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Eigenfunction Asymptotics in the Complex Domain for a Compact Lie Group

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Abstract
Let (G, κ) be a compact connected Lie group endowed with a biinvariant Riemannian metric, and let $\tilde{G}$ be the complexification of G. We apply Grauert tube techniques to the near-diagonal scaling asymptotics of certain operator kernels, which are defined in terms of the matrix elements of an irreducible representation drifting to infinity along a ray in weight space. These kernels are the equivariant components of Poisson and Szegő kernels on a fixed sphere bundle in $\tilde{G}$, when the latter is identified with the tangent bundle of G in an appropriate way.

Keywords Lie group · Complexified Laplacian eigenfunctions · Matrix elements · Compatible complex structures · Grauert tubes · Moment maps · Equivariant asymptotics

Mathematics Subject Classification 32M05 · 32V20 · 34L20 · 53D25 58J40

1 Introduction

Let G be a connected compact Lie group with Lie algebra of left-invariant vector fields $\mathfrak{g}$ and coalgebra $\mathfrak{g}^\vee$; we shall canonically identify $\mathfrak{g}$ with the tangent space of G at the identity $e \in G$, T_eG. The dimension of G and its rank (i.e., the dimension of a maximal torus $T \leqslant G$) will be denoted by, respectively, d and r_G.

Let κ_e be an Ad-invariant Euclidean product on $\mathfrak{g}$, where Ad denotes the adjoint representation; we shall denote by $\kappa_e^\vee$ the corresponding Euclidean product on $\mathfrak{g}^\vee$. Let κ be the induced biinvariant Riemannian metric on G, and let $L^2(G)$ be the Hilbert space of L^2-summable functions on G with respect to the associated Riemannian

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density $\mathrm{d}V_G$. Then G is unitarily represented on $L^2(G)$ by left translations:

$$\theta_l(g)(f)(h) := f\left(g^{-1}\,h\right) \quad (f \in L^2(G),\ g, h \in G). \tag{1}$$

Let $T \leqslant G$ be a maximal torus, with Lie algebra $\mathfrak{t}$ and corresponding set of roots $R \subset \mathfrak{t}^\vee$. As is well-known, choosing an ordering of R, i.e., a set of positive roots $R^+ \subset R$, determines a notion of dominant weight for $(\mathfrak{g}, \mathfrak{t})$; the set $\hat{G}$ of all irreducible representations of G is then in bijective correspondence with a certain subset $\mathcal{D}^G$ of the set $\mathcal{D}$ of all dominant weights. For instance, $\mathcal{D}^G = \mathcal{D}$ when G is simply connected. For every $\boldsymbol{\lambda} \in \mathcal{D}^G$, we shall denote by $V_{\boldsymbol{\lambda}}$ the corresponding representation space, by $d_{\boldsymbol{\lambda}}$ its dimension, and by $\chi_{\boldsymbol{\lambda}}$ its character. We shall also denote by $\boldsymbol{\lambda}^\vee$ the weight associated to the dual representation.

By the Theorem of Peter and Weyl, there is a unitary and equivariant decomposition of $L^2(G)$ as the Hilbert space direct sum of its isotypical components (see, e.g., [4], [44]). More precisely, let $L^2(G)_{\boldsymbol{\lambda}} \subset L^2(G)$ denote the span of the matrix elements of the representation corresponding to $\boldsymbol{\lambda}^\vee$, with respect to any given basis of $V_{\boldsymbol{\lambda}^\vee}$. Then there is an equivariant isomorphism

$$L^2(G)_{\boldsymbol{\lambda}} \cong V_{\boldsymbol{\lambda}}^{\oplus d_{\boldsymbol{\lambda}}};$$

hence $L^2(G)_{\boldsymbol{\lambda}}$ is the $\boldsymbol{\lambda}$-th isotypical component of $L^2(G)$, and there is an equivariant isomorphism of Hilbert spaces

$$L^2(G) \cong \bigoplus_{\boldsymbol{\lambda} \in \mathcal{D}^G} L^2(G)_{\boldsymbol{\lambda}}. \tag{2}$$

The subspaces $L^2(G)_{\boldsymbol{\lambda}}$ may also be described in terms of the positive Laplacian operator Δ on (G, κ). Namely, Δ acts on $L^2(G)_{\boldsymbol{\lambda}}$ as scalar multiplication by $c_{\boldsymbol{\lambda}}^2$, where

$$c_{\boldsymbol{\lambda}} := \sqrt{\|\boldsymbol{\lambda}^\vee + \boldsymbol{\delta}\|_\kappa^2 - \|\boldsymbol{\delta}\|_\kappa^2} = \kappa_e^\vee\left(\boldsymbol{\lambda}^\vee, \boldsymbol{\lambda}^\vee + 2\,\boldsymbol{\delta}\right)^{1/2}, \tag{3}$$

$\boldsymbol{\delta}$ being the half-sum of positive roots (see (59)); in other words, every matrix element of $V_{\boldsymbol{\lambda}^\vee}$ is an eigenfunction of Δ for the eigenvalue $c_{\boldsymbol{\lambda}}^2$.

The complexification $(\tilde{G}, J)$ of G is a connected complex d-dimensional Lie group, with complex structure J, in which G sits as a totally real submanifold and a maximal compact subgroup [4]; the Lie algebra $\tilde{\mathfrak{g}}$ of $\tilde{G}$ the complexification of $\mathfrak{g}$, i.e. $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}$. Any irreducible representation of G extends uniquely to a holomorphic irreducible representation of $\tilde{G}$ on the same representation space. Hence, the matrix elements of any irreducible representation of G admit holomorphic extensions to $\tilde{G}$.

There is a diffeomorphism $\gamma : G \times \mathfrak{g} \cong \tilde{G}$, which can be described as follows. Consider the exponential map $\exp^{\tilde{G}} : \tilde{\mathfrak{g}} \to \tilde{G}$. Then

$$\gamma(g, \boldsymbol{\xi}) := g\,\exp^{\tilde{G}}(\imath\,\boldsymbol{\xi}) \qquad (g \in G,\ \boldsymbol{\xi} \in \mathfrak{g}). \tag{4}$$

In the sequel, we shall use the short-hands

$$g \cdot \boldsymbol{\xi} = \mathrm{d}_e L_g(\boldsymbol{\xi}), \quad \boldsymbol{\xi} \cdot g = \mathrm{d}_e R_g(\boldsymbol{\xi}) \quad (g \in G,\ \boldsymbol{\xi} \in \mathfrak{g}),$$

where $L_g,\ R_g : G \to G$ denote left and right translations by $g \in G$, respectively. Let us compose γ with the trivialization of the tangent bundle TG given by left translations to the identity. If $v \in T_gG$, we have $v = g \cdot \boldsymbol{\xi}$ for a unique $\boldsymbol{\xi} \in \mathfrak{g}$. We obtain a diffeomorphism

$$E : (g, v) = (g, g \cdot \boldsymbol{\xi}) \in TG \mapsto g \exp^{\tilde{G}}(\imath\, \boldsymbol{\xi}) \in \tilde{G}. \tag{5}$$

Besides being a complex Lie group, $\tilde{G}$ also carries a built-in G-invariant Kähler structure, which is intrinsically determined by κ, and has an invariant global potential ρ (see (6) below). In terms of ρ, one can define a family of pseudoconvex boundaries X^τ in $\tilde{G}$, and we shall be concerned with certain operator kernels on X^τ. To clarify this, it is in order to make a brief recall on so-called *Grauert tubes* of compact real-analytic Riemannian manifolds (such as (G, κ)).

Let (M, β) be a compact real-analytic Riemannian manifold, and for $\tau > 0$ let $T^\tau M$ be the tubular neighbourhood of radius τ of the zero section in the tangent bundle. For sufficiently small τ, there exists an intrinsic so-call *adapted* complex structure J_{ad} on $T^\tau M$, which is uniquely characterized by the property that the standard parametrization of the leaves of the Riemann foliation are holomorphic maps from suitable strips in $\mathbb{C}$ ([36], [35], [22], [23], [31], [46]; see also [5], [6]). Furthermore, J_{ad} is compatible with the canonical symplectic structure Ω_{can} on TM, and the associated Riemannian metric restricts to β along M.

For a general (M, β), J_{ad} needn't be defined on the whole of TM: as proved in [31], this will certainly fail in the presence of negative scalar curvatures. In the present setting, however, the following holds.

Theorem 1.1 (Szöke, [47])) *Let (G, κ) be a compact Lie group with a biinvariant Riemannian metric. Then J_{ad} is defined on TG, and E in (5) is a biholomorphism from (TG, J_{ad}) to $(\tilde{G}, J)$.*

This result puts the theory of complex reductive groups in contact with the general theory of Grauert tubes.

On the other hand, the asymptotics of Poisson and Szegő kernels on Grauert tube boundaries have attracted considerable attention in recent years, sparked to a large extent by seminal insight of Boutet de Monvel on the extension properties of Laplacian eigenfunctions (see e.g. [1], [9], [10], [19], [21], [29], [42], [43], [54], [55], [58]). Grauert tube techniques also find applications in the study of nodal sets (see e.g. [7], [8], [48], [49]).

In light of these considerations, it seems natural to apply Grauert tube methods to the study the local and global asymptotics related to matrix elements of irrducible representations. To bring this theme into focus, let us remark that Theorem 1.1 has the following consequences, which are the basis for the present discussion:

1. $(TG, J_{\mathrm{ad}}, \Omega_{\mathrm{can}})$ is a Kähler manifold.

2. If $\Omega := (E^{-1})^*(\Omega_{\text{can}})$, then $(\tilde{G}, J, \Omega)$ is also a Kähler manifold, and E is an isomorphism of Kähler manifolds.
3. The induced Riemannian metric $\hat{\kappa}(\cdot,\cdot) := \Omega(\cdot, J(\cdot))$ on $\tilde{G}$ restricts to κ along G.
4. Consider the norm function $\|\cdot\|_\kappa$ on TG, which is the pull-back of the norm $\|\cdot\|_{\kappa_e}$ on $\mathfrak{g}$ under the previous trivialization:

$$\|(g, g\cdot\boldsymbol{\xi})\|_\kappa := \|g\cdot\boldsymbol{\xi}\|_{\kappa_g} = \|\boldsymbol{\xi}\|_{\kappa_e};$$

then $\|\cdot\|_\kappa^2$ is strictly plurisubharmonic with respect to J_{ad}, and is in fact a global Kähler potential for Ω_{can}.
5. Therefore, the composition

$$\rho := \|\cdot\|_\kappa^2 \circ E^{-1} : \tilde{G} \to [0, +\infty) \tag{6}$$

is strictly plurisubharmonic on $(\tilde{G}, J)$, and a global Kähler potential for Ω:

$$\Omega = \imath\,\partial\,\overline{\partial}\rho.$$

6. $\sqrt{\rho} = \|\cdot\|_\kappa \circ E^{-1}$ (a.k.a. the *tube function* on $\tilde{G}$) restricts on $\tilde{G}\setminus G$ to a solution of the homogeneous Monge-Ampère equation.
7. For $\tau > 0$, let $S^\tau(\mathfrak{g}) \subset \mathfrak{g}$ denote the sphere of radius τ centered at the origin. Then, with γ as in (4)),

$$X^\tau := \rho^{-1}(\tau^2) = \gamma\left(G \times S^\tau(\mathfrak{g})\right) \subset \tilde{G} \tag{7}$$

is the boundary of a strictly pseudoconvex domain, and as such is equipped with a natural contact structure and an induced volume form; we shall let $H(X^\tau) \subset L^2(X^\tau)$ denote its Hardy space, and $\Pi^\tau : L^2(X^\tau) \to H(X^\tau)$ the corresponding Szegö kernel (orthogonal projector).

Clearly, $E^{-1}(X^\tau) \subset TG$ is the sphere bundle of radius τ (see (5)).

Let $\tilde{L} : G \times \tilde{G} \to \tilde{G}$ be the holomorphic action given by left translations. For any $g \in G$, ρ is $\tilde{L}_g$-invariant, and $\tilde{L}_g$ is a Kähler automorphism of $(\tilde{G}, J, \Omega)$. Therefore, $\tilde{L}$ restricts for any $\tau > 0$ to a CR-holomorphic action

$$\mu^\tau := \tilde{L}\Big|_{G\times X^\tau} : G \times X^\tau \to X^\tau \tag{8}$$

(that is, $\mu^\tau_g : X^\tau \to X^\tau$ is given by restriction of $\tilde{L}_g : \tilde{G} \to \tilde{G}$, $\forall\, g \in G$). Then μ^τ induces in a standard manner a unitary representation of G on $H(X^\tau)$. By the Theorem of Peter and Weyl, there is a unitary and equivariant decomposition as a Hilbert space direct sum

$$H(X^\tau) = \bigoplus_{\boldsymbol{\lambda}\in\mathcal{D}^G} H(X^\tau)_{\boldsymbol{\lambda}}, \tag{9}$$

where $H(X^\tau)_{\boldsymbol{\lambda}} \subset H(X^\tau)$ is the $\boldsymbol{\lambda}$-th isotypical decomposition. The equivariant decompositions (2) and (9) are related by the following property: for any $\boldsymbol{\lambda}$, $H(X^\tau)_{\boldsymbol{\lambda}}$

consists of the holomorphic extensions of elements of $L^2(G)_{\boldsymbol{\lambda}}$, restricted to X^τ. Hence there is an algebraic (but non-unitary) isomorphism $L^2(G)_{\boldsymbol{\lambda}} \to H(X^\tau)_{\boldsymbol{\lambda}}$, given by holomorphic extension and restriction.

There are two natural smoothing operator kernels associated to this setting. One relates to the CR and metric structure of X^τ, and is the kernel of the orthogonal projector

$$\Pi^\tau_{\boldsymbol{\lambda}} : L^2(X^\tau) \to H(X^\tau)_{\boldsymbol{\lambda}}, \tag{10}$$

i.e. the $\boldsymbol{\lambda}$-th equivariant piece of the Szegö kernel Π^τ. If $(\sigma_{\boldsymbol{\lambda},j})_{j=1}^{d_{\boldsymbol{\lambda}}^2}$ is an orthonormal basis of $H(X^\tau)_{\boldsymbol{\lambda}}$, then the distributional kernel of (10) is

$$\Pi^\tau_{\boldsymbol{\lambda}}(x, y) = \sum_{j=1}^{d_{\boldsymbol{\lambda}}^2} \sigma_{\boldsymbol{\lambda},j}(x) \cdot \overline{\sigma_{\boldsymbol{\lambda},j}(y)} \qquad (x, y \in X^\tau). \tag{11}$$

The other operator kernel in point is related to the holomorphic extension property of the matrix elements of the irreducible representations of G, or equivalently of the eigenfunctions of the Laplacian of G. Let $(\sigma_{\boldsymbol{\lambda},j})_{j=1}^{d_{\boldsymbol{\lambda}}^2}$ be an orthonormal basis of $L^2(G)_{\boldsymbol{\lambda}}$; for any j, let $\tilde{\sigma}_{\boldsymbol{\lambda},j}$ be the holomorphic extension of $\sigma_{\boldsymbol{\lambda},j}$ to $\tilde{G}$, and denote by $\tilde{\sigma}^\tau_{\boldsymbol{\lambda},j}$ its restriction to X^τ. Thus $(\tilde{\sigma}^\tau_{\boldsymbol{\lambda},j})_{j=1}^{d_{\boldsymbol{\lambda}}^2}$ is also a basis of $H(X^\tau)_{\boldsymbol{\lambda}}$, albeit not an orthonormal one. The smoothing operator kernel related the complexified eigenfunctions for the eigenvalue $c_{\boldsymbol{\lambda}}$ (equivalently, of the matrix elements of $V_{\boldsymbol{\lambda}^\vee}$) is then

$$P^\tau_{\boldsymbol{\lambda}}(x, y) := e^{-2\,\tau\, c_{\boldsymbol{\lambda}}} \sum_j \tilde{\sigma}^\tau_{\boldsymbol{\lambda},j}(x) \cdot \overline{\tilde{\sigma}^\tau_{\boldsymbol{\lambda},j}(y)} \qquad (x, y \in X^\tau), \tag{12}$$

where $c_{\boldsymbol{\lambda}}$ is as in (3).

Let us briefly dwell to motivate the tempering factor $e^{-2\,\tau\, c_{\boldsymbol{\lambda}}}$. As discussed, say, in [54], [56] and [58], the operator $P^\tau : \mathcal{C}^\infty(X^\tau) \to \mathcal{C}^\infty(X^\tau)$ with Schwartz kernel

$$P^\tau(x, y) := \sum_{\boldsymbol{\lambda} \in \mathcal{D}^G} e^{-2\,\tau\, c_{\boldsymbol{\lambda}}} \sum_j \tilde{\sigma}^\tau_{\boldsymbol{\lambda},j}(x) \cdot \overline{\tilde{\sigma}^\tau_{\boldsymbol{\lambda},j}(y)} \qquad (x, y \in X^\tau) \tag{13}$$

is a Fourier integral operator of degree $-(d-1)/2$ with the same canonical relation as Π^τ. Furthermore, P^τ is closely related to the so-called Poisson-wave operator, which is obtained by holomorphically extending the kernel of the wave operator on G and governs the holomorphic extension of Laplacian eigenfunctions. For this reason, (13) plays a crucial role in the asymptotic study of complexified eigenfunction of compact real-analytic Riemannian manifolds (see also the surveys [55] and [57]). On the other hand, (12) is the $\boldsymbol{\lambda}$-th component of P^τ; hence it is a natural candidate for the asymptotic study of complexified isotypical eigenfunctions.

In this paper, we shall be concerned with the near-diagonal asymptotics of (11) and (12), when the weight $\boldsymbol{\lambda}$ drifts to infinity along a ray in weight space. In other words, for a fixed non-zero $\boldsymbol{\lambda} \in \mathcal{D}^G$, we restrict attention to the *ladder* of representations $V_{k\boldsymbol{\lambda}}$,

$k = 1, 2, \ldots$ (see [25]), and consider the asymptotics of the sequence of smoothing kernels $\Pi^\tau_{k\boldsymbol{\lambda}}$ and $P^\tau_{k\boldsymbol{\lambda}}$ when $k \to +\infty$.

A guiding heuristic aspect of asymptotics on Grauert tube boundaries is the balance between analogies and differences with Szegő kernel asymptotics in the setting of line bundles over complex projective manifolds or, more precisely, of the associated unit circle bundles (see, e.g., the discussion in [9] and [10]). In the latter context, analogues of the equivariant asymptotics studied in this paper were considered in [16], [17], and [33]. In particular, in spite of the rather different geometric context, the representation-theoretic side of the approach used here has formal analogies with the one in [33] and, correspondingly, the statements bear a formal similarity to those in *loc. cit.* (see the closing remark of this introduction).

In order to explain our first result in this direction, we need to premise some further notation. The isomorphism $\mathcal{L} : \mathfrak{g} \to \mathfrak{g}^\vee$ induced by κ_e intertwines the adjoint and coadjoint representations Ad and Coad of G. Hence the inverse image under $\mathcal{L}$ of a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^\vee$ is an adjoint orbit $\tilde{\mathcal{O}} \subset \mathfrak{g}$. Furthermore, since Ad and Coad are unitary, if non trivial, $\mathcal{O}$ and $\tilde{\mathcal{O}}$ lie in spheres centered at the origin in $\mathfrak{g}^\vee$ and $\mathfrak{g}$, respectively. For any $\tau > 0$, we shall denote by $\mathcal{O}^\tau$ (respectively, $\tilde{\mathcal{O}}^\tau$) the unique rescaling of $\mathcal{O}$ (respectively, $\tilde{\mathcal{O}}$) contained in the sphere of radius τ centered at the origin in $\mathfrak{g}^\vee$ (respectively, $\mathfrak{g}$); thus $\mathcal{L}(\tilde{\mathcal{O}}^\tau) = \mathcal{O}^\tau$.

Definition 1.2 Given a non-zero $\boldsymbol{\lambda} \in \mathcal{D}^G$, let $\mathcal{O} \subset \mathfrak{g}^\vee$ be its coadjoint orbit, and let $\mathcal{C}(\mathcal{O}) \subseteq \mathfrak{g}^\vee$ be the positive cone over $\mathcal{O}$.

1. With γ is as in (4), let us set

$$\tilde{G}_{\mathcal{O}} := \gamma\left(G \times \mathcal{C}(\tilde{\mathcal{O}})\right).$$

2. For any $\tau > 0$, let us define

$$X^\tau_{\mathcal{O}} := \gamma(G \times \tilde{\mathcal{O}}^\tau) \subseteq X^\tau \subset \tilde{G}.$$

The geometric significance of $X^\tau_{\mathcal{O}}$ is discussed in §4.1, where it is shown that it is the inverse image in X^τ of $\mathcal{C}(\tilde{\mathcal{O}})$ under the natural moment map for the action of G on $(\tilde{G}, \Omega)$ (see (30)); see also the concluding remarks of this Introduction.

Clearly, X^τ and $X^\tau_{\mathcal{O}}$ are G-invariant. If $x \in \tilde{G}$, let $G \cdot x \subset \tilde{G}$ denote its G-orbit under left translations.

Definition 1.3 In the same setting as in Definition 1.2, let us pose

$$\mathcal{Z}^\tau_{\mathcal{O}} := \left\{(x, y) \in X^\tau \times X^\tau \,:\, x \in X^\tau_{\mathcal{O}} \quad \text{and} \quad y \in G \cdot x\right\}.$$

Theorem 1.4 *Uniformly on compact subsets of* $X^\tau \times X^\tau \setminus \mathcal{Z}^\tau_{\mathcal{O}}$, *one has*

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) = O\left(k^{-\infty}\right) \qquad \textit{and} \qquad P^\tau_{k\boldsymbol{\lambda}}(x, y) = O\left(k^{-\infty}\right)$$

for $k \to +\infty$.

We next aim to show, by analogy with the results in [33], that rapid decay may be established at pairs $(x, y) \in X^\tau \times X^\tau \setminus \mathcal{Z}^\tau_{\mathcal{O}}$ approaching $\mathcal{Z}^\tau_{\mathcal{O}}$ at a controlled pace as $k \to +\infty$.

A first manifestation of this is the following result, which deals with pairs (x, y) with $y \to G \cdot x$ sufficiently slowly in k (with a restriction on $\mathcal{O}$).

Let dist_{X^τ} denote the Riemannian distance on X^τ, and let $\mathfrak{t}^0 \subset \mathfrak{g}^\vee$ denote the annhilator of $\mathfrak{t}$.

Theorem 1.5 *Suppose that $\mathcal{O} \cap \mathfrak{t}^0 = \emptyset$. Let $C, \epsilon > 0$ be fixed. Then, uniformly for*

$$\mathrm{dist}_{X^\tau}(y, G \cdot x) \geq C\, k^{\epsilon - \frac{1}{2}}, \tag{14}$$

one has

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) = O\left(k^{-\infty}\right) \quad \textit{and} \quad P^\tau_{k\boldsymbol{\lambda}}(x, y) = O\left(k^{-\infty}\right)$$

for $k \to +\infty$.

For instance, the previous assumption on $\mathcal{O}$ is satisfied when $G = U(n)$ and the matrices in $\tilde{\mathcal{O}}$ have non-zero trace. On the other hand, for $G = SU(2)$ the conclusion of Theorem 1.5 may be proved by adapting the *ad hoc* argument for Theorem 1.1 in [17]. Thus it seems natural to expect that the result should hold in greater generality.

The weight $\boldsymbol{\lambda}$ is called *regular* if its coadjoint orbit $\mathcal{O}$ has maximal dimension, or equivalently if its stabilizer is T.

Theorem 1.6 *In the situation of Theorem 1.5, let us assume that $\boldsymbol{\lambda}$ is regular. Let us a fix constants $C, \epsilon > 0$. Then, uniformly on*

$$\left\{(x, y) \in X^\tau \times X^\tau \,:\, \mathrm{dist}_{X^\tau}\left(x, X^\tau_{\mathcal{O}}\right) \geq C\, k^{\epsilon - \frac{1}{2}}\right\} \tag{15}$$

one has

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) = O\left(k^{-\infty}\right) \quad \textit{and} \quad P^\tau_{k\boldsymbol{\lambda}}(x, y) = O\left(k^{-\infty}\right)$$

for $k \to +\infty$.

Next we consider near-diagonal scaling asymptotics for $\Pi^\tau_{k\boldsymbol{\lambda}}$ along $X^\tau_{\mathcal{O}}$. To formulate this, we need to describe how the tangent space of X^τ at a given $x \in X^\tau_{\mathcal{O}}$ decomposes in terms of the local geometry.

Let $\jmath^\tau : X^\tau \hookrightarrow \tilde{G}$ be the inclusion, and set $\alpha^\tau := \jmath^{\tau *}(\alpha)$; thus (X^τ, α^τ) is a contact manifold, and $\mathcal{H}^\tau_x := \ker(\alpha^\tau_x)$ is the maximal complex subspace of $T_x X^\tau$ (and has complex dimension $d - 1$). Let $\mathcal{R}^\tau$ be the Reeb vector field of (X^τ, α^τ). Then

$$T_x X^\tau = \mathrm{span}_{\mathbb{R}}\big(\mathcal{R}^\tau(x)\big) \oplus_{\hat{\kappa}_x} \mathcal{H}^\tau_x, \tag{16}$$

where $\oplus_{\hat{\kappa}_x}$ denotes an orthogonal direct sum for $\hat{\kappa}_x$; (16) holds at any $x \in X^\tau$.

Assuming $x \in X^\tau_{\mathcal{O}}$, let $N(\tilde{G}_{\mathcal{O}}/\tilde{G})_x \subseteq T_x\tilde{G}$ be the normal space to $\tilde{G}_{\mathcal{O}}$ in $\tilde{G}$ at x. Its (real) dimension is $r_{\mathcal{O}} - 1$, where $r_{\mathcal{O}}$ is the dimension of the stabilizer of any element of $\mathcal{O}$ (that is, $r_{\mathcal{O}}$ is the codimension of $\mathcal{O}$ in $\mathfrak{g}^\vee$).

By the transversality of $\tilde{G}_{\mathcal{O}}$ and X^τ, $N(\tilde{G}_{\mathcal{O}}/\tilde{G})_x \subseteq T_x X^\tau$ is also the normal space to $X^\tau_{\mathcal{O}} = X^\tau \cap \tilde{G}_{\mathcal{O}}$ in X^τ, that is,

$$N(\tilde{G}_{\mathcal{O}}/\tilde{G})_x = N(X^\tau_{\mathcal{O}}/X^\tau)_x.$$

As discussed in §4.3, $N(X^\tau_{\mathcal{O}}/X^\tau)_x \subset \mathcal{H}_x$, and furthermore $N(X^\tau_{\mathcal{O}}/X^\tau)_x$ is $\hat{\kappa}_x$-orthogonal to $J_x\big(N(X^\tau_{\mathcal{O}}/X^\tau)_x\big)$. Let us consider the complex vector subspace

$$\mathcal{N}_x := N(X^\tau_{\mathcal{O}}/X^\tau)_x \oplus_{\hat{\kappa}_x} J_x\big(N(X^\tau_{\mathcal{O}}/X^\tau)_x\big) \subseteq \mathcal{H}_x, \tag{17}$$

and let $\mathcal{S}_x$ be its orthocomplement in $\mathcal{H}_x$. There is a direct sum of complex vector spaces

$$\mathcal{H}_x = \mathcal{S}_x \oplus_{\hat{\kappa}_x} \mathcal{N}_x, \quad \text{where} \quad \dim_{\mathbb{C}}(\mathcal{N}_x) = r_{\mathcal{O}} - 1,\ \dim_{\mathbb{C}}(\mathcal{S}_x) = d - r_{\mathcal{O}}. \tag{18}$$

Then:

1. $\tilde{G}_{\mathcal{O}}$ is a coisotropic submanifold of $(\tilde{G}, \Omega)$;
2. $J_x\big(N(X^\tau_{\mathcal{O}}/X^\tau)_x\big)$ is the tangent space to the leaf through x of the null foliation of $\tilde{G}_{\mathcal{O}}$;
3. the $\hat{\kappa}_x$-orthocomplement of $J_x\big(N(X^\tau_{\mathcal{O}}/X^\tau)_x\big)$ in $T_x X^\tau$ is

$$J_x\big(N(X^\tau_{\mathcal{O}}/X^\tau)_x\big)^{\perp_{\kappa_x}} = \mathrm{span}_{\mathbb{R}}\big(\mathcal{R}^\tau(x)\big) \oplus_{\hat{\kappa}_x} N(X^\tau_{\mathcal{O}}/X^\tau)_x \oplus_{\hat{\kappa}_x} \mathcal{S}_x; \tag{19}$$

4. there is an orthogonal direct sum of real vector spaces

$$T_x X^\tau_{\mathcal{O}} = \mathrm{span}_{\mathbb{R}}\big(\mathcal{R}^\tau(x)\big) \oplus_{\hat{\kappa}_x} J_x\big(N(X^\tau_{\mathcal{O}}/X^\tau)_x\big) \oplus_{\hat{\kappa}_x} \mathcal{S}_x. \tag{20}$$

We shall consider scaling asymptotics along directions normal to the null foliations, that is in (19).

Furthermore, these scaling asymptotics will be formulated in suitable sets of local coordinates on X^τ centered at x, adapted to the pseudoconvex geometry of X^τ, and called *normal Heisenberg local coordinates* (NHLC's). We refer to [34] for a detailed discussion building on [9], [10], [38], [14] and [15]; the introduction of this type of coordinates in pseudoconvex geometry goes back to Folland and Stein, and (to the best of our knowledge) they have first been to use in the specific Grauert tube setting by Chang and Rabinowitz. NHLC's at x on X^τ are constructed by first introducing NHLC's on $\tilde{G}$, and then 'projecting and restricting'.

In NHLC's on $\tilde{G}$, the local defining equation for X^τ admits an especially simple canonical form, and this allows for a relatively explicit approximation of the metric, CR and symplectic structures, and ultimatively of the phase of the Szegö and Poisson kernels (see §4.6).

NHLC's on X^τ centered at x will be denoted by a pair $(\theta, \mathbf{v}) \in \mathbb{R}\times\mathbb{R}^{2d-2}$ of suitably small norm. When convenient, we shall identify $\mathbb{R}^{2d-2} \cong \mathbb{C}^{d-1}$. A point x' close to x

having NHLC's $(\theta, \mathbf{v})$ will be denoted by the additive notation $x' = x + (\theta, \mathbf{v})$. Then

$$\left.\frac{\partial}{\partial\theta}\right|_x = \mathcal{R}^\tau(x), \qquad \left.\frac{\partial}{\partial\mathbf{v}}\right|_x \in \mathcal{H}_x, \tag{21}$$

where $\partial/\partial\mathbf{v}$ denotes the directional derivative along $\mathbf{v} \in \mathbb{R}^{2d-2}$.

For $\theta = 0$, we shall generally write $x + \mathbf{v}$ for $x + (0, \mathbf{v})$.

It follows that NHLC's at $x \in X^\tau$ determine an isomorphism $\mathbb{R} \times \mathbb{R}^{2d-2} \cong T_x X^\tau$, with $\mathbb{R} \times \{\mathbf{0}\}$ (and $\{0\} \times \mathbb{R}^{2d-2}$) mapping to, respectively, $\text{span}_{\mathbb{R}}\big(\mathcal{R}^\tau(x)\big)$ and $\mathcal{H}_x$. Furthermore, with the identification $\{0\} \times \mathbb{R}^{2d-2} \cong \mathbb{C}^{d-1}$ we obtain an isomorphism of complex vector spaces $\mathbb{C}^{d-1} \cong \mathcal{H}_x$.

Therefore, pulling back to $\mathbb{C}^{d-1}$ the direct sum decomposition (18) determines a corresponding orthogonal complex direct sum decomposition

$$\mathbb{C}^{d-1} = \mathbb{C}_{\mathcal{N}}^{r_{\mathcal{O}}-1} \oplus \mathbb{C}_{\mathcal{S}}^{d-r_{\mathcal{O}}} = \left[\mathbb{R}_N^{r_{\mathcal{O}}-1} \oplus \mathbb{R}_{JN}^{r_{\mathcal{O}}-1}\right] \oplus \mathbb{C}_{\mathcal{S}}^{d-r_{\mathcal{O}}},$$

where $\mathbb{C}_{\mathcal{N}}^{r_{\mathcal{O}}-1}$ and $\mathbb{C}_{\mathcal{S}}^{d-r_{\mathcal{O}}}$ are, respectively, the inverse image of $\mathcal{N}_x$ and $\mathcal{S}_x$; in the second equality, we have further used the splitting (17) to define the real summands $\mathbb{R}_N^{r_{\mathcal{O}}-1} \subseteq \mathbb{C}_{\mathcal{N}}^{r_{\mathcal{O}}-1}$ (corresponding to $N(X_{\mathcal{O}}^\tau/X^\tau)_x \subseteq \mathcal{N}_x$) and $\mathbb{R}_{JN}^{r_{\mathcal{O}}-1} \subseteq \mathbb{C}_{\mathcal{N}}^{r_{\mathcal{O}}-1}$ (corresponding to $J_x\big(N(X_{\mathcal{O}}^\tau/X^\tau)_x\big) \subseteq \mathcal{N}_x$) in a similar manner.

We shall focus on near-diagonal scaling asymptotics on the scale of $k^{\epsilon-1/2}$ along directions normal to the null foliation of $\tilde{G}_{\mathcal{O}}$, thus at pairs of points (x_{1k}, x_{2k}) near (x, x) (for a given $x \in X_{\mathcal{O}}^\tau$) of the form

$$x_{j,k} := x + \left(\frac{\theta_j}{\sqrt{k}}, \frac{\mathbf{n}_j + \mathbf{s}_j}{\sqrt{k}}\right) \qquad (j = 1, 2), \tag{22}$$

where

$$(\theta_j, \mathbf{n}_j, \mathbf{s}_j) \in \mathbb{R} \times \mathbb{R}_N^{r_{\mathcal{O}}-1} \times \mathbb{C}_{\mathcal{S}}^{d-r_{\mathcal{O}}} \cong \mathbb{R} \cdot \mathcal{R}_x \times N(X_{\mathcal{O}}^\tau/X^\tau)_x \times \mathcal{S}_x \tag{23}$$

has norm $O\left(k^\epsilon\right)$.

If $\mathcal{O} = \mathcal{O}_{\boldsymbol{\lambda}}$ is the coadjoint orbit of a regular weight, then $r_{\mathcal{O}} = r_G$.

In order to state the following Theorem, we need to define a certain number of local and global invariants of our geometric setting.

Definition 1.7 Let (V, h) be a complex d-dimensional Hermitian vector space, and set $\varphi := \Re(h)$ (an Euclidean scalar product on V) and $\gamma = -\Im(h)$ (a symplectic bilinear form on V), so that $h = \varphi - \imath\,\gamma$. Let us define $\psi_2^h : V \times \times V \to \mathbb{C}$ by setting

$$\psi_2^h(v, w) := -\imath\,\gamma(v, w) - \frac{1}{2}\,\|v - w\|_\varphi^2,$$

where $\|v\|_\varphi := \varphi(v, v)^{1/2}$; we shall equivalently write $\psi_2^h = \psi_2^\varphi = \psi_2^\gamma$. If $(V, h) = (\mathbb{C}^d, h_{st})$ (the standard Hermitian product), we shall write $\psi_2 = \psi_2^{h_{st}}$.

Definition 1.8 Let us adopt the following notation:

1. $\mathrm{vol}(\mathcal{O})$ is the symplectic volume of the coadjoint orbit $\mathcal{O} = \mathcal{O}_{\boldsymbol{\lambda}}$ through $\boldsymbol{\lambda}$ (for the Kirillov-Kostant-Souriau symplectic form);
2. $\mathrm{vol}^{k}(G)$ and $\mathrm{vol}^{\kappa}(T)$ are the volumes of G and T, respectively, for the bi-invariant Riemannian structures associated to κ;
3. If $\boldsymbol{\lambda}$ is a regular weight and $\boldsymbol{\lambda}^{\kappa} \in \mathfrak{t}$ corresponds to $\boldsymbol{\lambda}$ under κ_e, then the endomorphism
$$S_{\boldsymbol{\lambda}} := \mathrm{ad}_{\boldsymbol{\lambda}^{\kappa}}|_{\mathfrak{t}^{\perp_{\kappa_e}}} : \boldsymbol{\rho} \in \mathfrak{t}^{\perp_{\kappa_e}} \mapsto \left[\boldsymbol{\lambda}^{\kappa}, \boldsymbol{\rho}\right] \in \mathfrak{t}^{\perp_{\kappa_e}} \tag{24}$$
is a skew-symmetric automorphism. Thus $\mathfrak{d}_{\boldsymbol{\lambda}} := \det(S_{\boldsymbol{\lambda}}) > 0$.
4. Let $\omega := \frac{1}{2}\,\Omega$, so that $(\tilde{G}, J, \omega)$ is Kähler manifold with associated Riemannian metric $\tilde{\kappa} := \frac{1}{2}\,\hat{\kappa}$.
5. Given $x \in \tilde{G}$, let $\mathrm{val}_x : \mathfrak{g} \to T_x\tilde{G}$ be the map $\boldsymbol{\xi} \mapsto \boldsymbol{\xi}_{\tilde{G}}(x)$, where $\boldsymbol{\xi}_{\tilde{G}} \in \mathfrak{X}(\tilde{G})$ is the vector field induced by $\boldsymbol{\xi}$ under the action $\tilde{L}$; then $\mathrm{val}_x^t(\tilde{\kappa}_x)$ is an Euclidean scalar product on $\mathfrak{g}$.
6. Suppose that $x \in \tilde{G} \setminus G$ is such that $\Phi(x) \in \mathfrak{g}^{\vee}$ is regular. Let T_x be the stabilizer of $\Phi(x)$ (a maximal torus) and $\mathfrak{t}_x \subseteq \mathfrak{g}$ its Lie algebra. Let $\mathfrak{t}_x' := \mathfrak{t}_x \cap \Phi(x)^0$ (a hyperplane in $\mathfrak{t}_x$). Let $\mathfrak{B}_x$ be an orthonormal basis of $\mathfrak{t}_x'$ for κ_e, and let D_x be the matrix representing the restriction of $\mathrm{val}_x^t(\tilde{\kappa}_x)$ to $\mathfrak{t}_x'$ with respect to $\mathcal{B}_x$. Then $\det(D_x)$ only depends on x, and we may set
$$\mathfrak{D}^{\kappa}(x) := \sqrt{\det(D_x)}.$$

Theorem 1.9 *Let us fix constants $C > 0$ and $\epsilon \in (0, 1/6)$. Under the assumptions of Theorem 1.6, consider $x \in X^{\tau}_{\mathcal{O}}$ and choose a system of NHLC at x. Let us define x_{jk} as in (22) and (23). Then, uniformly for $\|(\theta_j, \mathbf{n}_j, \mathbf{s}_j)\| \leq C\,k^{\epsilon-1/2}$, the following asymptotic expansions hold for $k \to +\infty$:*

$$\begin{aligned}\Pi^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k}) \sim & \left(\frac{k\,\|\boldsymbol{\lambda}\|}{2\pi\,\tau}\right)^{d-1+\frac{1-r_G}{2}} \left(\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})}{\mathrm{vol}^{\kappa}(G)}\right)^2 \cdot \frac{\mathrm{vol}^{\kappa}(T)}{\mathfrak{D}^{\kappa}(x)\cdot \mathfrak{d}_{\boldsymbol{\lambda}}} \\ & \cdot \exp\left(\frac{\|\boldsymbol{\lambda}\|}{\tau}\left[\psi_2^{\omega_x}(\mathbf{s}_1, \mathbf{s}_2) - \|\mathbf{n}_1\|^2_{\tilde{\kappa}_x} - \|\mathbf{n}_2\|^2_{\tilde{\kappa}_x}\right]\right) \\ & \cdot \left[1 + \sum_{j\geq 1} k^{-j/2}\,R_j(\theta_1, \theta_2, \mathbf{s}_1, \mathbf{s}_2, \mathbf{n}_1, \mathbf{n}_2)\right],\end{aligned}$$

$$P^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k}) \sim \left(\frac{1}{2}\right)^{\frac{d-1}{2}} \left(\frac{k\,\|\boldsymbol{\lambda}\|}{2\pi\,\tau}\right)^{\frac{d-r_G}{2}} \left(\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})}{\mathrm{vol}^{\kappa}(G)}\right)^2 \cdot \frac{\mathrm{vol}^{\kappa}(T)}{\mathfrak{D}^{\kappa}(x)\cdot \mathfrak{d}_{\boldsymbol{\lambda}}}$$
$$\cdot \exp\left(\frac{\|\boldsymbol{\lambda}\|}{\tau}\left[\psi_2^{\omega_x}(\mathbf{s}_1, \mathbf{s}_2) - \|\mathbf{n}_1\|^2_{\tilde{\kappa}_x} - \|\mathbf{n}_2\|^2_{\tilde{\kappa}_x}\right]\right)$$
$$\cdot \left[1 + \sum_{j\geq 1} k^{-j/2}\, S_j(\theta_1, \theta_2, \mathbf{s}_1, \mathbf{s}_2, \mathbf{n}_1, \mathbf{n}_2)\right],$$

where R_j, S_j are polynomials of degree $\leq 3j$ and parity j.

1.1 Applications

Let us describe a sample of applications of the previous Theorems. To begin with, Theorem 1.6 and the second expansion of Theorem 1.9 yield an asymptotic estimate on the L^∞ norms of complexified matrix elements and their Husimi probability distributions (see [54], [58]).

If $\varphi \in L^2(G)$ is an eigenfunction of Δ, let $\tilde{\varphi}$ denote its complexification and set $\tilde{\varphi}^\tau := \tilde{\varphi}|_{X^\tau}$. The *Husimi distribution* $U^\tau_\varphi : X^\tau \to \mathbb{R}$ is given by

$$U^\tau_\varphi(x) := \frac{|\tilde{\varphi}^\tau(x)|^2}{\|\varphi^\tau\|^2_{L^2(X^\tau)}}.$$

The following Proposition specializes to the present setting the upper bounds in Theorem 0.1 in [58].

Theorem 1.10 *There exist constants $C(\tau, \boldsymbol{\nu}),\ C'(\tau, \boldsymbol{\nu}) > 0$ such that for any $k \gg 0$ the following holds.*

1. *Let $\varphi \in L^2(G)_{k\boldsymbol{\nu}}$ have unit $L^2(G)$-norm. Then*

$$\max_{x\in X^\tau}\left\{\left|\varphi^\tau(x)\right|\right\} \leq C(\tau, \boldsymbol{\nu})\, e^{\tau\, c_{k\lambda}}\, (c_{k\boldsymbol{\lambda}})^{\frac{d-r_G}{4}}.$$

2. *Under the same assumptions,*

$$\max_{x\in X^\tau}\left\{U^\tau_\varphi(x)^{\frac{1}{2}}\right\} \leq C'(\tau, \boldsymbol{\nu})\,(c_{k\boldsymbol{\lambda}})^{\frac{d}{2}-\frac{1+r_G}{4}}.$$

For the meaning of $c_{k\boldsymbol{\lambda}}$, see (3).

The second application concerns the norm of $\Pi^\tau_{k\boldsymbol{\lambda}}$ as a bounded operator $L^p(X^\tau) \to L^q(X^\tau)$. A similar estimate can be given for $P^\tau_{k\boldsymbol{\lambda}}$ and will be left to the reader. This result is the analogue of analogous operator estimates in [10] and [18] in the Grauert tube setting, and [39] in the line bundle context. On the other hand, these estimates are inspired by previous estimates in the real domain due to Sogge ([40] and [41]).

Theorem 1.11 *Suppose* $1 \leq p \leq q$, *and define* R *by setting*

$$\frac{1}{R} := 1 - \frac{1}{p} + \frac{1}{q}.$$

Then there is a constant $C > 0$ *such that for any* $\epsilon > 0$ *and* $k \gg 0$ *one has*

$$\| \Pi^{\tau}_{k\boldsymbol{\lambda}} \|_{L^p(M) \to L^q(M)} \leq C\, k^{\frac{1}{2R}[(d-1)\,(R-1)+R(d-r_G)]+\epsilon}.$$

1.2 Concluding Remarks

Let us close this introduction with some heuristic remarks, and a digression on possibile developments (we thank the referees for stimulating comments in this regard).

First, let us dwell on a slight difference in the representation-theoretic approach between [33], where similar asymptotics were considered in the line bundle setting, and the present paper. Given a dominant weight $\boldsymbol{\lambda}$, let us set $\boldsymbol{\nu} := \boldsymbol{\nu}_{\boldsymbol{\lambda}} = \boldsymbol{\lambda} + \boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is the half-sum of the positive roots - see (59) below; for a fixed dominant weight $\boldsymbol{\lambda}$, the equivariant asymptotics in [33] are those associated to the sequence of weights $\boldsymbol{\lambda}_k$ such that $\boldsymbol{\nu}_{\boldsymbol{\lambda}_k} = k\,\boldsymbol{\nu}$.

In this paper, given a fixed regular dominant weight $\boldsymbol{\lambda}$, we consider the asymptotics associated to the representations $k\,\boldsymbol{\lambda}$. This implies that the role played by $\mathcal{O}_{\boldsymbol{\nu}}$ in [33] is played here by $\mathcal{O}_{\boldsymbol{\lambda}}$.

This approach is perhaps more natural, and makes the character computations based on the Kirillov character formula only slightly more involved. The same arguments and change in approach could of course be applied to the setting of [33].

Let us also pause to give a heuristic explanation for the asymptotic concentration described in Theorem 1.4; this essentially boils down to the fact that $\mathcal{Z}^{\tau}_{\mathcal{O}}$ in Definition 1.3 is the singular support of the 'ladder projection' kernel associated to $\boldsymbol{\lambda}$ (for more details, see the discussions in §3.1.2 of [16] and §5.1 below). Let $L_{\boldsymbol{\lambda}} := (k\,\boldsymbol{\lambda})_{k=1}^{+\infty}$ denote the ladder of weights sprayed by $\boldsymbol{\lambda}$. With the foregoing notation for characters and dimensions, let us associate to L the distribution

$$\chi_{L_{\boldsymbol{\lambda}}} := \sum_{k=1}^{+\infty} d_{k\,\boldsymbol{\lambda}}\, \chi_{k\,\boldsymbol{\lambda}} \in \mathcal{D}'(G).$$

It was proved by Guillemin and Sternberg in [25] that $\chi_{L_{\boldsymbol{\lambda}}}$ is a Lagrangian distribution, and that its associated conic Lagrangian submanifold $\Lambda_{L_{\boldsymbol{\lambda}}} \subset T^{\vee}G \cong G \times \mathfrak{g}^{\vee}$ is a suitable incidence correspondence projecting down to $\mathcal{C}(\mathcal{O})$ in $\mathfrak{g}^{\vee}$. Let us consider the ladder subrepresentation

$$H(X^{\tau})_{L_{\boldsymbol{\lambda}}} := \bigoplus_{k=1}^{+\infty} H(X^{\tau})_{k\,\boldsymbol{\lambda}} \subseteq H(X),$$

and denote by $\Pi^\tau_{L_\lambda} : L^2(X^\tau) \to H(X^\tau)_{L_\lambda}$ the orthogonal projector. The distributional kernel of $\Pi^\tau_{L_\lambda}$ is given in terms of χ_{L_λ} and Π^τ by the relation

$$\Pi^\tau_{L_\lambda}(x, y) = \int_G \overline{\chi_{L_\lambda}(g)}\, \Pi^\tau \left(\mu^\tau_{g^{-1}}(x), y\right) \mathrm{d}V_G(g). \tag{25}$$

On the other hand, the wave front $\mathrm{WF}(\Pi^\tau)$ of Π^τ is the anti-diagonal of the cone sprayed by α^τ (see (49) below). Using (25), the wave front of $\Pi^\tau_{L_\lambda}$ can be then be worked out from Λ_{L_λ} and $\mathrm{WF}(\Pi^\tau)$ building on the functorial properties of distributions (see e.g. [12] and [24]). One then concludes that the singular support of $\Pi^\tau_{L_\lambda}(\cdot,\cdot)$ is contained in $\mathscr{Z}^\tau_{\mathcal{O}}$. It follows that $\Pi^\tau_{L_\lambda}(\cdot,\cdot)$ is $\mathcal{C}^\infty$ on $X^\tau \times X^\tau \setminus \mathscr{Z}^\tau_{\mathcal{O}}$. Since $\Pi^\tau_{k\boldsymbol{\lambda}}$ is obtained by composing $\Pi^\tau_{L_\lambda}$ with the projection onto the $k\,\boldsymbol{\lambda}$-th isotypical component, rapid decrease on compact subsets follows from classical properties of the group-theoretic Fourier transform [45].

Finally, let us discuss some possible interesting developments in relation to analytic microlocal analysis, suggested by the line bundle setting (and on which we hope to return in the future). Equivariant asymptotic expansions for Szegő kernels on Grauert tube boundaries have a heuristic precursor in the study of the Fourier components of Szegő kernels of positive complex line bundles, which has given rise to a vast literature in recent years. In fact, the dual unit circle bundle associated to a positive Hermitian line bundle (L, h) is a principal S^1-bundle, and its Hardy space splits in a direct sum of isotypical components, which may be identified with the spaces of holomorphic sections of powers of L.

The present arguments are grounded in the classical microlocal analysis of the Szegő kernel as a Fourier integral operator with complex phase of positive type [3]: this is the technical core on which the derivations of the above expansions are pivoted. In this sense, the present approach is the counterpart of the one in [52] and [38] for the line bundle case, adapted to general group actions.

On the other hand, in the line bundle context, when the Hermitian bundle metric h is real-analytic the description of the components of the Bergman kernel as semiclassical analytic Fourier integral operators has led to asymptotic expansions in the semiclassical limit with exponential off-diagonal estimates (with no pretense of completeness, see [59], [27], [37], [26], [11] and references therein for general background and precise statements of key results). It seems therefore natural to expect that the general theory of analytic Fourier integral operators can be profitably put to use in the present context, leading to off-diagonal estimates akin to those in the analytic line bundle setting. Let us mention two issues in this regard. First, general group characters are considerably more complicated than for S^1, and a good deal of the following technicalities relate to the application of character formulae. In particular, to write the isotypical kernels $\Pi^\tau_{k\boldsymbol{\lambda}}$ as oscillatory integrals in the semiclassical parameter $k = 1/h$ we shall make recourse to the Kirillov character formula. Since the latter holds on a neighbourhood of the unit in G, this entails introducing a $\mathcal{C}^\infty$ cut-off, rendering the final amplitude in the computation non-analytic (although the general setting is). A second issue is that in the present situation the isotypical components $H(X^\tau)_{\boldsymbol{\lambda}}$ are not interpretable as spaces of sections of powers of a given line bundle, hence not all the arguments for the line bundle setting may be applied directly (although some do hold for general phase spaces).

2 Notation

In this section, we collect some notation and conventions used in the paper for the reader's convenience.

We have adopted the notation $V^\vee$ for the dual of a vector space V, rather than the perhaps more common V^*, as the superscript $\cdot^*$ often appears with other meanings, such as the symbol of pull-back, or to denote the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. This notation is also in line with the one in [16], [17] and [33], to which we often refer.

We shall think of $\mathfrak{g}$ as the Lie algebra of left-invariant vector fields on G (that is, those vector fields associated to the right action of G on itself by right translations). Since G also acts on $\tilde{G}$ by right translations, any $\boldsymbol{\xi} \in \mathfrak{g}$ extends to a left-invariant vector field $\boldsymbol{\xi}^{\tilde{G}}$ on $\tilde{G}$.

On the other hand, G also acts on itself and $\tilde{G}$ by left translations. This action associates to each $\boldsymbol{\xi} \in \mathfrak{g}$ a right-invariant vector field $\boldsymbol{\xi}_{\tilde{G}}$ on $\tilde{G}$.

Thus $\boldsymbol{\xi}^{\tilde{G}}(x),\ \boldsymbol{\xi}_{\tilde{G}}(x) \in T_x\tilde{G}$ are given by

$$\boldsymbol{\xi}_{\tilde{G}}(x) := \frac{\mathrm{d}}{\mathrm{d}t} e^{t\,\boldsymbol{\xi}} \cdot x\Big|_{t=0}, \quad \boldsymbol{\xi}^{\tilde{G}}(x) := \frac{\mathrm{d}}{\mathrm{d}t} x \cdot e^{t\,\boldsymbol{\xi}}\Big|_{t=0} \quad (x \in \tilde{G}).$$

Since X^τ is G-invariant, when $x \in X^\tau$ we have $\boldsymbol{\xi}_{\tilde{G}}(x) \in T_x X^\tau$; we shall occasionally emphasize this by writing $\boldsymbol{\xi}_{X^\tau}(x)$ for $\boldsymbol{\xi}_{\tilde{G}}(x)$.

1. $\mathcal{O}_{\boldsymbol{\beta}}$: the coadjoint orbit through $\boldsymbol{\beta} \in \mathfrak{g}^\vee$; if $\boldsymbol{\beta} = \boldsymbol{\lambda}$ (the fixed regular weight in the positive Weyl chamber) we shall often write $\mathcal{O}$ for $\mathcal{O}_{\boldsymbol{\lambda}}$;
2. $\mathcal{O}^\tau$: the rescaling of $\mathcal{O}$ which is contained in the sphere of radius τ in $\mathfrak{g}^\vee$ ($\tau > 0$);
3. $\tilde{G}$: the complexification of G; $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \imath\,\mathfrak{g}$: its Lie algebra;
4. $\boldsymbol{\xi}_{\tilde{G}} \in \mathfrak{X}(\tilde{G})$: the vector field induced by $\boldsymbol{\xi} \in \mathfrak{g}$ where G acts on $\tilde{G}$ by left translations (see above);
5. $\mathrm{val}_x : \mathfrak{g} \to T_x\tilde{G}$: the evaluation map $\boldsymbol{\xi} \mapsto \boldsymbol{\xi}_{\tilde{G}}(x)$; its complex linear extension is the corresponding evaluation map for the holomorphic action of $\tilde{G}$ on itself by left translations, $\widehat{\mathrm{val}}_x : \tilde{\mathfrak{g}} \to T_x\tilde{G}$;
6. since $X^\tau \subset \tilde{G}$ is G-invariant for every $\tau > 0$, we have $\boldsymbol{\xi}_{\tilde{G}}(x) \in T_x X^\tau \subseteq T_x\tilde{G}$ if $x \in X^\tau$; thus there is no ambiguity in identifying $\boldsymbol{\xi}_{\tilde{G}}(x)$ and $\boldsymbol{\xi}_{X^\tau}(x)$ (similarly defined) and as mentioned we shall occasionally lighten notation (e.g. in the presence of subscripts) by simply writing $\boldsymbol{\xi}(x)$;
7. for a vector subspace $\mathfrak{a} \subseteq \mathfrak{g}$, we shall set $\mathfrak{a}_{\tilde{G}} \subseteq T\tilde{G}$ the (real) vector subbundle given by
$$\mathfrak{a}_{\tilde{G}}(x) := \mathrm{val}_x(\mathfrak{a}),$$
and similarly for its restriction $\mathfrak{a}_{X^\tau} \subset TX^\tau$ to X^τ; we shall again occasionally lighten notation and simply write $\mathfrak{a}(x)$ for $\mathfrak{a}_{\tilde{G}}(x)$;

8. when the previous vector subspace depends on $x \in X^\tau$, we shall emphasize this by writing $\mathfrak{a}_x$, and lighten notation by writing $\mathfrak{a}_x(x)$ instead of $(\mathfrak{a}_x)_{X^\tau}(x)$;
9. κ_e: the given Ad-invariant Euclidean product on $\mathfrak{g}$;
10. $\sigma_x := \mathrm{val}_x^t(\hat{\kappa}_x)$ (the Euclidean product on $\mathfrak{g}$ given by pull-back of $\hat{\kappa}$ under vector field evaluation at x - see §4.5);
11. ρ: the unique Kähler potential on $\tilde{G}$, such that $\sqrt{\rho}$ satisfies the homogeneous Monge-Ampère equation on $\tilde{G} \setminus G$ and the associated Kähler metric $\hat{\kappa}$ restricts to κ on G;
12. $\Omega := \imath\, \overline{\partial}\, \partial\, \rho$: the Kähler form on $\tilde{G}$; $\omega := \frac{1}{2}\,\Omega$, $\tilde{\kappa} := \frac{1}{2}\,\hat{\kappa}$.

3 An Example

Consider the compact torus $T^d = \mathbb{R}^d/(2\pi\,\mathbb{Z}^d)$ with the standard metric (*cfr.* §2 of [34]); in this case $d = r_{T^d}$. Let us identify $\mathbb{R}^d \cong (\mathbb{R}^d)^\vee$ by the standard Euclidean product. The the unitary dual $\hat{T}^d \cong \mathbb{Z}^d$, and for each $\boldsymbol{\lambda} \in \mathbb{Z}^d$ the subspace $L^2(T^d)_{\boldsymbol{\lambda}}$ is 1-dimensional. Furthermore, the cone $\mathcal{C}(\mathcal{O})$ is in this case just the positive ray $\mathbb{R}_+ \cdot \boldsymbol{\lambda}$.

More precisely, let $e^{\imath\,\boldsymbol{\vartheta}} = \left(e^{\imath\,\vartheta_1}\ \ldots\ e^{\imath\,\vartheta_d}\right)$ denote the general element of T^d, and for $\boldsymbol{\lambda} \in \mathbb{Z}^d$ let us set $\varphi_{\boldsymbol{\lambda}}(e^{\imath\,\boldsymbol{\vartheta}}) := c_d\, e^{-\imath\,\langle \boldsymbol{\lambda}, \boldsymbol{\vartheta}\rangle}$, where $c_d = (2\pi)^{-d/2}$. In view of (1), $(\varphi_{\boldsymbol{\lambda}})$ is an orthonormal basis of $L^2(T^d)_{\boldsymbol{\lambda}}$; furthermore, $\varphi_{\boldsymbol{\lambda}}$ is an eigenvector of the positive Laplacian corresponding to the eigenvalue $\|\boldsymbol{\lambda}\|^2$.

Let us identify $\tilde{T}^d = \mathbb{C}^d/(2\pi\,\mathbb{Z}^d) \cong (\mathbb{C}^*)^d$, and write the general element of $\tilde{T}^d$ as $e^{\imath\,\boldsymbol{\vartheta}+\mathbf{r}} = e^{\imath\,\boldsymbol{\vartheta}}\, e^{\imath\cdot\imath(-\mathbf{r})}$, where $\mathbf{r} \in \mathbb{R}^d$. The holomorphic extension of $\varphi_{\boldsymbol{\lambda}}$ is $\tilde{\varphi}_{\boldsymbol{\lambda}}(e^{\imath\,\boldsymbol{\vartheta}+\mathbf{r}}) := c_d\, e^{-\imath\,\langle\boldsymbol{\lambda},\boldsymbol{\vartheta}\rangle - \langle\boldsymbol{\lambda},\mathbf{r}\rangle}$.

On $\tilde{T}^d \setminus T$, we may pass to polar coordinates in the real component, and write the general element as in the form $e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}} = e^{\imath\,\boldsymbol{\vartheta}}\, e^{\imath\,\imath(-\tau\,\boldsymbol{\omega})}$, where $\tau > 0$ and $\boldsymbol{\omega} \in S^{d-1}$ (the unit sphere). Keeping τ fixed yields a parametrization of X^τ.

Thus on X^τ

$$e^{-2\,k\,\tau\,\|\boldsymbol{\lambda}\|}\,\left|\tilde{\varphi}_{k\boldsymbol{\lambda}}(e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}})\right|^2 = (2\pi)^{-d}\, e^{2\,\tau\,k\,[\langle\boldsymbol{\lambda},-\boldsymbol{\omega}\rangle - \|\boldsymbol{\lambda}\|]}.$$

We have $\langle\boldsymbol{\lambda}, -\boldsymbol{\omega}\rangle - \|\boldsymbol{\lambda}\| \le 0$ for any $\boldsymbol{\omega}$ of unit norm, and equality holds if and only if $-\boldsymbol{\omega} = \boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|$; in terms of the moment map $\tilde{\Phi}$ discussed in §4, this is the condition $e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}} \in \tilde{\Phi}^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\lambda})$.

Therefore, if $e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}} \notin \tilde{\Phi}^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\lambda}) \cap X^\tau$ then

$$e^{-2\,k\,\tau\,\|\boldsymbol{\lambda}\|}\,\left|\tilde{\varphi}_{k\boldsymbol{\lambda}}(e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}})\right|^2 = O\left(k^{-\infty}\right) \quad \text{for} \quad k \to +\infty;$$

on the other hand, if $e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}} \in \Phi^{-1}(\mathbb{R}_+ \cdot \boldsymbol{\lambda}) \cap X^\tau$ then

$$\left|\tilde{\varphi}_{k\boldsymbol{\lambda}}(e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}})\right| = (2\pi)^{-d/2}\, e^{\tau\,(k\,\|\boldsymbol{\lambda}\|)}, \quad e^{-2\,k\,\tau\,\|\boldsymbol{\lambda}\|}\,\left|\tilde{\varphi}_{k\boldsymbol{\lambda}}(e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}})\right|^2 = (2\pi)^{-d}.$$

This conclusion is in agreement with the statement of Theorem 1.9 for $P^{\tau}_{k\boldsymbol{\lambda}}$, since $\mathrm{vol}^{\kappa}(G) = (2\,\pi)^d$ and $D_x = 2^{-1}\,I_{d-1}$, so that

$$\mathfrak{D}^{\kappa}(x) = \sqrt{\det(D_x)} = \frac{1}{2^{(d-1)/2}}.$$

Let us determine the orthonormal basis of $L^2(X^\tau)_{k\boldsymbol{\lambda}}$. Let $S^{d-1}(\tau) \subseteq \mathbb{R}^d$ denote the sphere centered of radius τ at the origin. Then the L^2-norm of $\tilde{\varphi}^{\tau}_{k\boldsymbol{\lambda}}$ is

$$\begin{aligned}
\left\|\tilde{\varphi}^{\tau}_{k\boldsymbol{\lambda}}\right\|^2_{L^2(X^\tau)} &= (2\,\pi)^{-d}\int_{T^d}\mathrm{d}\boldsymbol{\vartheta}\int_{S^{d-1}(\tau)}\mathrm{d}\boldsymbol{\omega}\left[e^{2\,k\,\langle\boldsymbol{\lambda},\boldsymbol{\omega}\rangle}\right]\\
&= \tau^{d-1}\int_{S^{d-1}(1)}\mathrm{d}\boldsymbol{\omega}\left[e^{2\,\tau\,k\,\langle\boldsymbol{\lambda},\boldsymbol{\omega}\rangle}\right]\\
&= \tau^{d-1}\,e^{2\,k\,\tau\,\|\boldsymbol{\lambda}\|}\int_{S^{d-1}(1)}\mathrm{d}\boldsymbol{\omega}\left[e^{2\,\tau\,k\,[\langle\boldsymbol{\lambda},\boldsymbol{\omega}\rangle-\|\boldsymbol{\lambda}\|]}\right]\\
&= \tau^{d-1}\,e^{2\,k\,\tau\,\|\boldsymbol{\lambda}\|}\int_{S^{d-1}(1)}\mathrm{d}\boldsymbol{\omega}\left[e^{\imath\,k\,\Psi_{\boldsymbol{\lambda}}(\boldsymbol{\omega})}\right],
\end{aligned}$$

where

$$\Psi_{\boldsymbol{\lambda}}(\boldsymbol{\omega}) = \imath\,2\,\tau\,\left[\|\boldsymbol{\lambda}\| - \langle\boldsymbol{\lambda},\boldsymbol{\omega}\rangle\right].$$

We have $\Psi_{\boldsymbol{\lambda}} = \Im\Psi_{\boldsymbol{\lambda}} \geq 0$, and $\Im\Psi_{\boldsymbol{\lambda}} = 0$ if and only if $\boldsymbol{\omega} = \boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|$. In the neighbourhood of $\boldsymbol{\omega} = \boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|$, we can write

$$\boldsymbol{\omega}_{\boldsymbol{\eta}} := \sqrt{1-\|\boldsymbol{\eta}\|^2}\,\frac{\boldsymbol{\lambda}}{\|\boldsymbol{\lambda}\|} + \boldsymbol{\eta} = \left(1 - \frac{1}{2}\,\|\boldsymbol{\eta}\|^2 + R_3(\boldsymbol{\eta})\right)\frac{\boldsymbol{\lambda}}{\|\boldsymbol{\lambda}\|} + \boldsymbol{\eta},$$

where $\boldsymbol{\eta} \in \boldsymbol{\lambda}^\perp \cong \mathbb{R}^{2d-1}$ varies near the origin. Hence,

$$\begin{aligned}
\Psi_{\boldsymbol{\lambda}}(\boldsymbol{\omega}_{\boldsymbol{\eta}}) &= 2\,\imath\,\tau\left[\|\boldsymbol{\lambda}\| - \left(1 - \frac{1}{2}\,\|\boldsymbol{\eta}\|^2 + R_3(\boldsymbol{\eta})\right)\|\boldsymbol{\lambda}\|\right]\\
&= \imath\,\tau\left[\|\boldsymbol{\eta}\|^2\cdot\|\boldsymbol{\lambda}\| + R_3(\boldsymbol{\eta})\right].
\end{aligned}$$

Hence $\boldsymbol{\eta} = \mathbf{0}$ is a non-degenerate critical point, with Hessian matrix $2\,\imath\,\tau\,\|\boldsymbol{\lambda}\|\,I_{d-1}$. Thus

$$\sqrt{\det\left(\frac{1}{2\,\pi\,\imath}\,2\,\imath\,\tau\,\|\boldsymbol{\lambda}\|\,I_{d-1}\right)} = \sqrt{\left(\frac{\tau\,\|\boldsymbol{\lambda}\|}{\pi}\right)^{d-1}} = \left(\frac{\tau\,\|\boldsymbol{\lambda}\|}{\pi}\right)^{\frac{d-1}{2}}.$$

Hence, there is an asymptotic expansion for $k \to +\infty$

$$\left\|\tilde{\varphi}^{\tau}_{k\boldsymbol{\lambda}}\right\|^2_{L^2(X^\tau)} \sim \tau^{d-1}\, e^{2\,k\,\tau\,\|\boldsymbol{\lambda}\|}\, \left(\frac{\pi}{k\,\tau\,\|\boldsymbol{\lambda}\|}\right)^{\frac{d-1}{2}} \cdot \left(1+\sum_j k^{-j}\,a_j\right)$$
$$= e^{2\,k\,\tau\,\|\boldsymbol{\lambda}\|}\, \left(\frac{\tau\,\pi}{k\,\|\boldsymbol{\lambda}\|}\right)^{\frac{d-1}{2}} \cdot \left(1+\sum_j k^{-j}\,a_j\right).$$

Thus the L^2-normalization of $\tilde{\varphi}^{\tau}_{k\boldsymbol{\lambda}}$ is

$$\rho^{\tau}_{k\boldsymbol{\lambda}}\left(e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}}\right) \sim \frac{1}{(2\,\pi)^{d/2}}\,\left(\frac{k\,\|\boldsymbol{\lambda}\|}{\tau\,\pi}\right)^{\frac{d-1}{4}}\, e^{\imath\,k\,\langle\boldsymbol{\lambda},\boldsymbol{\vartheta}\rangle+k\,\tau\,[\langle\boldsymbol{\lambda},-\boldsymbol{\omega}\rangle-\|\boldsymbol{\lambda}\|]}.$$

Therefore, to leading order,

$$\left|\rho^{\tau}_{k\boldsymbol{\lambda}}\left(e^{\imath\,\boldsymbol{\vartheta}+\tau\,\boldsymbol{\omega}}\right)\right|^2 \sim \frac{1}{(2\,\pi)^{d}}\,\left(\frac{k\,\|\boldsymbol{\lambda}\|}{\tau\,\pi}\right)^{\frac{d-1}{2}}\, e^{2\,k\,\tau\,[\langle\boldsymbol{\lambda},-\boldsymbol{\omega}\rangle-\|\boldsymbol{\lambda}\|]}, \tag{26}$$

which is also rapidly decreasing unless $-\boldsymbol{\omega} = \boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|$, and in that case it agrees with the first asymptotic expansion in Theorem 1.9.

Let us consider a rescaled displacement in a normal direction to Z^τ, which amounts in this case to replacing (to first order) $\tau\,\boldsymbol{\omega} = \tau\,(-\boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|)$ by $\tau\,\boldsymbol{\omega} = \tau(-\boldsymbol{\lambda}/\|\boldsymbol{\lambda}\|) + \mathbf{n}/\sqrt{k}$, where $\mathbf{n} \in \boldsymbol{\lambda}^{\perp}$ is fixed. In the previous general notation, we consider the asymptotics at a diagonal pair $(x+\mathbf{n}/\sqrt{k},\, x+\mathbf{n}/\sqrt{k})$, with $x \in X^{\tau}_{\mathcal{O}}$ and $\mathbf{n}$ normal to $X^{\tau}_{\mathcal{O}}$. Since $\boldsymbol{\omega} \in S^{d-1}$,

$$\boldsymbol{\omega} = \sqrt{1-\frac{1}{\tau^2\,k}\|\mathbf{n}\|^2}\,\left(-\frac{\boldsymbol{\lambda}}{\|\boldsymbol{\lambda}\|}\right) + \frac{1}{\tau\,\sqrt{k}}\,\mathbf{n}.$$

Hence, the exponent in (26) is

$$2\,k\,\tau\,\,[\langle\boldsymbol{\lambda},-\boldsymbol{\omega}\rangle - \|\boldsymbol{\lambda}\|] = 2\,k\,\tau\,\|\boldsymbol{\lambda}\|\,\left[\sqrt{1-\frac{1}{\tau^2\,k}\|\mathbf{n}\|^2}-1\right]$$
$$= 2\,k\,\tau\,\|\boldsymbol{\lambda}\|\,\left[-\frac{1}{2\,\tau^2\,k}\|\mathbf{n}\|^2 + R_2\left(\frac{\|\mathbf{n}\|^2}{k}\right)\right] = \frac{\|\boldsymbol{\lambda}\|}{\tau}\,\left[-\|\mathbf{n}\|^2 + O\left(k^{-1}\right)\right]$$
$$= \frac{\|\boldsymbol{\lambda}\|}{\tau}\,\left[-2\,\|\mathbf{n}\|^2_{\tilde{\kappa}} + O\left(k^{-1}\right)\right],$$

where $\|\cdot\|^2_{\tilde{\kappa}} = \|\cdot\|^2/2$ in the notation of the introduction. The same argument applies for, say, $\|\mathbf{n}\| = O\left(k^{1/6}\right)$, with a remainder term $O\left(k^{-1/3}\right)$.

4 Preliminaries

In this section, we collect some basic results that will be used in the following proofs. More specific preliminaries will be given in the subsections devoted to the proofs.

4.1 The Hamiltonian Structure of $\tilde{L}$

As above, let $L : G \times G \to G$ and $\tilde{L} : G \times \tilde{G} \to \tilde{G}$ be the actions given by left translations. Then $\tilde{L}$ is intertwined by E in (5) with the tangent lift $\mathrm{d}L : G \times TG \to TG$ of L.

In turn, $\tilde{L}$ extends to the holomorphic action $\hat{L}$ of $\tilde{G}$ on itself given by left translations. For any $x \in \tilde{G}$, the induced evaluation $\widehat{\mathrm{val}}_x : \tilde{\mathfrak{g}} \to T_x\tilde{G}$ is an isomorphism of complex vector spaces. Since $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \imath\, \mathfrak{g}$, we conclude that $(\mathfrak{g}_{\tilde{G}})_x \cap J_x\big((\mathfrak{g}_{\tilde{G}})_x\big) = (0)$ (heuristically, the action of G on $\tilde{G}$ and on TG is 'totally real').

Since $\mathrm{d}L$ is Hamiltonian for Ω_{can}, so is $\tilde{L}$ for Ω, with moment map

$$\tilde{\Phi} := \Phi \circ E^{-1} : \tilde{G} \to \mathfrak{g}^\vee,$$

where $\Phi : TG \to \mathfrak{g}^\vee$ is the moment map for $\mathrm{d}L$.

Let $\mathcal{L} : \mathfrak{g} \to \mathfrak{g}^\vee$ be the isomorphism induced by κ_e. Then, for any $\boldsymbol{\xi} \in \mathfrak{g}$, $\boldsymbol{\beta} \in \mathfrak{g}^\vee$ let us set

$$\boldsymbol{\xi}_\kappa := \mathcal{L}(\boldsymbol{\xi}) \in \mathfrak{g}^\vee, \qquad \boldsymbol{\beta}^\kappa := \mathcal{L}^{-1}(\boldsymbol{\beta}). \tag{27}$$

Thus the annhilator subspace of $\boldsymbol{\beta}$ is the orthocomplement of $\boldsymbol{\beta}^\kappa$ for κ_e:

$$\boldsymbol{\beta}^0 = (\boldsymbol{\beta}^\kappa)^{\perp_{\kappa_e}} \subseteq \mathfrak{g}.$$

Then Φ can be written equivalently as

$$\Phi(g, \boldsymbol{\xi} \cdot g) = \boldsymbol{\xi}_\kappa, \qquad \Phi(g, g \cdot \boldsymbol{\xi}) = \mathrm{Ad}_g(\boldsymbol{\xi})_\kappa.$$

Hence

$$\tilde{\Phi}\big(g\ \exp^{\tilde{G}}(\imath\, \boldsymbol{\xi})\big) = \mathrm{Ad}_g(\boldsymbol{\xi})_\kappa = \mathrm{Coad}_g(\boldsymbol{\xi}_\kappa). \tag{28}$$

In other words, if we identify $\mathfrak{g}$ and $\mathfrak{g}^\vee$ by $\mathcal{L}$ and TG with $G \times \mathfrak{g} \cong G \times \mathfrak{g}^\vee$ by right translations, then Φ is the projection on the second factor.

Let now

$$\mathcal{C}(\mathcal{O}) := \{r\, \boldsymbol{\beta}\ :\ r > 0,\ \boldsymbol{\beta} \in \mathcal{O}\} \subset \mathfrak{g}^\vee, \quad \mathcal{C}(\tilde{\mathcal{O}}) := \left\{r\, \boldsymbol{\xi}\ :\ r > 0,\ \boldsymbol{\xi} \in \tilde{\mathcal{O}}\right\} \subseteq \mathfrak{g}$$

be the cone over $\tilde{\mathcal{O}} = \mathcal{L}^{-1}(\mathcal{O})$. Then, recalling Definition 1.2,

$$\tilde{\Phi}^{-1}\left(\mathcal{C}(\tilde{\mathcal{O}})\right) = \gamma\left(G \times \mathcal{C}(\tilde{\mathcal{O}})\right) = \tilde{G}_{\mathcal{O}}. \tag{29}$$

Therefore, if $\Phi^\tau := \tilde{\Phi}\Big|_{X^\tau} : X^\tau \to \mathfrak{g}^\vee$ then

$$\Phi^{\tau\,-1}\big(\mathcal{C}(\mathcal{O})\big) = X^\tau_{\mathcal{O}} = \tilde{G}_{\mathcal{O}} \cap X^\tau. \tag{30}$$

(see Definition 1.2), and the intersection is transverse.

4.2 The Splitting of $T\tilde{G}$

The previous setting determines a natural splitting of the tangent bundle $T\tilde{G}$ on $\tilde{G} \setminus G$. Let $\upsilon_{\sqrt{\rho}}$ be the Hamiltonian vector field of $\sqrt{\rho}$ with respect to Ω. Then

$$\mathcal{T} := \mathrm{span}_{\mathbb{R}}\left(\upsilon_{\sqrt{\rho}}\right), \quad \mathcal{V} := \mathrm{span}_{\mathbb{C}}\left(\upsilon_{\sqrt{\rho}}\right) = \mathrm{span}_{\mathbb{R}}\left(\upsilon_{\sqrt{\rho}},\ J\left(\upsilon_{\sqrt{\rho}}\right)\right) \tag{31}$$

are a real and a complex line subbundle of $T\tilde{G}$, respectively; the symplectic orthocomplement of $\mathcal{V}$ is a complex subbundle $\mathcal{H}$ of $T\tilde{G}$. Thus $T\tilde{G} = \mathcal{V} \oplus \mathcal{H}$.

The previous splitting may be related to the exact symplectic structure of $(\tilde{G}, \Omega)$, as follows. Let λ_{can} denote the canonical 1-form on TG (identified with $T^\vee G$ by κ), so that $\Omega_{\mathrm{can}} := -\mathrm{d}\lambda_{\mathrm{can}}$, and set

$$\lambda := (E^{-1})^*(\lambda_{\mathrm{can}}), \quad \alpha := -\lambda.$$

Thus $\Omega = -\mathrm{d}\lambda = \mathrm{d}\alpha$. Then

$$\ker(\alpha) = J(\mathcal{T}) \oplus \mathcal{H}, \quad \mathcal{T} = \ker(\alpha)^{\perp_{\hat{\kappa}}},$$

on $\tilde{G} \setminus G$, where the suffix $\perp_{\hat{\kappa}}$ denotes the Riemannian orthocomplement with respect to $\hat{\kappa}$.

Furthermore, $\mathcal{H}$ and $\mathcal{T}$ are tangent to X^τ for every $\tau > 0$; therefore they restrict to subbundles $\mathcal{T}^\tau$ and $\mathcal{H}^\tau$ of TX^τ, and we have the $\hat{k}$-orthogonal direct sum of vector bundles

$$TX^\tau = \mathcal{T}^\tau \oplus \mathcal{H}^\tau, \quad \mathcal{H}^\tau = \ker(\alpha^\tau) \tag{32}$$

where α^τ is the pull-back of α to X^τ. Furthermore, $\mathcal{H}^\tau$ is the maximal complex subbundle of TX^τ.

For any $\boldsymbol{\xi} \in \mathfrak{g}$, let $\boldsymbol{\xi}_{\tilde{G}} \in \mathfrak{X}(\tilde{G})$ denote the induced vector field on $\tilde{G}$ under $\tilde{L}$. Then $\boldsymbol{\xi}_{\tilde{G}}$ is Hamiltonian and is tangent to X^τ for every τ. In view of this and (32), by the discussion in §3.2 of [Pao 24] on $\tilde{G} \setminus G$ we have

$$\boldsymbol{\xi}_{\tilde{G}} = \boldsymbol{\xi}^\sharp_{\tilde{G}} - \tilde{\varphi}^{\boldsymbol{\xi}}\,\mathcal{R}, \text{ where } \boldsymbol{\xi}^\sharp_{\tilde{G}} \text{ is tangent to } \mathcal{H} \text{ and } \mathcal{R} := -\frac{1}{\sqrt{\rho}}\,\upsilon_{\sqrt{\rho}}. \tag{33}$$

Here $\tilde{\varphi}^{\boldsymbol{\xi}} := \langle \tilde{\Phi}, \boldsymbol{\xi} \rangle$. We shall call $\mathcal{R} \in \mathfrak{X}(\tilde{G} \setminus G)$ in (33) the *Reeb vector field* of $\tilde{G}$; its restriction $\mathcal{R}^\tau \in \mathfrak{X}(X^\tau)$ is the genuine Reeb vector field of (X^τ, α^τ) for every $\tau > 0$. In particular, $\alpha(\mathcal{R}) \equiv 1$ on $\tilde{G} \setminus G$.

It follows from (28) and (33) that if $x = g\, \exp^{\tilde{G}}(\imath\, \boldsymbol{\xi}) \in \tilde{G}$ and $\boldsymbol{\eta} \in \mathfrak{g}$ then

$$\boldsymbol{\eta}_{\tilde{G}}(x) = \boldsymbol{\eta}_{\tilde{G}}(x)^{\sharp} - \kappa_e \left(\boldsymbol{\eta}, \mathrm{Ad}_g(\boldsymbol{\xi})\right) \mathcal{R}_x.$$

Hence $\boldsymbol{\eta}_{\tilde{G}}(x) \in \mathcal{H}_x$ if and only if $\boldsymbol{\eta} \in \mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}$. We conclude the following.

Lemma 4.1 *Suppose that $x = g\, \exp^{\tilde{G}}(\imath\, \boldsymbol{\xi}) \in \tilde{G}$. Then*

$$\mathcal{H}_x = \widehat{\mathrm{val}}_x \left(\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}} \oplus \imath\, \mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}\right).$$

Proof We have seen that $\widehat{\mathrm{val}}_x \left(\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}\right) \subseteq \mathcal{H}_x$. Since $\mathcal{H}_x \subseteq T_x\tilde{G}$ is a complex vector subspace,

$$\widehat{\mathrm{val}}_x \left(\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}\right) \oplus J_x \left(\widehat{\mathrm{val}}_x \left(\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}\right)\right) \subseteq \mathcal{H}_x.$$

However $\widehat{\mathrm{val}}_x$ is $\mathbb{C}$-linear, hence $\widehat{\mathrm{val}}_x \circ M_\imath = J_x \circ \widehat{\mathrm{val}}_x$ ($M_\imath$ denoting multiplication by $\imath$). Thus

$$\mathcal{H}_x \supseteq \widehat{\mathrm{val}}_x \left(\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}} \oplus \imath\, \mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}\right);$$

since both spaces have complex dimension $d - 1$, equality holds. □

4.3 The Normal Bundle to $\tilde{G}_{\mathcal{O}}$ and $X^\tau_{\mathcal{O}}$

We assume here that $\mathcal{O}$ is the coadjoint orbit (not necessarily of maximal dimension) through a given $\boldsymbol{\lambda} \neq 0 \in \mathfrak{g}^\vee$.

By (30), the normal bundle $N(X^\tau_{\mathcal{O}}/X^\tau)$ of $X^\tau_{\mathcal{O}}$ in X^τ is the restriction to X^τ of the normal bundle $N(\tilde{G}_{\mathcal{O}}/\tilde{G})$ to $\tilde{G}_{\mathcal{O}}$ in $\tilde{G}$. If the stabilizer $T_{\boldsymbol{\lambda}} \leqslant T$ of $\boldsymbol{\lambda}$ has dimension $r_{\boldsymbol{\lambda}}$, then $\mathcal{O} \cong G/T_{\boldsymbol{\lambda}}$ has dimension $d - r_{\boldsymbol{\lambda}}$; hence $\dim\left(\mathcal{C}(\mathcal{O})\right) = d - r_{\boldsymbol{\lambda}} + 1$. By (29), $\dim(\tilde{G}_{\mathcal{O}}) = 2d - r_{\boldsymbol{\lambda}} + 1$; thus $N(\tilde{G}_{\mathcal{O}}/\tilde{G})$ has rank $r_{\boldsymbol{\lambda}} - 1$. In the sequel we shall also set $r_{\mathcal{O}} := r_{\boldsymbol{\lambda}}$ if $\boldsymbol{\lambda} \in \mathcal{O}$.

Definition 4.2 For $\boldsymbol{\lambda} \in \mathfrak{g}^\vee$, let us set set

$$\mathfrak{t}'_{\boldsymbol{\lambda}} := \mathfrak{t}_{\boldsymbol{\lambda}} \cap \boldsymbol{\lambda}^0 = \left\{\boldsymbol{\eta} \in \mathfrak{g} \,:\, \left[\boldsymbol{\eta}, \boldsymbol{\lambda}^\kappa\right] = 0,\ \kappa_e\left(\boldsymbol{\eta}, \boldsymbol{\lambda}^\kappa\right) = 0\right\}.$$

We shall denote by $T'_{\boldsymbol{\lambda}} \leqslant T_{\boldsymbol{\lambda}}$ the connected subgroup with Lie algebra $\mathfrak{t}'_{\boldsymbol{\lambda}}$.

Remark 4.3 If $\boldsymbol{\lambda} \in \mathcal{C}(\mathcal{O})$, where $\mathcal{O}$ is an integral orbit [25], then $T'_{\boldsymbol{\lambda}}$ is closed. In particular, this is the case if $\boldsymbol{\lambda} \in \mathcal{D}^G$.

Remark 4.4 For $x \in \tilde{G}$, we shall abridge notation by setting

$$T_x := T_{\tilde{\Phi}(x)}, \quad \mathfrak{t}_x := \mathfrak{t}_{\tilde{\Phi}(x)}, \quad \mathfrak{t}'_x := \mathfrak{t}'_{\tilde{\Phi}(x)}, \quad T'_x := T'_{\tilde{\Phi}(x)}.$$

Hence if $x = g \cdot \exp^{\tilde{G}}(\imath\, \boldsymbol{\xi})$ then $T_x = T_{\mathrm{Coad}_g(\boldsymbol{\xi}_\kappa)} = T_{\mathrm{Ad}_g(\boldsymbol{\xi})}$, and so forth.

Thus $\dim(\mathfrak{t}'_x) = r_{\tilde{\Phi}(x)} - 1$; since the G action on $\tilde{G}$ is free, for any $y \in \tilde{G}$ we also have $\dim\big((\mathfrak{t}'_x)_{\tilde{G}}(y)\big) = r_{\tilde{\Phi}(x)} - 1$. In the following, we shall abridge notation and simply write

$$\mathfrak{t}'_x(x) := (\mathfrak{t}'_x)_{\tilde{G}}(x). \tag{34}$$

This does not define a vector bundle on $\tilde{G}$, given that $\mathfrak{t}'_x$ has variable dimension. On the other hand, (34) does define a vector bundle on $\tilde{G}_{\mathcal{O}}$, with the following geometric significance: $\tilde{G}_{\mathcal{O}}$ is a coisotropic submanifold of $(\tilde{G}, \Omega)$, and $\mathfrak{t}'_x(x) \subseteq T_x\tilde{G}_{\mathcal{O}}$ is the tangent space to its null fibration.

Lemma 4.5 *For every $x \in \tilde{G}$, the following holds:*

1. $\mathfrak{t}'_x(x) \oplus J_x\left(\mathfrak{t}'_x(x)\right) \subseteq \mathcal{H}_x$;
2. *if $x \in \tilde{G}_{\mathcal{O}}$, then $N(\tilde{G}_{\mathcal{O}}/\tilde{G})_x = J_x\big(\mathfrak{t}'_x(x)\big)$.*

Proof To begin with, $\mathfrak{g}_{\tilde{G}}(x) \cap J_x\left(\mathfrak{g}_{\tilde{G}}(x)\right) = (0)$ at any $x \in \tilde{G}$, because $\tilde{G}$ acts freely on itself by left translations.

Suppose $\boldsymbol{\xi} \in \mathfrak{t}'_x$, and consider the induced vector field (see (33)) $\boldsymbol{\xi}_{\tilde{G}} = \boldsymbol{\xi}^{\sharp}_{\tilde{G}} - \tilde{\varphi}^{\boldsymbol{\xi}}\,\mathcal{R}$. As $\boldsymbol{\xi} \in \tilde{\Phi}(x)^0$, $\tilde{\varphi}^{\boldsymbol{\xi}}(x) = \langle\tilde{\Phi}(x), \boldsymbol{\xi}\rangle = 0$; hence $\boldsymbol{\xi}_{\tilde{G}}(x) \in \mathcal{H}_x$. Thus $\mathfrak{t}'_x(x) \subseteq \mathcal{H}_x$ by definition. Since $\mathcal{H}_x$ is a complex subspace of $T_p\tilde{G}$, the proof of 1. is complete.

Since both $N(\tilde{G}_{\mathcal{O}}/\tilde{G})_x$ and $J_p\big(\mathfrak{t}'_x(p)\big)$ have dimension $r_{\mathcal{O}}-1$, to prove 2. it suffices to show that $N(\tilde{G}_{\mathcal{O}}/\tilde{G})_x \supseteq J_x\big(\mathfrak{t}'_x(x)\big)$. If $w \in T_x\tilde{G}_{\mathcal{O}}$, then

$$\mathrm{d}_x\tilde{\Phi}(w) \in T_{\tilde{\Phi}(x)}\mathcal{C}(\mathcal{O}) = \mathbb{R}\,\tilde{\Phi}(x) \oplus T_{\tilde{\Phi}(x)}\mathcal{O} \subseteq \mathfrak{g}^{\vee}.$$

Hence there exist $a \in \mathbb{R}$ and $\boldsymbol{\eta} \in \mathfrak{g}$ such that

$$\mathrm{d}_x\tilde{\Phi}(w) = a\,\tilde{\Phi}(x) + \mathrm{coad}_{\boldsymbol{\eta}}\big(\tilde{\Phi}(x)\big). \tag{35}$$

Suppose $\boldsymbol{\xi} \in \mathfrak{t}'_x$. Then

$$\begin{aligned}
\hat{\kappa}_x\left(J_x\big(\boldsymbol{\xi}_{\tilde{G}}(x)\big), w\right) &= \Omega_x\left(\boldsymbol{\xi}_{\tilde{G}}(x), w\right) = \mathrm{d}_x\tilde{\varphi}^{\boldsymbol{\xi}}(w) = \left\langle \mathrm{d}_x\tilde{\Phi}(w), \boldsymbol{\xi}\right\rangle \\
&= \left\langle a\,\tilde{\Phi}(x) + \mathrm{coad}_{\boldsymbol{\eta}}\big(\tilde{\Phi}(x)\big), \boldsymbol{\xi}\right\rangle \\
&= \kappa_e\left(a\,\tilde{\Phi}(x)^{\kappa} + \left[\boldsymbol{\eta}, \tilde{\Phi}(x)^{\kappa}\right], \boldsymbol{\xi}\right) \\
&= a\,\kappa_e\left(\tilde{\Phi}(x)^{\kappa}, \boldsymbol{\xi}\right) + \kappa_e\left(\boldsymbol{\eta}, \left[\tilde{\Phi}(x)^{\kappa}, \boldsymbol{\eta}\right]\right) = 0.
\end{aligned}$$

Hence $J_x\big(\boldsymbol{\xi}_{\tilde{G}}(x)\big) \in N(\tilde{G}_{\mathcal{O}}/\tilde{G})_x$. □

Corollary 4.6 *The normal bundle of $X^{\tau}_{\mathcal{O}}$ in X^{τ} is given by*

$$N\left(X^{\tau}_{\mathcal{O}}/X^{\tau}\right)_x = J_x\left(\mathfrak{t}'_x(x)\right) \qquad (x \in X^{\tau}_{\mathcal{O}}).$$

4.4 The Geodesic Flow on *TG* and $\tilde{G}$

Let us consider the trivialization of TG given by *right* translations:

$$\Psi : (g, \boldsymbol{\xi}) \in G \times \mathfrak{g} \mapsto (g, \boldsymbol{\xi} \cdot g) \in TG.$$

Passing to the differential, we obtain a diffeomorphism

$$\mathrm{d}\Psi^{-1} : T(TG) \to TG \times T\mathfrak{g} \cong TG \times (\mathfrak{g} \times \mathfrak{g}),$$

Composing with $\Psi^{-1} \times \mathrm{id}_{\mathfrak{g}\times\mathfrak{g}} : TG \times (\mathfrak{g} \times \mathfrak{g}) \to (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g})$ yields another diffeomorphism

$$(\Psi^{-1} \times \mathrm{id}_{\mathfrak{g}\times\mathfrak{g}}) \circ \mathrm{d}\Psi^{-1} : T(TG) \to (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g}).$$

Explicitly, the element of $T(TG)$ corresponding to $\big((g, \boldsymbol{\eta}'), (\boldsymbol{\xi}, \boldsymbol{\eta}'')\big)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(e^{t\,\boldsymbol{\eta}'}\, g, \left(\boldsymbol{\xi} + t\,\boldsymbol{\eta}''\right) \cdot e^{t\,\boldsymbol{\eta}'}\, g \right)\Bigg|_{t=0}.$$

The geodesic in G with initial condition $(g, \boldsymbol{\xi} \cdot g) \in TG$ is

$$\gamma_{(g,\boldsymbol{\xi})} : t \in \mathbb{R} \mapsto e^{t\,\boldsymbol{\xi}} \cdot g \in G.$$

Its velocity lift $\dot{\gamma}_{(g,\boldsymbol{\xi})} : \mathbb{R} \to TG$ is

$$\dot{\gamma}_{(g,\boldsymbol{\xi})}(t) = \left(e^{t\,\boldsymbol{\xi}} \cdot g, \boldsymbol{\xi} \cdot \left(e^{t\,\boldsymbol{\xi}} \cdot g \right) \right) \quad (t \in \mathbb{R}),$$

and is the integral curve of the geodesic vector field $\Gamma^G \in \mathfrak{X}(TG)$ passing through $(g, \boldsymbol{\xi} \cdot g)$ for $t = 0$. In particular,

$$\Gamma^G(g, \boldsymbol{\xi} \cdot g) = \frac{\mathrm{d}}{\mathrm{d}t} \dot{\gamma}_{(g,\boldsymbol{\xi})}(t)\Bigg|_{t=0}.$$

On the other hand, $\Upsilon_{(g,\boldsymbol{\xi})} := \Psi^{-1} \circ \dot{\gamma}_{(g,\boldsymbol{\xi})} : \mathbb{R} \to G \times \mathfrak{g}$ is given by

$$\Upsilon_{(g,\boldsymbol{\xi})}(t) = \left(e^{t\,\boldsymbol{\xi}} \cdot g, \boldsymbol{\xi} \right).$$

Hence,

$$\dot{\Upsilon}_{(g,\boldsymbol{\xi})}(0) = \mathrm{d}\Psi^{-1} \left(\Gamma^G(g, \boldsymbol{\xi} \cdot g) \right) = ((g, \boldsymbol{\xi} \cdot g)\,, (\boldsymbol{\xi}, 0)) \in TG \times (\mathfrak{g} \times \mathfrak{g}).$$

Therefore,

$$(\Psi^{-1} \times \mathrm{id}) \circ \mathrm{d}\Psi^{-1} \left(\Gamma^G(g, \boldsymbol{\xi} \cdot g) \right) = ((g, \boldsymbol{\xi}), (\boldsymbol{\xi}, 0)) \in (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g}). \quad (36)$$

Since Ψ intertwines the action $\mathrm{d}L$ of G on TG with the action on $G \times \mathfrak{g}$ given by

$$\hat{L}_h(g, \boldsymbol{\xi}) := \big(h\,g, \mathrm{Ad}_h(\boldsymbol{\xi})\big),$$

the composed diffeomorphism

$$(\Psi^{-1} \times \mathrm{id}) \circ \mathrm{d}\Psi^{-1} : T(TG) \to (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g})$$

is also G-equivariant. Thus for any $\boldsymbol{\eta} \in \mathfrak{g}$ the induced vector fields $\boldsymbol{\eta}_{TG}$ and $\boldsymbol{\eta}_{G\times\mathfrak{g}}$ are Ψ-correlated, and abusing notation we may view the latter as a map $G \times \mathfrak{g} \to (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g})$. Hence

$$\begin{aligned}(\Psi^{-1} \times \mathrm{id}) \circ \mathrm{d}\Psi^{-1}\big(\boldsymbol{\eta}_{TG}(g, \boldsymbol{\xi} \cdot g)\big) &= (\Psi^{-1} \times \mathrm{id})\big(\boldsymbol{\eta}_{G\times\mathfrak{g}}(g, \boldsymbol{\xi})\big) \\ = (\Psi^{-1} \times \mathrm{id})\big((g, \boldsymbol{\eta} \cdot g), (\boldsymbol{\xi}, [\boldsymbol{\eta}, \boldsymbol{\xi}])\big) &= \big((g, \boldsymbol{\eta}), (\boldsymbol{\xi}, [\boldsymbol{\eta}, \boldsymbol{\xi}])\big) \in (G \times \mathfrak{g}) \times (\mathfrak{g} \times \mathfrak{g}).\end{aligned} \tag{37}$$

In particular, with $\boldsymbol{\xi} = \boldsymbol{\eta}$ in view of (36) we obtain

$$\begin{aligned}(\Psi^{-1} \times \mathrm{id}) \circ \mathrm{d}\Psi^{-1}\big(\boldsymbol{\xi}_{TG}(g, \boldsymbol{\xi} \cdot g)\big) &= \big((g, \boldsymbol{\xi}), (\boldsymbol{\xi}, 0)\big) \\ &= (\Psi^{-1} \times \mathrm{id}) \circ \mathrm{d}\Psi^{-1}\left(\Gamma^G(g, \boldsymbol{\xi} \cdot g)\right).\end{aligned} \tag{38}$$

We conclude the following.

Lemma 4.7 *For any $(g, \boldsymbol{\xi} \cdot g) \in TG$, we have*

$$\Gamma^G(g, \boldsymbol{\xi} \cdot g) = \boldsymbol{\xi}_{TG}(g, \boldsymbol{\xi} \cdot g),$$

where Γ^G, $\boldsymbol{\xi}_{TG} \in \mathfrak{X}(TG)$ are, respectively, the geodesic vector field and the vector field induced by $\boldsymbol{\xi}$ under the action $\mathrm{d}L$. In particular, Γ^G is tangent to the G-orbits and the geodesic flow of any $p \in TG$ is contained in its G-orbit.

Corollary 4.8 *The $\mathbb{R}$-orbit of any non-zero $p \in TG$ under the homogeneous geodesic flow is contained in its G-orbit under $\mathrm{d}L$.*

Remark 4.9 Lemma 4.7 may be reformulated as saying that

$$\Gamma^G(g, g \cdot \boldsymbol{\xi}) = \mathrm{Ad}_g(\boldsymbol{\xi})_{TG}(g, g \cdot \boldsymbol{\xi})$$

The G-equivariant diffemorphism $E : TG \to \tilde{G}$ in (5) intertwines the homogeneous geodesic flow on TG (which is the Hamiltonian flow of $\|\cdot\|_\kappa$ with respect to Ω_{can}) with the Hamiltonian flow of $\sqrt{\rho}$ with respect to Ω. Therefore, Corollary 4.8 may be restated as follows.

Corollary 4.10 *The $\mathbb{R}$-orbit of any $x \in \tilde{G} \setminus G$ under the flow of $\upsilon_{\sqrt{\rho}}$ is contained in its G-orbit under $\tilde{L}$.*

Somewhat more precisely, let $\tilde{\Gamma}^G \in \mathfrak{X}(\tilde{G})$ be the vector field on $\tilde{G}$ correlated to Γ^G by E. We shall refer to $\tilde{\Gamma}^G$ as the geodesic vector field on $\tilde{G}$. By Remark 4.9,

$$\tilde{\Gamma}^G\left(g\cdot \exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})\right)=\mathrm{Ad}_g(\boldsymbol{\xi})_{\tilde{G}}\left(g\cdot \exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})\right), \tag{39}$$

for all $g\in G$ and $\boldsymbol{\xi}\in\mathfrak{g}$. Furthermore, $\tilde{\Gamma}^G$ is a multiple of $\upsilon_{\sqrt{\rho}}$, hence of $\mathcal{R}$ in (33); in particular, it spans $\mathcal{V}^\tau$. It follows from (28), (33), and (39) that

$$\begin{aligned}&\mathrm{Ad}_g(\boldsymbol{\xi})_{\tilde{G}}\left(g\ \exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})\right)\\&=-\left\langle\tilde{\Phi}\left(g\cdot\exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})\right),\mathrm{Ad}_g(\boldsymbol{\xi})_{\tilde{G}}\right\rangle\mathcal{R}\left(g\cdot\exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})\right)\\&=-\|\boldsymbol{\xi}\|_\kappa^2\,\mathcal{R}\left(g\cdot\exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})\right)\end{aligned} \tag{40}$$

(notice that $\|\boldsymbol{\xi}\|_\kappa=\tau$ if $g\cdot\exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})\in X^\tau$).

4.5 Direct Sum Decompositions of $\mathfrak{g}$

In §4.3, we have denoted by $T_{\boldsymbol{\lambda}}\leqslant G$ the stabilizer subgroup of a given $\boldsymbol{\lambda}\in\mathfrak{g}^\vee$ under the coadjoint action, and by $\mathfrak{t}_{\boldsymbol{\lambda}}\leqslant\mathfrak{g}$ its Lie algebra. With slight ambiguity, we shall also denote by $T_{\boldsymbol{\xi}}\leqslant G$ the stabilizer subgroup of a given $\boldsymbol{\xi}\in\mathfrak{g}$ under the adjoint action, by $r_{\boldsymbol{\xi}}$ its dimension, and by $\mathfrak{t}_{\boldsymbol{\xi}}\leqslant\mathfrak{g}$ its Lie algebra.

Similarly, we shall set

$$\mathfrak{t}'_{\boldsymbol{\xi}}:=\mathfrak{t}_{\boldsymbol{\xi}}\cap\boldsymbol{\xi}^{\perp_{\kappa_e}}=\{\boldsymbol{\eta}\in\mathfrak{g}:[\boldsymbol{\eta},\boldsymbol{\xi}]=0,\ \kappa_e(\boldsymbol{\eta},\boldsymbol{\xi})=0\}. \tag{41}$$

For any $x\in\tilde{G}$, there are two natural choices of a Euclidean product on $\mathfrak{g}$. One is the given κ_e, and the other is the pull-back $\sigma_x:=\mathrm{val}_x^t(\hat{\kappa}_x)$ of $\hat{\kappa}_x$ under the injective linear map given by vector field evaluation at x:

$$\mathrm{val}_x:\boldsymbol{\eta}\in\mathfrak{g}\mapsto\boldsymbol{\eta}_{\tilde{G}}(x)\in T_x\tilde{G}.$$

If $x=g\cdot\exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})$, where $g\in G$ and $\boldsymbol{\xi}\in\mathfrak{g}$, then σ_x depends only on $\boldsymbol{\xi}$, since G acts isometrically on $\tilde{G}$. In particular, $\kappa_e=\sigma_x$ when $x\in G$ (i.e., when $\boldsymbol{\xi}=0$).

Given $\boldsymbol{\eta}\neq 0\in\mathfrak{g}$, the orthocomplements $\boldsymbol{\eta}^{\perp_{\kappa_e}}$, $\boldsymbol{\eta}^{\perp_{\sigma_x}}\subset\mathfrak{g}$ are two *a priori* distinct hyperplanes.

Lemma 4.11 *If $g\in G$, $\boldsymbol{\xi}\in\mathfrak{g}\setminus\{0\}$ and $x=g\cdot\exp^{\tilde{G}}(\imath\,\boldsymbol{\xi})$, then we have the following equality of Euclidean orthocomplements:*

$$\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}=\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\sigma_x}}.$$

Proof Let us set $\tau := \|\boldsymbol{\xi}\|_{\kappa_e}$, so that $x \in X^\tau$. By (33), for any $\boldsymbol{\eta} \in \mathfrak{g}$

$$\boldsymbol{\eta}_{\tilde{G}}(x) = \boldsymbol{\eta}^\sharp_{\tilde{G}}(x) - \kappa_e\big(\mathrm{Ad}_g(\boldsymbol{\xi}), \boldsymbol{\eta}\big)\,\mathcal{R}(x). \tag{42}$$

Thus one the one hand $\mathrm{Ad}_g(\boldsymbol{\xi})_{\tilde{G}}(x) \in \mathcal{T}^\tau_x$ by (40), and on the other $\boldsymbol{\eta}_{\tilde{G}}(x) = \boldsymbol{\eta}^\sharp_{\tilde{G}}(x) \in \mathcal{H}^\tau_x$ for any $\boldsymbol{\eta} \in \mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}$. Since (32) is an orthogonal direct sum, the claim follows. □

By its definition in (41), $\mathfrak{t}'_{\mathrm{Ad}_g(\boldsymbol{\xi})} \subset \mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}$.

Definition 4.12 If $x = g \cdot \exp^{\tilde{G}}(\imath\, \boldsymbol{\xi}) \in \tilde{G} \setminus G$, so that $\tilde{\Phi}(x) = \mathrm{Coad}_g(\boldsymbol{\xi}_\kappa)$,

1. we shall set $\mathfrak{r}_x := (\mathfrak{t}_x)^{\perp_{\kappa_e}}$, so that (recalling Remark 4.4)

$$\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}} = \mathfrak{t}'_{\mathrm{Ad}_g(\boldsymbol{\xi})} \oplus_{\kappa_e} \mathfrak{r}_x,$$

   that is, the direct sum is κ_e-orthogonal;
2. we shall denote by $\mathfrak{s}_x \subseteq \mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}$ the vector subspace such that

$$\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}} = \mathfrak{t}'_{\mathrm{Ad}_g(\boldsymbol{\xi})} \oplus_{\sigma_x} \mathfrak{s}_x,$$

   that is, the direct sum is σ_x-orthogonal.

Clearly,

$$\dim \mathfrak{s}_x = \dim(\mathrm{Ad}_g(\boldsymbol{\xi})^{\perp_{\kappa_e}}) - \dim(\mathfrak{t}'_{\mathrm{Ad}_g(\boldsymbol{\xi})}) = (d-1) - (r_{\boldsymbol{\xi}} - 1) = d - r_{\boldsymbol{\xi}}.$$

Lemma 4.13 *Under the previous assumptions,*

$$\mathfrak{s}_x(x) \oplus J_x\big(\mathfrak{s}_x(x)\big) = \big[\mathfrak{t}_x(x) \oplus J_x\big(\mathfrak{t}_x(x)\big)\big]^{\perp_{\hat{\kappa}_x}} \cap \mathcal{H}_x.$$

Proof We have $\mathfrak{t}_x(x) = \mathrm{span}\big(\mathrm{Ad}_g(\boldsymbol{\xi})_{X^\tau}(x)\big) \oplus \mathfrak{t}'_x(x)$. Therefore, by Definition 4.12 $\mathfrak{s}_x(x) \subseteq \mathfrak{t}_x(x)^{\perp_{\hat{\kappa}_x}}$. On the other hand, since $X^\tau_{\mathcal{O}}$ is G-invariant $\mathfrak{g}_{X^\tau}(x) \subseteq T_x X^\tau_{\mathcal{O}}$. Hence, $\mathfrak{s}_x(x) \subseteq \mathfrak{g}_{X^\tau}(x) \subseteq J_x\big(\mathfrak{t}'_x(x)\big)^{\perp_{\hat{\kappa}_x}}$ by Corollary 4.6. Thus

$$\begin{aligned} &\mathfrak{s}_x(x) \subseteq \big[\mathfrak{t}'_x(x) \oplus J_x\big(\mathfrak{t}'_x(x)\big)\big]^{\perp_{\hat{\kappa}_x}} \\ &\Rightarrow \mathfrak{s}_x(x) \oplus J_x\big(\mathfrak{s}_x(x)\big) \subseteq \big[\mathfrak{t}'_x(x) \oplus J_x\big(\mathfrak{t}'_x(x)\big)\big]^{\perp_{\hat{\kappa}_x}} \cap \mathcal{H}_x. \end{aligned} \tag{43}$$

On the other hand, $\mathfrak{s}_x(x) \oplus J_x\big(\mathfrak{s}_x(x)\big)$ has real dimension $2\,\dim \mathfrak{s}_x = 2\,(d - r_{\boldsymbol{\xi}})$, and $\big[\mathfrak{t}'_x(x) \oplus J_x\big(\mathfrak{t}'_x(x)\big)\big]^{\perp_{\hat{\kappa}_x}} \cap \mathcal{H}_x$ has real dimension $2d - 2 - 2\,(r_{\boldsymbol{\xi}} - 1) = 2\,(d - r_{\boldsymbol{\xi}})$. Hence equality holds in (43), and the claim follows since $\mathfrak{t}'_x(x) = \mathfrak{t}_x(x) \cap \mathcal{H}_x$. □

Thus we have direct sum decompositions

$$\begin{aligned}\mathcal{H}_x &= \left[\mathfrak{t}'_x(x) \oplus_{\hat{\kappa}_x} J_x\big(\mathfrak{t}'_x(x)\big)\right] \oplus_{\hat{\kappa}_x} \left[\mathfrak{s}_x(x) \oplus J_x\big(\mathfrak{s}_x(x)\big)\right] \\ &= (\tilde{\Phi}(x)^{\perp_{\kappa_e}}_{\tilde{G}})(x) \oplus J_x\left((\tilde{\Phi}(x)^{\perp_{\kappa_e}}_{\tilde{G}})(x)\right), \\ T_x\tilde{G}_{\mathcal{O}} \cap \mathcal{H}_x &= \mathfrak{t}'_x(x) \oplus_{\hat{\kappa}_x} \left[\mathfrak{s}_x(x) \oplus J_x\big(\mathfrak{s}_x(x)\big)\right],\end{aligned} \tag{44}$$

where $\oplus_{\hat{\kappa}_x}$ denotes $\hat{\kappa}_x$-orthogonality.

Remark 4.14 In terms of (18), we have

$$\mathcal{N}_x = \mathfrak{t}'_x(x) \oplus_{\hat{\kappa}_x} J_x\big(\mathfrak{t}'_x(x)\big) = \widetilde{\mathfrak{t}'}_x(x), \qquad \mathcal{S}_x = \mathfrak{s}_x(x) \oplus J_x\big(\mathfrak{s}_x(x)\big) = \widetilde{\mathfrak{s}}_x(x),$$

where $\widetilde{\mathfrak{t}'}_x$ and $\widetilde{\mathfrak{s}}_x$ are the complexifications of $\mathfrak{t}'_x$ and $\mathfrak{s}_x$, respectively.

4.6 Π^τ and P^τ

Let us dwell on the key properties of the operators Π^τ and P^τ discussed in the Introduction that will be used in the following arguments.

4.6.1 The Description by Fourier Integral Operators

The Szegö projector $\Pi^\tau : L^2(X^\tau) \to H(X^\tau)$ is a Fourier integral operator with complex phase of positive type, whose microlocal structure has been precisely determined in [3] (see also [13], [32], [2], [52], [38]). In particular, up to a smoothing term its Schartz kernel $\Pi^\tau \in \mathcal{D}'(X^\tau \times X^\tau)$ can be written in the form

$$\Pi^\tau(x, y) \sim \int_0^{+\infty} e^{\imath\, u\, \psi^\tau(x,y)}\, s^\tau(x, y, u)\, \mathrm{d}u, \tag{45}$$

where the phase ψ^τ has non-negative imaginary part and the amplitude s^τ is a semiclassical symbol admitting an asymptotic expansion of the form

$$s^\tau(x, y, u) \sim \sum_{j\geq 0} u^{d-1-j}\, s^\tau_j(x, y). \tag{46}$$

The phase ψ^τ is only determined up to a function vanishing to infinite order along the diagonal. In particular, for any $x \in X^\tau$ we have

$$\mathrm{d}_{(x,x)}\psi^\tau = (\alpha^\tau_x, -\alpha^\tau_x). \tag{47}$$

Furthermore, the imaginary part of ψ^τ may be assumed to satisfy

$$\Im\left(\psi^\tau(x, y)\right) \geq C^\tau\, \mathrm{dist}_{X^\tau}(x, y)^2 \quad (x, y \in X^\tau) \tag{48}$$

for an appropriate constant $C^\tau > 0$.

Let Σ^τ be the closed symplectic cone in $T X^\tau \setminus (0)$ ((0) denoting the zero section) sprayed by the contact form α^τ (see §4.2):

$$\Sigma^\tau := \left\{ \left(x, r\,\alpha^\tau_x\right) \,:\, x \in X^\tau,\ r > 0 \right\}.$$

The wave front of Π^τ is the anti-diagonal of Σ^τ:

$$\mathrm{WF}(\Pi^\tau) = \Sigma^{\tau\,\sharp} := \left\{ \left(x, r\,\alpha^\tau_x, x, -r\,\alpha^\tau_x\right) : x \in X^\tau,\ r > 0 \right\} \tag{49}$$

Therefore, the singular support is the diagonal in $X^\tau \times X^\tau$:

$$\mathrm{S.S.}(\Pi^\tau) = \mathrm{diag}(X^\tau \times X^\tau)$$

(see [3] for discussion and derivation of these properties)

As proved by Zelditch, the operator P^τ in (13) is also a Fourier integral operator with complex phase, associated to the same canonical relation as Π^τ - hence with the same wave front and singular support - and of degree $-(d-1)/2$ (see e.g. [54], [56] and [58], and the discussions in [9] and [10]). In fact $P^\tau = U_{\mathbb{C}}(2\,\imath\,\tau)$, where for general $t \in \mathbb{R}$ one denotes $U_{\mathbb{C}}(t + 2\,\imath\,\tau) : \mathcal{O}(X^\tau) \to \mathcal{O}(X^\tau)$ the complexified Poisson-wave operators obtained (roughly speaking) by holomorphically extending the operator kernel of the wave operator. Zelditch showed that complexified Poisson-wave operators may described in terms of dynamical Toeplitz operators; one can then conclude that P^τ is a Toeplitz operator of degree $-(d-1)/2$, given by the compression by Π^τ of a pseudodifferential operator Q^τ of the same degree. To leading order, Q^τ has the form $(\pi\,\tau)^{\frac{d-1}{2}} \cdot D_{\sqrt{\rho}}^{-(d-1)/2}$ (*cfr.* the discussion in §5 of [18]). It follows that the operator kernel of P^τ has the form

$$P^\tau(x, y) \sim \int_0^{+\infty} e^{\imath\,u\,\psi^\tau(x,y)}\, q^\tau(x, y, u)\,\mathrm{d}u, \tag{50}$$

where

$$q^\tau(x, y, u) \sim \sum_{j \ge 0} u^{\frac{d-1}{2}-j}\, q_j(x, y), \quad \text{where} \quad q_0^\tau(x, y) = \pi^{\frac{d-1}{2}}\, s_0^\tau(x, y). \tag{51}$$

In the present real-analytic setting, there is a natural choice for ψ^τ. Namely, $\phi^\tau := \rho - \tau^2 : \tilde{G} \to \mathbb{R}$ is real-analytic defining funtion for X^τ; let $\tilde{\phi}^\tau$ denote the holomorphic extension of ϕ^τ to $\tilde{G} \times \overline{\tilde{G}}$ (defined at least on a neighbourhood of the diagonal). Then we can set

$$\psi^\tau := \frac{1}{\imath}\, \tilde{\phi}^\tau\Big|_{X^\tau \times X^\tau}. \tag{52}$$

4.6.2 Normal Heisenberg Local Coordinates on X^τ

To perform computations involving Π^τ and P^τ, it is convenient to work in specific systems of local coordinates on X^τ. For any $\tau > 0$ and $x \in X^\tau$, there exist systems of holomorphic local coordinates for $\tilde{G}$ centered at x in which the defining equation of X^τ has a certain canonical form; in turn, these induce local coordinates on X^τ centered at x, in which (given (52)) ψ^τ admits a relatively explicit form.

These coordinates are called *normal Heisenberg local coordinates* and will be referred to as NHLC's in the following (here we shall follow the notation and conventions in [34], and refer the reader to [14], [15], [9], [10], [38] for general background and discussion).

We shall refer as needed to §3 of [34], in particular Propositions 34 and 48. Furthermore, in NHLC's centered at x, s_0^τ in (46) satisfies

$$s_0^\tau(x, x) = \frac{\tau}{(2\pi)^d} \tag{53}$$

(Theorem 51 of [Pao 2024]). NHLC's centered at $x \in X^\tau$ will be denoted in additive notation, in the form $x + (\theta, \mathbf{v})$, where $(\theta, \mathbf{v}) \in \mathbb{R} \times \mathbb{R}^{2d-2}$ belong to a small ball centered at the origin.

5 Proofs of the Theorems

Let us premise the following general remark. Let $\mathcal{Q}_{\boldsymbol{\lambda}}^\tau : L^2(X^\tau) \to L^2(X^\tau)_{\boldsymbol{\lambda}}$ be the orthogonal projector, so that $\Pi_{\boldsymbol{\lambda}}^\tau = \mathcal{Q}_{\boldsymbol{\lambda}}^\tau \circ \Pi^\tau$, and let μ^τ be as in (8). Then the relation between the Schwartz kernels of Π^τ and $\Pi_{\boldsymbol{\lambda}}^\tau$ is given by

$$\Pi_{\boldsymbol{\lambda}}^\tau(x, y) := d_{\boldsymbol{\lambda}} \cdot \int_G \overline{\chi_{\boldsymbol{\lambda}}(g)}\, \Pi^\tau \left(\mu_{g^{-1}}^\tau(x), y\right) \mathrm{d}^H V_G(g), \tag{54}$$

where $\mathrm{d}^H V_G$ denotes the Haar measure on G. A similar relation holds between P^τ and $P_{\boldsymbol{\lambda}}^\tau$.

Notational Caveat 5.1 We shall generally denote the coupling of elements in $\boldsymbol{\beta} \in \mathfrak{g}^\vee$ and $\boldsymbol{\xi} \in \mathfrak{g}$ by $\boldsymbol{\beta}(\boldsymbol{\xi})$, and of elements $\boldsymbol{\lambda} \in \mathfrak{t}^\vee$ and $\boldsymbol{\vartheta} \in \mathfrak{t}$ by by $\langle \boldsymbol{\lambda}, \boldsymbol{\vartheta} \rangle$.

5.1 Proof of Theorem 1.4

Proof Given $(x, y) \in X^\tau \times X^\tau$, set

$$\delta(x, y) := \mathrm{dist}_{X^\tau}(x, G \cdot y) + \mathrm{dist}_{X^\tau}\left(x, X_{\mathcal{O}}^\tau\right).$$

If $K \Subset X^\tau \times X^\tau \setminus \mathcal{Z}_{\mathcal{O}}^\tau$, then

$$\delta_K := \min\{\delta(x, y) \,:\, (x, y) \in K\} > 0.$$

Thus it suffices to show that for any given $\delta_0 > 0$ the conclusion of the Theorem holds uniformly on the locus where $\delta(x, y) \geq \delta_0$.

Set $K_{\delta_0} := \{(x, y) \in X^\tau \times X^\tau \, : \, \delta(x, y) \geq \delta_0\}$ and

$$U_1 := \left\{(x, y) \in X^\tau \times X^\tau \, : \, \mathrm{dist}_{X^\tau}(x, G \cdot y) > \frac{1}{2}\,\delta_0\right\},$$

$$U_2 := \left\{(x, y) \in X^\tau \times X^\tau \, : \, \mathrm{dist}_{X^\tau}\left(x, X^\tau_{\mathcal{O}}\right) > \frac{1}{2}\,\delta_0\right\};$$

then $\{U_1, U_2\}$ is an open cover of K_{δ_0}, and we need only prove that the statement holds uniformly on each U_j.

Let us consider U_1. Since the singular support of Π^τ is the diagonal in $X^\tau \times X^\tau$, $f_{x,y}(g) := \Pi^\tau\big(\mu^\tau_{g^{-1}}(x), y\big)$ is a uniformly smooth function of $g \in G$ for $(x, y) \in U_1$. By a theorem of Sugiura [45], its group-theoretic Fourier transform $\mathcal{F}(f_{x,y})$ is uniformly rapidly decreasing as a matrix-valued function on $\mathcal{D}^G$; therefore, so is its trace.

Let $\rho_{\boldsymbol\lambda}(g) \in \mathrm{GL}(V_{\boldsymbol\lambda})$ be the automorphism associated to $g \in G$ in the representation $V_{\boldsymbol\lambda}$; assuming the choice of an orthonormal basis, we may view it as a unitary matrix. In view of (54)

$$\begin{aligned}
\Pi^\tau_{k\boldsymbol\lambda}(x, y) &= d_{k\boldsymbol\lambda} \int_G \overline{\chi_{k\boldsymbol\lambda}(g)}\, \Pi^\tau\left(\mu^\tau_{g^{-1}}(x), y\right)\, \mathrm{d}^H V_G(g) \\
&= d_{k\boldsymbol\lambda}\, \mathrm{trace}\left(\int_G \rho_{k\boldsymbol\lambda}(g^{-1})\, \Pi^\tau\left(\mu^\tau_{g^{-1}}(x), y\right)\, \mathrm{d}^H V_G(g)\right) \\
&= d_{k\boldsymbol\lambda}\, \mathrm{trace}\left(\mathcal{F}(f_{x,y})(k\,\boldsymbol\lambda)\right) = O\left(k^{-\infty}\right).
\end{aligned} \tag{55}$$

Let us consider U_2. Let $L_{\boldsymbol\lambda} = (\boldsymbol\lambda,\, 2\,\boldsymbol\lambda,\, \ldots)$ denote the *ladder* of irreducible representations generated by $\boldsymbol\lambda$ [25]. Let us consider the corresponding subspaces

$$L^2(X^\tau)_{L_{\boldsymbol\lambda}} := \bigoplus_{k=1}^{+\infty} L^2(X^\tau)_{k\boldsymbol\lambda}, \qquad H(X^\tau)_{L_{\boldsymbol\lambda}} := \bigoplus_{k=1}^{+\infty} H(X^\tau)_{k\boldsymbol\lambda}$$

with orthogonal projectors

$$\mathcal{Q}^\tau_{L_{\boldsymbol\lambda}} : L^2(X^\tau) \to L^2(X^\tau)_{L_{\boldsymbol\lambda}}, \qquad \Pi^\tau_{L_{\boldsymbol\lambda}} = \mathcal{Q}^\tau_{L_{\boldsymbol\lambda}} \circ \Pi^\tau : L^2(X^\tau) \to H^2(X^\tau)_{L_{\boldsymbol\lambda}}.$$

Therefore the wave front set $\mathrm{WF}(\Pi^\tau_{L_{\boldsymbol\lambda}}) \subseteq \big(T^\vee X^\tau \setminus (0)\big) \times \big(T^\vee X^\tau \setminus (0)\big)$ of $\Pi^\tau_{L_{\boldsymbol\lambda}}$ is obtained by composing those of $\mathcal{Q}^\tau_{L_{\boldsymbol\lambda}}$ and Π^τ. More precisely, the arguments in §3.1.2 of [16] (based on the theory of [25]) imply that

$$\mathrm{WF}(\Pi^\tau_{L_{\boldsymbol\lambda}}) \subseteq \left\{\Big((x, r\,\alpha^\tau_x), (y, -r\,\alpha^\tau_y)\Big) : x \in X^\tau_{\mathcal{O}},\ r > 0,\ y \in T'_{\tilde\Phi(x)} \cdot x\right\} \tag{56}$$

(recall Definition 4.2). Hence, $\Pi^\tau_{L_{\boldsymbol\lambda}}$ is $\mathcal{C}^\infty$ on $\overline{U}_2$; given that U_2 is G-invariant, $h_{x,y}(g) := \Pi^\tau_{L_{\boldsymbol\lambda}}\left(\mu^\tau_{g^{-1}}(x), y\right)$ is $\mathcal{C}^\infty$ on G when $(x, y) \in \overline{U}_2$. On the other hand,

$\Pi^\tau_{k\boldsymbol{\lambda}} = \mathcal{Q}^\tau_{k\boldsymbol{\lambda}} \circ \Pi^\tau_{L_{\boldsymbol{\lambda}}}$, since we can first project onto the full ladder of isotypical components and then to the $k\boldsymbol{\lambda}$-th. Hence, again by Sugiura's Theorem ([45]),

$$\begin{aligned}\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) &= d_{k\boldsymbol{\lambda}} \int_G \overline{\chi_{k\boldsymbol{\lambda}}(g)}\, \Pi^\tau_{L_{\boldsymbol{\lambda}}} \left(\mu^\tau_{g^{-1}}(x), y\right) \mathrm{d}^H V_G(g) \\ &= d_{k\boldsymbol{\lambda}} \operatorname{trace} \left(\int_G \rho_{k\boldsymbol{\lambda}}(g^{-1})\, \Pi^\tau_{L_{\boldsymbol{\lambda}}} \left(\mu^\tau_{g^{-1}}(x), y\right) \mathrm{d}^H V_G(g) \right) \\ &= d_{k\boldsymbol{\lambda}} \operatorname{trace} \left(\mathcal{F}(h_{x,y})(k\,\boldsymbol{\lambda})\right) = O\left(k^{-\infty}\right). \end{aligned} \tag{57}$$

Since Π^τ and P^τ are Fourier integral operators associated to the same canonical relation (§4.6), the previous arguments apply *verbatim* to the case of P^τ. □

5.2 Proof of Theorem 1.5

Before attacking the proof, we need to lay down some basic facts from Lie theory (standard references are [4], [51], [50]).

There exists a a finite covering of Lie groups

$$\mathfrak{p} : \hat{G} \to G, \tag{58}$$

where $\hat{G}$ is isomorphic to the direct product of a compact torus and a simply connected compact semisimple Lie group; in particular, $\hat{G}$ is also compact (§V.8 of [4]). The Lie algebras of $\hat{G}$ and G may be identified by means of the differential $\mathrm{d}_{\hat{e}}\mathfrak{p} : T_{\hat{e}}\hat{G} \to T_e G$ ($\hat{e} \in \hat{G}$ and $e \in G$ are the identity elements).

The inverse image $\hat{T} = \mathfrak{p}^{-1}(T)$ is a maximal torus of $\hat{G}$, and $\mathfrak{p}$ restricts to a covering map $\mathfrak{q} : \hat{T} \to T$ of the same degree as $\mathfrak{p}$ (this is clear of the connected component through the identity $\hat{T}_0 \subseteq \hat{T}$; on the other hand, $\ker(\mathfrak{p})$ is normal and discrete, hence central, so that $\ker(\mathfrak{p}) \subset \hat{T}_0 = \hat{T}$). We can similarly identify the Lie algebras of $\hat{T}$ and T by $\mathrm{d}_{\hat{e}}\mathfrak{q}$.

Let $\hat{\Lambda}$, $\Lambda \subset \mathfrak{t}$ be the lattices of $\hat{T}$ and T, respectively. Then $\hat{\Lambda} \subseteq \Lambda$, and $\hat{\Lambda}$ has index in Λ equal to the degree of $\mathfrak{p}$. Thus $\mathcal{D}^{\hat{G}} \supseteq \mathcal{D}^G$. Any representation of G pulls back to a representation of $\hat{G}$, and the isotypical decomposition is the same over G and $\hat{G}$, in view of the previous inclusion.

Let $R_+ \subset R$ denote the collection of positive roots of the pair $(\mathfrak{g}, \mathfrak{t}) = (\hat{\mathfrak{g}}, \hat{\mathfrak{t}})$, and set

$$\boldsymbol{\delta} := \frac{1}{2} \sum_{\boldsymbol{\alpha} \in R_+} \boldsymbol{\alpha}. \tag{59}$$

Then $\boldsymbol{\delta} \in \mathcal{D}^{\hat{G}}$.

We shall view the elements of Weyl group as group automorphisms of $\hat{T}$ (or, depending on the context of T, $\mathfrak{t}$,...). If $f : \hat{T} \to \mathbb{C}$, we shall set $f^w := f \circ w^{-1}$.

Definition 5.2 Given $f : \hat{T} \to \mathbb{C}$, let us set

$$\mathrm{Alt}_W(f) := \sum_{w \in W} (-1)^w f^w,$$

where $(-1)^w := \det(w) \in \{\pm 1\}$ is the determinant of $w \in W$ as a linear automorphism of $\mathfrak{t}$.

Remark 5.3 For any $f : \hat{T} \to \mathbb{C}$ and $w \in W$,

$$\mathrm{Alt}_W(f^w) = (-1)^w \, \mathrm{Alt}_W(f).$$

Definition 5.4 Any $\boldsymbol{\eta} \in \mathcal{D}^{\hat{G}}$ corresponds to a group character $E_{\boldsymbol{\eta}} : \hat{T} \to S^1$. Let us define $A_{\boldsymbol{\eta}}, \ \Delta : \hat{T} \to \mathbb{C}$ by setting

$$A_{\boldsymbol{\eta}} := \mathrm{Alt}_W(E_{\boldsymbol{\eta}}) = \sum_{w \in W} (-1)^w \, E_{w(\boldsymbol{\eta})}, \quad \Delta := A_{\boldsymbol{\delta}}$$

Then $\Delta \neq 0$ on the dense open subset $\hat{T}' \subseteq \tilde{T}$ of regular elements; $\hat{T}'$ is the inverse image in $\hat{T}$ of the open dense subset $T' \subseteq T$ of regular elements of T.

5.2.1 The Weyl Character Formula

Suppose $\boldsymbol{\lambda} \in \mathcal{D}^G \subseteq \mathcal{D}^{\hat{G}}$. The Weyl character formula describes the restriction $\chi_{\boldsymbol{\lambda}}|_T : T \to \mathbb{C}$ in terms of alternating sums of irreducible characters on T or, more precisely, on the covering $\hat{T}$. Set $\boldsymbol{\nu}_{\boldsymbol{\lambda}} := \boldsymbol{\lambda} + \boldsymbol{\delta} \in \mathcal{D}^{\hat{G}}$. Thus we can consider the ratio

$$\frac{A_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}}{\Delta} : \hat{T}' \to \mathbb{C}. \tag{60}$$

Theorem 5.5 (Weyl character formula) *The function in (60) admits a unique continuous extension to $\hat{T}$; furthermore, this extended function is the pull-back of $\chi_{\boldsymbol{\lambda}}|_T$ by the covering map $\hat{T} \to T$.*

(See, e.g., [4], §VI.1).

5.2.2 The Weyl Integration Formula

Given any compact Lie group K, $\mathrm{d}^H_K V$ denotes its Haar measure. If $K' \leqslant K$ is an inclusion of compact Lie groups, there is an induced Haar measure $\mathrm{d}^H V_{K/K'}$ on the homogeneous space K/K'. If $f : T \to \mathbb{C}$ is continuous, let us define $f_T : T \to \mathbb{C}$ by setting

$$f_T(t) := \int_{G/T} f\left(g\, t\, g^{-1}\right) \, \mathrm{d}^H V_{G/T}(g\,T).$$

Let us consider the κ-orthogonal direct sum $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp$. For any $t \in T$, the orthogonal automorphism $\mathrm{Ad}_{t^{-1}} : \mathfrak{g} \to \mathfrak{g}$ leaves both $\mathfrak{t}$ and $\mathfrak{t}^\perp$ invariant; hence, it restricts to an orthogonal endomorphism $(\mathrm{Ad}_{\mathfrak{t}^\perp})_{t^{-1}} : \mathfrak{t}^\perp \to \mathfrak{t}^\perp$, which doesn't have the eigenvalue 1 whenever t is regular. Furthermore, $\det\left((\mathrm{Ad}_{\mathfrak{t}^\perp})_{t^{-1}} - \mathrm{Id}_{\mathfrak{t}^\perp}\right) > 0$ for any $t \in T'$.

Theorem 5.6 (Weyl integration formula) *Under the previous assumptions,*

$$\int_G f(g)\,\mathrm{d}^H V_G(g) = \frac{1}{|W|}\int_T \det\left((\mathrm{Ad}_{\mathfrak{t}^\perp})_{t^{-1}} - \mathrm{Id}_{\mathfrak{t}^\perp}\right) f_T(t)\,\mathrm{d}^H V_T(t).$$

In addition, when $(G, T) = (\tilde{G}, \tilde{T})$

$$\det\left((\mathrm{Ad}_{\mathfrak{t}^\perp})_{t^{-1}} - \mathrm{Id}_{\mathfrak{t}^\perp}\right) = |\Delta(t)|^2.$$

5.2.3 The Proof

Proof of Theorem 1.5 We shall give the argument for Π^τ, the one for P^τ being essentially the same. Our starting point is again the relation

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) = d_{k\boldsymbol{\lambda}} \int_G \overline{\chi_{k\boldsymbol{\lambda}}(g)}\,\Pi^\tau\left(\mu^\tau_{g^{-1}}(x), y\right)\,\mathrm{d}^H V_G(g). \tag{61}$$

By composing with the covering map $\mathfrak{p}$ in (58), we can pull-back μ^τ to an action $\hat{\mu}^\tau$ of $\hat{G}$ on X^τ, and $\chi_{k\boldsymbol{\lambda}}$ to the character $\hat{\chi}_{k\boldsymbol{\lambda}}$ on $\hat{G}$. We can rewrite (61) as

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) = d_{k\boldsymbol{\lambda}} \int_{\hat{G}} \overline{\hat{\chi}_{k\boldsymbol{\lambda}}(g)}\,\Pi^\tau\left(\hat{\mu}^\tau_{g^{-1}}(x), y\right)\,\mathrm{d}^H V_{\hat{G}}(g). \tag{62}$$

By Theorem 1.4, we may restrict to the case where (x, y) belongs to a small neighbourhood of $\mathcal{Z}^\tau_{\mathcal{O}} \subset X^\tau \times X^\tau$; hence we may assume that x and y belong to a small neighbourhood of $X^\tau_{\mathcal{O}}$, and therefore that μ^τ is locally free at x and y.

Perhaps after replacing (x, y) by $\left(\mu^\tau_h(x), \mu^\tau_h(y)\right)$ for some $h \in G$, we may assume that $\Phi^\tau(y)$ belongs to a small conic neighbourhood in $\mathfrak{g}^\vee$ of the ray $\mathbb{R}_+\,\boldsymbol{\lambda}$. Hence

$$\Phi^\tau(y) = a\,\boldsymbol{\lambda} + \boldsymbol{\beta}, \tag{63}$$

where $\boldsymbol{\beta} \in \boldsymbol{\lambda}^\perp$ and $\|\boldsymbol{\beta}\| \ll \|\Phi^\tau(y)\|$.

Furthermore, let us fix a suitably small $\delta > 0$, and consider a cut-off function $\rho_1(g, x, y)$ on $\hat{G} \times X^\tau \times X^\tau$ which is identically equal to one where $\mathrm{dist}_{X^\tau}\left(\hat{\mu}^\tau_{g^{-1}}(x), y\right) < \delta$, and vanishes identically where $\mathrm{dist}_{X^\tau}\left(\hat{\mu}^\tau_{g^{-1}}(x), y\right) > 2\,\delta$. We can rewrite the left hand side of (62) as

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) = \Pi^\tau_{k\boldsymbol{\lambda}}(x, y)' + \Pi^\tau_{k\boldsymbol{\lambda}}(x, y)'',$$

where $\Pi^\tau_{k\boldsymbol{\lambda}}(x, y)'$ (respectively, $\Pi^\tau_{k\boldsymbol{\lambda}}(x, y)''$) is as in (62), but with the integrand multiplied by $\rho_1(g, x, y)$ (respectively, by $1 - \rho_1(g, x, y)$). Arguing as in §5.1, one can check that $\Pi^\tau_{k\boldsymbol{\lambda}}(x, y)'' = O\left(k^{-\infty}\right)$, so that

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) \sim \Pi^\tau_{k\boldsymbol{\lambda}}(x, y)'.$$

On the domain of integration of the integrand of $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,y)'$, we can represent Π^{τ} as a Fourier integral operator with complex phase, as discussed in §4.6.

In view of the Weyl integration formula in §5.2.2

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,y) \sim \frac{d_{k\boldsymbol{\lambda}}}{|W|} \int_{\hat{T}} \mathrm{d}^H V_{\hat{T}}(t) \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \left[\overline{\hat{\chi}_{k\boldsymbol{\lambda}}(t)}\,\rho_1\left(g\,t\,g^{-1},x,y\right)\,\Pi^{\tau}\left(\hat{\mu}^{\tau}_{g\,t^{-1}\,g^{-1}}(x),y\right)|\Delta(t)|^2\right]. \tag{64}$$

Combining (64) with the Weyl character formula in §5.2.1 we obtain

$$\begin{aligned}\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,y) &\sim \frac{d_{k\boldsymbol{\lambda}}}{|W|} \sum_{w\in W} \int_{\hat{T}} \mathrm{d}^H V_{\tilde{T}}(t) \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \\ &\quad \left[(-1)^w\,\overline{E_{w(\boldsymbol{\nu}_{k\boldsymbol{\lambda}})}(t)}\,\rho_1\left(g\,t\,g^{-1},x,y\right)\,\Pi^{\tau}\left(\tilde{\mu}^{\tau}_{g\,t^{-1}\,g^{-1}}(x),y\right)\,\Delta(t)\right] \\ &= \frac{d_{k\boldsymbol{\lambda}}}{|W|} \sum_{w\in W} \int_{\hat{T}} \mathrm{d}^H V_{\hat{T}}(t) \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \\ &\quad \left[(-1)^w\,\overline{E_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}\big(w(t)\big)}\,\rho_1\left(g\,t\,g^{-1},x,y\right)\,\Pi^{\tau}\left(\hat{\mu}^{\tau}_{g\,t^{-1}\,g^{-1}}(x),y\right)\,\Delta(t)\right]\end{aligned} \tag{65}$$

where $\boldsymbol{\nu}_{k\boldsymbol{\lambda}} := k\boldsymbol{\lambda} + \boldsymbol{\delta}$.

For any $w \in W$, there is $g_w \in N(\hat{T})$ (the normalizer of $\hat{T}$ in $\hat{G}$) such that $w(t) := g_w\,t\,g_w^{-1}$. Furthermore, $\Delta(g_w\,t\,g_w^{-1}) = (-1)^w\,\Delta(t)$ by Remark 5.3. We may then rewrite (65) as follows:

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,y) \sim \frac{d_{k\boldsymbol{\lambda}}}{|W|} \sum_{w\in W} \int_{\hat{T}} \mathrm{d}^H V_{\hat{T}}(t) \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \left[\overline{E_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}\left(g_w\,t\,g_w^{-1}\right)}\,\rho_1\left(g\,t\,g^{-1},x,y\right)\,\Pi^{\tau}\left(\hat{\mu}^{\tau}_{g\,t^{-1}\,g^{-1}}(x),y\right)\,\Delta(g_w\,t\,g_w^{-1})\right]. \tag{66}$$

Observe that $w \in W$ also induces a measure preserving diffeomorphism

$$\alpha_w : g\,\hat{T} \in \hat{G}/\hat{T} \mapsto g\,g_w\,\hat{T} \in \hat{G}/\hat{T}.$$

Hence (66) may be rewritten

$$\begin{aligned}&\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,y) \\ &\sim \frac{d_{k\boldsymbol{\lambda}}}{|W|} \sum_{w\in W} \int_{\hat{T}} \mathrm{d}^H V_{\hat{T}}(t) \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \\ &\quad \left[\overline{E_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}\left(g_w\,t\,g_w^{-1}\right)}\,\rho_1\left(g\,g_w\,t\,g_w^{-1}\,g^{-1},x,y\right)\,\Pi^{\tau}\left(\hat{\mu}^{\tau}_{g\,g_w\,t^{-1}\,g_w^{-1}\,g^{-1}}(x),y\right)\,\Delta(g_w\,t\,g_w^{-1})\right] \\ &= d_{k\boldsymbol{\lambda}} \int_{\hat{T}} \mathrm{d}^H V_{\hat{T}}(t) \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \\ &\quad \left[\overline{E_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}(t)}\,\rho_1\left(g\,t\,g^{-1},x,y\right)\,\Pi^{\tau}\left(\tilde{\mu}^{\tau}_{g\,t^{-1}\,g^{-1}}(x),y\right)\,\Delta(t)\right];\end{aligned} \tag{67}$$

to obtain the last equality, in the w-th summand we have performed the measure preserving change of variable $t' = g_w\, t\, g_w^{-1}$ in $\hat{T}$.

Let us insert in (67) the description of Π^τ as a Fourier integral operator recalled in §4.6. Furthermore, let us fix an isomorphism $\hat{T} \cong (S^1)^{r_G}$ and trigonometric coordinates $\boldsymbol{\vartheta} = (\vartheta_1, \ldots, \vartheta_{r_G})$ with $\vartheta_j \in (-\pi, \pi)$. We may view $\mathcal{D}^{\tilde{G}} \subset \mathbb{Z}^{r_G}$. Thus we have the replacements

$$t = e^{\imath\, \boldsymbol{\vartheta}}, \quad \mathrm{d}^H V_{\tilde{T}}(t) = \frac{1}{(2\,\pi)^{r_G}}\, \mathrm{d}\boldsymbol{\vartheta}, \quad E_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}\big(t\big) = e^{\imath\, \langle k\boldsymbol{\lambda}+\boldsymbol{\delta}, \boldsymbol{\vartheta}\rangle}.$$

We obtain

$$\begin{aligned}
&\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) \\
&\sim d_{k\boldsymbol{\lambda}} \int_{\hat{T}} \mathrm{d}^H V_{\hat{T}}(t) \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \int_0^{+\infty} \mathrm{d}u \\
&\quad \left[\overline{E_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}\big(t\big)}\, \rho_1\left(g\,t\,g^{-1}, x, y\right) e^{\imath\, u\, \psi^\tau\left(\hat{\mu}^\tau_{g\,t^{-1}\,g^{-1}}(x), y\right)} s\left(\hat{\mu}^\tau_{g\,t^{-1}\,g^{-1}}(x), y, u\right) \Delta(t)\right] \\
&= \frac{d_{k\boldsymbol{\lambda}}}{(2\,\pi)^{r_G}} \int_{(-\pi,\pi)^{r_G}} \mathrm{d}\boldsymbol{\vartheta} \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \int_0^{+\infty} \mathrm{d}u \\
&\quad \left[e^{-\imath\, \langle k\boldsymbol{\lambda}+\boldsymbol{\delta}, \boldsymbol{\vartheta}\rangle}\, \rho_1\left(g\,e^{\imath\,\boldsymbol{\vartheta}}\,g^{-1}, x, y\right) e^{\imath\, u\, \psi^\tau\left(\hat{\mu}^\tau_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x), y\right)} s\left(\hat{\mu}^\tau_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x), y, u\right) \Delta(t)\right]
\end{aligned} \tag{68}$$

With the rescaling $u \mapsto k\,u$, we may rewrite (68) as an oscillatory integral

$$\begin{aligned}
&\Pi^\tau_{k\boldsymbol{\lambda}}(x, y) \\
&\sim \frac{k\,d_{k\boldsymbol{\lambda}}}{(2\,\pi)^{r_G}} \int_{(-\pi,\pi)^{r_G}} \mathrm{d}\boldsymbol{\vartheta} \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \int_0^{+\infty} \mathrm{d}u \\
&\quad \left[e^{-\imath\, \langle k\boldsymbol{\lambda}+\boldsymbol{\delta}, \boldsymbol{\vartheta}\rangle}\, \rho_1\left(g\,e^{\imath\,\boldsymbol{\vartheta}}\,g^{-1}, x, y\right) e^{\imath\, k\, u\, \psi^\tau\left(\hat{\mu}^\tau_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x), y\right)} s\left(\hat{\mu}^\tau_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x), y, k\,u\right) \Delta(e^{\imath\,\boldsymbol{\vartheta}})\right] \\
&= \frac{k\,d_{k\boldsymbol{\lambda}}}{(2\,\pi)^{r_G}} \int_{(-\pi,\pi)^{r_G}} \mathrm{d}\boldsymbol{\vartheta} \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T}) \int_0^{+\infty} \mathrm{d}u \left[e^{\imath\, k\, \Psi_{x,y}(u,g,\boldsymbol{\vartheta})}\, \tilde{\mathcal{A}}_{x,y,k}(u, g, \boldsymbol{\vartheta})\right],
\end{aligned} \tag{69}$$

where

$$\Psi_{x,y}(u, g\,\hat{T}, \boldsymbol{\vartheta}) := u\,\psi^\tau\left(\hat{\mu}^\tau_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x), y\right) - \langle \boldsymbol{\lambda}, \boldsymbol{\vartheta}\rangle, \tag{70}$$

$$\tilde{\mathcal{A}}_{x,y,k}(u, g\,\hat{T}, \boldsymbol{\vartheta}) := e^{-\imath\, \langle \boldsymbol{\delta}, \boldsymbol{\vartheta}\rangle}\, \rho_1\left(g\,e^{\imath\,\boldsymbol{\vartheta}}\,g^{-1}, x, y\right) s\left(\hat{\mu}^\tau_{g\,e^{-\imath\,\boldsymbol{\vartheta}}\,g^{-1}}(x), y, k\,u\right) \Delta(e^{\imath\,\boldsymbol{\vartheta}}).$$

Let us consider a fixed but general pair $(g_0\,\hat{T}, t_0) \in (\hat{G}/\hat{T}) \times \hat{T}$, where $t_0 = e^{\imath\,\boldsymbol{\vartheta}_0}$, in the domain of integration (that is, such that $\left(g_0\,e^{\imath\,\boldsymbol{\vartheta}_0}\,g_0^{-1}, x, y\right) \in \mathrm{supp}(\rho'))$.

Near t_0, we can write $t = e^{\imath\,(\boldsymbol{\vartheta}+\boldsymbol{\vartheta}_0)}$ with $\boldsymbol{\vartheta} \sim \mathbf{0}$. Let us set $x_0 := \tilde{\mu}^{\tau}_{g_0\,e^{-\imath\,\boldsymbol{\vartheta}_0}\,g_0^{-1}}(x)$. Then $x_0 \sim y$ (since we are on the support of ρ'), whence (recalling (63))

$$\Phi^{\tau}(x_0) \sim \Phi^{\tau}(y) \sim a\,\boldsymbol{\lambda}, \tag{71}$$

where $\sim$ provisionally stands for 'is very close to' (say at distance $O(\delta)$). Thus for any $\boldsymbol{\vartheta}$

$$\hat{\mu}^{\tau}_{g_0\,e^{-\imath\,(\boldsymbol{\vartheta}+\boldsymbol{\vartheta}_0)}\,g_0^{-1}}(x) = \hat{\mu}^{\tau}_{g_0\,e^{-\imath\,\boldsymbol{\vartheta}}\,g_0^{-1}}(x_0).$$

Therefore, by the results in §2.5 of [18], in NHLC's centered at x_0 (§4.6.2) we have

$$\begin{aligned}\tilde{\mu}^{\tau}_{g_0\,e^{-\imath\,(\boldsymbol{\vartheta}+\boldsymbol{\vartheta}_0)}\,g_0^{-1}}(x) &= x_0 + \Big(\big\langle\Phi(x_0), \mathrm{Ad}_{g_0}(\boldsymbol{\vartheta})\big\rangle + R_3(\boldsymbol{\vartheta}), \mathbf{R}_1(\boldsymbol{\vartheta})\Big)\\ &= x_0 + \Big(\big\langle\mathrm{Ad}_{g_0^{-1}}\big(\Phi(x_0)\big), \boldsymbol{\vartheta}\big\rangle + R_3(\boldsymbol{\vartheta}), \mathbf{R}_1(\boldsymbol{\vartheta})\Big),\end{aligned} \tag{72}$$

where R_j (respectively, $\mathbf{R}_j$) generically denotes a real-valued (respectively, vector valued) function defined on a neighbourhood of the origin of some Euclidean space and vanishing to j-th order at the origin.

In view of (47), (70) and (71) above, and Proposition 34 of [34], (72) implies

$$\partial_{\boldsymbol{\vartheta}}\Psi_{x,y}(u, g\,\tilde{T}, \boldsymbol{\vartheta}) \sim \Big[u\,\mathrm{Ad}_{g_0^{-1}}\big(\Phi(x_0)\big) - \boldsymbol{\lambda}\Big]\Big|_{\mathfrak{t}} \sim \Big[u\,a\,\mathrm{Ad}_{g_0^{-1}}\big(\boldsymbol{\lambda}\big) - \boldsymbol{\lambda}\Big]\Big|_{\mathfrak{t}}. \tag{73}$$

On the other hand, by the assumption on $\mathcal{O}$ there exist a constant $c > 0$ such that $\Big\|\mathrm{Ad}_{g_0^{-1}}\big(\boldsymbol{\lambda}\big)\Big|_{\mathfrak{t}}\Big\| \geq c$ for all $g_0 \in G$. Therefore, (73) implies the following.

Lemma 5.7 *Under the previous assumptions, there exist constants $D \gg 0$, $c > 0$ such that $\|\partial_{\boldsymbol{\vartheta}}\Psi_{x,y}(u, g\,\tilde{T}, \boldsymbol{\vartheta})\| \geq c$ whenever $u \notin (1/D, D)$.*

We can then 'integrate by parts' in $\boldsymbol{\vartheta}$ to reduce to the case where the integrand in (69) is compactly supported in u. More precisely, by a standard argument Lemma 5.7 implies the following.

Corollary 5.8 *The asymptotics of (68) are unchanged, if the integrand is multiplied by a compactly supported cut-off function $\rho_2 \in \mathcal{C}^{\infty}_c\big(1/(2\,D), 2\,D)\big)$ such that $\rho_2 \equiv 1$ on $(1/D, D)$.*

Thus the asymptotics remain unchanged if the amplitude $\tilde{\mathcal{A}}_{x,y,k}(u, g\,\tilde{T}, \boldsymbol{\vartheta})$ in (70) is replaced by

$$\mathcal{A}_{x,y,k}(u, g\,\tilde{T}, \boldsymbol{\vartheta}) := \tilde{\mathcal{A}}_{x,y,k}(u, g\,\tilde{T}, \boldsymbol{\vartheta}) \cdot \rho_2(u).$$

Hence,

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}(x, y) \sim \frac{k\,d_{k\boldsymbol{\lambda}}}{(2\,\pi)^{r_G}} \int_{(-\pi,\pi)^{r_G}} \mathrm{d}\boldsymbol{\vartheta} \int_{\hat{G}/\hat{T}} \mathrm{d}^H V_{\hat{G}/\hat{T}}(g\,\hat{T})$$

$$\int_0^{+\infty} \mathrm{d}u \left[e^{\imath\, k\, \Psi_{x,y}(u,g,\boldsymbol{\vartheta})}\, \mathcal{A}_{x,y,k}(u,g,\boldsymbol{\vartheta})\right], \tag{74}$$

where the integrand is now compactly supported in u. In particular, it is legitimate to integrate by parts in u.

Under hypothesis (14) of the Theorem, in view of (48) for some constant $D^\tau > 0$ we have

$$\left|\partial_u \Psi_{x,y}(u, g\,\hat{T}, \boldsymbol{\vartheta})\right| = \left|\psi^\tau(x,y)\right| \geq \Im\left(\psi^\tau(x,y)\right) \geq D^\tau\, k^{2\epsilon-1}. \tag{75}$$

Writing

$$e^{\imath\, k\, \Psi_{x,y}(u,g\,\hat{T},\boldsymbol{\vartheta})} = -\frac{\imath}{k\,\psi^\tau(x,y)}\, \partial_u \left(e^{\imath\, k\, \Psi_{x,y}(u,g\,\hat{T},\boldsymbol{\vartheta})}\right),$$

we can then iteratively integrate by parts in u, introducing at each step a factor which is bounded by $D_1^\tau\, k^{-2\epsilon}$ for some constant $D_1^\tau > 0$. We conclude that $\Pi^\tau_{k\boldsymbol{\lambda}}(x,y) = O\left(k^{-\infty}\right)$ in the given range. □

5.3 Proof of Theorem 1.6

The proof of Theorem 1.6 relies on the Kirillov character formula (see [28]), which we briefly recall.

5.3.1 The Kirillov Character Formula

Let us fix an Ad-invariant Euclidean product κ^H on $\mathfrak{g}$ whose associated Riemannian density is the Haar measure $\mathrm{d}^H V_G$, and let $\mathrm{d}^H V_{\mathfrak{g}}$ be the corresponding Lebesgue measure on $\mathfrak{g}$.

Let $\exp_G : \mathfrak{g} \to G$ denote the exponential map, and let $\mathfrak{g}' \subseteq \mathfrak{g}$ be an open neighbourhood of the origin such that

$$\exp'_G := \exp_G\big|_{\mathfrak{g}'} : \mathfrak{g}' \to G' := \exp_G(\mathfrak{g}') \tag{76}$$

is a diffeomorphism. Let $\mathcal{P} : \mathfrak{g}' \to (0, +\infty)$ be the smooth function defined by

$$\exp_G^*\left(\mathrm{d}^H V_G\right) = \mathcal{P}^2\, \mathrm{d}^H V_{\mathfrak{g}}; \tag{77}$$

thus $\mathcal{P}(0) = 1$. We shall usually write $\exp_G(\boldsymbol{\xi}) = e^{\boldsymbol{\xi}}$ ($\boldsymbol{\xi} \in \mathfrak{g}$).

Furthermore, let $\mathcal{O}_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}} \subseteq \mathfrak{g}^\vee$ be the coadjoint orbit through $\boldsymbol{\nu}_{\boldsymbol{\lambda}} := \boldsymbol{\lambda} + \boldsymbol{\delta}$, and let $\sigma_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}$ denote its Konstant-Kirillov symplectic structure. The symplectic volume form on $\mathcal{O}_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}$ is then

$$\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}} := \frac{\sigma_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}^{\wedge n_G}}{n_G!}, \quad \text{where} \quad n_G := \frac{1}{2}\dim(\mathcal{O}_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}) = \frac{1}{2}(d - r_G),$$

where the latter equality holds because $\boldsymbol{\nu}_{\boldsymbol{\lambda}}$ is a regular weight.

The Kirillov character formula expresses the restriction of $\chi_{\boldsymbol{\lambda}}$ to G' in terms of an integral on the symplectic manifold $(\mathcal{O}_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}, \sigma_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}})$. Namely, if $\boldsymbol{\xi} \in \mathfrak{g}'$ then

$$\chi_{\boldsymbol{\lambda}}\left(e^{\boldsymbol{\xi}}\right) = \frac{1}{(2\,\pi)^{n_G}}\,\frac{1}{\mathcal{P}(\boldsymbol{\xi})}\int_{\mathcal{O}_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}} e^{\imath\,\boldsymbol{\beta}(\boldsymbol{\xi})}\,\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}_{\boldsymbol{\lambda}}}}(\boldsymbol{\beta}). \tag{78}$$

5.3.2 The Proof

We shall first prove the following apparently weaker statement.

Theorem 5.9 *Under the hypothesis of Theorem 1.6, uniformly for*

$$x \in K \ :\ \mathrm{dist}_{X^\tau}\left(x, X^\tau_{\mathcal{O}}\right) \geq C\,k^{\epsilon-\frac{1}{2}} \tag{79}$$

one has

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x,x) = O\left(k^{-\infty}\right) \qquad \text{and} \qquad P^\tau_{k\boldsymbol{\lambda}}(x,x) = O\left(k^{-\infty}\right)$$

for $k \to +\infty$.

Proof Let us start with (61) with $x = y$. We may assume without loss that x belongs to a small tubular neighbourhood of $X^\tau_{\mathcal{O}}$. Perhaps replacing x by $\mu^\tau_h(x)$ for some $h \in G$, we may further assume that $\Phi^\tau(x)$ belongs to a small conic neighourhood of $\mathbb{R}_+ \cdot \boldsymbol{\lambda}$, hence that (63) holds with $y = x$.

Furthermore, fix $r > 0$ sufficiently small (how small may depend on K) and let $\rho_2 \in \mathcal{C}^\infty(G)$ be supported in the ball $G_r \subseteq G$ centered at e of radius $2\,r$ (say, in the Riemannian metric κ) and identically equal to 1 on the ball of radius r. Since $\tilde{\mu}^\tau$ is free on K, the asymptotics of $\Pi^\tau_{k\boldsymbol{\lambda}}(x,x)$ are unchanged, if the integrand in (61) is multiplied by $\rho_2(g)$. Thus

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x,x) \sim d_{k\boldsymbol{\lambda}} \int_{G_r} \overline{\chi_{k\boldsymbol{\lambda}}(g)}\,\rho_2(g)\,\Pi^\tau\left(\mu^\tau_{g^{-1}}(x), x\right)\,\mathrm{d}^H V_G(g). \tag{80}$$

We may assume without loss that $G_r \subseteq G'$; hence integration over G_r may be transferred to $\mathfrak{g}_r \subseteq \mathfrak{g}$, the open ball centered at the origin for κ_e, by the diffeomorphism (76). In view of (77), (80) may be rewritten

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x,x) \sim d_{k\boldsymbol{\lambda}} \int_{\mathfrak{g}_r} \overline{\chi_{k\boldsymbol{\lambda}}(e^{\boldsymbol{\xi}})}\,\rho_2(e^{\boldsymbol{\xi}})\,\Pi^\tau\left(\mu^\tau_{e^{-\boldsymbol{\xi}}}(x), x\right)\,\mathcal{P}(\boldsymbol{\xi})^2\,\mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi}). \tag{81}$$

Next let us make use of (78), with $k\boldsymbol{\lambda}$ in place of $\boldsymbol{\lambda}$. Namely, for $\boldsymbol{\xi} \in \mathfrak{g}'$ we have

$$\chi_{k\boldsymbol{\lambda}}\left(e^{\boldsymbol{\xi}}\right) = \frac{1}{(2\,\pi)^{n_G}}\,\frac{1}{\mathcal{P}(\boldsymbol{\xi})}\int_{\mathcal{O}_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}} e^{\imath\,\boldsymbol{\beta}(\boldsymbol{\xi})}\,\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}}(\boldsymbol{\beta}). \tag{82}$$

We have

$$\mathcal{O}_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}} = \mathcal{O}_{k\boldsymbol{\lambda}+\boldsymbol{\delta}} = k\,\mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\,\boldsymbol{\delta}} \subseteq \mathfrak{g}^\vee.$$

We can therefore transfer integration on $\mathcal{O}_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}$ to $\mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}}$ by the dilation $\mathfrak{d}_k : \boldsymbol{\beta} \in \mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}} \mapsto k\boldsymbol{\beta} \in \mathcal{O}_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}$. Since

$$\mathfrak{d}_k^*(\sigma_{\mathcal{O}_{\boldsymbol{\nu}_{k\boldsymbol{\lambda}}}}) = k\,\sigma_{\mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}}},$$

we can rewrite (78) as

$$\chi_{k\boldsymbol{\lambda}}\left(e^{\boldsymbol{\xi}}\right) = \left(\frac{k}{2\pi}\right)^{n_G} \frac{1}{\mathcal{P}(\boldsymbol{\xi})} \int_{\mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}}} e^{\imath\, k\boldsymbol{\beta}(\boldsymbol{\xi})}\, \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}}}(\boldsymbol{\beta}). \tag{83}$$

Finally, we transfer integration from the variable orbit $\mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}}$ to the fixed orbit $\mathcal{O} = \mathcal{O}_{\boldsymbol{\lambda}}$. Since both $\boldsymbol{\lambda}$ and $\boldsymbol{\delta}$ are regular weights, there is a well-defined equivariant diffeomorphism $L_k : \mathcal{O}_{\boldsymbol{\lambda}} \rightarrow \mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}}$, defined as follows. If $\boldsymbol{\beta} = \mathrm{Coad}_g(\boldsymbol{\lambda})$ for some $g \in G$, let us set

$$\begin{aligned} \boldsymbol{\delta}_{\boldsymbol{\beta}} &:= \mathrm{Coad}_g(\boldsymbol{\delta}) \in \mathcal{O}_{\boldsymbol{\delta}} \\ L_k(\boldsymbol{\beta}) &:= \mathrm{Coad}_g\left(\boldsymbol{\lambda} + \frac{1}{k}\boldsymbol{\delta}\right) = \boldsymbol{\beta} + \frac{1}{k}\boldsymbol{\delta}_{\boldsymbol{\beta}} \in \mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}}. \end{aligned} \tag{84}$$

Then

$$L_k^*(\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}+\frac{1}{k}\boldsymbol{\delta}}}) = \mathcal{V}_k\, \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}, \quad \text{where} \quad \mathcal{V}_k(\boldsymbol{\beta}) \sim 1 + \sum_{j\geq 1} k^{-j}\, V_j(\boldsymbol{\beta}). \tag{85}$$

Thus (83) can be rewritten

$$\begin{aligned} \chi_{k\boldsymbol{\lambda}}\left(e^{\boldsymbol{\xi}}\right) &= \left(\frac{k}{2\pi}\right)^{n_G} \frac{1}{\mathcal{P}(\boldsymbol{\xi})} \int_{\mathcal{O}_{\boldsymbol{\lambda}}} e^{\imath\, k\, L_k(\boldsymbol{\beta})(\boldsymbol{\xi})}\, \mathcal{V}_k(\boldsymbol{\beta})\, \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}) \\ &= \left(\frac{k}{2\pi}\right)^{n_G} \frac{1}{\mathcal{P}(\boldsymbol{\xi})} \int_{\mathcal{O}_{\boldsymbol{\lambda}}} e^{\imath\, [k\boldsymbol{\beta}(\boldsymbol{\xi})+\boldsymbol{\delta}_{\boldsymbol{\beta}}(\boldsymbol{\xi})]}\, \mathcal{V}_k(\boldsymbol{\beta})\, \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}). \end{aligned} \tag{86}$$

Let us insert (86) in (81), and then make use of the description of Π^τ as an FIO (§4.6). We obtain

$$\begin{aligned} \Pi^\tau_{k\boldsymbol{\lambda}}(x,x) &\sim d_{k\boldsymbol{\lambda}} \left(\frac{k}{2\pi}\right)^{n_G} \int_{\mathfrak{g}} \mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi}) \int_{\mathcal{O}_{\boldsymbol{\lambda}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}) \\ &\quad \left[e^{-\imath\, [k\boldsymbol{\beta}(\boldsymbol{\xi})+\boldsymbol{\delta}_{\boldsymbol{\beta}}(\boldsymbol{\xi})]}\, \rho_2(e^{\boldsymbol{\xi}})\, \Pi^\tau\left(\mu^\tau_{e^{-\boldsymbol{\xi}}}(x), y\right) \mathcal{V}_k(\boldsymbol{\beta})\, \mathcal{P}(\boldsymbol{\xi})\right] \\ &= d_{k\boldsymbol{\lambda}} \left(\frac{k}{2\pi}\right)^{n_G} \int_{\mathfrak{g}} \mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi}) \int_{\mathcal{O}_{\boldsymbol{\lambda}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}) \int_0^{+\infty} \mathrm{d}u \\ &\quad \left[e^{\imath\left[u\,\psi^\tau\left(\mu^\tau_{e^{-\boldsymbol{\xi}}}(x),x\right) - k\boldsymbol{\beta}(\boldsymbol{\xi}) - \boldsymbol{\delta}_{\boldsymbol{\beta}}(\boldsymbol{\xi})\right]}\, \rho_2(e^{\boldsymbol{\xi}})\, s^\tau\left(\mu^\tau_{e^{-\boldsymbol{\xi}}}(x), x, u\right) \mathcal{V}_k(\boldsymbol{\beta})\, \mathcal{P}(\boldsymbol{\xi})\right]. \end{aligned} \tag{87}$$

Performing the change of variable $u \mapsto k\,u$, we finally obtain

$$\begin{aligned}
\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x) &\sim k\,d_{k\boldsymbol{\lambda}} \left(\frac{k}{2\pi}\right)^{n_G} \int_{\mathfrak{g}} \mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi}) \int_{\mathcal{O}_{\boldsymbol{\lambda}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}) \int_0^{+\infty} \mathrm{d}u \\
&\quad \left[e^{\imath\,k\left[u\,\psi^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x\right)-\boldsymbol{\beta}(\boldsymbol{\xi})\right]}\, e^{-\boldsymbol{\delta}_{\boldsymbol{\beta}}(\boldsymbol{\xi})} \rho_2(e^{\boldsymbol{\xi}})\; s^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x,k\,u\right) \mathcal{V}_k(\boldsymbol{\beta})\,\mathcal{P}(\boldsymbol{\xi})\right] \\
&= k\,d_{k\boldsymbol{\lambda}} \left(\frac{k}{2\pi}\right)^{n_G} \int_{\mathfrak{g}} \mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi}) \int_{\mathcal{O}_{\boldsymbol{\lambda}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}) \int_0^{+\infty} \mathrm{d}u \\
&\quad \left[e^{\imath\,k\,\Gamma_x(u,\boldsymbol{\beta},\boldsymbol{\xi})}\,\tilde{\mathcal{B}}_{x,k}(u,\boldsymbol{\beta},\boldsymbol{\xi})\right],
\end{aligned} \tag{88}$$

where

$$\begin{aligned}
\Gamma_x(u,\boldsymbol{\beta},\boldsymbol{\xi}) &:= u\,\psi^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x\right) - \boldsymbol{\beta}(\boldsymbol{\xi}) \\
\tilde{\mathcal{B}}_{x,k}(u,\boldsymbol{\beta},\boldsymbol{\xi}) &:= e^{-\boldsymbol{\delta}_{\boldsymbol{\beta}}(\boldsymbol{\xi})}\,\tilde{\rho}(\boldsymbol{\xi})\,s^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x,k\,u\right)\mathcal{V}_k(\boldsymbol{\beta})\,\mathcal{P}(\boldsymbol{\xi}),
\end{aligned} \tag{89}$$

where we have set $\tilde{\rho}(\boldsymbol{\xi}) := \rho_2(e^{\boldsymbol{\xi}})$, and $\boldsymbol{\delta}_{\boldsymbol{\beta}}$ is as in (84).

The right hand side of (88) is an oscillatory integral with phase Γ_x and amplitude $\tilde{\mathcal{B}}_{x,k}$. Our next goal is to prove that we may reduce to a compact domain of integration without altering the asymptotics.

We have $x = g\,\exp^{\tilde{G}}(\imath\,\boldsymbol{\eta})$, for some $\boldsymbol{\eta} \in \mathfrak{g}$ of norm τ. Then by the discussion of §4.2

$$\boldsymbol{\xi}_{\tilde{G}}(x) = \boldsymbol{\xi}^{\sharp}_{\tilde{G}}(x) - \tilde{\varphi}^{\boldsymbol{\xi}}(x)\,\mathcal{R}(x) = \boldsymbol{\xi}^{\sharp}_{\tilde{G}}(x) - \kappa_e\big(\mathrm{Ad}_g(\boldsymbol{\eta}),\boldsymbol{\xi}\big)\,\mathcal{R}(x).$$

Hence,

$$\alpha_x\big(\boldsymbol{\xi}_{\tilde{G}}(x)\big) = -\kappa_e\big(\mathrm{Ad}_g(\boldsymbol{\eta}),\boldsymbol{\xi}\big) = -\mathrm{Ad}_g(\boldsymbol{\eta})_{\kappa}\big(\boldsymbol{\xi}\big).$$

On the support of $\tilde{\rho}$ we have $\mathrm{dist}_{X^{\tau}}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x\right) = O(r)$. Therefore, in view of (47), we conclude that

$$\partial_{\boldsymbol{\xi}}\psi^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x\right) = \mathrm{Ad}_g(\boldsymbol{\eta})_{\kappa} + O(r) \quad \text{if} \quad x = g\,\exp^{\tilde{G}}(\imath\,\boldsymbol{\eta}). \tag{90}$$

In view of (89), on the domain of integration of (88) we have

$$\partial_{\boldsymbol{\xi}}\Gamma_x(u,\boldsymbol{\beta},\boldsymbol{\xi}) = u\left[\mathrm{Ad}_g(\boldsymbol{\eta})_{\kappa} + O(r)\right] - \boldsymbol{\beta}. \tag{91}$$

Here $\boldsymbol{\beta} = \mathrm{Coad}_h(\boldsymbol{\lambda})$ for some $h \in G$, and we may assume that $0 < r \ll \|\boldsymbol{\lambda}\|$. We conclude that for some $\epsilon_0 > 0$ we have $\|\partial_{\boldsymbol{\xi}}\Gamma_x(u,\boldsymbol{\beta},\boldsymbol{\xi})\| \geq \epsilon_0$ if $0 < u \ll 1$ and $\|\partial_{\boldsymbol{\xi}}\Gamma_x(u,\boldsymbol{\beta},\boldsymbol{\xi})\| \geq u\,\epsilon_0$ if $u \gg 1$. Using a standard argument based on 'integration by parts' in the compactly supported variable $\boldsymbol{\xi}$, we conclude the following.

Lemma 5.10 *Suppose $D \gg 0$ and let $\rho_3 \in \mathcal{C}^{\infty}_0(1/(2\,D),2\,D)$ such that $\rho_3 \equiv 1$ on $(1/D,D)$. Then the asymptotics of (88) are unchanged, if the amplitude in (89) is replaced by*

$$\mathcal{B}_{x,k}(u,\boldsymbol{\beta},\boldsymbol{\xi}) := \tilde{\mathcal{B}}_{x,k}(u,\boldsymbol{\beta},\boldsymbol{\xi})\,\rho_3(u).$$

Let us define, for $k \gg 0$,

$$\tilde{\rho}_k(\boldsymbol{\xi}) := \tilde{\rho}\left(k^{\frac{1}{2}-\epsilon}\,\boldsymbol{\xi}\right). \tag{92}$$

Thus $\tilde{\rho}_k$ is supported where $\|\boldsymbol{\xi}\| \le 2\,r\,k^{\epsilon-\frac{1}{2}}$, and identically $\equiv 1$ where $\|\boldsymbol{\xi}\| \le r\,k^{\epsilon-\frac{1}{2}}$. We conclude from (88) that

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x) \sim \Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x)_1 + \Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x)_2,$$

where $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x)_1$ (respectively, $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x)_2$) is as in (88), except that $\mathcal{B}_{x,k}$ in (89) has been multiplied by $\tilde{\rho}_k(\boldsymbol{\xi})$ (respectively, by $1-\tilde{\rho}_k(\boldsymbol{\xi})$).

Lemma 5.11 $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x)_2 = O\left(k^{-\infty}\right)$ *as* $k \to +\infty$.

In the following argument, C^{τ}_j will denote suitable positive constants (uniform on $X^{\tau}_{\mathcal{O}}$ for $\boldsymbol{\xi}$ small).

Proof In view of (48) and (89),

$$\begin{aligned}|\partial_u \Gamma_x(u,\boldsymbol{\beta},\boldsymbol{\xi})| &= \left|\psi^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x\right)\right| \ge \Im\left(\psi^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x\right)\right)\\ &\ge C^{\tau}\,\mathrm{dist}_{X^{\tau}}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}}}(x),x\right)^2 \ge C^{\tau}_1\,\|\boldsymbol{\xi}\|^2.\end{aligned}$$

Hence, on the support of $1-\tilde{\rho}_k(\boldsymbol{\xi})$ we have

$$|\partial_u \Gamma_x(u,\boldsymbol{\beta},\boldsymbol{\xi})| \ge C^{\tau}_2\,k^{2\epsilon-1}.$$

Thus, by iteratively integrating by parts in $\mathrm{d}u$ (which is now legitimate by Lemma 5.10) we introduce at each step a factor which is bounded by $C^{\tau}_3\,k^{-2\epsilon}$. The claim follows. □

We conclude that

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x) \sim \Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x)_1. \tag{93}$$

Let us perform the rescaling $\boldsymbol{\xi} \mapsto k^{-1/2}\,\boldsymbol{\xi}$. We can rewrite (88) as follows:

$$\begin{aligned}\Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x) &\sim k^{1-\frac{d_G}{2}}\,d_{k\boldsymbol{\lambda}}\left(\frac{k}{2\pi}\right)^{n_G}\int_{\mathfrak{g}}\mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi})\int_{\mathcal{O}_{\boldsymbol{\lambda}}}\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta})\int_0^{+\infty}\mathrm{d}u\\ &\left[e^{\imath\,k\,\Gamma_x(u,\boldsymbol{\beta},\boldsymbol{\xi}/\sqrt{k})}\,\mathcal{B}_{x,k}\left(u,\boldsymbol{\beta},\frac{\boldsymbol{\xi}}{\sqrt{k}}\right)\,\tilde{\rho}\left(k^{-\epsilon}\,\boldsymbol{\xi}\right)\right];\end{aligned} \tag{94}$$

integration in $\boldsymbol{\xi}$ is now over an expanding ball of radius $O\left(k^{\epsilon}\right)$, and integration in u is over $(1/D, D)$.

Let us fix a system of NHLC's at $x = g\,\exp^{\tilde{G}}(\imath\,\boldsymbol{\eta})$. By the discussion in §2.5 of [18], we have

$$\begin{aligned} &\mu^{\tau}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x) \\ &= x + \left(\frac{1}{\sqrt{k}}\,\kappa_e\big(\mathrm{Ad}_g(\boldsymbol{\eta}),\boldsymbol{\xi}\big) + R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}\right), -\boldsymbol{\xi}^{\sharp}_{\tilde{G}}(x) + \mathbf{R}_2\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}\right)\right). \end{aligned} \tag{95}$$

In view of Proposition 48 of [34], we obtain

$$\begin{aligned} &\imath\,k\,\psi^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x), x\right) \\ &= \imath\,\sqrt{k}\cdot\kappa_e\big(\mathrm{Ad}_g(\boldsymbol{\eta}),\boldsymbol{\xi}\big) - \frac{1}{4\,\tau^2}\,\kappa_e\big(\mathrm{Ad}_g(\boldsymbol{\eta}),\boldsymbol{\xi}\big)^2 - \frac{1}{2}\,\|\boldsymbol{\xi}^{\sharp}_{\tilde{G}}(x)\|^2 + k\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}\right). \end{aligned} \tag{96}$$

In particular,

$$\Re\left(\imath\,k\,\psi^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x), x\right)\right) \le -C'^{\tau}\,\|\boldsymbol{\xi}\|^2_{\kappa_e} \tag{97}$$

for some constant $C'^{\tau} > 0$.

We can then rewrite (94) in the following form:

$$\begin{aligned} \Pi^{\tau}_{k\boldsymbol{\lambda}}(x,x) \sim k^{1-\frac{d_G}{2}}\,d_{k\boldsymbol{\lambda}}\left(\frac{k}{2\,\pi}\right)^{n_G}\int_{\mathfrak{g}} \mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi})\int_{\mathcal{O}_{\boldsymbol{\lambda}}}\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta})\int_0^{+\infty}\mathrm{d}u \\ \left[e^{\imath\,\sqrt{k}\,\Upsilon_x(u,\boldsymbol{\beta},\boldsymbol{\xi})}\,\mathcal{H}_{x,k}\,(u,\boldsymbol{\beta},\boldsymbol{\xi})\right], \end{aligned} \tag{98}$$

where

$$\begin{aligned} \Upsilon_x(u,\boldsymbol{\beta},\boldsymbol{\xi}) &:= u\,\kappa_e\big(\mathrm{Ad}_g(\boldsymbol{\eta}),\boldsymbol{\xi}\big) - \boldsymbol{\beta}(\boldsymbol{\xi}) = \left(u\,\tilde{\Phi}^{\tau}(x) - \boldsymbol{\beta}\right)(\boldsymbol{\xi}), \\ \mathcal{H}_{x,k}\,(u,\boldsymbol{\beta},\boldsymbol{\xi}) &:= e^{-\frac{u}{4\,\tau^2}\,\kappa_e\left(\mathrm{Ad}_g(\boldsymbol{\eta}),\boldsymbol{\xi}\right)^2 - \frac{u}{2}\,\|\boldsymbol{\xi}^{\sharp}_{\tilde{G}}(x)\|^2 + k\,u\,R_3\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}\right)} \\ &\quad\cdot\mathcal{B}_{x,k}\left(u,\boldsymbol{\beta},\frac{\boldsymbol{\xi}}{\sqrt{k}}\right)\,\tilde{\rho}\left(k^{-\epsilon}\,\boldsymbol{\xi}\right). \end{aligned} \tag{99}$$

Under the hypothesis of the Theorem, $\mathrm{dist}_{X^{\tau}}(x, X^{\tau}_{\mathcal{O}}) \ge C\,k^{\epsilon-\frac{1}{2}}$, and therefore for some constant $C' > 0$

$$\left\|\tilde{\Phi}^{\tau}(x) - \boldsymbol{\beta}\right\|_{\kappa^{\vee}} = \|\mathrm{Coad}_g(\boldsymbol{\eta}) - \boldsymbol{\beta}\|_{\kappa^{\vee}} \ge C'\,k^{\epsilon-\frac{1}{2}}$$

for any $\boldsymbol{\beta} \in \mathcal{O}^{\tau}$ and $g \in G$. It follows that (perhaps changing $C' > 0$) we also have

$$\|u\,\mathrm{Coad}_g(\boldsymbol{\eta}) - \boldsymbol{\beta}\| \ge C'\,k^{\epsilon-\frac{1}{2}} \quad \text{for any} \quad \boldsymbol{\beta} \in \mathcal{O}_{\boldsymbol{\lambda}},\ u \in (1/D, D),\ g \in G. \tag{100}$$

In other words, by (99) we have

$$\left\|\partial_{\boldsymbol{\xi}}\Upsilon_x(u,\boldsymbol{\beta},\boldsymbol{\xi})\right\| \ge C'\,k^{\epsilon-\frac{1}{2}}. \tag{101}$$

Hence, by a standard argument, iteratively integrating by parts in $\boldsymbol{\xi}$ we introduce at each step a factor which bounded by $C''\,k^{-\epsilon+\frac{1}{2}}/\sqrt{k} = C''\,k^{-\epsilon}$. Then claim follows.□

Proof of Theorem 1.6 To see why Theorem 1.6 follows from Theorem 5.9, recall that in view of the results in [9] and (3) one has an *a priori* uniform bound $\Pi^{\tau}_{k\boldsymbol{\lambda}}(y, y) \leq C^{\tau}_{\boldsymbol{\lambda}}\,k^{N}$ $(y \in X^{\tau})$ for some N ($N = 2d - 2$ will do). On the other hand,

$$\left|\Pi^{\tau}_{k\boldsymbol{\lambda}}(x, y)\right| \leq \Pi^{\tau}_{k\boldsymbol{\lambda}}(x, x)^{\frac{1}{2}}\,\Pi^{\tau}_{k\boldsymbol{\lambda}}(y, y)^{\frac{1}{2}}.$$

Therefore, since , $\Pi^{\tau}_{k\boldsymbol{\lambda}}(y, y)$ has at most polynomial growth and $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x, x) = O\left(k^{-\infty}\right)$, we also have $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x, y) = O\left(k^{-\infty}\right)$ for $k \to +\infty$. The argument for $P^{\tau}_{k\boldsymbol{\lambda}}$ is similar. □

5.4 Proof of Theorem 1.9

5.4.1 Preliminaries

If $x \in X^{\tau}$ and $h \in G$, composing a system of NHLC's at $x \in X^{\tau}$ with μ^{τ}_{h} yields a system of NHLC's at $\mu^{\tau}_{h}(x) \in X^{\tau}$. Furthermore, since α^{τ}, $\hat{\kappa}$, and $X^{\tau}_{\mathcal{O}}$ are G-invariant, μ_h preserves the relevant local decompositions of $T_x X^{\tau}$ and $\mathcal{H}_x$ (see (18), (20), (32)).

On the other hand, for any $(x, y) \in X^{\tau}_{\mathcal{O}}$ and $h \in G$, we have

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}\left(\mu^{\tau}_{h}(x), \mu^{\tau}_{h}(y)\right) = \Pi^{\tau}_{k\boldsymbol{\lambda}}(x, y).$$

This remark allows for the following preliminary reduction. Suppose $x = g \cdot \exp^{\tilde{G}}(\imath\,\boldsymbol{\eta}) \in X^{\tau}_{\mathcal{O}}$, so that $\Phi^{\tau}(x) = \mathrm{Ad}_g(\boldsymbol{\eta})_{\kappa} \in \mathcal{O}^{\tau}$ (§4.1). Perhaps replacing x with $\mu^{\tau}_{h}(x) = h\,g\cdot\exp^{\tilde{G}}(\imath\,\boldsymbol{\eta})$ for a suitable $h \in G$, we may assume without loss of generality that $\Phi^{\tau}(x) = a^{\tau}\,\boldsymbol{\lambda}$, where $a^{\tau} = \tau/\|\boldsymbol{\lambda}\|_{\kappa_e^{\vee}}$.

On the other hand, if $\Phi^{\tau}(x) = a^{\tau}\,\boldsymbol{\lambda}$ then by Lemma 4.11 for any $\boldsymbol{\xi} \in \mathfrak{g}$ we have

$$\boldsymbol{\xi} = \boldsymbol{\xi}^{\parallel} + \boldsymbol{\xi}^{\perp}, \quad \text{where} \quad \boldsymbol{\xi}^{\parallel} \in \mathrm{span}(\boldsymbol{\lambda}), \quad \boldsymbol{\xi}^{\perp} \in \mathrm{span}(\boldsymbol{\lambda}^{\kappa})^{\perp_{\kappa_e}} = \mathrm{span}(\boldsymbol{\lambda}^{\kappa})^{\perp_{\kappa_x}}.$$

Thus

$$\boldsymbol{\xi}^{\parallel} = b_{\boldsymbol{\xi}}\,\boldsymbol{\lambda}^{\kappa}_{u} \quad \text{where} \quad \boldsymbol{\lambda}^{\kappa}_{u} := \frac{1}{\|\boldsymbol{\lambda}^{\kappa}\|_{\kappa_e}}\,\boldsymbol{\lambda}^{\kappa}, \quad b_{\boldsymbol{\xi}} := \kappa_e\left(\boldsymbol{\xi}, \boldsymbol{\lambda}^{\kappa}_{u}\right). \tag{102}$$

Recalling the discussion in §4.5 (especially (44)),

$$\boldsymbol{\xi}^{\parallel}_{\tilde{G}}(x) \in \mathrm{span}\left(\mathcal{R}^{\tau}(x)\right), \quad \boldsymbol{\xi}^{\perp}_{\tilde{G}}(x) = \boldsymbol{\xi}_{\tilde{G}}(x)^{\sharp} \in \mathcal{H}^{\tau}(x). \tag{103}$$

We can further decompose $\boldsymbol{\xi}^{\perp}$ according to the two alternative direct sums in Definition 4.12. Namely, we have the κ_e-orthogonal direct sum

$$\boldsymbol{\xi}^{\perp} = \boldsymbol{\xi}' + \boldsymbol{\xi}'', \quad \text{where} \quad \boldsymbol{\xi}' \in \mathfrak{t}'_{\boldsymbol{\lambda}},\ \boldsymbol{\xi}'' \in \mathfrak{r}_x = \mathfrak{t}_{\boldsymbol{\lambda}}^{\perp_{\kappa_e}} = \mathfrak{t}^{\perp_{\kappa_e}}, \tag{104}$$

where we use that $\boldsymbol{\lambda}$ is a regular weight, and the σ_x-orthogonal one

$$\boldsymbol{\xi}^{\perp} = \boldsymbol{\xi}_{\mathfrak{t}} + \boldsymbol{\xi}_{\mathfrak{s}}, \quad \text{where} \quad \boldsymbol{\xi}_{\mathfrak{t}} \in \mathfrak{t}'_{\boldsymbol{\lambda}},\ \boldsymbol{\xi}_{\mathfrak{s}} \in \mathfrak{s}_x. \tag{105}$$

Remark 5.12 If $\boldsymbol{\xi}'' = \mathbf{0}$, then $\boldsymbol{\xi}^{\perp} = \boldsymbol{\xi}' = \boldsymbol{\xi}_{\mathfrak{t}} \in \mathfrak{t}'_{\boldsymbol{\lambda}}$, hence $\boldsymbol{\xi}_{\mathfrak{s}} = \mathbf{0}$ (and conversely).

We aim to study the asymptotics of $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k})$, where x_{jk} is as in (22). In view of Lemma 4.5 and Remark 4.14, there exist unique $\boldsymbol{\rho}_j \in \mathfrak{t}'_{\boldsymbol{\lambda}}$, $\boldsymbol{\eta}_j \in \tilde{\mathfrak{s}}_x$ such that

$$\mathbf{n}_j = J_x\big(\boldsymbol{\rho}_{j\,\tilde{G}}(x)\big), \quad \mathbf{s}_j = \boldsymbol{\eta}_{j\,\tilde{G}}(x). \tag{106}$$

5.4.2 Proof of Theorem 1.9

In the following, we assume that $\Phi^{\tau}(x) = a^{\tau}\,\boldsymbol{\lambda}$ and fix NHLC's on X^{τ} at x; In particular, for any $\boldsymbol{\xi} \in \mathfrak{g}$ we have

$$\varphi^{\boldsymbol{\xi}}(x) = \left\langle \tilde{\Phi}^{\tau}(x), \boldsymbol{\xi} \right\rangle = a^{\tau}\ \langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle = a^{\tau}\,\kappa_e\left(\boldsymbol{\lambda}^{\kappa}, \boldsymbol{\xi}\right). \tag{107}$$

The definition of x_{jk} is in (22). Thus $\mathbf{n}_j \in \mathbb{R}^{r_{\mathcal{O}}-1}_N \cong N(X^{\tau}_{\mathcal{O}}/X^{\tau})_x$ and, by Lemma 4.5 and (107), $\mathbf{n}_j = J_x\big(\boldsymbol{\rho}_{j\,\tilde{G}}(x)\big)$ for a unique $\boldsymbol{\rho}_j \in \mathfrak{t}'_x = \mathfrak{t}'_{\boldsymbol{\lambda}}$. Similarly, $\mathbf{s}_j \in \mathbb{C}^{d-r_{\mathcal{O}}}_{\mathcal{S}} \cong \mathcal{S}_x$; by Remark 4.14, there exists a unique $\boldsymbol{\eta}_j \in \tilde{\mathfrak{s}}_x = \mathfrak{s}_x \oplus \imath\,\mathfrak{s}_x$ such that $\mathbf{s}_j = \boldsymbol{\eta}_{j\,\tilde{G}}(x)$.

Proof By a minor modification of the arguments leading to (94) (perhaps with a different choice of the radius $r > 0$ involved in the definition of ρ_2 in (80), hence of $\tilde{\rho}$ in (89) and $\tilde{\rho}_k$ in (92)), one obtains

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k}) \sim k^{1-\frac{d_G}{2}}\, d_{k\boldsymbol{\lambda}} \left(\frac{k}{2\,\pi}\right)^{n_G} \int_{\mathfrak{g}} \mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi}) \int_{\mathcal{O}_{\boldsymbol{\lambda}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}) \int_0^{+\infty} \mathrm{d}u \left[e^{\imath\,k\,\Lambda_{x,k}(u,\boldsymbol{\beta},\boldsymbol{\xi}/\sqrt{k})}\,\mathcal{D}_{x,k}\,(u, \boldsymbol{\beta}, \boldsymbol{\xi})\right], \tag{108}$$

where

$$\Lambda_{x,k}\,(u, \boldsymbol{\beta}, \boldsymbol{\xi}) := u\,\psi^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x_{1k}), x_{2k}\right) - \frac{1}{\sqrt{k}}\,\boldsymbol{\beta}(\boldsymbol{\xi}) \tag{109}$$

$$\mathcal{D}_{x,k}\,(u, \boldsymbol{\beta}, \boldsymbol{\xi}) := e^{-\delta_{\boldsymbol{\beta}}(\boldsymbol{\xi})/\sqrt{k}}\, s^{\tau}\left(\mu^{\tau}_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x_{1k}), x_{2k}, k\,u\right) \cdot \mathcal{V}_k(\boldsymbol{\beta})\,\mathcal{P}\left(\frac{\boldsymbol{\xi}}{\sqrt{k}}\right)\,\tilde{\rho}\left(k^{-\epsilon}\,\boldsymbol{\xi}\right).$$

In order to expand $\Lambda_{x,k}(u, \boldsymbol{\beta}, \boldsymbol{\xi})$, consider first $\mu^\tau_{e^{-\xi/\sqrt{k}}}(x_{1k})$. In view of Lemma 64 of [18], we have the following refinement of (95):

$$\begin{aligned}
&\mu^\tau_{e^{-\xi/\sqrt{k}}}(x_{1k}) \\
&= x + \left(\frac{1}{\sqrt{k}} \left(\theta_1 + a^\tau \, \langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle\right) + \frac{1}{k}\, \omega_x \left(\boldsymbol{\xi}_{\tilde{G}}(x)^\sharp, \mathbf{n}_j + \mathbf{s}_j\right) + \mathbf{R}_3 \left(\frac{\bullet}{\sqrt{k}}\right), \right. \\
&\quad \left. \frac{1}{\sqrt{k}} \left(\mathbf{n}_1 + \mathbf{s}_1 - \boldsymbol{\xi}_{\tilde{G}}(x)^\sharp\right) + R_2 \left(\frac{\bullet}{\sqrt{k}}\right)\right),
\end{aligned} \tag{110}$$

where '$\bullet$' collectively denotes the variables involved. In view of the initial discussion in §5.4.1 and (102),

$$a^\tau \, \langle \boldsymbol{\lambda}, \boldsymbol{\xi} \rangle = \frac{\tau}{\|\boldsymbol{\lambda}^\kappa\|_{\kappa_e}} \kappa_e \left(\boldsymbol{\lambda}^\kappa, \boldsymbol{\xi}\right) = \tau \, \kappa_e \left(\boldsymbol{\lambda}^\kappa_u, \boldsymbol{\xi}\right) = \tau \, b_{\boldsymbol{\xi}}. \tag{111}$$

By (103) and (105), we have

$$\begin{aligned}
&\omega_x \left(\boldsymbol{\xi}_{\tilde{G}}(x)^\sharp, \mathbf{n}_1 + \mathbf{s}_1\right) \\
&= \omega_x \Big(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x) + \boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x), \, J_x\big(\boldsymbol{\rho}_{1\tilde{G}}(x)\big) + \boldsymbol{\eta}_{1\tilde{G}}(x)\Big) \\
&= \omega_x \Big(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x), \, J_x\big(\boldsymbol{\rho}_{1\tilde{G}}(x)\big)\Big) + \omega_x \Big(\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x), \boldsymbol{\eta}_{1\tilde{G}}(x)\Big) \\
&= \tilde{\kappa}_x \Big(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x), \boldsymbol{\rho}_{1\tilde{G}}(x)\Big) + \omega_x \Big(\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x), \boldsymbol{\eta}_{1\tilde{G}}(x)\Big).
\end{aligned} \tag{112}$$

Given Proposition 48 of [34], we obtain

$$\begin{aligned}
&\imath \, \psi^\tau \left(\mu^\tau_{e^{-\xi/\sqrt{k}}}(x_{1k}), x_{2k}\right) \\
&= \frac{\imath}{\sqrt{k}} \left(\theta_1 - \theta_2 + \tau \, b_{\boldsymbol{\xi}}\right) + \frac{\imath}{k} \left[\tilde{\kappa}_x \Big(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x), \boldsymbol{\rho}_{1\tilde{G}}(x)\Big) + \omega_x \Big(\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x), \boldsymbol{\eta}_{1\tilde{G}}(x)\Big)\right] \\
&\quad - \frac{1}{4\,\tau^2\, k} \left(\theta_1 - \theta_2 + \tau \, b_{\boldsymbol{\xi}}\right)^2 \\
&\quad + \frac{1}{k}\, \psi_2^{\omega_x} \Big(J_x\big(\boldsymbol{\rho}_{1\tilde{G}}(x)\big) + \boldsymbol{\eta}_{1\tilde{G}}(x) - \boldsymbol{\xi}_{\tilde{G}}(x)^\sharp, \, J_x\big(\boldsymbol{\rho}_{2\tilde{G}}(x)\big) + \boldsymbol{\eta}_{2\tilde{G}}(x)\Big) + R_3 \left(\frac{\bullet}{\sqrt{k}}\right).
\end{aligned} \tag{113}$$

We have

$$\begin{aligned}
&\psi_2^{\omega_x} \Big(J_x\big(\boldsymbol{\rho}_{1\tilde{G}}(x)\big) + \boldsymbol{\eta}_{1\tilde{G}}(x) - \boldsymbol{\xi}_{\tilde{G}}(x)^\sharp, \, J_x\big(\boldsymbol{\rho}_{2\tilde{G}}(x)\big) + \boldsymbol{\eta}_{2\tilde{G}}(x)\Big) \\
&= -\imath \, \omega_x \Big(J_x\big(\boldsymbol{\rho}_{1\tilde{G}}(x)\big) + \boldsymbol{\eta}_{1\tilde{G}}(x) - \boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x) - \boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x), \, J_x\big(\boldsymbol{\rho}_{2\tilde{G}}(x)\big) + \boldsymbol{\eta}_{2\tilde{G}}(x)\Big) \\
&\quad - \frac{1}{2} \left\| J_x\big(\boldsymbol{\rho}_{1\tilde{G}}(x) - \boldsymbol{\rho}_{2\tilde{G}}(x)\big) + \big(\boldsymbol{\eta}_{1\tilde{G}}(x) - \boldsymbol{\eta}_{2\tilde{G}}(x) - \boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x)\big) - \boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x)\right\|^2.
\end{aligned} \tag{114}$$

Furthermore,

$$\begin{aligned}&\omega_x\Big(J_x\big(\boldsymbol{\rho}_{1\tilde{G}}(x)\big)+\boldsymbol{\eta}_{1\tilde{G}}(x)-\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x)-\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x),\,J_x\big(\boldsymbol{\rho}_{2\tilde{G}}(x)\big)+\boldsymbol{\eta}_{2\tilde{G}}(x)\Big)\\&=\omega_x\Big(\boldsymbol{\eta}_{1\tilde{G}}(x),\,\boldsymbol{\eta}_{2\tilde{G}}(x)\Big)-\omega_x\Big(\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x),\,\boldsymbol{\eta}_{2\tilde{G}}(x)\Big)-\tilde{\kappa}_x\Big(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x),\,\boldsymbol{\rho}_{2\tilde{G}}(x)\Big),\end{aligned}$$

$$\begin{aligned}&\Big\|J_x\big(\boldsymbol{\rho}_{1\tilde{G}}(x)-\boldsymbol{\rho}_{2\tilde{G}}(x)\big)+\big(\boldsymbol{\eta}_{1\tilde{G}}(x)-\boldsymbol{\eta}_{2\tilde{G}}(x)-\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x)\big)-\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x)\Big\|^2\\&=\Big\|\boldsymbol{\rho}_{1\tilde{G}}(x)-\boldsymbol{\rho}_{2\tilde{G}}(x)\Big\|^2+\Big\|\boldsymbol{\eta}_{1\tilde{G}}(x)-\boldsymbol{\eta}_{2\tilde{G}}(x)-\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x)\Big\|^2+\Big\|\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x)\Big\|^2.\end{aligned}$$

Inserting this in (113)

$$\begin{aligned}&\imath\,\psi^\tau\left(\mu^\tau_{e^{-\boldsymbol{\xi}/\sqrt{k}}}(x_{1k}),\,x_{2k}\right)\\&=\frac{\imath}{\sqrt{k}}\,\big(\theta_1-\theta_2+\tau\,b_{\boldsymbol{\xi}}\big)+\frac{\imath}{k}\,\Big[\tilde{\kappa}_x\Big(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x),\,\boldsymbol{\rho}_{1\tilde{G}}(x)\Big)+\omega_x\Big(\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x),\,\boldsymbol{\eta}_{1\tilde{G}}(x)\Big)\Big]\\&\quad-\frac{1}{4\,\tau^2\,k}\,\big(\theta_1-\theta_2+\tau\,b_{\boldsymbol{\xi}}\big)^2\\&\quad+\frac{\imath}{k}\Big[-\omega_x\Big(\boldsymbol{\eta}_{1\tilde{G}}(x),\,\boldsymbol{\eta}_{2\tilde{G}}(x)\Big)+\omega_x\Big(\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x),\,\boldsymbol{\eta}_{2\tilde{G}}(x)\Big)+\tilde{\kappa}_x\Big(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x),\,\boldsymbol{\rho}_{2\tilde{G}}(x)\Big)\Big]\\&\quad-\frac{1}{2\,k}\,\Big(\Big\|\boldsymbol{\rho}_{1\tilde{G}}(x)-\boldsymbol{\rho}_{2\tilde{G}}(x)\Big\|^2+\Big\|\boldsymbol{\eta}_{1\tilde{G}}(x)-\boldsymbol{\eta}_{2\tilde{G}}(x)-\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x)\Big\|^2+\Big\|\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x)\Big\|^2\Big)\\&\quad+R_3\left(\frac{\bullet}{\sqrt{k}}\right)\\&=\frac{\imath}{\sqrt{k}}\,\big(\theta_1-\theta_2+\tau\,b_{\boldsymbol{\xi}}\big)\\&\quad+\frac{\imath}{k}\,\Big[\tilde{\kappa}_x\Big(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x),\,\boldsymbol{\rho}_{1\tilde{G}}(x)+\boldsymbol{\rho}_{2\tilde{G}}(x)\Big)+\omega_x\Big(\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x),\,\boldsymbol{\eta}_{1\tilde{G}}(x)+\boldsymbol{\eta}_{2\tilde{G}}(x)\Big)\Big]\\&\quad-\frac{1}{4\,\tau^2\,k}\,\big(\theta_1-\theta_2+\tau\,b_{\boldsymbol{\xi}}\big)^2-\frac{\imath}{k}\,\omega_x\Big(\boldsymbol{\eta}_{1\tilde{G}}(x),\,\boldsymbol{\eta}_{2\tilde{G}}(x)\Big)\\&\quad-\frac{1}{2\,k}\,\Big(\Big\|\boldsymbol{\rho}_{1\tilde{G}}(x)-\boldsymbol{\rho}_{2\tilde{G}}(x)\Big\|^2+\Big\|\boldsymbol{\eta}_{1\tilde{G}}(x)-\boldsymbol{\eta}_{2\tilde{G}}(x)-\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x)\Big\|^2+\Big\|\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x)\Big\|^2\Big)\\&\quad+R_3\left(\frac{\bullet}{\sqrt{k}}\right).\end{aligned}\tag{115}$$

In view of (109), we obtain

$$\imath\,k\,\Lambda_{x,k}\,(u,\boldsymbol{\beta},\boldsymbol{\xi})=\imath\,\sqrt{k}\,\Psi(u,\boldsymbol{\xi},\boldsymbol{\beta})+u\,\Lambda(\boldsymbol{\xi}),\tag{116}$$

where

$$\Psi(u,\boldsymbol{\xi},\boldsymbol{\beta})=\Psi_{\theta_1,\theta_2}(u,\boldsymbol{\xi},\boldsymbol{\beta}):=u\,\left(\theta_1-\theta_2+\tau\,b_{\boldsymbol{\xi}}\right)-\boldsymbol{\beta}(\boldsymbol{\xi}),\tag{117}$$

$$\Lambda(\boldsymbol{\xi}) = \Lambda_2(\boldsymbol{\xi}) + k\, R_3\left(\frac{\bullet}{\sqrt{k}}\right), \tag{118}$$

with

$$\begin{aligned}
&\Lambda_2(\boldsymbol{\xi}) = \Lambda_2(\theta_j, \boldsymbol{\eta}_j, \boldsymbol{\rho}_j; \boldsymbol{\xi}) \\
&:= \imath\, \left[\tilde{\kappa}_x\left(\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x), \boldsymbol{\rho}_{1\tilde{G}}(x) + \boldsymbol{\rho}_{2\tilde{G}}(x)\right) + \omega_x\left(\boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x), \boldsymbol{\eta}_{1\tilde{G}}(x) + \boldsymbol{\eta}_{2\tilde{G}}(x)\right)\right] \\
&\quad -\frac{1}{4\,\tau^2}\, \left(\theta_1 - \theta_2 + \tau\, b_{\boldsymbol{\xi}}\right)^2 - \imath\, \omega_x\left(\boldsymbol{\eta}_{1\tilde{G}}(x), \boldsymbol{\eta}_{2\tilde{G}}(x)\right) \\
&\quad -\frac{1}{2}\, \left(\left\|\boldsymbol{\rho}_{1\tilde{G}}(x) - \boldsymbol{\rho}_{2\tilde{G}}(x)\right\|^2 + \left\|\boldsymbol{\eta}_{1\tilde{G}}(x) - \boldsymbol{\eta}_{2\tilde{G}}(x) - \boldsymbol{\xi}_{\mathfrak{s}\tilde{G}}(x)\right\|^2 + \left\|\boldsymbol{\xi}_{\mathfrak{t}\tilde{G}}(x)\right\|^2\right);
\end{aligned} \tag{119}$$

dependence of Ψ and Λ on $(\theta_j, \boldsymbol{\eta}_j, \boldsymbol{\rho}_j)$ will be left implicit in the following.

In view of (102) and (117),

$$\partial_{\boldsymbol{\xi}} \Psi = u\, \tau\, \frac{\boldsymbol{\lambda}}{\|\boldsymbol{\lambda}\|} - \boldsymbol{\beta}. \tag{120}$$

Let $\varrho_2 : \mathcal{O}_{\boldsymbol{\lambda}} \to \mathbb{R}$ be $\mathcal{C}^\infty$ function which is supported in a suitably small neighborhood of $\boldsymbol{\lambda}$ and identically equal to 1 near $\boldsymbol{\lambda}$. Since integration in $\boldsymbol{\xi}$ is compactly supported, we conclude from (108), (116), and (120) that the following holds.

Lemma 5.13 *The asymptotics of (108) are unaltered, if the integrand is multiplied by $\varrho_2(\boldsymbol{\beta})$.*

Furthermore, recall from §5.3.1 that $\mathrm{d}^H V_{\mathfrak{g}}$ is the Lebesgue measure on $\mathfrak{g}$ associated to κ^H. Its relation to the Lebesgue measure $\mathrm{d}^\kappa V_{\mathfrak{g}}$ associated to κ is given by

$$\mathrm{d}^H V_{\mathfrak{g}}(\boldsymbol{\xi}) = \frac{1}{\mathrm{vol}^\kappa(G)}\, \mathrm{d}^\kappa V_{\mathfrak{g}}(\boldsymbol{\xi}).$$

Furthermore, (86) with $\boldsymbol{\xi} = \mathbf{0}$ yields

$$\begin{aligned}
d_{k\boldsymbol{\lambda}} &= \left(\frac{k}{2\,\pi}\right)^{n_G} \int_{\mathcal{O}_{\boldsymbol{\lambda}}} \mathcal{V}_k(\boldsymbol{\beta})\, \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}) \\
&= \left(\frac{k}{2\,\pi}\right)^{n_G}\, \mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})\, \left[1 + O\left(k^{-1}\right)\right],
\end{aligned} \tag{121}$$

the previous expression being a polynomial in k. Summing up, we can rewrite (108) in the following form:

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k}) \sim k^{1-d_G/2}\, \left(\frac{k}{2\,\pi}\right)^{2\,n_G}\, \frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})}{\mathrm{vol}^\kappa(G)} \int_{\mathfrak{g}} \mathrm{d}^\kappa V_{\mathfrak{g}}(\boldsymbol{\xi}) \int_{\mathcal{O}_{\boldsymbol{\lambda}}} \mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}}(\boldsymbol{\beta}) \int_0^{+\infty} \mathrm{d}u$$

$$\left[e^{\imath\sqrt{k}\,\Psi(u,\boldsymbol{\xi},\boldsymbol{\beta})}\,e^{u\,\Lambda(\boldsymbol{\xi})}\,\rho_2(\boldsymbol{\beta})\,\mathcal{D}_{x,k}\,(u,\boldsymbol{\beta},\boldsymbol{\xi})\right]. \tag{122}$$

In (122), integration in $\boldsymbol{\beta}$ has been restricted to a small neighbourhood of $\boldsymbol{\lambda}$. To proceed, we need a parametrization of $\mathcal{O}_{\boldsymbol{\lambda}}$ near $\boldsymbol{\lambda}$. Consider the smooth map

$$f:\boldsymbol{\gamma}\in\mathfrak{t}^{\perp_{\kappa_e}}\mapsto e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\lambda}:=\mathrm{Coad}_{e^{\boldsymbol{\gamma}}}(\boldsymbol{\lambda})\in\mathcal{O}_{\boldsymbol{\lambda}}. \tag{123}$$

Lemma 5.14 *Let $\mathrm{d}^{\kappa}\boldsymbol{\gamma}$ denote the Lebesgue measure on $\mathfrak{t}^{\perp_{\kappa_e}}$ associated to the restriction of κ_e. Then*

$$f^*(\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}})=\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})\,\frac{\mathrm{vol}^{\kappa}(T)}{\mathrm{vol}^{\kappa}(G)}\,\mathcal{R}(\boldsymbol{\gamma})\,\mathrm{d}^{\kappa}\boldsymbol{\gamma},$$

where $\mathcal{R}\in\mathcal{C}^{\infty}(\mathfrak{t}^{\perp_{\kappa_e}})$ and $\mathcal{R}(0)=1$. Furthermore, on a suitably small open neighbourhood V of $0\in\mathfrak{t}^{\perp_{\kappa_e}}$, f restricts to a diffeomorphism onto its image, which is an open neighbourhood of $\boldsymbol{\lambda}$ in $\mathcal{O}_{\boldsymbol{\lambda}}$.

The proof was given in the discussion following Lemma 7.4 in [33] (with some notational differences), but we reproduce it here for the reader's convenience.

Let $\mathrm{d}^{\kappa}V_{G/T}$ be the volume density induced by κ on G/T. Similarly, let $\mathrm{vol}^{H}(G/T)$ be the Haar volume density. Then

$$\mathrm{vol}^{\kappa}(G/T)=\frac{\mathrm{vol}^{\kappa}(G)}{\mathrm{vol}^{\kappa}(T)},\quad \mathrm{d}^{\kappa}V_{G/T}=\mathrm{vol}^{\kappa}(G/T)\,\mathrm{d}^{H}V_{G/T}$$

Proof If the first statement holds, then f is a local diffeomorphism at the origin, and the second statement follows. To verify the former statement, let $f':G/T\to\mathcal{O}_{\boldsymbol{\lambda}}$ be given by

$$f'(g\,T):=g\cdot\boldsymbol{\lambda}:=\mathrm{Coad}_g(\boldsymbol{\lambda}).$$

Then f' is an equivariant diffeomorphism and

$$f'^*(\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}})=\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})\,\mathrm{d}V^{H}_{G/H}=\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})}{\mathrm{vol}^{\kappa}(G/T)}\,\mathrm{d}^{\kappa}V_{G/H}=\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})\,\mathrm{vol}^{\kappa}(T)}{\mathrm{vol}^{\kappa}(G)}\,\mathrm{d}V^{\kappa}_{G/H}.$$

Furthermore, define $f'':\mathfrak{t}^{\perp_{\kappa}}\to G/T$ by $f''(\boldsymbol{\gamma}):=e^{\boldsymbol{\gamma}}\,T$. Then

$$f''^*(\mathrm{d}^{\kappa}V_{G/T})=\mathcal{R}(\boldsymbol{\gamma})\,\mathrm{d}^{\kappa}\boldsymbol{\gamma},\quad\text{where}\quad\mathcal{R}(0)=1.$$

Furthermore, $f=f'\circ f''$. Thus

$$\begin{aligned}f^*(\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}})&=f''^*\left(f'^*(\mathrm{d}V_{\mathcal{O}_{\boldsymbol{\lambda}}})\right)\\&=\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})\,\mathrm{vol}^{\kappa}(T)}{\mathrm{vol}^{\kappa}(G)}\cdot f''^*\left(\mathrm{d}V^{\kappa}_{G/H}\right)=\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})\,\mathrm{vol}^{\kappa}(T)}{\mathrm{vol}^{\kappa}(G)}\cdot\mathcal{R}(\boldsymbol{\gamma})\,\mathrm{d}^{\kappa}\boldsymbol{\gamma}.\end{aligned}$$

□

In the following, $V \subset \mathfrak{t}^{\perp_\kappa}$ will denote a neighbourhood of the origin such that f induces a diffeomorphism $V \cong f(V)$. We may assume without loss that $\mathrm{supp}(\rho_2) \subset V$. Hence (122) may be rewritten as follows:

$$\Pi^\tau_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k}) \sim k^{1-d_G/2} \left(\frac{k}{2\pi}\right)^{2n_G} \left(\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})}{\mathrm{vol}^\kappa(G)}\right)^2 \cdot \mathrm{vol}^\kappa(T) \cdot \int_{\mathfrak{g}} \mathrm{d}^\kappa V_{\mathfrak{g}}(\boldsymbol{\xi}) \int_{\mathfrak{t}^{\perp_{\kappa_e}}} \mathrm{d}^\kappa \boldsymbol{\gamma} \int_0^{+\infty} \mathrm{d}u \left[e^{\imath\sqrt{k}\,\Psi(u,\boldsymbol{\xi},e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\lambda})}\, e^{u\,\Lambda(\boldsymbol{\xi})}\, \mathcal{R}(\boldsymbol{\gamma})\, \rho_2(e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\lambda})\, \mathcal{D}_{x,k}\left(u, e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\lambda}, \boldsymbol{\xi}\right)\right]. \tag{124}$$

Recalling (123), for $\boldsymbol{\gamma} \sim \mathbf{0}$ we have

$$\left(e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\lambda}\right)^\kappa = \mathrm{Ad}_{e^{\boldsymbol{\gamma}}}(\boldsymbol{\lambda}^\kappa) = \boldsymbol{\lambda}^\kappa + \left[\boldsymbol{\gamma}, \boldsymbol{\lambda}^\kappa\right] + R_2(\boldsymbol{\gamma}).$$

Hence, in view of (117),

$$\begin{aligned}\Psi\left(u, \boldsymbol{\xi}, e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\lambda}\right) &= u\,\left(\theta_1 - \theta_2 + \tau\, b_{\boldsymbol{\xi}}\right) - \kappa_e\left(\left(e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\lambda}\right)^\kappa, \boldsymbol{\xi}\right)\\ &= u\,\left(\theta_1 - \theta_2 + \tau\, b_{\boldsymbol{\xi}}\right) - \kappa_e\left(\boldsymbol{\lambda}^\kappa + \left[\boldsymbol{\gamma}, \boldsymbol{\lambda}^\kappa\right] + R_2(\boldsymbol{\gamma}), \boldsymbol{\xi}\right).\end{aligned} \tag{125}$$

Recalling (102), $\kappa_e(\boldsymbol{\lambda}^\kappa, \boldsymbol{\xi}) = \|\boldsymbol{\lambda}\|\, b_{\boldsymbol{\xi}}$. Let us decompose $\boldsymbol{\xi}$ according to (102) and (104). Then

$$-\kappa_e\left(\left[\boldsymbol{\gamma}, \boldsymbol{\lambda}^\kappa\right], \boldsymbol{\xi}\right) = \kappa_e\left(\left[\boldsymbol{\lambda}^\kappa, \boldsymbol{\gamma}\right], \boldsymbol{\xi}\right) = -\kappa_e\left(\boldsymbol{\gamma}, \left[\boldsymbol{\lambda}^\kappa, \boldsymbol{\xi}\right]\right) = -\kappa_e\left(\boldsymbol{\gamma}, \left[\boldsymbol{\lambda}^\kappa, \boldsymbol{\xi}''\right]\right).$$

Therefore, (125) may be rewritten

$$\begin{aligned}&\Psi\left(u, \boldsymbol{\xi}, e^{\boldsymbol{\gamma}}\cdot\boldsymbol{\lambda}\right)\\ &= u\,(\theta_1 - \theta_2) + (u\,\tau - \|\boldsymbol{\lambda}\|)\, b_{\boldsymbol{\xi}} - \kappa_e\left(\boldsymbol{\gamma}, \left[\boldsymbol{\lambda}^\kappa, \boldsymbol{\xi}''\right]\right) + \kappa_e\left(R_2(\boldsymbol{\gamma}), \boldsymbol{\xi}\right).\end{aligned} \tag{126}$$

Let us choose basis of $\mathfrak{t}'_{\boldsymbol{\lambda}}$ and $\mathfrak{t}^{\perp_{\kappa_e}}$ that are orthonormal with respect to κ_e, so as to unitarily identify $\mathfrak{t}'_{\boldsymbol{\lambda}} \cong \mathbb{R}^{r_G-1}$ and $\mathfrak{t}^{\perp_{\kappa_e}} \cong \mathbb{R}^{2n_G}$. Together with $\boldsymbol{\lambda}^\kappa_u$, these form an orthonormal basis of $(\mathfrak{g}, \kappa_e)$.

Accordingly, we shall replace $\boldsymbol{\xi}' \in \mathfrak{t}'_{\boldsymbol{\lambda}}$ with $\mathbf{r} \in \mathbb{R}^{r_G-1}$, $\boldsymbol{\xi}'' \in \mathfrak{t}^{\perp_{\kappa_e}}$ with $\boldsymbol{\rho} \in \mathbb{R}^{2n_G}$, and $\boldsymbol{\xi}^\parallel \in \mathrm{span}\,(\boldsymbol{\lambda}^\kappa)$ with $b = b_{\boldsymbol{\xi}}$ as in (102), and substitute

$$\int_{\mathfrak{g}} \mathrm{d}^\kappa V_{\mathfrak{g}}(\boldsymbol{\xi}) \quad \text{with} \quad \int_{\mathbb{R}^{r_G-1}} \mathrm{d}\mathbf{r} \int_{\mathbb{R}^{2n_G}} \mathrm{d}\boldsymbol{\rho} \int_{-\infty}^{+\infty} \mathrm{d}b.$$

We shall simply identify $\boldsymbol{\gamma} \in \mathfrak{t}^{\perp_{k_e}}$ with its image in $\mathbb{R}^{2n_G}$.

Let $Z_{\boldsymbol{\lambda}}$ be the $(2n_G) \times (2n_G)$ nondegenerate skew-symmetric matrix representing the endomorphism $S_{\boldsymbol{\lambda}} : \mathfrak{t}^{\kappa_e} \to \mathfrak{t}^{\kappa_e}$ in (24).

Let us denote by $\cdot_{\mathrm{st}}$ the standard scalar product (on the appropriate Euclidean space); in terms of these identifications, (126) may be rewritten

$$\begin{aligned}\Psi_{\mathbf{r}}(u, b, \boldsymbol{\rho}, \boldsymbol{\gamma}) &:= \Psi\left(u, \boldsymbol{\xi}, e^{\boldsymbol{\gamma}} \cdot \boldsymbol{\lambda}\right) \\ &= u(\theta_1 - \theta_2) + (u\,\tau - \|\boldsymbol{\lambda}\|)\, b - \boldsymbol{\gamma} \cdot_{\mathrm{st}} Z_{\boldsymbol{\lambda}}\, \boldsymbol{\rho} + R_2(\boldsymbol{\gamma}) \cdot_{\mathrm{st}} (b, \mathbf{r}, \boldsymbol{\rho}).\end{aligned} \tag{127}$$

With these replacements, (124) becomes

$$\begin{aligned}\Pi^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k}) \sim k^{1-d_G/2} \left(\frac{k}{2\pi}\right)^{2n_G} \left(\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})}{\mathrm{vol}^{\kappa}(G)}\right)^2 \cdot \mathrm{vol}^{\kappa}(T) \\ \cdot \int_{\mathbb{R}^{r_G-1}} I_{x,k}(\mathbf{r})\, \mathrm{d}\mathbf{r},\end{aligned} \tag{128}$$

where

$$\begin{aligned}I_{x,k}(\mathbf{r}) := \int_0^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}b \int_{\mathbb{R}^{2n_G}} \mathrm{d}\boldsymbol{\rho} \int_{\mathbb{R}^{2n_G}} \mathrm{d}\boldsymbol{\gamma} \\ \left[e^{\imath\sqrt{k}\,\Psi_{\mathbf{r}}(u,b,\boldsymbol{\rho},\boldsymbol{\gamma})}\, \mathcal{K}_{\mathbf{r}}(u, b, \boldsymbol{\rho}, \boldsymbol{\gamma})\right].\end{aligned} \tag{129}$$

where

$$\mathcal{K}_{\mathbf{r}}(u, b, \boldsymbol{\rho}, \boldsymbol{\gamma}) := e^{u\,\Lambda(\boldsymbol{\xi})}\, \mathcal{R}(\boldsymbol{\gamma})\, \rho_2(e^{\boldsymbol{\gamma}} \cdot \boldsymbol{\lambda})\, \mathcal{D}_{x,k}\left(u, e^{\boldsymbol{\gamma}} \cdot \boldsymbol{\lambda}, \boldsymbol{\xi}\right),$$

where $\boldsymbol{\xi}$ is expressed in terms of $(b, \mathbf{r}, \boldsymbol{\rho})$, and dependence of $\mathcal{K}_r$ on $(\theta_j, \boldsymbol{\eta}_j, \boldsymbol{\rho}_j)$ is left implicit.

In view of (109) and (119), $\mathcal{K}_{\mathbf{r}}(u, b, \boldsymbol{\rho}, \boldsymbol{\gamma})$ admits an asymptotic expansion of the form

$$\begin{aligned}\mathcal{K}_{\mathbf{r}}(u, b, \boldsymbol{\rho}, \boldsymbol{\gamma}) \sim (k\,u)^{d-1}\, e^{u\,\Lambda_2(\boldsymbol{\xi})}\, \mathcal{R}(\boldsymbol{\gamma})\, \rho_2(e^{\boldsymbol{\gamma}} \cdot \boldsymbol{\lambda})\; \tilde{\rho}\left(k^{-\epsilon}\, \boldsymbol{\xi}\right)\, s_0^{\tau}(x, x) \\ \cdot \left[1 + \sum_{j \geq 1} k^{-j/2}\, P_j(\theta_j, \boldsymbol{\eta}_j, \boldsymbol{\rho}_j; u, \boldsymbol{\xi}, \boldsymbol{\gamma})\right],\end{aligned} \tag{130}$$

where each P_j is a polynomial in the rescaled variables $(\theta_j, \boldsymbol{\eta}_j, \boldsymbol{\rho}_j, \boldsymbol{\xi})$ of degree $\leq 3\,j$ and parity j, and coefficients depending smoothly on the remaining variables (cfr. the similar proof of Theorem 1.3 of [33], or the arguments in the proof of Theorem 25 of [18]). Furthermore, $s_0^{\tau}(x, x)$ is given by (53).

Integration is supported on an expanding ball centered at the origin and of radius $O(k^{\epsilon})$ in the variables $(b, \mathbf{r}, \boldsymbol{\rho}) \in \mathbb{R} \times \mathbb{R}^{r_G-1} \times \mathbb{R}^{2n_G}$ (recall the last factor in the amplitude $\mathcal{D}_{x,k}$ in (109)), and compactly supported in $(u, \boldsymbol{\gamma}) \in \mathbb{R} \times \mathbb{R}^{2n_G}$. The expansion (130) can be integrated term by term.

Lemma 5.15 $\Psi_{\mathbf{r}}$ *has a unique critical point* P_0 *on the given domain of integration, given by*

$$P_0 = \left(u_0, b_0, \boldsymbol{\rho}_0, \boldsymbol{\gamma}_0\right) := \left(\frac{\|\boldsymbol{\lambda}\|}{\tau}, \frac{\theta_2 - \theta_1}{\tau}, \mathbf{0}, \mathbf{0}\right).$$

The Hessian matrix at the critical point is given by

$$H_{P_0}(\Psi_{\mathbf{r}}) = \begin{pmatrix} 0 & \tau & \mathbf{0}^t & \mathbf{0}^t \\ \tau & 0 & \mathbf{0}^t & \mathbf{0}^t \\ \mathbf{0} & \mathbf{0} & [0] & Z_{\boldsymbol{\lambda}} \\ \mathbf{0} & \mathbf{0}^t & -Z_{\boldsymbol{\lambda}} & \partial^2_{\boldsymbol{\lambda},\boldsymbol{\lambda}} \Psi_{\mathbf{r}}\Big|_{P_0} \end{pmatrix},$$

where $[0]$ *denotes the* $(2\,n_G) \times (2\,n_G)$ *zero matrix. Furthermore,* $H_{P_0}(\Psi_{\mathbf{r}})$ *has signature uqual to zero, and its determinant is*

$$\det\left(H_{P_0}(\Psi_{\mathbf{r}})\right) = -\tau^2 \,\det\left(Z_{\boldsymbol{\lambda}}\right)^2.$$

In particular, the critical point is non degenerate. Finally, the critical value is

$$\Psi_{\mathbf{r}}(P_0) = \frac{\|\boldsymbol{\lambda}\|}{\tau}\,(\theta_1 - \theta_2).$$

Proof By (127),

$$\partial_{\boldsymbol{\rho}} \Psi_{\mathbf{r}} = Z_{\boldsymbol{\lambda}}\,\boldsymbol{\gamma} + R_2(\boldsymbol{\gamma}). \tag{131}$$

Since $Z_{\boldsymbol{\lambda}}$ is non-degenerate and $\boldsymbol{\gamma}$ ranges in a small ball centered at the origin in $\mathbb{R}^{2\,n_G}$, this forces $\boldsymbol{\gamma} = \mathbf{0}$ at any critical point. Given this, at any critical point we also need to have

$$\partial_{\boldsymbol{\gamma}} \Psi_{\mathbf{r}} = -Z_{\boldsymbol{\lambda}}\,\boldsymbol{\rho} + R_1(\boldsymbol{\gamma}) \cdot_{\mathrm{st}} (b, \mathbf{r}, \boldsymbol{\rho}) = -Z_{\boldsymbol{\lambda}}\,\boldsymbol{\rho},$$

whence $\boldsymbol{\rho} = \mathbf{0}$.

Furthermore,

$$\partial_b \Psi_{\mathbf{r}} = u\,\tau - \|\boldsymbol{\lambda}\|, \quad \partial_u \Psi_{\mathbf{r}} = \theta_1 - \theta_2 + \tau\, b,$$

which imply that at the critical point $u = \|\boldsymbol{\lambda}\|/\tau$ and $b = (\theta_2 - \theta_1)/\tau$.

The computation of $H_{P_0}(\Psi_{\mathbf{r}})$ and $\det\left(H_{P_0}(\Psi_{\mathbf{r}})\right)$ is straightforward. Regarding the signature, multiplying $\partial^2_{\boldsymbol{\lambda},\boldsymbol{\lambda}} \Psi_{\mathbf{r}}\Big|_{P_0}$ by a factor $t \in [0, 1]$ we obtain a homotopy of non-degenerate symmetric matrices, which for $t = 1$ is $H_{P_0}(\Psi_{\mathbf{r}})$, and for $t = 0$ is a matrix which is easily seen to have zero signature, since it is in block-diagonal form with each block having zero signature. The claim follows. □

Let $\varphi \in \mathcal{C}^\infty_c\left(\mathbb{R}^{2\,n_G}\right)$ be compactly supported in an open ball centered at the origin, and identically equal to 1 on some smaller neighbourhood of $\mathbf{0}$.

Proposition 5.16 *The asymptotics of (128) are unchanged, if the integrand is multiplied by* $\varphi(\boldsymbol{\rho})$.

Proof The Proposition is a consequence of the following Lemma.

Lemma 5.17 *The asymptotics of (128) are unchanged, if the integrand is multiplied by* $\varphi_k(\boldsymbol{\gamma}) := \varphi\left(k^{1/3}\,\boldsymbol{\gamma}\right)$.

Proof of the Lemma The proof is by a standard argument based on 'integration by parts' in $\boldsymbol{\rho}$, but we outline it for the reader's convenience. By (131), where $\varphi_k \neq 1$ for $k \gg 0$ one has $\|\partial_{\boldsymbol{\rho}} \Psi_{\mathbf{r}}\| \geq D\, k^{-1/3}$ for some fixed $D > 0$; furthermore, $\partial_{\boldsymbol{\rho}} \Psi_{\mathbf{r}}$ does not depend on $\boldsymbol{\rho}$. Let us consider the differential operator (well defined for $\boldsymbol{\gamma} \neq \mathbf{0}$)

$$L := \frac{1}{\|\partial_{\boldsymbol{\rho}} \Psi_{\mathbf{r}}\|^2} \sum_j (\partial_{\boldsymbol{\rho}_j} \Psi_{\mathbf{r}})\, \partial_{\rho_j}.$$

Then

$$-\frac{\imath}{\sqrt{k}}\, L\left(e^{\imath\,\sqrt{k}\,\Psi_{\mathbf{r}}(u,b,\boldsymbol{\rho},\boldsymbol{\gamma})}\right) = e^{\imath\,\sqrt{k}\,\Psi_{\mathbf{r}}(u,b,\boldsymbol{\rho},\boldsymbol{\gamma})}$$

Thus, looking only at the integral in $\mathrm{d}\boldsymbol{\rho}$,

$$\begin{aligned}
&\int_{\mathbb{R}^{2n_G}} \left[e^{\imath\,\sqrt{k}\,\Psi_{\mathbf{r}}(u,b,\boldsymbol{\rho},\boldsymbol{\gamma})}\, \mathcal{K}_{\mathbf{r}}\,(u, b, \boldsymbol{\rho}, \boldsymbol{\gamma})\right] \mathrm{d}\boldsymbol{\rho} \qquad (132)\\
&= -\frac{\imath}{\sqrt{k}}\, \frac{1}{\|\partial_{\boldsymbol{\rho}} \Psi_{\mathbf{r}}\|^2} \sum_j (\partial_{\boldsymbol{\rho}_j} \Psi_{\mathbf{r}}) \int_{\mathbb{R}^{2n_G}} \left[\partial_{\rho_j}\left(e^{\imath\,\sqrt{k}\,\Psi_{\mathbf{r}}(u,b,\boldsymbol{\rho},\boldsymbol{\gamma})}\right) \mathcal{K}_{\mathbf{r}}\,(u, b, \boldsymbol{\rho}, \boldsymbol{\gamma})\right] \mathrm{d}\boldsymbol{\rho}\\
&= \frac{\imath}{\sqrt{k}}\, \frac{1}{\|\partial_{\boldsymbol{\rho}} \Psi_{\mathbf{r}}\|^2} \sum_j (\partial_{\boldsymbol{\rho}_j} \Psi_{\mathbf{r}}) \int_{\mathbb{R}^{2n_G}} \left[e^{\imath\,\sqrt{k}\,\Psi_{\mathbf{r}}(u,b,\boldsymbol{\rho},\boldsymbol{\gamma})}\, \partial_{\rho_j}\left(\mathcal{K}_{\mathbf{r}}\,(u, b, \boldsymbol{\rho}, \boldsymbol{\gamma})\right)\right] \mathrm{d}\boldsymbol{\rho}
\end{aligned}$$

We have $\left|(\partial_{\boldsymbol{\rho}_j} \Psi_{\mathbf{r}})/\|\partial_{\boldsymbol{\rho}} \Psi_{\mathbf{r}}\|^2\right| \leq C'\, k^{1/3}$ for some $C' > 0$.

On the other hand, in view of the exponential factors in (119), the integrand in (128) may expanded as a linear combination of terms, each of which is bounded by a product of the form $k^N\, |\boldsymbol{\eta}^I|\, |P(\theta, u, \mathbf{r})|\, \left|\tilde{\boldsymbol{\rho}}^J\right|\, e^{-\frac{1}{2}\|\boldsymbol{\rho}\|^2}$, where N is uniformly bounded from above over all the summands, I and J are multi-indexes, and $\tilde{\boldsymbol{\rho}} := \Re(\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) - \boldsymbol{\rho}$; recall that $\boldsymbol{\rho} \in \mathbb{R}^{2n_G}$ is the coordinate expression for $\boldsymbol{\xi}_s$ and $\mathbf{r} \in \mathbb{R}^{r_G - 1}$ is the one for $\boldsymbol{\xi}_t$ in (105). Iteratively integrating by parts as in (132) r times, each such term is trasnformed in a linear combination of terms, each of which is bounded by an expression of the form $k^{N - r/6}\, |\boldsymbol{\eta}^I|\, |P(\theta, u, \mathbf{r})|\, \left|\tilde{\boldsymbol{\rho}}^{J'}\right|\, e^{-\frac{1}{2}\|\boldsymbol{\rho}\|^2}$, where $|J'| = |J| + r$. The claim follows. □

Let us conclude the proof of the Proposition. By the Lemma, we may assume without loss that the integrand has been multiplied by $\varphi_k(\boldsymbol{\gamma})$. Where $1 - \varphi(\boldsymbol{\rho}) \neq 0$, we have $\|\boldsymbol{\rho}\| \geq C'$ for some $C' > 0$. Then on the domain of integration for some constants $C'',\ C''' > 0$ we have

$$\|\partial_{\boldsymbol{\gamma}} \Psi_{\mathbf{r}}\| = \|Z_{\boldsymbol{\lambda}}\, \boldsymbol{\rho} + R_1(\boldsymbol{\gamma}) \cdot_{\mathrm{st}} (b, \mathbf{r}, \boldsymbol{\rho})\| \geq C''\, \|\boldsymbol{\rho}\| + O\left(k^{-\frac{1}{3}+\epsilon}\right) \geq C'''.$$

The statement follows by integration by parts in $\mathrm{d}\boldsymbol{\gamma}$, by a modification of the previous argument. □

By Proposition 5.16, in the asymptotic evaluation of (129) we may assume without loss that all variables are compactly supported, hence we are in a position to apply the Stationary Phase Lemma. On the critical locus, $b = b_0 = (\theta_2 - \theta_1)/\tau$ and $\boldsymbol{\rho} = \mathbf{0}$. Hence $\boldsymbol{\xi}'' = \boldsymbol{\xi}_{\mathfrak{s}} = \mathbf{0}$ (Remark 5.12), thus $\boldsymbol{\xi}^{\perp} = \boldsymbol{\xi}_{\mathfrak{t}} = \boldsymbol{\xi}'$. Using coordinates with respect to the given orthonormal basis with respect to κ_e, we shall identify $\boldsymbol{\xi}_{\mathfrak{t}}$ with $\mathbf{r} \in \mathbb{R}^{r_G - 1}$. Let us set

$$L_x(\mathbf{r}) := \imath\, \tilde{\kappa}_x\Big(\mathbf{r}_{\tilde{G}}(x), \boldsymbol{\rho}_{1\tilde{G}}(x) + \boldsymbol{\rho}_{2\tilde{G}}(x)\Big) - \frac{1}{2}\left\|\mathbf{r}_{\tilde{G}}(x)\right\|^2.$$

Recalling (119), we have

$$\begin{aligned}
& I_{x,k}(\mathbf{r}) = I_{x,k}(\theta_1, \theta_2, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \mathbf{r}) \\
& \sim e^{\imath\sqrt{k}\,\frac{\|\boldsymbol{\lambda}\|}{\tau}\,(\theta_1 - \theta_2)} \left(\frac{2\pi}{\sqrt{k}}\right)^{1+2n_G} \frac{1}{\tau\,|\det(Z_{\boldsymbol{\lambda}})|} \cdot \left(\frac{k\,\|\boldsymbol{\lambda}\|}{\tau}\right)^{d-1} \\
& \cdot e^{\frac{\|\boldsymbol{\lambda}\|}{\tau}\left[\psi_2\left(\boldsymbol{\eta}_{1\tilde{G}}(x), \boldsymbol{\eta}_{2\tilde{G}}(x)\right) - \frac{1}{2}\left\|\boldsymbol{\rho}_{1\tilde{G}}(x) - \boldsymbol{\rho}_{2\tilde{G}}(x)\right\|^2\right]} \cdot e^{\frac{\|\boldsymbol{\lambda}\|}{\tau} L_x(\mathbf{r})} \\
& \cdot \tilde{\rho}\left(k^{-\epsilon}\,\mathbf{r}\right) \frac{\tau}{(2\pi)^d} \cdot \left[1 + \sum_{j\geq 1} k^{-j/2} F_j(\theta_1, \theta_2, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \mathbf{r})\right] \\
& = (2\pi)^{1+2n_G-d} \frac{1}{|\det(Z_{\boldsymbol{\lambda}})|} e^{\imath\sqrt{k}\,\frac{\|\boldsymbol{\lambda}\|}{\tau}\,(\theta_1-\theta_2)}\, k^{d-n_G-\frac{3}{2}} \left(\frac{\|\boldsymbol{\lambda}\|}{\tau}\right)^{d-1} \cdot \tilde{\rho}\left(k^{-\epsilon}\,\mathbf{r}\right) \\
& \cdot e^{\frac{\|\boldsymbol{\lambda}\|}{\tau}\left[\psi_2\left(\boldsymbol{\eta}_{1\tilde{G}}(x), \boldsymbol{\eta}_{2\tilde{G}}(x)\right) - \frac{1}{2}\left\|\boldsymbol{\rho}_{1\tilde{G}}(x) - \boldsymbol{\rho}_{2\tilde{G}}(x)\right\|^2\right]} \cdot e^{\frac{\|\boldsymbol{\lambda}\|}{\tau} L_x(\mathbf{r})} \\
& \cdot \left[1 + \sum_{j\geq 1} k^{-j/2} F_j(\theta_1, \theta_2, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \boldsymbol{\rho}_1, \boldsymbol{\rho}_2; \mathbf{r})\right],
\end{aligned}$$

where $L_x(\mathbf{r}) = \Lambda_2(\boldsymbol{\xi})$ with $\boldsymbol{\xi}$ corresponding to $(b_0, \mathbf{r}, \mathbf{0}) \in \mathbb{R} \times \mathbb{R}^{r_G - 1} \times \mathbb{R}^{2n_G}$, and F_j is a polynomial of degree $\leq 3\,j$ and parity j. In view of (128), integrating the previous asymptotic expansion term by term yields an asymptotic expansion for $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k})$.

Let us consider the leading order term.

Integration in (128) is with respect to the standard measure associated to the Euclidean structure of $\mathfrak{t}'_{\boldsymbol{\lambda}} \cong \mathbb{R}^{r_G - 1}$ induced by κ_e; on the other hand, the norms and products in (133) are with respect to the Euclidean product on $\mathfrak{t}'_{\boldsymbol{\lambda}\tilde{G}}(x)$ induced by $\tilde{\kappa}_x$. Let D_x be the matrix representing the pull-back of $\tilde{\kappa}_x$ (under the evaluation $\mathfrak{t}'_{\boldsymbol{\lambda}} \rightarrow \mathfrak{t}'_{\boldsymbol{\lambda}\tilde{G}}(x)$) with respect to an orthonormal basis of $\mathfrak{t}'_{\boldsymbol{\lambda}}$. If we set $\mathbf{a} := \mathbf{r}_{\tilde{G}}(x)$, then $d\mathbf{r} = \det(D_x)^{-1/2}\, d\mathbf{a} = \mathfrak{D}^{\kappa}(x)^{-1}\, d\mathbf{a}$. Hence

$$\begin{aligned}
& \int_{\mathbb{R}^{r_G-1}} e^{\frac{\|\boldsymbol{\lambda}\|}{\tau} L_x(\mathbf{r})}\, d\mathbf{r} \\
& = \frac{1}{\sqrt{\det(D_x)}} \int_{\mathbb{R}^{r_G-1}} e^{\frac{\|\boldsymbol{\lambda}\|}{\tau}\left[\imath\,\langle \mathbf{a}, \boldsymbol{\rho}_{1\tilde{G}}(x) + \boldsymbol{\rho}_{2\tilde{G}}(x)\rangle - \frac{1}{2}\|\mathbf{a}\|^2\right]}\, d\mathbf{a} \\
& = \frac{1}{\mathfrak{D}^{\kappa}(x)} \left(\frac{2\pi\,\tau}{\|\boldsymbol{\lambda}\|}\right)^{\frac{r_G-1}{2}} e^{-\frac{1}{2}\frac{\|\boldsymbol{\lambda}\|}{\tau}\left\|\boldsymbol{\rho}_{1\tilde{G}}(x) + \boldsymbol{\rho}_{2\tilde{G}}(x)\right\|^2}.
\end{aligned} \tag{133}$$

Inserting this in (128), we obtain an asymptotic expansion of the form

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k}) \sim \left(\frac{k\,\|\boldsymbol{\lambda}\|}{2\pi\,\tau}\right)^{d-1+\frac{1-r_G}{2}} \left(\frac{\mathrm{vol}(\mathcal{O}_{\boldsymbol{\lambda}})}{\mathrm{vol}^{\kappa}(G)}\right)^2 \cdot \frac{\mathrm{vol}^{\kappa}(T)}{\mathfrak{D}^{\kappa}(x)\cdot|\det(Z_{\boldsymbol{\lambda}})|}$$
$$\cdot e^{\frac{\|\boldsymbol{\lambda}\|}{\tau}\left[\psi_2\left(\boldsymbol{\eta}_{1\tilde{G}}(x),\boldsymbol{\eta}_{2\tilde{G}}(x)\right)-\left\|\boldsymbol{\rho}_{1\tilde{G}}(x)\right\|^2-\left\|\boldsymbol{\rho}_{2\tilde{G}}(x)\right\|^2\right]}$$
$$\cdot\left[1+\sum_{j\geq 1}k^{-j/2}\,R_j(\theta_1,\theta_2,\boldsymbol{\eta}_1,\boldsymbol{\eta}_2,\boldsymbol{\rho}_1,\boldsymbol{\rho}_2)\right],$$

where the R_j's are as stated.

The argument for $P^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k})$ is essentially the same, in view of the considerations in §4.6. However, by (51) the leading order term in the amplitude of the FOI representation of P^{τ} is the the one for Π^{τ} multiplied by $(\pi/u)^{(d-1)/2}$. Since the previous arguments involve the rescaling $u \mapsto k\,u$, to leading order there is an additional factor $(\pi/k\,u)^{(d-1)/2}$ in the asymptotic expansion corresponding to (130). Evaluating at the critical point of Lemma 5.15, we obtain an asymptotic expansion formally similar to the one for $\Pi^{\tau}_{k\boldsymbol{\lambda}}(x_{1k}, x_{2k})$, but multiplied by $(\tau\,\pi/k\,\|\boldsymbol{\lambda}\|)^{(d-1)/2} = (k\,\|\boldsymbol{\lambda}\|/\tau\,\pi)^{-(d-1)/2}$. □

6 Proofs of the Applications

6.1 Theorem 1.10

The proof of Theorem 1.10 rests on the previous asymptotic expansions and on an L^2-norm asymptotic estimate for restrictions of complexified eigenfunctions, which is the specialization in the present setting of a basic result of Zelditch (Lemma 0.2 in [58]).

Lemma 6.1 (Zelditch) *There exists a universal constant $C(d,\tau) > 0$ such that the following holds. Let $\varphi \in L^2(G)_{k\boldsymbol{\lambda}}$ have unit L^2-norm. Let $\tilde{\varphi}$ be its complexification and $\tilde{\varphi}^{\tau} := \tilde{\varphi}|_{X^{\tau}}$. Then*

$$\left\|\tilde{\varphi}^{\tau}\right\|^2_{L^2(X^{\tau})} = C(d,\tau)\,e^{2\,\tau\,c_{k\boldsymbol{\lambda}}}\,(c_{k\boldsymbol{\lambda}})^{-\frac{d-1}{2}}\cdot\left(1+O\left(\frac{1}{k\,\|\boldsymbol{\lambda}\|}\right)\right).$$

Proof of Theorem 1.10 Let $\varphi \in L^2(G)_{k\boldsymbol{\lambda}}$ have unit L^2-norm. Suppose $x \in X^{\tau}_{\mathcal{O}}$, choose a system of NHLC's on X^{τ} centered at x, and let $\mathbf{n} \in N(X^{\tau}_{\mathcal{O}}/X^{\tau})_x$ be of norm $C\,k^{\epsilon}$. Since φ can be extended to an L^2-orthonormal basis of $L^2(G)_{k\boldsymbol{\lambda}}$, by (12)

$$e^{-2\,\tau\,c_{k\boldsymbol{\lambda}}}\,\left|\tilde{\varphi}^{\tau}(y)\right|^2 \leq P^{\tau}_{k\boldsymbol{\lambda}}(y,y) \quad \forall\, y \in X^{\tau}.$$

Choose C, ϵ as in the statement of Theorem 1.6.

In view of Theorem 1.6, we conclude that

$$e^{-2\,\tau\, c_{k\lambda}}\,\left|\tilde{\varphi}^{\tau}\,(y)\right|^{2}=O\left(k^{-\infty}\right), \tag{134}$$

uniformly for $\mathrm{dist}_{X^{\tau}},\left(y, X^{\tau}_{\mathcal{O}}\right)\geq\, C\,k^{\epsilon-\frac{1}{2}}$.

On the other hand, any point y in a tubular neighbourhood or radius $O\left(k^{\epsilon-\frac{1}{2}}\right)$ of $X^{\tau}_{\mathcal{O}}$ can be written in the form

$$y=x+\frac{\mathbf{n}}{\sqrt{k}} \tag{135}$$

with $x\in X^{\tau}_{\mathcal{O}}$ and $\mathbf{n}\in N(X^{\tau}_{\mathcal{O}}/X^{\tau})_{x}$ of norm $O\,(k^{\epsilon})$, for some choice of NHLC's centered at x. Since we may locally smoothly vary systems of NHLC's centered at moving points in $X^{\tau}_{\mathcal{O}}$, this is indeed a local parametrization of a shrinking neighbourhood of $X^{\tau}_{\mathcal{O}}$.

In view of Theorem 1.9, we obtain that for certain constants $C'_{d,\tau},\ C''_{d,\tau}>0$

$$\begin{aligned} e^{-2\,\tau\, c_{k\lambda}}\,\left|\tilde{\varphi}^{\tau}\left(x+\frac{\mathbf{n}}{\sqrt{k}}\right)\right|^{2} &\leq P^{\tau}_{k\lambda}\left(x+\frac{\mathbf{n}}{\sqrt{k}}, x+\frac{\mathbf{n}}{\sqrt{k}}\right)\\ &\leq C'_{d,\tau}\;(k\,\|\boldsymbol{\lambda}\|)^{\frac{d-r_G}{2}}\;e^{-2\,\frac{\|\boldsymbol{\lambda}\|}{\tau}\,\|\mathbf{n}\|^{2}}\leq C'_{d,\tau}\;(k\,\|\boldsymbol{\lambda}\|)^{\frac{d-r_G}{2}}\\ &\leq C''_{d,\tau}\,(c_{k\lambda})^{\frac{d-r_G}{2}}, \end{aligned} \tag{136}$$

since $c_{k\lambda}\sim k\boldsymbol{\lambda}$ for $k\to+\infty$ by (3). Pairing (134) and (136), we conclude that some constant $C'''_{d,\tau}>0$

$$\left|\tilde{\varphi}^{\tau}\,(x)\right|^{2}\leq e^{2\,\tau\, c_{k\lambda}}\,C'''_{d,\tau}\,c_{k\lambda}^{\;\frac{d-r_G}{2}},\quad \forall\, x\in X^{\tau}.$$

This proves the first statement of Theorem 1.10.

The second statement follows from the first and Lemma 6.1. □

6.2 Theorem 1.11

Following arguments in [39] and [10], we shall make recourse to the Shur-Young inequality ([41]): given a Riemannian manifold (M,β) with Riemannian density $\mathrm{d}V_M$ and $q\geq p\geq 1$, there is a constant $C_p>0$ such that for any integral self-adjoint operator kernel K on M we have

$$\|K\|_{L^{p}(M)\to L^{q}(M)}\leq C_{p}\left[\sup_{y\in M}\int\left|K(y,y')\right|^{R}\mathrm{d}V_{M}(y')\right]^{\frac{1}{R}},\quad \frac{1}{R}:=1-\frac{1}{p}+\frac{1}{q}.$$

Proof of Theorem 1.11 We need to estimate

$$\sup_{y\in M}\left[\int\left|\Pi^{\tau}_{k\lambda}(y,y')\right|^{R}\mathrm{d}V_{M}(y')\right]. \tag{137}$$

Let us fix C, ϵ as in Theorem 1.9. By Theorem 1.6,

$$\Pi^{\tau}_{k\boldsymbol{\lambda}}(y, y') = O\left(k^{-\infty}\right)$$

uniformly for $\mathrm{dist}_{X^\tau}(y, X^\tau_{\mathcal{O}}) \geq C\, k^{\epsilon - \frac{1}{2}}$. We may thus reduce to the case where y is given by (135). Given this, by Theorems 1.6 and 1.5, a non-negligible contribution to the integral in (137) only comes from the locus where both the conditions

$$\mathrm{dist}_{X^\tau}(y', X^\tau_{\mathcal{O}}),\ \mathrm{dist}_{X^\tau}(y', G \cdot y) \leq C\, k^{\epsilon - \frac{1}{2}}$$

are met. Thus, perhaps replacing C with a bigger constant C', we also have

$$\mathrm{dist}_{X^\tau}(y', G \cdot x) \leq C'\, k^{\epsilon - \frac{1}{2}}.$$

Any such y' has the form

$$y' = \mu_g(x) + \frac{1}{\sqrt{k}}\,(\mathbf{n}' + \mathbf{s}'), \tag{138}$$

for suitable $\mathbf{n}' \in N(X^\tau_{\mathcal{O}}/X^\tau)_{\mu_g(x)}$, $\mathbf{s}' \in \mathcal{S}_{\mu_g(x)}$ of norm $O\left(k^{\epsilon}\right)$. The system of NHLC's at $\mu_g(x)$ may be taken to be the μ_g-translate of the one at x.

However, in view of Remark 4.14, (4.14) is not a parametrization, since the real summand $\mathfrak{s}_x(\mu_g(x))$ in $\mathcal{S}_{\mu_g(x)}$ is contained in the tangent space to the G-orbit. To obtain an effective parametrization, we restrict $\mathbf{s}'$ to be an imaginary vector, i.e. assume $\mathbf{s}' \in J_{\mu_g(x)}\big(\mathfrak{s}_x(\mu_g(x))\big)$.

Thus, up to a negligible contribution, for a suitable $D > 0$ we may restrict integration to the image in X^τ of the immersion

$$\begin{aligned} \Lambda_{x,k} : (g, \mathbf{n}', \mathbf{s}') \in G \times B_{r_G - 1}\left(\mathbf{0}, D\, k^{\epsilon}\right) \times B_{d - r_G}\left(\mathbf{0}, D\, k^{\epsilon}\right) \\ \mapsto \quad y' = \mu_g(x) + \frac{1}{\sqrt{k}}\left(\mathbf{n}' + J_{\mu_g(x)}(\mathbf{s}')\right) \in X^\tau. \end{aligned} \tag{139}$$

Then

$$\Lambda^*_{x,k}(\mathrm{d}V_M) = \frac{1}{k^{\frac{d-1}{2}}}\, \mathcal{V}_k(g, \mathbf{n}', \mathbf{s}')\, \mathrm{d}^H V_G(g)\, \mathrm{d}\mathbf{n}'\, \mathrm{d}\mathbf{s}', \tag{140}$$

where

$$\mathcal{V}_k(g, \mathbf{n}', \mathbf{s}') \sim V_0(g) + \sum_{j \geq 1} k^{-j/2}\, V_j(g, \mathbf{n}', \mathbf{s}'),$$

with $V_0(g) > 0$ and V_j homogeneous of degree j in $(\mathbf{n}', \mathbf{s}')$.

On the other hand, by Theorem 1.9 and the Cauchy-Schwartz inequality, if y and y' are given by (135) and (138) we have

$$\left|\Pi^{\tau}_{k\boldsymbol{\lambda}}(y, y')\right| \leq \sqrt{\Pi^{\tau}_{k\boldsymbol{\lambda}}(y, y)\, \Pi^{\tau}_{k\boldsymbol{\lambda}}(y', y')} \tag{141}$$

$$\leq C_{\boldsymbol{\lambda},\tau}\, k^{d-1+\frac{1-r_G}{2}}\, e^{-\frac{\|\boldsymbol{\lambda}\|}{2\tau}\left(\|\mathbf{n}\|^2+\|\mathbf{n}'\|^2+\frac{1}{2}\,\|\mathbf{s}'\|^2\right)}$$
$$\leq C_{\boldsymbol{\lambda},\tau}\, k^{d-1+\frac{1-r_G}{2}}\, e^{-\frac{\|\boldsymbol{\lambda}\|}{2\tau}\,\|\mathbf{n}'\|^2}.$$

Hence, allowing the constant to vary from line to line,

$$\left|\Pi^{\tau}_{k\boldsymbol{\lambda}}(y, y')\right|^R \leq C_{\boldsymbol{\lambda},\tau}\, k^{R\left(d-1+\frac{1-r_G}{2}\right)}\, e^{-R\,\frac{\|\boldsymbol{\lambda}\|}{2\tau}\,\|\mathbf{n}'\|^2}. \tag{142}$$

Using this and (140), we conclude that

$$\int \left|\Pi^{\tau}_{k\boldsymbol{\lambda}}(y, y')\right|^R \mathrm{d}V_M(y') \tag{143}$$
$$\leq C_{\boldsymbol{\lambda},\tau}\, k^{R\left(d-1+\frac{1-r_G}{2}\right)}\, k^{-\frac{d-1}{2}}\, k^{\epsilon\cdot(d-r_G)}.$$

Thus we conclude that for some constant $C > 0$ (depending on τ, $\boldsymbol{\lambda}$, p and q), if $\epsilon' > 0$ then for $k \gg 0$ we have

$$\|\Pi^{\tau}_{k\boldsymbol{\lambda}}\|_{L^p(M)\to L^q(M)} \leq C_{\boldsymbol{\lambda},\tau,}\, k^{\frac{d-1}{2}\left(1-\frac{1}{R}\right)+\frac{d-r_G}{2}+\epsilon'}.$$

□

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