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Integral normal Cayley graphs

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Abstract
Given a finite group G, we say that a subset C of G is power-closed if, for every $x \in C$ and $y \in \langle x \rangle$ with $\langle x \rangle = \langle y \rangle$, we have $y \in C$. In this paper, we are interested in finite Cayley digraphs $\mathrm{Cay}(G, C)$ over G with connection set C, where C is a union of conjugacy classes of G. We show that each eigenvalue of $\mathrm{Cay}(G, C)$ is integral if and only if C is power-closed. This result will follow from a more general result.

Keywords Conjugacy classes · Eigenvalues · Irreducible characters

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1 Introduction

Let G be a finite group, and let C be a subset of G. The *Cayley digraph* $\mathrm{Cay}(G, C)$ over G with connection set C is the digraph with vertex set G and with (g, h) being a directed arc if and only if $gh^{-1} \in C$. The *eigenvalues* of a digraph are the eigenvalues of its adjacency matrix. We recall that the adjacency matrix of a digraph is the $\{0, 1\}$-matrix where the rows and the columns are indexed by the vertices of the graph, with an entry of 1 in position (g, h) if and only if there is a directed arc from g to h.

In this paper, we are concerned with some rationality conditions on the eigenvalues of $\mathrm{Cay}(G, C)$ when C is a union of G-conjugacy classes. (Cayley digraphs of this form are sometimes called *normal*.) In particular, we are interested in the case that each eigenvalue of $\mathrm{Cay}(G, C)$ is rational. Observe that since the eigenvalues of a digraph are algebraic integers (being the zeros of the characteristic polynomial of a matrix with integer coefficients), we see that if λ is a rational eigenvalue of $\mathrm{Cay}(G, C)$, then λ is actually an integer. Such Cayley graphs are usually referred to as *integral* Cayley graphs.

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We say that $C \subseteq G$ is *power-closed* if, for every $x \in C$ and $y \in \langle x \rangle$ with $\langle y \rangle = \langle x \rangle$, we have $y \in C$.

Theorem 1.1 *Let G be a finite group and let C be a union of conjugacy classes of G. Then each eigenvalue of* $\mathrm{Cay}(G, C)$ *is an integer if and only if C is power-closed.*

One implication of Theorem 1.1 is known. Indeed, it was proved in [5] that, if C is a union of conjugacy classes and C is power-closed, then each eigenvalue of $\mathrm{Cay}(G, C)$ is an integer. Our proof, however, is independent from the arguments in [5]. The integrality problem for the eigenvalues of Cayley graphs appears in the Kourovka Notebook, see [9, 19.50].

As every power-closed subset C is inverse-closed (that is, $C = C^{-1}$), if follows that if each eigenvalue of $\mathrm{Cay}(G, C)$ is an integer, then $\mathrm{Cay}(G, C)$ is an undirected graph. Theorem 1.1 gives a rather efficient (and linear-algebra-free) test to check when a Cayley digraph has only integer eigenvalues.

We note that, aside from its inherent interest, there are other reasons to consider this question. Let X be a graph on n vertices with adjacency matrix A. A *continuous quantum walk* on X is specified by the family of matrices

$$U(t) := \exp(itA), \quad (t \in \mathbb{R}).$$

If $u \in V(X)$, we use e_u to denote the standard basis vector in $\mathbb{R}^n$ indexed by u. We say that X is *periodic* at u if there are a complex scalar γ of norm 1 and a positive time t such that

$$U(t)e_u = \gamma e_u.$$

For surveys on this topic, see, e.g., [7, 8]. In [10], Saxena, Severini and Shparlinski showed that if X was a circulant, then X was periodic at a vertex if and only if the eigenvalues of X were integers. Subsequently it was shown in [6] that this conclusion held for any vertex-transitive graph, not just for circulants. This work has motivated the search for nice classes of vertex-transitive graphs with integer eigenvalues.

For abelian groups, our theorem corresponds to a well-known result of Bridges and Mena; see [3, Theorem 2.4]. Note that for an abelian group G, every subset of G is a union of G-conjugacy classes. Alperin and Peterson [1, Corollary 7.2] independently proved the same result as Bridges and Mena, albeit in a different context, obtaining it as a corollary. [1] In particular, Theorem 1.1 extends the work of Bridges and Mena (as well as that of Alperin and Peterson) by removing the assumption that G is abelian and replacing it with a natural condition on the connection set.

Theorem 1.1 will follow at once from a slightly more general theorem. Before giving its statement, we need some preliminary notation, which we will use throughout the whole paper, and some observations. Here we follow closely [11].

[1] Alperin and Peterson do not appear to have been aware that their corollary had already been established by Bridges and Mena. Although they cite some of the works of Bridges and Mena in their bibliography, they do not reference the relevant work on the spectra of Cayley graphs over abelian groups [3].

Let G be a finite group and let C be a union of conjugacy classes of G. From [2] or [4, Theorem 11.12.3], we get that the eigenvalues of $\mathrm{Cay}(G, C)$ are

$$\frac{1}{\chi(1)}\sum_{x\in C}\chi(x), \tag{1}$$

as χ runs through the set of irreducible complex characters of G. (We denote this set by $\mathrm{Irr}_{\mathbb{C}}(G)$.)

Following Serre [11, Section 9.1], we denote by $R_{\mathbb{C}}(G)$ the subring of the class functions of G generated by $\mathrm{Irr}_{\mathbb{C}}(G)$, that is,

$$R_{\mathbb{C}}(G) = \bigoplus_{\chi\in\mathrm{Irr}_{\mathbb{C}}(G)} \mathbb{Z}\chi.$$

More generally, given a field K with $\mathbb{Q} \le K \le \mathbb{C}$, we denote by $R_K(G)$ the subring of $R_{\mathbb{C}}(G)$ generated by the characters of the representations of G over K. Clearly, $R_K(G)$ is indeed a ring, for details see [11].

We let m be the least common multiple of the order of the elements of G, $\mathbb{Q}(m)$ the algebraic field obtained by adjoining the mth roots of unity to $\mathbb{Q}$ and $\Gamma_{\mathbb{Q}}$ the Galois group of $\mathbb{Q}(m)$ over $\mathbb{Q}$. By a well-known theorem of Brauer [11, Theorem 24], we have $R_{\mathbb{C}}(G) = R_{\mathbb{Q}(m)}(G)$, that is, every complex irreducible representation of G is realizable over $\mathbb{Q}(m)$. In particular, every $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$ has values in $\mathbb{Q}(m)$, and hence, from (1), every normal Cayley digraph $\mathrm{Cay}(G, C)$ has all of its eigenvalues in $\mathbb{Q}(m)$.

Now, let ε be a primitive mth root of unity. From a celebrated theorem of Gauss, the mth cyclotomic polynomial is irreducible over $\mathbb{Q}$, and hence, $\Gamma_{\mathbb{Q}} \cong (\mathbb{Z}/m\mathbb{Z})^*$ (where $(\mathbb{Z}/m\mathbb{Z})^*$ denotes the invertible elements of the ring $\mathbb{Z}/m\mathbb{Z}$). Here we identify $\Gamma_{\mathbb{Q}}$ with $(\mathbb{Z}/m\mathbb{Z})^*$ under this isomorphism. More precisely, for $\sigma \in \Gamma_{\mathbb{Q}}$, there exists a unique $t \in (\mathbb{Z}/m\mathbb{Z})^*$ with $\sigma(\varepsilon) = \varepsilon^t$.

Finally, given a field K with $\mathbb{Q} \le K \le \mathbb{Q}(m)$, we denote by Γ_K the image of $\mathrm{Gal}(\mathbb{Q}(m)/K)$ in $(\mathbb{Z}/m\mathbb{Z})^*$, and if $t \in \Gamma_K$, we let σ_t denote the corresponding element of $\mathrm{Gal}(\mathbb{Q}(m)/K)$.

For $s \in G$ and for an integer n, the element $s^n \in G$ depends only on the residue class of n modulo the order of s, and hence only on n modulo m. Therefore, s^t is defined for each $t \in \Gamma_K$, and the group Γ_K induces an action on the underlying set of G.

Definition 1.2 We say that $g, h \in G$ are Γ_K-conjugate, if there exists $t \in \Gamma_K$ such that g and h^t are conjugate in G. Clearly, being Γ_K-conjugate is an equivalence relation in G, and we call Γ_K-conjugacy classes its equivalence classes.

Observe that when $K = \mathbb{Q}(m)$, we have $\Gamma_K = 1$, and hence, the Γ_K-conjugacy classes coincide with the G-conjugacy classes. Moreover, when $K = \mathbb{Q}$, we have $\Gamma_K = (\mathbb{Z}/m\mathbb{Z})^*$, and hence, two elements g and h of G are Γ_K-conjugate if there exists $t \in (\mathbb{Z}/m\mathbb{Z})^*$ with g conjugate to h^t in G.

We are finally ready to state the main result of this paper.

Theorem 1.3 *Let G be a finite group, let C be a union of G-conjugacy classes, let m be the least common multiple of the order of the elements of G, and let K be a field with $\mathbb{Q} \le K \le \mathbb{Q}(m)$. Then each eigenvalue of $\mathrm{Cay}(G, C)$ lies in K if and only if C is a union of Γ_K-conjugacy classes.*

2 Proofs

Theorem 1.1 follows from Theorem 1.3 (applied with $K = \mathbb{Q}$) and the following lemma.

Lemma 2.1 *Let G be a finite group, and let C be a union of G-conjugacy classes. Then C is power-closed if and only if C is a union of $\Gamma_{\mathbb{Q}}$-conjugacy classes.*

Proof We first suppose that C is power-closed, and we show that C is a union of $\Gamma_{\mathbb{Q}}$-conjugacy classes. Let $x \in C$, and let $y \in G$ be $\Gamma_{\mathbb{Q}}$-conjugate to x. Then, by definition, there exists $t \in (\mathbb{Z}/m\mathbb{Z})^*$ with y^t conjugate to x in G, that is, $y^t = x^g$ for some $g \in G$. Now, $x^g \in C$ and $\langle y \rangle = \langle y^t \rangle = \langle x^g \rangle$; thus, $y \in C$ because C is power-closed.

Conversely, we suppose that C is a union of $\Gamma_{\mathbb{Q}}$-conjugacy classes and we show that C is power-closed. Let $x \in C$ and $y \in \langle x \rangle$ with $\langle y \rangle = \langle x \rangle$. Then $y = x^{t'}$, for some integer t' coprime to the order $|x|$ of x. From Dirichlet's theorem on primes in arithmetic progression, there exists a prime $t \in \{t' + \ell|x| \mid \ell \in \mathbb{Z}\}$ with $t > m$. We get that the residue class of t in $\mathbb{Z}/m\mathbb{Z}$ is invertible. Now $x^t = x^{t'} = y$, and hence, x and y are $\Gamma_{\mathbb{Q}}$-conjugate. Thus, $y \in C$. □

Proof of Theorem 1.3 Suppose that C is a union $C_1 \cup \cdots \cup C_\ell$ of Γ_K-conjugacy classes. From (1), we need to show that $\sum_{x \in C} \chi(x)/\chi(1) \in K$, for every $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$. For simplicity, we write $e_\chi = \sum_{x \in C} \chi(x)/\chi(1)$. As

$$e_\chi = \frac{1}{\chi(1)} \sum_{x \in C} \chi(x) = \left(\frac{1}{\chi(1)} \sum_{x \in C_1} \chi(x) \right) + \cdots + \left(\frac{1}{\chi(1)} \sum_{x \in C_\ell} \chi(x) \right),$$

it suffices to consider the case that $C = C_1$ is a Γ_K-conjugacy class. In particular, from the definition of Γ_K-conjugacy class we get $C = (x^{t_0})^G \cup \cdots \cup (x^{t_\ell})^G$, for some $x \in G$ and some $t_0, \ldots, t_\ell \in \Gamma_K$. (We denote by x^G the conjugacy class of x under G.) Observe that the action of the group Γ_K on C induces a transitive action of Γ_K on $\{(x^{t_0})^G, \ldots, (x^{t_\ell})^G\}$.

Fix $\chi \in \mathrm{Irr}_{\mathbb{C}}(G)$, and let ρ be a representation of G affording the character χ. Let $t \in \Gamma_K$, and let σ be the corresponding element in $\mathrm{Gal}(\mathbb{Q}(m)/K)$. For $s \in G$, let $\omega_1, \ldots, \omega_{\chi(1)}$ be the eigenvalues of $\rho(s)$. As $|s|$ is a divisor of m, we get that ω_i is an mth root of unity, and hence, the eigenvalues of $\rho(s^t)$ are the $\omega_1^t, \ldots, \omega_{\chi(1)}^t$. Thus, we have

$$(\chi(s))^\sigma = \left(\sum_{i=1}^{\chi(1)} \omega_i \right)^\sigma = \sum_{i=1}^{\chi(1)} \omega_i^t = \chi(s^t). \tag{2}$$

Now applying σ to e_χ, using (2) and recalling that the set C is invariant under taking tth powers, we get $e_\chi^\sigma = e_\chi$. In particular, $e_\chi^\sigma = e_\chi$ for every $\sigma \in \mathrm{Gal}(\mathbb{Q}(m)/K)$. Since $\mathbb{Q}(m)/K$ is a Galois extension, we have $e_\chi \in K$.

Conversely, suppose that each eigenvalue of $\mathrm{Cay}(G, C)$ lies in K. Since C is a union of G-conjugacy classes, for showing that C is also a union of Γ_K-conjugacy classes it suffices to prove that, for each $x \in C$ and for each $t \in \Gamma_K$, we have $x^t \in C$. Clearly, if $|x| = 1$, then there is nothing to prove. Now assume that $|x| > 1$. Let $\eta \in \mathbb{C}$ be a primitive $|x|$th root of unity, let $\ell \in \mathbb{N}$, let $\theta_\ell : \langle x \rangle \to \mathbb{C}$ be the irreducible character of $\langle x \rangle$ with $\theta_\ell(x) = \eta^\ell$, and let $\Theta_\ell = \mathrm{Ind}_{\langle x \rangle}^G(\theta_\ell)$, that is, Θ_ℓ is the character of G obtained by inducing θ_ℓ from $\langle x \rangle$ to G. From [11, page 55], we have

$$\Theta_\ell(s) = \frac{1}{|x|} \sum_{\substack{y \in G \\ y^{-1}sy \in \langle x \rangle}} \theta_\ell(y^{-1}sy). \tag{3}$$

Since Θ_ℓ is a character of G, Θ_ℓ is an integral linear combination of the irreducible characters of G. Moreover, since every eigenvalue of $\mathrm{Cay}(G, C)$ lies in K, from (1) we obtain $\sum_{z \in C} \Theta_\ell(z) \in K$. Write

$$e_{\Theta_\ell} := \frac{|x|}{|G|} \sum_{z \in C} \Theta_\ell(z).$$

From (3), we get

$$\begin{aligned} e_{\Theta_\ell} &= \frac{1}{|G|} \sum_{z \in C} \sum_{\substack{y \in G \\ y^{-1}zy \in \langle x \rangle}} \theta_\ell(y^{-1}zy) = \frac{1}{|G|} \sum_{z \in C} \sum_{i=0}^{|x|-1} \sum_{\substack{y \in G \\ y^{-1}zy = x^i}} \theta_\ell(x^i) \\ &= \frac{1}{|G|} \sum_{z \in C} \sum_{i=0}^{|x|-1} \sum_{\substack{y \in G \\ y^{-1}zy = x^i}} \eta^{\ell i} = \frac{1}{|G|} \sum_{i=0}^{|x|-1} \sum_{z \in C} \sum_{\substack{y \in G \\ y^{-1}zy = x^i}} \eta^{\ell i} \\ &= a_0 \eta^{\ell \cdot 0} + a_1 \eta^{\ell \cdot 1} + \cdots + a_{|x|-1} \eta^{\ell \cdot (|x|-1)}, \end{aligned} \tag{4}$$

where

$$a_i := \frac{|\{(z, y) \in C \times G \mid y^{-1}zy = x^i\}|}{|G|} = \begin{cases} 1 & \text{if } x^i \in C, \\ 0 & \text{if } x^i \notin C. \end{cases} \tag{5}$$

Now, let $t \in \Gamma_K$ and let σ be its corresponding element in $\mathrm{Gal}(\mathbb{Q}(m)/K)$. Applying σ on both sides of (4), we get

$$\begin{aligned} e_{\Theta_\ell}^\sigma &= a_0 \eta^{\ell \cdot 0 \cdot t} + a_1 \eta^{\ell \cdot 1 \cdot t} + \cdots + a_{|x|-1} \eta^{\ell \cdot (|x|-1) \cdot t} \\ &= a_{0 \cdot t^{-1}} \eta^{\ell \cdot 0} + a_{1 \cdot t^{-1}} \eta^{\ell \cdot 1} + \cdots + a_{(|x|-1)t^{-1}} \eta^{\ell \cdot (|x|-1)}. \end{aligned} \tag{6}$$

In particular, subtracting (6) from (4), we get

$$(a_0 - a_{0\cdot t^{-1}})\eta^{\ell\cdot 0} + (a_1 - a_{1\cdot t^{-1}})\eta^{\ell\cdot 1} + \cdots + (a_{|x|-1} - a_{(|x|-1)\cdot t^{-1}})\eta^{\ell\cdot(|x|-1)} = 0,$$

for every ℓ.

Let us consider the polynomial

$$f(T) := (a_0 - a_{0\cdot t^{-1}}) + (a_1 - a_{1\cdot t^{-1}})T + \cdots + (a_{|x|-1} - a_{(|x|-1)\cdot t^{-1}})T^{|x|-1} \in \mathbb{Z}[T].$$

We have shown that every $|x|$th root of unity is a root of f, and hence, $f(T)$ is divisible by $T^{|x|}-1$. Since the degree of f is less than $|x|$, we get $f(T)=0$, that is, $a_i = a_{i\cdot t^{-1}}$. From (5), this proves that $x^i \in C$ if and only if $x^{it} \in C$. □

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