\nu^\beta$ and let $(\mu, \nu) \in O$. Trivially, $\mu(u) \leq \alpha$ and $\nu(u) \geq 0$ for each $u \in U$. Moreover, $0 \leq \mu(u)$ and $\beta \geq \nu(u)$ for each $u \in U$. Thus, $O \preceq_O O_{(\alpha,0)}$ and $O_{(\beta,0)} \preceq_O O$. Therefore, we can conclude that $O_{(0,\beta)}$ and $O_{(\alpha,0)}$ are respectively the bottom and the top of $OR_\mu^\alpha \cap OR_\nu^\beta$. □

Remark 19. The structure $(OR_\mu^\alpha \cap OR_\nu^\beta, \preceq_O)$ [7] is a lattice equivalent to (12). This means that let $O_1, O_2 \in OR_\mu^\alpha \cap OR_\nu^\beta$, $O_1 \cap_O O_2 = O_1$ if and only if $O_1 \preceq_O O_2$.

5. Entropy measures on generalized fuzzy orthopartitions

In this section, we firstly extend the intuitionistic entropy measures of Definition 5 to generalized fuzzy orthopartitions. After that, we explain the meaning of the proposed entropy measures by focusing on their property of monotonicity.

5.1. Definition of entropies of generalized fuzzy orthopartitions

Let $O = \{(\mu_i, \nu_i) \mid i \in I\}$ be a generalized fuzzy orthopartition, we propose to define entropies of O in two steps:

1. by calculating $m(\mu_i, \nu_i)$ for each IFS (μ_i, ν_i) of O, where $m \in \{\eta_1, \eta_2\}$ is an entropy measure given in Definition 5;
2. by aggregating all the entropies $m(\mu_1, \nu_1), \dots, m(\mu_n, \nu_n)$ by means of the minimum, the maximum, or the arithmetic mean.

This leads to the following formal definition of entropy measures on generalized fuzzy orthopartitions.

Definition 15. Let $O = \{(\mu_i, \nu_i) \mid i \in I\} \in OR$ and let $k \in \{1, 2\}$. Then,

(i) $\mathcal{H}_{*k}(O) = \min\{\eta_k(\mu_i, \nu_i) \mid i \in I\}$;
(ii) $\mathcal{H}_k^*(O) = \max\{\eta_k(\mu_i, \nu_i) \mid i \in I\}$;
(iii) $\tilde{\mathcal{H}}_k(O) = \frac{1}{n} \sum_{i \in I} \eta_k(\mu_i, \nu_i)$.

Example 5. Considering the generalized fuzzy orthopartition O defined by Tables 4, then, by Definition 5, we get

1. $\eta_1(\mu_1, \nu_1) = 1 - \frac{0.1 + 0.2}{4} = 0.925$
   and $\eta_1(\mu_2, \nu_2) = 1 - \frac{0.4 + 0.7}{4} = 0.725$;
2. $\eta_2(\mu_1, \nu_1) = 0.1 + 0.2 = 0.3$ and $\eta_2(\mu_2, \nu_2) = 0.2 + 03 = 0.5$.

Hence, by Definition 15, we can compute the entropies of O:

1. $\mathcal{H}_{*1}(O) = \min\{0.925, 0.725\} = 0.725$,
   $\mathcal{H}_1^*(O) = \max\{0.925, 0.725\} = 0.925$,
   and $\tilde{\mathcal{H}}_1(O) = \frac{0.925 + 0.725}{2} = 0.825$;

2. $\mathcal{H}_{*2}(O) = \min\{0.3, 0.5\} = 0.3$,
$\mathcal{H}^*_2(O) = \max\{0.3, 0.5\} = 0.5$,
and $\tilde{\mathcal{H}}_2(O) = \dfrac{0.3 + 0.5}{2} = 0.4$.

5.2. Properties of the entropies of generalized fuzzy orthopartitions

In the sequel, let us consider two special cases of generalized fuzzy orthopartitions:

a) the null orthopartition $O_0 = \{(\mu_i, \nu_i) \mid i \in I\}$ such that

$$h_i(u) = 1 \text{ for each } i \in I \text{ and } u \in U. \tag{13}$$

b) the set OR^* of orthopartitions defined as

$$OR^* = \{O = \{(\mu_i, \nu_i) \mid i \in I\} \in OR \mid \mu_i(u) = \nu_i(u), \forall i \in I, \forall u \in U\}. \tag{14}$$

It is easy to see that (13) is equivalent to $\mu_i(u) = \nu_i(u) = 0$ for each $i \in I$ and $u \in U$. Thus O_0 does not give us any information about the relationship between the elements and the classes.

The special collection of generalized fuzzy orthopartitions OR^* is composed of all generalized fuzzy orthopartitions of OR that capture no difference between the membership and non-membership degrees of elements to the classes. Indeed, let $O \in OR^*$, we do not have significant information about the relationship between elements of the universe and classes of the partition: let $(\mu_i, \nu_i) \in O$, we know that "u belongs to the class i" and "u does not belong to the class i" with the same truth degree.

Furthermore, let us recall that by Remark 5, a generalized fuzzy orthopartition $O = \{(\mu_i, \nu_i) \mid i \in I\}$ is a standard partition if and only if

$$\text{"}\mu_i(u) = 1 \text{ and } \nu_i(u) = 0\text{" or "}\mu_i(u) = 0 \text{ and } \nu_i(u) = 1\text{"}, \quad \text{for each } u \in U \text{ and } i \in I. \tag{15}$$

By Definition 15, the entropies of generalized fuzzy orthopartitions are real numbers. More precisely, the following propositions show

- the membership interval of the entropies of a generalized fuzzy orthopartition;
- the characteristics that a generalized fuzzy orthopartition needs to have so that its entropies are minimum or maximum (i.e. they coincide with one of the extremes of the interval).

Proposition 2. *Let $O = \{(\mu_i, \nu_i) \mid i \in I\}$ be a generalized fuzzy orthopartition. Then,*

(a) $\mathcal{H}_{*1}(O), \mathcal{H}^*_1(O), \tilde{\mathcal{H}}_1(O) \in \left[\frac{1}{2}, 1\right]$;
(b) $\mathcal{H}^*_1(O) = \tilde{\mathcal{H}}_1(O) = \frac{1}{2}$ *if and only if O is a standard partition;*
(c) $\mathcal{H}_{*1}(O) = \tilde{\mathcal{H}}_1(O) = 1$ *if and only if $O \in OR^*$.*

Proof. (a) By Definition 5 (i), it is easy to see that, for all $i \in I$, we get $\eta_1(\mu_i, \nu_i) \in \left[\frac{1}{2}, 1\right]$. Thus, according to Definition 15, the thesis (a) clearly holds.

(b) ($\Rightarrow$). By Definition 15 and by the previous point, $\mathcal{H}^*_1(O) = \tilde{\mathcal{H}}_1(O) = \frac{1}{2}$ implies that $\eta_1(\mu_i, \nu_i) = \frac{1}{2}$ for each $i \in I$. By Definition 5 (i), the latter is possible when $|\mu_i(u) - \nu_i(u)| = 1$ for each $u \in U$ and $i \in I$; namely, "$\mu_i(u) = 1$ and $\nu_i(u) = 0$" or "$\mu_i(u) = 0$ and $\nu_i(u) = 1$" for each $u \in U$ and $i \in I$. Finally, by (15), O is a standard partition.
($\Leftarrow$). By (15), when O is a standard partition, it is true that "$\mu_i(u) = 1$ and $\nu_i(u) = 0$" or "$\mu_i(u) = 0$ and $\nu_i(u) = 1$", for each $i \in I$ and $u \in U$. Then, by Definition 5(i), $\eta_1(\mu_i, \nu_i) = \frac{1}{2}$ for each $i \in I$. Ultimately, by Definition 15, it is clear that $\mathcal{H}^*_1(O) = \tilde{\mathcal{H}}_1(O) = \frac{1}{2}$.

(c) ($\Rightarrow$). By Definition 15 and by item (a), $\mathcal{H}_{*1}(O) = \tilde{\mathcal{H}}_1(O) = 1$ implies that $\eta_1(\mu_i, \nu_i) = 1$ for each $i \in I$. By Definition 5 (i), the latter is possible when $|\mu_i(u) - \nu_i(u)| = 0$ for each $u \in U$ and $i \in I$; namely, $\mu_i(u) = \nu_i(u)$ for each $u \in U$ and $i \in I$. Thus, by (14), $O \in OR^*$.
($\Leftarrow$). Suppose that $O \in OR^*$. Therefore, by (14), we know that $\mu_i(u) = \nu_i(u)$ for each $i \in I$ and $u \in U$. Then, by Definition 5(i), $\eta_1(\mu_i, \nu_i) = 1$ for each $i \in U$. Hence, by Definition 15, we can easily conclude $\mathcal{H}_{*1}(O) = \tilde{\mathcal{H}}_1(O) = 1$. □

Remark 20. It can be observed that when $O \in OR$ is a standard partition, it is true that $\mathcal{H}_{*1}(O) = \frac{1}{2}$. Analogously, when $O \in OR^*$, we get $\mathcal{H}^*_1(O) = 1$.

Proposition 3. *Let $O = \{(\mu_i, \nu_i) \mid i \in I\}$ be a generalized fuzzy orthopartition. Then,*

(a) $\mathcal{H}_{*2}(O), \mathcal{H}^*_2(O), \tilde{\mathcal{H}}_2(O) \in [0, |U|]$;
(b) $\mathcal{H}^*_2(O) = \tilde{\mathcal{H}}_2(O) = 0$ *if and only if* O *is a Ruspini partition;*
(c) $\mathcal{H}_{*2}(O) = \tilde{\mathcal{H}}_2(O) = |U|$ *if and only if* $O = O_0$.

Proof. (a) From Definition 5 (ii), we get $\eta_2(\mu_i, \nu_i) \in [0, |U|]$ for all $i \in I$. Thus, according to Definition 15, the thesis (a) clearly holds.

(b) ($\Leftarrow$). By Definition 5 and by the previous point, $\mathcal{H}^*_2(O) = \tilde{\mathcal{H}}_2(O) = 0$ implies that $\eta_2(\mu_i, \nu_i) = 0$ for each $i \in I$. By Definition 5 (ii), the latter is possible when $h_i(u) = 0$ for each $u \in U$ and $i \in I$. By (3), O is a Ruspini partition.
($\Rightarrow$). By (3), if O is a Ruspini partition, then $h_i(u) = 0$ for each $i \in I$ and $u \in U$. Hence, by Definition 5 (ii), $\eta_2(\mu_i, \nu_i) = 0$ for each $i \in I$. Then, by Definition 15, $\mathcal{H}^*_2(O) = \tilde{\mathcal{H}}_2(O) = 0$.

(c) ($\Rightarrow$). By Definition 15 and by item (a), $\mathcal{H}_{*2}(O) = \tilde{\mathcal{H}}_2(O) = |U|$ implies that $\eta_2(\mu_i, \nu_i) = |U|$ for each $i \in I$. By Definition 5 (ii), the latter is possible when $h_i(u) = 1$ for each $u \in U$ and $i \in I$. Finally, by (13), we can conclude that $O = O_0$.
($\Leftarrow$). By (13), if $O = O_0$, then $h_i(u) = 1$ for each $i \in I$ and $u \in U$. Hence, $\eta_2(\mu_i, \nu_i) = 1$ for each $i \in U$ from Definition 5(ii). Then, by Definition 15, it is clear that $\mathcal{H}_{*2}(O) = \tilde{\mathcal{H}}_2(O) = 1$. □

Remark 21. It can be observed that when $O \in OR$ is a Ruspini partition, it is true that $\mathcal{H}_{*2}(O) = 0$. Analogously, when $O = O_0$, we get $\mathcal{H}^*_2(O) = |U|$.

5.3. Monotonicity of $\mathcal{H}_{*1}$, $\mathcal{H}^*_1$ and $\tilde{\mathcal{H}}_1$

In order to study the monotonicity of the measures introduced by Definition 15, we need the orderings $\preceq_O$ and $\preceq^*_O$ introduced by Definition 13 and the relation $\preceq^1_O$ defined as follows.

Definition 16. Let $O_1, O_2 \in OR$, then

$$O_1 \preceq^1_O O_2 \text{ if and only if } (\mu^1_i \nu^1_i) \preceq^1 (\mu^2_i, \nu^2_i), \text{ for each } i \in I,$$

where $\preceq^1$ is given by Definition 3 (iii).

The following theorem states that the entropy measures $\mathcal{H}_{*1}$, $\mathcal{H}^*_1$, and $\tilde{\mathcal{H}}_1$ satisfy the O-monotonicity (see Section 2) w.r.t. $\preceq^1_O$.

Theorem 15.

(a) $\mathcal{H}_{*1}$ *is O-monotonous w.r.t.* $\preceq^1_O$, *i.e.*

$$\textit{let } O_1, O_2 \in OR, \textit{ if } O_1 \preceq^1_O O_2, \textit{ then } \mathcal{H}_{*1}(O_1) \le \mathcal{H}_{*1}(O_2);$$

(b) $\mathcal{H}^*_1$ *is O-monotonous w.r.t.* $\preceq^1_O$, *i.e.*

$$\textit{let } O_1, O_2 \in OR, \textit{ if } O_1 \preceq^1_O O_2, \textit{ then } \mathcal{H}^*_1(O_1) \le \mathcal{H}^*_1(O_2);$$

(c) $\tilde{\mathcal{H}}_1$ *is O-monotonous w.r.t.* $\preceq^1_O$, *i.e.*

$$\textit{let } O_1, O_2 \in OR, \textit{ if } O_1 \preceq^1_O O_2, \textit{ then } \tilde{\mathcal{H}}_1(O_1) \le \tilde{\mathcal{H}}_1(O_2).$$

Proof. (a) Let $(\mu^1_i, \nu^1_i) \in O_1$ and let $(\mu^2_j, \nu^2_j) \in O_2$ with $\mathcal{H}_{*1}(O_1) = \eta_1(\mu^1_i, \nu^1_i)$ and $\mathcal{H}_{*1}(O_2) = \eta_1(\mu^2_j, \nu^2_j)$. We want to show that the inequality
$\mathcal{H}_{*1}(O_1) > \mathcal{H}_{*1}(O_2)$, namely $\eta_1(\mu^1_i, \nu^1_i) > \eta_1(\mu^2_j, \nu^2_j)$, leads to a contradiction:
(1) $\eta_1(\mu^1_i, \nu^1_i) > \eta_1(\mu^2_j, \nu^2_j)$ implies that $\eta_1(\mu^1_k, \nu^1_k) > \eta_1(\mu^2_j, \nu^2_j)$ for each $k \in I$.
(2) As a consequence, $\eta_1(\mu^1_j, \nu^1_j) > \eta_1(\mu^2_j, \nu^2_j)$.
(3) This means that $(\mu^1_j, \nu^1_j) \npreceq^1 (\mu^2_j, \nu^2_j)$ from the O-monotonicity of η_1 w.r.t. $\preceq^1$ (see Theorem 2(i)).
(4) This contradicts the hypothesis that $O_1 \preceq^1_O O_2$.
Therefore, it must be true that $\mathcal{H}_{*1}(O_1) \le \mathcal{H}_{*1}(O_2)$.

(b) The proof is similar to that of point (a) and is based on the O-monotonicity of η_1 w.r.t. $\preceq^1$ (Theorem 2(i)) and the hypothesis $O_1 \preceq^1_O O_2$.

(c) The thesis clearly follows from the hypothesis $O_1 \preceq^1_O O_2$ and the well-known properties of the arithmetic mean. □

Remark 22. Notice that $\preceq_O$ is a restriction of $\preceq_O^1$: let $O_1, O_2 \in OR$, if $O_1 \preceq_O O_2$ then $O_1 \preceq_O^1 O_2$. As a consequence, the measures $\mathcal{H}_{*1}$, $\mathcal{H}_1^*$, and $\tilde{\mathcal{H}}_1$ must satisfy the O-monotonicity w.r.t. $\preceq_O$ too. Therefore, let $O_1, O_2 \in OR$ such that $O_1 \preceq_O O_2$, the following inequalities hold:

$$\mathcal{H}_{*1}(O_1) \leq \mathcal{H}_{*1}(O_2),\ \mathcal{H}_*^1(O_1) \leq \mathcal{H}_*^1(O_2),\ \text{and } \tilde{\mathcal{H}}_1(O_1) \leq \tilde{\mathcal{H}}_1(O_1).$$

Let us reflect on the meaning of the measures $\mathcal{H}_{*1}$, $\mathcal{H}_1^*$, and $\tilde{\mathcal{H}}_1$, which arises from Proposition 2 and Theorem 15.

Let O be a generalized fuzzy orthopartition of the universe U, $\mathcal{H}_{*1}(O)$, $\mathcal{H}_1^*(O)$, and $\tilde{\mathcal{H}}_1(O)$ quantify the capacity of O to classify the elements of U. Let us explain this concept in more detail.

- We have discovered that $\mathcal{H}_{*1}(O)$, $\mathcal{H}_1^*(O)$, and $\tilde{\mathcal{H}}_1(O)$ are between $\frac{1}{2}$ and 1.
- Moreover, the closer $\mathcal{H}_{*1}(O)$, $\mathcal{H}_1^*(O)$, and $\tilde{\mathcal{H}}_1(O)$ are to $\frac{1}{2}$, the better (with more precision) O describes the relationship between elements and classes. In other words, O is closer to being a standard partition. In fact, we have proved that O is a standard partition when $\mathcal{H}_{*1}(O)$, $\mathcal{H}_1^*(O)$, and $\tilde{\mathcal{H}}_1(O)$ precisely coincide with $\frac{1}{2}$.
- Conversely, the closer $\mathcal{H}_{*1}(O)$, $\mathcal{H}_1^*(O)$, and $\tilde{\mathcal{H}}_1(O)$ are to 1, the worse (with less precision) O describes the relationship between elements and classes. More exactly, in case $\mathcal{H}_{*1}(O)$, $\mathcal{H}_1^*(O)$, and $\tilde{\mathcal{H}}_1(O)$ are close to 1, then O is far from being a standard partition and close to being a generalized fuzzy orthopartition of OR^*. This reveals the inadequacy of O to distinguish if a given element belongs or does not belong to a given class. In fact, we have proved that $O \in OR^*$ if and only if $\mathcal{H}_{*1}(O)$, $\mathcal{H}_1^*(O)$, and $\tilde{\mathcal{H}}_1(O)$ precisely coincide with 1.

Therefore, given $O_1, O_2 \in OR$ such that $O_1 \preceq_O^1 O_2$, we get that the distance between the membership and non-membership degrees in O_2 is less than that in O_1. Consequently, O_1 is better than O_2 to classify the elements of the universe by means of its IFSs. Mathematically, the behavior of O_1 and O_2 is described by the inequalities $\mathcal{H}_{*1}(O_1) \leq \mathcal{H}_{*1}(O_2)$, $\mathcal{H}_*^1(O_1) \leq \mathcal{H}_*^1(O_2)$, and $\tilde{\mathcal{H}}_1(O_1) \leq \tilde{\mathcal{H}}_1(O_1)$.

5.4. Monotonicity of $\mathcal{H}_{*2}$, $\mathcal{H}_2^*$ and $\tilde{\mathcal{H}}_2$

Similarly to the previous case, the monotonicity of entropies based on η_2 can be stated.

Theorem 16.

(a) $\mathcal{H}_{*2}$ *is O-monotonous w.r.t.* $\preceq_O^*$, *i.e.*

$$\text{let } O_1,\ O_2 \in OR,\ \text{if } O_1 \preceq_O^* O_2,\ \text{then } \mathcal{H}_{*2}(O_2) \leq \mathcal{H}_{*2}(O_1);$$

(b) $\mathcal{H}_2^*$ *is O-monotonous w.r.t.* $\preceq_O^*$, *i.e.*

$$\text{let } O_1,\ O_2 \in OR,\ \text{if } O_1 \preceq_O^* O_2,\ \text{then } \mathcal{H}_2^*(O_2) \leq \mathcal{H}_2^*(O_1);$$

(c) $\tilde{\mathcal{H}}_2$ *is O-monotonous w.r.t.* $\preceq_O^*$, *i.e.*

$$\text{let } O_1,\ O_2 \in OR,\ \text{if } O_1 \preceq_O^* O_2,\ \text{then } \tilde{\mathcal{H}}_2(O_2) \leq \tilde{\mathcal{H}}_2(O_1).$$

Proof. The proof is similar to that of Theorem 15 and it is based on Theorem 2 (b). □

Remark 23. Let us reflect on the meaning of the measures $\mathcal{H}_{*2}$, $\mathcal{H}_2^*$, and $\tilde{\mathcal{H}}_2$, which arises from Proposition 3 and Theorem 16.

Let O be a generalized fuzzy orthopartition of the universe U, $\mathcal{H}_{*2}(O)$, $\mathcal{H}_2^*(O)$, and $\tilde{\mathcal{H}}_2(O)$ quantify the fuzziness of O. Let us explain this concept in more detail.

- We have discovered that $\mathcal{H}_{*2}(O)$, $\mathcal{H}_2^*(O)$, and $\tilde{\mathcal{H}}_2(O)$ are between 0 and $|U|$.
- Moreover, the closer $\mathcal{H}_{*2}(O)$, $\mathcal{H}_2^*(O)$, and $\tilde{\mathcal{H}}_2(O)$ are to 0, the better (with more precision) O describes the relationship between elements and classes. In other words, O is closer to being a Ruspini partition. In fact, we have proved that O is a Ruspini partition when $\mathcal{H}_{*2}(O)$, $\mathcal{H}_2^*(O)$, and $\tilde{\mathcal{H}}_2(O)$ precisely coincide with 0.
- Conversely, the closer $\mathcal{H}_{*2}(O)$, $\mathcal{H}_2^*(O)$, and $\tilde{\mathcal{H}}_2(O)$ are to $|U|$, the worse (with more uncertainty) O describes the relationship between elements and classes. More exactly, in case $\mathcal{H}_{*2}(O)$, $\mathcal{H}_2^*(O)$, and $\tilde{\mathcal{H}}_2(O)$ are close to $|U|$, then O is far from being a Ruspini partition and closer to O_0, which is a trivial generalized fuzzy orthopartition that does not give us any type of information about the classification of the elements. Also, we have proved that $O = O_0$ if and only if $\mathcal{H}_{*2}(O)$, $\mathcal{H}_2^*(O)$, and $\tilde{\mathcal{H}}_2(O)$ precisely coincide with $|U|$.

Therefore, given $O_1, O_2 \in OR$ such that $O_1 \preceq_O^* O_2$, we are sure that the degrees of uncertainty in O_2 are less than those in O_1. Consequently, O_2 is better than O_1 to classify the elements of the universe by means of its IFSs. Mathematically, the behavior of O_1 and O_2 is described by the inequalities $\mathcal{H}_{*2}(O_2) \leq \mathcal{H}_{*2}(O_1)$, $\mathcal{H}_*^2(O_2) \leq \mathcal{H}_*^2(O_1)$, and $\tilde{\mathcal{H}}_2(O_2) \leq \tilde{\mathcal{H}}_2(O_1)$.

Thinking of a dynamic situation, where the information about the classification of the elements increases, O can be transformed into a new generalized fuzzy partition O' with lower entropies: $\mathcal{H}_{*2}(O') \leq \mathcal{H}_{*2}(O)$, $\mathcal{H}_2^*(O') \leq \mathcal{H}_2^*(O)$, and $\tilde{\mathcal{H}}_2(O') \leq \tilde{\mathcal{H}}_2(O)$.

Furthermore, as shown by the following theorem, the R-monotonicity (see Section 2) holds for $\mathcal{H}_{*2}, \mathcal{H}_2^*$, and $\tilde{\mathcal{H}}_2$ w.r.t. $\preceq_O^*$.

Theorem 17.

(a) $\mathcal{H}_{*2}$ is R-monotonous w.r.t. $\preceq_O^*$, i.e. let $O \in OR$, $\mathcal{H}_{*2}(O_{(u_1,u_2)}) \leq \mathcal{H}_{*2}(O)$;
(b) $\mathcal{H}_2^*$ is R-monotonous w.r.t. $\preceq_O^*$, i.e. let $O \in OR$, $\mathcal{H}_2^*(O_{(u_1,u_2)}) \leq \mathcal{H}_2^*(O)$;
(c) $\tilde{\mathcal{H}}_2$ is R-monotonous w.r.t. $\preceq_O^*$, i.e. let $O \in OR$, $\tilde{\mathcal{H}}_2(O_{(u_1,u_2)}) \leq \tilde{\mathcal{H}}_2(O)$;

where $O_{(u_1,u_2)}$ is the generalized fuzzy orthopartition on $U \setminus \{u_1\}$ defined in Theorem 2 as follows: let $(\mu_i^*, \nu_i^*) \in O_{(u_1,u_2)}$, then

- *(μ_i^*, ν_i^*) is equal to (μ_i, ν_i) on the elements of $U \setminus \{u_1, u_2\}$;*
- *(μ_i^*, ν_i^*) is defined on u_2 by*

$$(\mu_i^*(u_2), \nu_i^*(u_2)) = (\max\{\mu_i(u_1), \mu_i(u_2)\}, \min\{\nu_i(u_1), \nu_i(u_2)\}).$$

Proof. (a) By Definition 15 (i),

$$\mathcal{H}_{*2}(O_{(u_1,u_2)}) = \min\{\eta_2(\mu_i^*, \nu_i^*) \mid i \in I\} \text{ and } \mathcal{H}_{*2}(O) = \min\{\eta_2(\mu_i, \nu_i) \mid i \in I\}.$$

By Theorem 2 (c), $\eta_3(\mu_i^*, \nu_i^*) \leq \eta_3(\mu_i, \nu_i)$ for each $i \in I$. Then, we have $\mathcal{H}_{*3}(O_{(u_1,u_2)}) \leq \mathcal{H}_{*3}(O)$.

(b) The poof is similar to that of point (a).

(c) The poof is similar to that of point (a). □

Concretely, according to the previous theorem, the entropies of a generalized fuzzy orthopartition O decrease, when O is reconsidered on a smaller universe.

6. Conclusion and future directions

Generalized fuzzy orthopartitions are mathematical objects recently introduced to formalize the concept of partitioning with vagueness and uncertainty. In this work, we have carried out their study initiated in previous articles, by introducing and analyzing operations and entropy measures.

In the future, we propose to extend our results by following several directions, thus let us mention some of them here. First of all, we plan to enrich the list of operations and entropy measures provided in this article, by selecting other operators and entropies on intuitionistic fuzzy sets existing in the literature (some examples are found in [15,17,34,37,42]). Moreover, we intend to discover algebraic structures that can be represented by generalized fuzzy orthopartitions equipped by various operations. Also, we will reconsider and study generalized fuzzy orthopartitions when their intuitionistic fuzzy sets have a different meaning as that shown in [37] and [13]. Finally, we intend to apply operations and entropies on generalized fuzzy orthopartitions in group decision making by generalizing the results shown in [1,24,28,35,36] and other articles.

CRediT authorship contribution statement

Stefania Boffa: Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Davide Ciucci:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization. **Christophe Marsala:** Writing – review & editing, Writing – original draft, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Table A.7
Definition of the elements of O_1 and O_2.

u	$\mu_1^1(u)$	$\nu_1^1(u)$	$\mu_2^1(u)$	$\nu_2^1(u)$	u	$\mu_1^2(u)$	$\nu_1^2(u)$	$\mu_2^2(u)$	$\nu_2^2(u)$
u_1	0.1	0.3	0.4	0.6	u_1	0	0.6	0.3	0.5
u_2	0.3	0.6	0.5	0.1	u_2	0.4	0.4	0.3	0.3

Table A.8
Definition of the elements of $O_1 \cap_O O_2$.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	$\min\{0.1, 0\}$	$\max\{0.3, 0.6\}$	$\min\{0.4, 0.3\}$	$\max\{0.6, 0.5\}$
u_2	$\min\{0.3, 0.4\}$	$\max\{0.6, 0.4\}$	$\min\{0.5, 0.3\}$	$\max\{0.1, 0.3\}$

Table A.9
Definition of the elements of O_3 and O_4.

u	$\mu_1^3(u)$	$\nu_1^3(u)$	$\mu_2^3(u)$	$\nu_2^3(u)$	u	$\mu_1^4(u)$	$\nu_1^4(u)$	$\mu_2^4(u)$	$\nu_2^4(u)$
u_1	0.1	0.3	0.7	0.2	u_1	0.4	0.5	0.5	0.3
u_2	0.3	0.6	0.5	0.1	u_2	0.4	0.4	0.3	0.3

Appendix A. Counterexamples of Section 4.1

Given $\& \in \{\cap, \cup, +, \odot, -\}$ and given $O_1, O_2 \in OR$, it is not always true that $O_1 \;\&_O\; O_2$ is a generalized fuzzy orthopartition, according to Definition 8. Thus, in what follows, for each $\& \in \{\cap, \cup, +, \odot, -\}$, we present an example of a pair of generalized fuzzy orthopartitions O_1 and O_2 such that $O_1 \&_O O_2 \notin OR$.

Example 6. Let $O_1 = \{(\mu_1^1, \nu_1^1), (\mu_2^1, \nu_2^1)\}$ and $O_2 = \{(\mu_1^2, \nu_1^2), (\mu_2^2, \nu_2^2)\}$ be the generalized fuzzy orthopartitions of $\{u_1, u_2\}$ defined by Table A.7.

Then, $O_1 \cap_O O_2 = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ is defined by Table A.8.

Clearly, $O_1 \cap_O O_2$ is not a generalized fuzzy orthopartition, considering that Axiom (ii) of Definition 8 is not satisfied by $O_1 \cap_O O_2$. Indeed,

$$(\mu_1(u_1) + h_1(u_1)) + (\mu_2(u_1) + h_2(u_1)) = (1 - \nu_1(u_1)) + (1 - \nu_2(u_1)) =$$
$$(1 - \max\{0.3, 0.6\}) + (1 - \max\{0.6, 0.5\}) = (1 - 0.6) + (1 - 0.6) = 0.4 + 0.4 = 0.8,$$

which is less than 1.

Example 7. Let $O_3 = \{(\mu_1^3, \nu_1^3), (\mu_2^3, \nu_2^3)\}$ and $O_4 = \{(\mu_1^4, \nu_1^4), (\mu_2^4, \nu_2^4)\}$ be the generalized fuzzy orthopartitions of $\{u_1, u_2\}$ defined by Table A.9.

We can easily verify that $O_3 \cup_O O_4 = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ is not a generalized fuzzy orthopartition. Indeed,

$$\mu_1(u_1) = \max\{\mu_1^3(u_1), \mu_1^4(u_1)\} = \max\{0.1, 0.4\} = 0.4,$$
$$\text{and} \quad \mu_2(u_1) = \max\{\mu_2^3(u_1), \mu_2^4(u_1)\} = \max\{0.7, 0.5\} = 0.7.$$

Thus, $\mu_1(u_1) + \mu_2(u_1)$ is greater than 1, which means that $O_3 \cup_O O_4$ does not satisfy Axiom (i) of Definition 8.

Example 8. Consider the generalized fuzzy orthopartitions O_3 and O_4 of $\{u_1, u_2\}$ defined by Table A.9. Then, $O_3 +_O O_4 = \{(\mu_1^*, \nu_1^*), (\mu_2^*, \nu_2^*)\}$ does not satisfy Axiom (i) of Definition 8. In fact, we get

$$\mu_1^*(u_1) + \mu_2^*(u_1) = (\mu_1^3(u_1) + \mu_1^4(u_1) - \mu_1^3(u_1)\mu_1^4(u_1)) + (\mu_2^3(u_1) + \mu_2^4(u_1) -$$
$$\mu_2^3(u_1)\mu_2^4(u_1)) = (0.1 + 0.4 - 0.04) + (0.7 + 0.5 - 0.35) = 0.46 + 0.85 = 1.31,$$

which is greater than 1. Hence, $O_3 +_O O_4$ is not a generalized fuzzy orthopartition of $\{u_1, u_2\}$.

Example 9. Consider the generalized fuzzy orthopartitions O_1 and O_2 of $\{u_1, u_2\}$ given by Table A.7. Then, $O_1 \odot_O O_2 = \{(\tilde{\mu}_1, \tilde{\nu}_1), (\tilde{\mu}_2, \tilde{\nu}_2)\}$ does not satisfy Axiom (ii) of Definition 8. In fact,

- $1 - \tilde{\nu}_1(u_1) = 1 - (\nu_1^1(u_1) + \nu_1^2(u_1) - \nu_1^1(u_1)\nu_1^2(u_1)) = 1 - (0.3 + 0.6 - 0.18) = 1 - 0.72 = 0.28;$
- $1 - \tilde{\nu}_2(u_1) = 1 - (\nu_2^1(u_1) + \nu_2^2(u_1) - \nu_2^1(u_1)\nu_2^2(u_1)) = 1 - (0.6 + 0.5 - 0.3) = 1 - 0.8 = 0.2.$

Table A.10
Definition of the elements $\neg_O O_2$.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0.6	0	0.5	0.3
u_2	0.4	0.4	0.3	0.3

Table A.11
Definition of the elements $\Diamond_O O_1$.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0.7	0.3	0.4	0.6
u_2	0.4	0.6	0.9	0.1

Table A.12
Definition of the elements O.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
v u_1	0.8	0.3	0.1	0.5
u_2	0.2	0.4	0.6	0

Table A.13
Definition of the elements $C_O O$.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0.8	0.3	0.6	0
u_2	0.8	0.3	0.6	0

Hence, $(1-\tilde{\nu}_1(u_1)) + (1-\tilde{\nu}_2(u_1)) = 0.28 + 0.2 = 0.48$, which is less than 1. We can conclude that $O_3 \odot_O O_4$ is not a generalized fuzzy orthopartition of $\{u_1, u_2\}$.

Example 10. Consider the generalized fuzzy orthopartitions O_5 and O_6 of a universe U, which are defined on $u \in U$ as follows:

- $\mu_1^5(u) = 0.1,\ \nu_1^5(u) = 0.7,\ \mu_2^5(u) = 0.7,\ \nu_2^5(u) = 0.2$;
- $\mu_1^6(u) = 0.2,\ \nu_1^6(u) = 0.5,\ \mu_2^6(u) = 0.6,\ \nu_2^6(u) = 0.3$.

Then, $O_5 -_O O_6 = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ is not a generalized fuzzy orthopartition of U because it does not satisfy Axiom (ii) of Definition 8. Indeed,

$$(1-\nu_1(u)) + (1-\nu_2(u)) = (1-\max\{\nu_1^5(u), \mu_1^6(u)\}) + (1-\max\{\nu_2^5(u), \mu_2^6(u)\})$$
$$= (1-\max\{0.7, 0.2\}) + (1-\max\{0.2, 0.6\}) = (1-0.7) + (1-0.6) = 0.3 + 0.4 = 0.7,$$

which is less than 1.

Let $\triangle \in \{\neg, \Diamond, \mathcal{C}, \mathcal{I}\}$ and let $O \in OR$, it is not always true that $\triangle O \in OR$. In what follows, for each $\& \in \{\neg, \Diamond, \mathcal{C}, \mathcal{I}\}$, we present an example of a generalized fuzzy orthopartition O such that $\triangle O \notin OR$.

Example 11. Let O_2 be the generalized fuzzy orthopartition of the universe $\{u_1, u_2\}$ defined by Table A.7. So, $\neg_O O_2 = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ is defined by Table A.10.

Of course, $\neg_O O_2$ is not a generalized fuzzy orthopartition of $\{u_1, u_2\}$. Indeed, since $\mu_1(u_1) + \mu_2(u_1) = 0.6 + 0.5 = 1.1$, Axiom (i) of Definition 8 is not satisfied by $\neg_O O_2$.

Example 12. Consider the generalized fuzzy orthopartition O_1 of $\{u_1, u_2\}$ defined by Table A.7. Then, we can see that $\Diamond O_1$ defined by Table A.11, does not belong to OR. In fact, Axiom (i) is not satisfied because

$$\mu_1(u_1) + \mu_2(u_1) = 0.7 + 0.4 = 1.1 \text{ and } \mu_1(u_2) + \mu_2(u_2) = 0.4 + 0.9 = 1.3,$$

which is greater than 1.

Example 13. Consider the generalized fuzzy orthopartition O of the universe $\{u_1, u_2\}$ defined by Table A.12.
According to Definition 4 (xi), $C_O O = \{(\mu_1^*, \nu_1^*), (\mu_2^*, \nu_2^*)\}$ is given by Table A.13.

Table A.14
Definition of the elements $\mathcal{I}_O O_1$.

u	$\mu_1(u)$	$\nu_1(u)$	$\mu_2(u)$	$\nu_2(u)$
u_1	0.1	0.6	0.4	0.6
u_2	0.1	0.6	0.4	0.6

Of course, $\mathcal{C}_O O$ is not a generalized fuzzy orthopartition of $\{u_1, u_2\}$ by considering that Axiom (i) of Definition 8 is not verified by $\mathcal{C}_O O$:

$$\mu_1(u_1) + \mu_2(u_1) = 0.8 + 0.6 = 1.4 > 1.$$

Example 14. Consider the generalized fuzzy orthopartition O_1 of $\{u_1, u_2\}$ defined by Table A.7. Then, $\mathcal{I}_O O_1 = \{(\mu_1, \nu_1), (\mu_2, \nu_2)\}$ is shown by Table A.14.

$\mathcal{I}_O O_1$ does not satisfy Axiom (ii) of Definition 8. In fact, we can see that

$$(1 - \nu_1(u_1)) + (1 - \nu_2(u_1)) = (1 - 0.6) + (1 - 0.6) = 0.4 + 0.4 = 0.8 \leq 1.$$

Data availability

No data was used for the research described in the article.

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