A note on generalized Nash games played on networks*

Mauro Passacantando and Fabio Raciti

Abstract We investigate a generalized Nash equilibrium problem where players are modeled as nodes of a network and the utility function of each player depends on his/her own action as well as on the actions of his/her neighbors in the network. In the case of a quadratic reference model with shared constraints we are able to derive the variational solution of the game as a series expansion which involves the powers of the adjacency matrix, thus extending a previous result. Our analysis is illustrated by means of some numerical examples.

Key words: Network Games; Generalized Nash equilibrium; Variational inequalities; Network centrality measures.

1 Introduction

Economic and social sciences are probably the areas that have benefited the most from the mathematical development of Game Theory (GT), although in the last decades, specific game-theoretical models have also been applied to various problems from engineering, transportation and communication networks, biology, and other fields (see, e.g., [2, 19, 21]. The pervasive role of physical and virtual social interactions in the actions taken by individuals or groups, described through a network model, has led to consider Network Games as a powerful tool to describe the

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process of decision making. Indeed, in many social or economic environments our actions are influenced by the actions of our friends, acquaintances or colleagues. In network game models, each individual (player) is identified with the node of a graph and the players that can interact directly are connected through links of the graph. The specificity of these games is the central role played by the graph structure in the description of the patterns of interactions, and in the final social or economic outcome. The mathematical consequence of this description is that some interesting results depend on quantities such as the spectral radius of the adjacency matrix, its minimum eigenvalue, and its powers. With the huge variety of possible networks and interactions, it is difficult to make progress in the analysis without positing some specific structure of the problems under consideration. In these regards it is interesting to investigate the two classes of *games with strategic complements and substitutes*. Roughly speaking, in the first case, the incentive for a player to take an action increases when the number of his/her social contacts who take the action increases while in the second case this monotonic relation in reversed. The linear-quadratic model, although its simplicity, deserves particular attention and has been investigated in detail because it can be solved exactly and the solution formula can be nicely interpreted from a graph-theoretical point of view. Among the numerous references we suggest that the reader who wishes to become familiar with the topic of network games see the beautiful survey by Jackson and Zenou [9], along with the seminal paper by Ballester et al. [3]. Very recently, a different approach, based on variational inequality theory, has been put forward to tackle this kind of problems and, in this respect, the interested reader can refer to the interesting and very detailed paper by Parise and Ozdaglar [20].

In this paper we investigate, within the simple frame of the linear-quadratic model, a generalized Nash equilibrium problem (GNEP) with shared constraints. This class of games was proposed a long time ago by Rosen [22], but the last decade has witnessed a renewed interest in the subject, due to its wide range of applications and to its connection with the theory of variational inequalities [5, 6, 16, 18]. By adding a shared constrained to the original quadratic problem we thus obtain a GNEP, and loose uniqueness of the solution. Among the solutions of the new problem, we select the so called *variational solution* and provide a closed form expression for it. Furthermore, the new formula can be written by a series expansion of the adjacency matrix, thus extending one of the results in [3] and allowing for a graph-theoretic (as well as socio-economic) interpretation. Namely, this expansion shows that although players only interact with their neighbors, the solution also depends, to a certain extent, on indirect contacts (i.e., neighbors of neighbors, neighbors of neighbors of neighbors, and so on).

The paper is organized as follows. In the following Section 2 we summarize the basic background material on network games and focus on the exactly solvable linear-quadratic model. Section 3 is devoted to a brief description of generalized Nash equilibrium problems with shared constrained, and to the solution of our specific problem, while Section 4 is dedicated to illustrate our result by means of two worked-out examples. The paper ends with a concluding section where we outline some promising avenues of research.

2 Network games

2.1 Elements of Graph Theory and game classes

We begin this section by recalling a few concepts and definitions of graph theory that will be used in the sequel. We warn the reader that the terminology is not uniform in the related literature. Formally, a graph g is a pair of sets (V,E), where V is the set of nodes (or vertexes) and E is the set of arcs (or edges), formed by pairs of nodes (v,w). Arcs which have the same end nodes are called parallel, while arcs of the form (v,v) are called loops. We consider here *simple* graphs, that is graphs with no parallel arcs or loops. In our setting, the players will be represented by the n nodes in the graph. Moreover, we consider here indirect graphs: arcs (v,w) and (w,v) are the same. Two nodes v and w are adjacent if they are connected by an arc, i.e., if (v,w) is an arc. The information about the adjacency of nodes can be stored in the adjacency matrix G whose elements g_{ij} are equal to 1 if (v_i,v_j) is an arc, 0 otherwise. G is thus a symmetric and zero diagonal matrix. Given a node v, the nodes connected to v with an arc are called the *neighbors* of v and are grouped in the set $N_v(g)$. The number of elements of $N_v(g)$ is the *degree* of v. A *walk* in the graph g is a finite sequence of the form

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, \dots, e_{j_k}, v_{j_k},$$

which consists of alternating nodes and arc of the graph, such that $v_{i_{t-1}}$ and v_{i_t} are end nodes of e_{j_t}. The *length* of a walk is the number of its arcs. Let us remark that it is allowed to visit a node or go through an arc more than once. A *path* is a walk with all different nodes (except possibly the initial and terminal ones if the walk is closed). The indirect connections between any two nodes in the graph are described by means of the powers of the adjacency matrix G. Indeed, it can be proved that the element $g_{ij}^{[k]}$ of G^k gives the number of walks of length k between v_i and v_j.

We now proceed to specify the type of game that we will consider. For simplicity, the set of players will be denoted by $\{1,2,\dots,n\}$ instead of $\{v_1,v_2,\dots,v_n\}$. We denote with $A_i \subset \mathbb{R}$ the action space of player i, while $A = A_1 \times \cdots \times A_n$ and the notation $a = (a_i, a_{-i})$ will be used when we want to distinguish the action of player i from the action of all the other players. Each player i is endowed with a payoff function $u_i : A \to \mathbb{R}$ that he/she wishes to maximize. The notation $u_i(a,g)$ is often utilized when one wants to emphasize the influence of the graph structure (e.g., when studying perturbation with respect to the removal of an arc). The solution concept that we consider here is the Nash equilibrium of the game, that is, we seek an element $a^* \in A$ such that for each $i \in \{1,\dots,n\}$:

$$u_i(a_i^*, a_{-i}^*) \geq u_i(a_i, a_{-i}^*), \qquad \forall\, a_i \in A_i. \tag{1}$$

A peculiarity of network games is that the vector a_{-i} is only made up of components a_j such that $j \in N_i(g)$, that is, j is a neighbor of i.

We mentioned in the introduction that it is convenient to consider two specific classes of games which allow a deeper investigation of the patterns of interactions among players. For any given player i it is interesting to distinguish how variations of the actions of player's i neighbors affect his/her marginal utility. In the case where the utility functions are twice continuously differentiable the following definitions clarify this point.

Definition 1 We say that the network game has the property of strategic substitutes if for each player i the following condition holds:

$$\frac{\partial^2 u_i(a_i, a_{-i})}{\partial a_j \partial a_i} < 0, \qquad \forall (i,j) : g_{ij} = 1, \; \forall\, a \in A.$$

Definition 2 We say that the network game has the property of strategic complements if for each player i the following condition holds:

$$\frac{\partial^2 u_i(a_i, a_{-i})}{\partial a_j \partial a_i} > 0, \qquad \forall (i,j) : g_{ij} = 1, \forall\, a \in A.$$

For the subsequent development it is important to recall that if the u_i are continuously differentiable functions on A, the Nash equilibrium problem is equivalent to the variational inequality $VI(F,A)$: find $a^* \in A$ such that

$$F(a^*)^\top (a - a^*) \geq 0, \qquad \forall\, a \in A, \tag{2}$$

where

$$[F(a)]^\top := -\left(\frac{\partial u_1}{\partial a_1}(a), \dots, \frac{\partial u_n}{\partial a_n}(a) \right) \tag{3}$$

is also called the pseudo-gradient of the game, according to the terminology introduced by Rosen. For an account of variational inequalities the interested reader can refer to [13, 17, 7]. We recall here some useful monotonicity properties.

Definition 3 $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be monotone on A iff:

$$[F(x) - F(y)]^\top (x - y) \geq 0, \qquad \forall\, x, y \in A.$$

If the equality holds only when $x = y$, F is said to be strictly monotone.

A stronger type of monotonicity is given by the following

Definition 4 $F : \mathbb{R}^n \to \mathbb{R}^n$ is said to be β-strongly monotone on A iff:

$$[F(x) - F(y)]^\top (x - y) \geq \beta \|x - y\|^2, \qquad \forall\, x, y \in A.$$

For linear operators on $\mathbb{R}^n$ the two concepts of strict and strong monotonicity coincide and are equivalent to the positive definiteness of the corresponding matrix.

Conditions that ensure the unique solvability of a variational inequality problem are given by the following theorem (see, e.g. [13, 17, 7]).

Theorem 1 *If $K \subset \mathbb{R}^n$ is a compact convex set and $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on K, then the variational inequality problem $VI(F,K)$ admits at least one solution. In* *the case that K is unbounded, existence of a solution may be established under the following coercivity condition:*

$$\lim_{\|x\|\to+\infty} \frac{[F(x)-F(x_0)]^\top (x-x_0)}{\|x-x_0\|} = +\infty,$$

for $x \in K$ and some $x_0 \in K$.
Furtheremore, if F is strictly monotone on K the solution is unique.

In the following subsection, we describe in detail the linear-quadratic reference model on which we will build our generalized Nash equilibrium problem.

2.2 The linear-quadratic model

Let $A_i = \mathbb{R}_+$ for any $i \in \{1,\dots,n\}$, hence $A = \mathbb{R}^n_+$. The payoff of player i is given by:

$$u_i(a,g) = -\frac{1}{2}a_i^2 + \alpha a_i + \phi \sum_{j=1}^n g_{ij} a_i a_j, \qquad \alpha, \phi > 0. \tag{4}$$

In this simplified model α and ϕ take on the same value for all players, which then only differ according to their position in the network. The last term describes the interaction between neighbors and since $\phi > 0$ this interaction falls in the class of strategic complements. The pseudo-gradient's components of this game are easily computed as:

$$F_i(a) = a_i - \alpha - \phi \sum_{j=1}^n g_{ij} a_j, \qquad i \in \{1,\dots,n\},$$

which can be written in compact form as:

$$F(a) = (I - \phi G)a - \alpha \mathbf{1},$$

where $\mathbf{1} = (1,\dots,1)^\top \in \mathbb{R}^n$. We will seek Nash equilibrium points by solving the variational inequality:

$$F(a^*)^\top (a - a^*) \geq 0, \qquad \forall\, a \in \mathbb{R}^n_+. \tag{5}$$

Since the constraint set is unbounded, to ensure solvability we require that F be strongly monotone, which (implying coercivity, for linear operators) also guarantees the uniqueness of the solution.

Lemma 1 (see e.g. [9])

The matrix $I - \phi G$ is positive definite iff $\phi \rho(G) < 1$, where $\rho(G)$ is the spectral radius of G.

Proof The symmetric matrix $I-\phi G$ is positive definite if and only if $\lambda_{min}(I-\phi G)>0$. On the other hand, $\lambda_{min}(I-\phi G)=1-\phi\lambda_{max}(G)$. Since G is a symmetric non-negative matrix, the Perron-Frobenius Theorem guarantees that $\lambda_{max}(G)=\rho(G)$, hence $I-\phi G$ is positive definite if and only if $\phi\rho(G)<1$. □

To be self consistent, in the next lemma we recall the following result about series of matrices.

Lemma 2 (see e.g. [1])

Let T be a square matrix and consider the series:

$$I+T+T^2+\cdots+T^k+\ldots$$

The series converges provided that $\lim_k T^k=0$, which is equivalent to $\rho(T)<1$. In such case the matrix $I-T$ is non singular and we have:

$$(I-T)^{-1}=I+T+T^2+\cdots+T^k+\ldots$$

Theorem 2 (see e.g. [9])

If $\phi\rho(G)<1$, then the unique Nash equilibrium is

$$a^*=\alpha(I-\phi G)^{-1}\mathbf{1}=\alpha\sum_{p=0}^{\infty}\phi^p G^p\mathbf{1}. \tag{6}$$

Proof Since $\phi\rho(G)<1$, Lemma 1 guarantees that F is strongly monotone. Hence, Theorem 1 applies and we get a unique solution of (5). On the other hand, Lemma 2 implies that the matrix $I-\phi G$ is non singular, thus the linear system $F(a)=0$, which reads

$$(I-\phi G)a=\alpha\mathbf{1},$$

has a unique solution a^* given by (6). Moreover, looking at the expansion we get, by construction, that any component of a^* is strictly positive. Therefore, a^* is the unique solution of (5), thus it is the unique Nash equilibrium. □

Remark 1 The expansion in (6) suggests an interesting interpretation. Indeed, it can be shown that the (i,j) entry, $g_{ij}^{[p]}$, of the matrix G^p gives the number of walks of length p between nodes i and j. Based on this observation, a measure of centrality on the network was proposed by Katz and Bonacich (see e.g. [4]). Specifically, for any weight $w\in\mathbb{R}^n_+$, the weighted vector of Katz-Bonacich is given by:

$$b_w(G,\phi)=M(G,\phi)=(I-\phi G)^{-1}w=\sum_{p=0}^{\infty}\phi^p G^p w.$$

In the case where $w=\mathbf{1}$, the (non weighted) centrality measure of Katz-Bonacich of node i is given by:

$$b_{1,i}(G,\phi)=\sum_{j=1}^{n}M_{ij}(G,\phi)$$

and counts the total number of walks in the graph, which start at node i, exponentially damped by ϕ.

Remark 2 The game under consideration also falls in the class of potential games according to the definition introduced by Monderer and Shapley [15]. Indeed, a potential function is given by:

$$P(a,G,\phi) = \sum_{i=1}^{n} u_i(a,G) - \frac{\phi}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}a_i a_j.$$

Monderer and Shapley have proved that, in general, the solutions of the problem

$$\max_{a\in A}\; P(a,G,\phi)$$

form a subset of the solution set of the Nash game. Because under the condition $\phi\rho(G) < 1$ both problems have a unique solution, it follows that the two problems share the same solution.

3 Generalized Nash equilibrium problems on networks

3.1 An overview of GNEPs and the variational inequality approach to their solution

In GNEPs each player's startegy set may depend on the strategies of the other players. We consider here the simplified framework where $A_i \subseteq \mathbb{R}_+$ and we are given a function $g : \mathbb{R}^n \to \mathbb{R}^m$ which describes the shared constraints. The strategy set of player i is then written as

$$K_i(a_{-i}) = \{a_i \in \mathbb{R}_+ :\; g(a) = g(a_i, a_{-i}) \le 0\}.$$

Thus, players share a common constraint g and have an additional individual nonnegativity constraint. With these ingredients, the GNEP is the problem of finding $a^* \in \mathbb{R}^n$ such that, for any $i \in \{1,\dots,n\}$, $a_i^* \in K_i(a_{-i}^*)$ and

$$u_i(a_i^*, a_{-i}^*) \ge u_i(a_i, a_{-i}^*), \qquad \forall\, a_i \in K_i(a_{-i}^*). \tag{7}$$

We will work under the common (although not minimal) assumptions that for each fixed $a_{-i} \in A_{-i}$, the functions $u_i(\cdot, a_{-i})$ are concave and continuously differentiable, and the components of g are convex and continuously differentiable. As a consequence, a necessary and sufficient condition for $a_i^* \in K_i(a_{-i}^*)$ to satisfy (7) is

$$-\frac{\partial u_i(a_i^*, a_{-i}^*)}{\partial a_i}(a_i - a_i^*) \ge 0, \qquad \forall\, a_i \in K_i(a_{-i}^*). \tag{8}$$

Thus, if we define $F(a)$ as in (3), and

$$K(a) = K_1(a_{-1}) \times \cdots \times K_n(a_{-n}),$$

it follows that a^* is a GNE if and only if $a^* \in K(a^*)$ and

$$[F(a^*)]^\top (a - a^*) \geq 0, \qquad \forall\, a \in K(a^*). \tag{9}$$

The problem above, where also the feasible set depends on the solution, is called a quasi-variational inequality and its solution is as difficult as the original GNEP.

Assume now that a^* is a solution of GNEP. Hence, for each i, a_i^* solves the maximization problem

$$\max_{a_i} \{u_i(a_i, a^*_{-i}) :\ g(a_i, a^*_{-i}) \leq 0,\ a_i \geq 0\}.$$

Under some standard constraint qualification we can then write the KKT conditions for each maximization problem. We then introduce the Lagrange multiplier $\lambda^i \in \mathbb{R}^m$ associated with the costraint $g(a_i, a^*_{-i}) \leq 0$ and the multiplier $\mu_i \in \mathbb{R}$ associated with the nonnegativity constraint $a_i \geq 0$. The Lagrangian function for each player i reads as:

$$L_i(a_i, a^*_{-i}, \lambda^i, \mu_i) = u_i(a_i, a^*_{-i}) - [g(a_i, a^*_{-i})]^\top \lambda^i + \mu_i a_i$$

and the KKT conditions for all players are given by:

$$\nabla_{a_i} L_i(a_i^*, a^*_{-i}, \lambda^{i*}, \mu_i^*) = 0, \qquad i = 1, \ldots, n, \tag{10}$$

$$\lambda_j^{i*} g_j(a^*) = 0,\ \lambda_j^{i*} \geq 0,\ g_j(a^*) \leq 0, \qquad i = 1, \ldots, n,\ j = 1, \ldots, m \tag{11}$$

$$\mu_i^* a_i^* = 0,\ \mu_i^* \geq 0,\ a_i^* \geq 0, \qquad i = 1, \ldots, n. \tag{12}$$

Conversely, under the assumptions made, if a^*, λ, μ^*, where $\lambda^* = (\lambda^{1*}, \ldots, \lambda^{n*})$ and $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$, satisfy the KKT system (10)–(12), then a^* is a GNE.

Definition 5 Let a^* be a GNE which together with the Lagrange multipliers $\lambda^* = (\lambda^{1*}, \ldots, \lambda^{n*})$ and $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ satisfies the KKT system of all players. We call a^* a *normalized equilibrium* if there exists a vector $r \in \mathbb{R}^n_{++}$ and a vector $\bar{\lambda} \in \mathbb{R}^m_+$ such that

$$\lambda^{i*} = \frac{\bar{\lambda}}{r_i}, \qquad \forall\, i = 1, \ldots, n,$$

which means that, for a normalized equilibrium, the multipliers of the constraints shared by all players are proportional to a common multiplier. In the special case $r_i = 1$ for any i, i.e., the multipliers coincide for each player, a^* is called *variational equilibrium* (VE). Rosen [22] proved that if the feasible set, which in our case is:

$$K = \{a \in \mathbb{R}^n_+ :\ g(a) \leq 0\}$$

is compact and convex, then there exists a normalized equilibrium for each $r \in \mathbb{R}^n_{++}$.

Now, let us define, for each $r \in \mathbb{R}^n_{++}$, the vector function $F^r : \mathbb{R}^n \to \mathbb{R}^n$ as follows:

$$[F^r(a)]^\top := -\left(r_1 \frac{\partial u_1}{\partial a_1}(a), \ldots, r_n \frac{\partial u_n}{\partial a_n}(a)\right).$$

The variational inequality approach for finding the normalized equilibria of the GNEP is expressed by the following theorem which can be viewed as a special case of proposition 3.2 in [18] or of theorem 6.1 in [14].

Theorem 3

1. *Suppose that a^* is a solution of $VI(F^r, K)$, where $r \in \mathbb{R}^n_{++}$, a constraint qualification holds at a^* and $(\bar{\lambda}, \bar{\mu}) \in \mathbb{R}^m \times \mathbb{R}^n$ are the multipliers associated to a^*. Then, a^* is a normalized equilibrium such that the multipliers (λ^{i*}, μ_i^*) of each player i satisfy the following conditions:*

$$\lambda^{i*} = \frac{\bar{\lambda}}{r_i}, \quad \mu_i^* = \frac{\bar{\mu}_i}{r_i}, \qquad \forall\, i = 1, \ldots, n.$$

2. *If a^* is a normalized equilibrium such that the multipliers (λ^{i*}, μ_i^*) of each player i satisfy the following conditions:*

$$\lambda^{i*} = \frac{\bar{\lambda}}{r_i}, \qquad \forall\, i = 1, \ldots, n,$$

for some vector $\bar{\lambda} \in \mathbb{R}^m_+$ and $r \in \mathbb{R}^n_{++}$, then a^* is a solution of $VI(F^r, K)$ and $(\bar{\lambda}, r_1\mu_1^*, \ldots, r_n\mu_n^*)$ are the corresponding multipliers.

3.2 A linear-quadratic network GNEP

In this section we investigate an extension to a GNEP of the linear-quadratic network game described in Section 2.2. Specifically, we assume the same network structure given by the adjacency matrix G and the same payoff functions defined as in (4), while the strategy set of player i is given by the usual individual constraint $a_i \geq 0$ and an additional constraint, shared by all the players, on the total quantity of activities of all players, that is

$$K_i(a_{-i}) = \left\{ a_i \in \mathbb{R}_+ : \sum_{i=1}^n a_i \leq C \right\}, \qquad i = 1, \ldots, n,$$

where $C > 0$ is a given parameter. Depending on the specific application, the additional constraint can have the meaning of a collective budget upper bound or of a limited availability of a certain commodity.

We know from Theorem 2 that if $\phi\rho(G) < 1$, then the linear-quadratic network game (without the new shared constraint) has a unique Nash equilibrium a^* given by (6). However, if a^* does not satisfy the shared constraint, i.e., it does not belong to the set

$$K = \left\{ a \in \mathbb{R}^n_+ : \sum_{i=1}^{n} a_i \leq C \right\},$$

it cannot be a GNE for the new game. On the other hand, under the assumption $\phi\rho(G) < 1$, Theorem 1 guarantees that the pseudo-gradient F of the game, defined as

$$F(a) = (I - \phi G)a - \alpha \mathbf{1},$$

is strongly monotone, hence there exists a unique solution of $VI(F,K)$, i.e., there exists a unique variational equilibrium of the linear-quadratic network GNEP. The following result gives an explicit formula for such variational equilibrium and an expansion similar to (6).

Theorem 4 *If $\phi\rho(G) < 1$, then the unique variational equilibrium $\bar{a}$ of the linear-quadratic network GNEP, that is the unique solution of $VI(F,K)$, is given by the following formula:*

$$\bar{a} = \begin{cases} a^* = \alpha \sum_{p=0}^{\infty} \phi^p G^p \mathbf{1} & \text{if } \sum_{i=1}^{n} a_i^* \leq C, \\ \dfrac{C a^*}{\sum_{i=1}^{n} a_i^*} = \dfrac{C \sum_{p=0}^{\infty} \phi^p G^p \mathbf{1}}{\sum_{p=0}^{\infty} \phi^p \mathbf{1}^\top G^p \mathbf{1}} & \text{if } \sum_{i=1}^{n} a_i^* > C, \end{cases} \tag{13}$$

where $a^* = \alpha(I - \phi G)^{-1}\mathbf{1}$ is the Nash equilibrium of the linear-quadratic network game.

Proof Theorem 1 guarantees that the matrix $I - \phi G$ is positive definite and the map F is strongly monotone. Therefore, $VI(F,K)$ has a unique solution. If $a^* \in K$, then a^* solves $VI(F,K)$ since $F(a^*) = 0$. Otherwise, if $a^* \notin K$, then $\bar{a} = Ca^* / \sum_{i=1}^{n} a_i^* \in K$ since $\bar{a}_i > 0$ for any $i = 1, \ldots, n$ and $\sum_{i=1}^{n} \bar{a}_i = C$. Moreover $\bar{a}$ is a solution of the KKT system related to $VI(F,K)$ with multipliers

$$\bar{\lambda} = \alpha \left(1 - \frac{C}{\sum_{i=1}^{n} a_i^*} \right) > 0,$$

associated to the shared constraint, and $\bar{\mu}_i = 0$ associated to $a_i \geq 0$ for any $i = 1, \ldots, n$. In fact, we have

$$F(\bar{a}) + \bar{\lambda}\mathbf{1} - \bar{\mu} = \frac{C\alpha(I - \phi G)(I - \phi G)^{-1}\mathbf{1}}{\sum_{i=1}^{n} a_i^*} - \alpha\mathbf{1} + \alpha \left(1 - \frac{C}{\sum_{i=1}^{n} a_i^*} \right) \mathbf{1} = 0$$

$$\bar{\lambda} \geq 0, \quad \sum_{i=1}^{n} \bar{a}_i \leq C, \quad \bar{\lambda} \left(C - \sum_{i=1}^{n} \bar{a}_i \right) = 0$$

$$\bar{\mu}_i \geq 0, \quad \bar{a}_i \geq 0, \quad \bar{\mu}_i \bar{a}_i = 0, \quad \forall\, i = 1, \ldots, n.$$

Therefore, $\bar{a}$ solves $VI(F,K)$. Finally, the ratio between the expansions in (13) follows from the one for a^* given in (6):

$$\bar{a} = \frac{Ca^*}{\mathbf{1}^\top a^*} = \frac{C\alpha \sum_{p=0}^{\infty} \phi^p G^p \mathbf{1}}{\alpha \sum_{p=0}^{\infty} \phi^p \mathbf{1}^\top G^p \mathbf{1}} = \frac{C \sum_{p=0}^{\infty} \phi^p G^p \mathbf{1}}{\sum_{p=0}^{\infty} \phi^p \mathbf{1}^\top G^p \mathbf{1}}.$$

This concludes the proof. □

Notice that if $a^* \in K$, then the formula giving $\bar{a}$ contains α but does not C, while if $a^* \notin K$, it contains C but not α.

4 Numerical experiments

In this section, we show some preliminary numerical experiments for the linear-quadratic network GNEP described in Section 3.2 by means of two small-size test problems.

Example 1. We consider the network shown in Fig. 1 (see also [3]) with 11 nodes (players). The spectral radius of the adjacency matrix G is $\rho(G) \simeq 4.4040$. We set

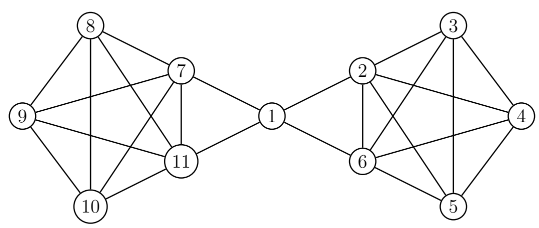

Fig. 1 Network topology of Example 1.

parameter $\alpha = 1$ and chose five different values for ϕ:

$$\phi = 0.3/\rho(G), \;\; \phi = 0.5/\rho(G), \;\; \phi = 0.7/\rho(G), \;\; \phi = 0.9/\rho(G), \;\; \phi = 0.95/\rho(G),$$

to guarantee the assumption of Theorem 4 holds. Moreover, we set $C = 20$ in the shared constraint so that the variational equilibrium $\bar{a}$ of the GNEP is different from the Nash equilibrium a^* of the classical network game. It follows from expansions in (13) that a^* and $\bar{a}$ can be approximated by the sequences $\{a_k^*\}$ and $\{\bar{a}_k\}$, respectively:

$$a_k^* = \alpha \sum_{p=0}^{k} \phi^p G^p \mathbf{1}, \qquad \bar{a}_k = \frac{C \sum_{p=0}^{k} \phi^p G^p \mathbf{1}}{\sum_{p=0}^{k} \phi^p \mathbf{1}^\top G^p \mathbf{1}}.$$

Table 1 shows, for both sequences, the number of sums needed to get an approximation error less than 10^{-t} for any $t = 1, \dots, 10$. Specifically, for any value of ϕ, the numbers

$$\min\left\{k: \ \|a_k^* - a^*\|_\infty < 10^{-t}\right\} \qquad \text{and} \qquad \min\left\{k: \ \|\bar{a}_k - \bar{a}\|_\infty < 10^{-t}\right\}$$

are reported in the first (NE) and second (VE) column, respectively, for any $t = 1, \dots, 10$.

	$\phi = \frac{0.3}{\rho(G)}$		$\phi = \frac{0.5}{\rho(G)}$		$\phi = \frac{0.7}{\rho(G)}$		$\phi = \frac{0.9}{\rho(G)}$		$\phi = \frac{0.95}{\rho(G)}$	
Error	NE	VE	NE	VE	NE	VE	NE	VE	NE	VE
10^{-1}	2	1	4	1	10	1	44	1	105	1
10^{-2}	4	2	7	3	16	5	66	9	150	11
10^{-3}	6	4	11	6	23	11	88	27	194	43
10^{-4}	8	5	14	9	29	17	110	48	239	86
10^{-5}	9	7	17	13	35	23	132	70	284	131
10^{-6}	11	9	21	16	42	30	153	92	329	176
10^{-7}	13	11	24	19	48	36	175	114	374	221
10^{-8}	15	13	27	23	55	43	197	136	419	265
10^{-9}	17	15	31	26	61	49	219	157	464	310
10^{-10}	19	17	34	29	68	56	241	179	509	355

Table 1 Speed of convergence of the sequences $\{a_k^*\}$ and $\{\bar{a}_k\}$ to a^* and $\bar{a}$, respectively. For any value of ϕ, the first column (NE) reports $\min\left\{k: \ \|a_k^* - a^*\|_\infty < 10^{-t}\right\}$, while the second column (VE) reports $\min\{k: \ \|\bar{a}_k - \bar{a}\|_\infty < 10^{-t}\}$, for any $t = 1, \dots, 10$.

The results in Table 1 show that the convergence of $\{\bar{a}_k\}$ to the variational equilibrium $\bar{a}$ seems to be faster than the convergence of $\{a_k^*\}$ to the Nash equilibrium a^*. Moreover, the more the value of ϕ is close to $1/\rho(G)$, the more this gap is evident.

Example 2. We now consider the network shown in Fig. 2 with 3 nodes (players). The spectral radius of the adjacency matrix G is $\rho(G) = 2$. We set parameter $\alpha = 1$, $\phi = 0.25$ and $C = 3$, so that the Nash equilibrium is $a^* = (2,2,2)^\top$ and the variational equilibrium is $\bar{a} = (1,1,1)^\top$. We exploit Theorem 3 to approximate the set of normalized equilibria. Consider the simplex

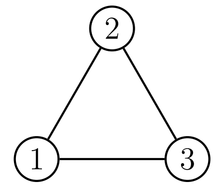

Fig. 2 Network topology of Example 2.

$$W = \left\{ r \in \mathbb{R}^3_{++} : \; r_1 + r_2 + r_3 = 1 \right\}$$

of weights of the parametrized $VI(F^r, K)$ and its discretization given by the finite set of vectors

$$\left(\frac{q_1}{D}, \frac{q_2}{D}, \frac{q_3}{D} \right)$$

such that q_1, q_2, q_3 and D are positive integers and $q_1 + q_2 + q_3 = D$. Figure 3 shows the set of normalized equilibria, projected on the plane (a_1, a_2), for different values of D. Notice that all the found normalized equilibria belong to the plane $a_1 + a_2 + a_3 = 3$.

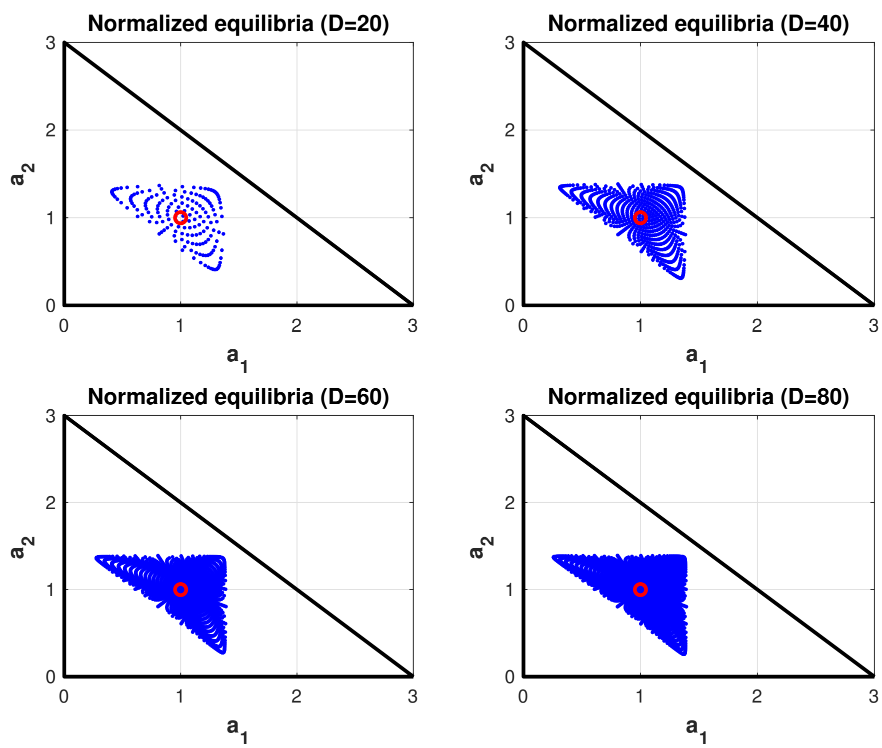

Fig. 3 Normalized equilibria (blue points) and variational equilibrium (red circle) of the linear-quadratic network GNEP.

The results in Figure 3 suggest that the set of normalized equilibria is equal to

$$\left\{a \in \mathbb{R}^3_+ :\ a_1 + a_2 + a_3 = 3, \quad a_i \leq 1.3924 \quad i = 1,2,3\right\}$$
$$= \operatorname{conv}\left\{ \begin{pmatrix} 1.3924 \\ 1.3924 \\ 0.2152 \end{pmatrix}, \begin{pmatrix} 1.3924 \\ 0.2152 \\ 1.3924 \end{pmatrix}, \begin{pmatrix} 0.2152 \\ 1.3924 \\ 1.3924 \end{pmatrix} \right\}.$$

Notice that, due to the symmetry of the considered network, the variational equilibrium $\bar{a}$ is equal to the barycenter of the set of normalized equilibria.

5 Conclusions and further research perspectives

In this note, we dealt with a network GNEP and derived a closed formula for its solution which involves the powers of the adjacency matrix, thus extending a previous result. To the best of our knowledge, this kind of formulas have been derived only in a few special cases and, because of their very interesting interpretation, it would be desirable to obtain similar results for more general problem classes. Another promising direction of research is the inclusion of random data in the model (see e.g. [10, 11]), which could be done by using tools from infinite-dimensional duality theory (see e.g. [8, 12, 14]).

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