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# CMB Multipole Expansion in a Frame Dragging-Sustained Milky Way

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**Abstract:** We study the impact on the cosmic microwave background (CMB) landscape of peculiar rotational general relativistic effects. These effects, on galactic scales, do not possess a Newtonian analogue, and therefore could a priori impact CMB analysis. We find that the velocity inferred from the CMB dipole, under the kinematic interpretation, coincides with that measured by a stationary observer within the Milky Way and not with the one measured by the zero angular momentum observer. We show that the galaxy peculiar frame-dragging effects do not impact the standard CMB analysis, as these modify the multipole coefficients only at higher orders with respect to the dominant terms. Moreover, we prove that no general relativistic framework at the galactic scale patched within the standard cosmological model can account for the current tension on the CMB quadrupole amplitude.

**Keywords:** General Relativity; cosmic microwave background; galactic dynamics; frame dragging; CMB quadrupole tension



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## 1. Introduction

General Relativity (GR) is the best theory of gravity we have and represents one of the crowning achievements of theoretical physics of the 20th century. After more than a century since its conception, GR has passed all the experimental and observational tests that the scientific community could devise [1]. Indeed, even its most outlandish predictions, such as neutron stars [2,3], black holes [4–6], and gravitational waves [7,8], have been confirmed by astrophysical observations. The picture of reality emerging from GR has deeply changed our knowledge of the local and far Universe, introducing novel elements with no direct analogues in the Newtonian theory of gravity, e.g., a curved spacetime.

It is clear that new phenomena, which do not possess any Newtonian analogue, arise in extreme astrophysical environments. This is, for example, the case for the coalescence of black hole binaries via gravitational wave emissions. However, even in the case of weak-gravitational fields, purely GR effects exist that do not possess a Newtonian counterpart, e.g., the gravitomagnetic effects related to mass currents and spacetime rotation [9,10].

We note that in general, it is false that GR effects are always negligible with respect to the corresponding Newtonian ones since the latter at times might simply not exist. Moreover, even if Newtonian analogues do exist, the situation is not always straightforward. In fact, if we consider the gravitational lensing by a point-like source, such as the Sun over cosmological scales, we find that the bending angle calculated within the Newtonian framework differs by a factor of two from the general relativistic one [11]. Hence, in this simple case, both the Newtonian and GR effects are of the same order of magnitude. Therefore, it is natural to wonder in which novel, and perhaps unexpected, astrophysical setting GR effects could significantly differ once more from Newtonian predictions.

The role of off-diagonal components in the global galactic metric—i.e., of a strong dragging vortex surrounding disc galaxies, an effect peculiar to GR—non-negligible even in the low-energy regime, has been recently considered [12–26].[1] Empirical evidence for such a purely GR effect was shown in [13,22], by employing the first, very simplified, rigidly rotating full GR disc galaxy model [12]. This was later generalised to differentially rotating models in [15], followed by [17–19,23–25], and further developed for the non-zero pressure case in [26].

Within the context of the frame-dragging sustained galaxies hypothesis, the novel problem of how to correctly identify the observationally inferred rotation speed of disc galaxies in the novel mathematical framework has to be considered. Indeed, due to the strong frame dragging, there is no unique, "natural" choice of general—relativistic reference frame. In particular, the stationary observer has a non-zero angular momentum, which is required to counter the local spacetime rotation [22,23,27], and the observer with zero angular momentum is not stationary, as it must rotate with the local geometry [13,22,28,29]. The debate about which reference system measures the speed we observationally infer—namely, those velocities that are identified with the disc galaxy rotation curves—is of crucial importance for interpreting the physics of such systems. In previous work [17–19,23,25,26], it was shown that the rotation speed of a far away disc galaxy inferred via frequency shift measurements (inclusive of both kinematic Doppler shift and GR corrections) actually coincides with the speed measured by the stationary observer, potentially allowing one to constrain the GR metric of disc galaxies through observations.

This is, however, just one of the many possible techniques for measuring the rotation of disc galaxies. In this paper, we investigate the impact of the existence of a frame-dragging vortex on the observed cosmic microwave background (CMB) anisotropy [30–34]. Assuming the kinematic dipole interpretation [30,35], the dipole anisotropy allows us to evaluate our own speed with respect to the CMB reference frame. Thus, CMB dipole measurements constitute another technique for measuring the rotation speed of our own galaxy. It is a priori an independent measurement technique with respect to the previously mentioned frequency shift method. The latter is the only viable method for measuring the rotation speed of most of the faraway galaxies, whilst the former is applicable only to measure the Milky Way's (MW) rotation, and only in our neighborhood.

Moreover, in principle, a strong dragging vortex could modify the CMB multipole coefficients with respect to the conventional $\Lambda$ cold dark matter ($\Lambda$CDM) expectations. If this were to be the case, then the CMB data analysis could give a direct test for the strong-dragging hypothesis within the MW and lead to a resolution of known tensions around CMB systematic multipole coefficients arising from purely special–relativistic kinematic effects. In particular, the tension on the quadrupole represents a long-standing issue [36–39], whose resolution has interestingly been already investigated within general relativistic modelling of local cosmological structures, but only above the galactic scale (see, e.g., [40,41]).

In our work, we show that: (i) the speed measured via the CMB dipole analysis coincides with that measured by the stationary observer, thus enforcing its physical interpretation; (ii) strong dragging correction to the CMB multipole coefficients results to be negligible by two orders of magnitude with respect to the dominant terms. Hence, neither the quadrupole tension nor any other CMB anomalies can be explained via strong frame-dragging corrections to the CMB multipole.

The rest of the paper is structured as follows. In Section 2, we introduce the mathematical framework of the dragging metrics and define the relevant observers to our investigation; in Section 3, we construct from first principles the relation between CMB observations and the presence of a dragging metric describing the MW geometry; in Section 4, we calculate the impact of a strong frame dragging on the multipole coefficients of the CMB spectrum; Section 5 is dedicated to a brief overview of the results and the discussion of future perspectives.

## 2. Dragging Metrics and Relevant Observers

We consider an idealised disc galaxy in stationary motion, with exact symmetry around its rotation axis and with respect to the galactic plane. Hence, we neglect the morphological evolution of the galaxy, as well as any structure that breaks the axisymmetry (bars or spirals). The spacetime line element describing the local geometry of the disc galaxy, in cylindrical coordinates, takes the form [42]

$$\mathrm{d}s^2 = -c^2 e^{2\Phi(r,z)/c^2}(\mathrm{d}t + A(r,z)\,\mathrm{d}\phi)^2 + e^{-2\Phi(r,z)/c^2}[W(r,z)^2\,\mathrm{d}\phi^2 + e^{2k(r,z)}(\mathrm{d}r^2 + \mathrm{d}z^2)]. \quad (1)$$

Here, the field $\Phi$ plays a role analogous to the Newtonian potential, $A$ represents the spacetime frame dragging, $W$ specifies the angular geometry, and $k$ is a conformal factor on the space of the orbits in the $(r,z)$ plane. The line element in Equation (1) is not in the Newtonian gauge, due to the off-diagonal term, so that we cannot expect to find the Newtonian equation of dynamics. The essentially non-Newtonian feature of the spacetime dragging can be then described by its angular speed $\chi = -g_{t\phi}/g_{\phi\phi}$, as well as by the field $A$ itself.

By coupling the geometry of Equation (1) to an idealised matter content of collisionless particles (i.e., without intrinsic pressure), we could represent stars and galactic gas.[2] However, the same average dynamics of the system can be more simply expressed by substituting the stochastic, collisionless source with a perfect fluid with effective pressure $p(r,z) := \sigma(r,z)^2\rho(r,z)$, where $\rho(r,z)$ and $\sigma(r,z)$ are the density and velocity dispersion of the particles, respectively. For a typical disc galaxy, we have $\sigma \sim 10\%\, v$, where $v/c \sim 10^{-3}$ is the typical speed of matter [43]. The effective perfect fluid thus has an angular velocity $\Omega(r,z)$ uniquely determined at each point, without dispersion, and an energy–momentum tensor given by

$$T^{\mu\nu} = (\rho(r,z) + p(r,z)/c^2)U^\mu U^\nu + p(r,z)g^{\mu\nu}, \quad (2)$$

where $\mathbf{U} := U^\mu\partial_\mu$ represents the fluid element four-velocity defined as

$$\mathbf{U} := U^\mu\partial_\mu = (-H(r,z))^{-1/2}(\partial_t + \Omega(r,z)\partial_\phi) \quad (3)$$

where $H(r,z)$ is a normalisation factor, such that $U^\mu U_\mu = -c^2$. We then note that, by employing the energy–momentum tensor in Equation (2), a central galactic bulge—which is pressure supported—is allowed within our disc galaxy description.

Here, we are considering a physical system that exhibits two preferential systems of reference for the study of its physics: (i) the stationary one, whose tetrad expresses the metric (1), and (ii) the one built by the zero angular momentum observer (ZAMO), which rotates with respect to the stationary observer (SO) with the relative angular speed $\chi(r,z)$. In GR, different classes of observers are mathematically represented by point-wise defined coframes. From the coframes defining the observers and the dust four-velocity, we can directly calculate the matter speed as measured by SO and ZAMO; see the exact calculations in [17,18,23,25]. The fluid element speed as measured by SO is

$$v_K = \frac{W\Omega}{\gamma_K} e^{-\Phi/c^2}\,, \tag{4}$$

where $\gamma_K$ is the time component of the four-velocity of the fluid elements according to SO, i.e., the Lorentz factor for SO. We notice that Equation (4) shows that, in (3), $\Omega = U^\phi/U^t$ identifies the angular velocity of the fluid elements as measured by SO and not ZAMO. In particular, ZAMO measures a matter speed

$$v_Z = \frac{W}{-H\gamma_Z^2}(\Omega - \chi)\,, \tag{5}$$

where $\gamma_Z$ is the effective Lorentz factor for ZAMO. Furthermore, the dragging speed can be defined as the relative speed between these two reference frames, so that

$$v_D = \frac{g_{\phi\phi}}{W}\chi\,. \tag{6}$$

What we have previously called "the typical speed of matter" $v$ can now be identified with $v_K$ or $v_Z$. Thus, in real disc galaxies, we find subrelativistic speeds, i.e., $v_K/c$, $v_Z/c \le v/c \sim 10^{-3}$, and weak gravitational accelerations, namely such that a nonrelativistic particle will not be accelerated to relativistic speeds over the typical scale length of the galaxy. Conventionally, imposing these conditions is taken to be equivalent to assert that the GR field equations, as well as the equations of motions, can be approximated with the Newtonian ones, plus some post-Newtonian correction, which can contribute only for a fraction $v/c \sim 10^{-3}$ anyway. Nevertheless, this is not necessarily the case [12,13,15,17–20,22–26]. Here, we perform the low-energy limit (LEL) by expanding the above equations in terms of $v_K/c, v_Z/c$ and by considering them at the first orders. The LEL is found for a $\Phi$ and a $k$ field of the same order of magnitude of $v^2$, and substituting $W \approx r$, $A \approx r^2\chi/c^2$, where the latter approximations can be derived under the LEL assumption directly from Eintein's field equations for the geometry in Equation (1) and the energy–momentum tensor in Equation (2) (see, e.g., [26]).[3] By imposing the normalisation of the four-velocity, one then finds [23,25,26]

$$H \approx -1 + \frac{v_K^2}{c^2} - 2\frac{v_K v_D}{c^2} - 2\frac{\Phi}{c^2}. \tag{7}$$

The three characteristic speeds of the model instead reduce to $v_K \approx r\Omega$, $v_Z \approx r(\Omega - \chi)$, and $v_D \approx r\chi$. Thus, we find that Equation (5) takes the (approximate) form

$$v_Z \approx v_K - v_D. \tag{8}$$

Let us now suppose real disc galaxies to be indeed surrounded by a dragging vortex of speed $v_D(r,z)$ with an order of magnitude comparable to $v_K$ and $v_Z$. In [23,26], from the non-Newtonian terms in the equations of motion and Einstein's equations, the authors estimate the $v_D(r,0)$ profile required to sustain a relevant fraction of the rotation curve of a MW-like disc galaxy—conventionally fully attributed to the gravitational contribution of dark matter. Such a fraction cannot exceed one, or it would impact baryonic contributions, i.e., would imply less mass in baryons than that directly measured. This implies $v_D \lesssim 10^{-4}c$. Then, Equation (8) shows that the two speeds naturally defined in the mathematical framework, $v_K$ and $v_Z$, display a non-negligible difference. Therefore, the metrological question arises of what is the "physical speed", i.e., whether the speed we usually infer for the rotation of stars and gas in galaxies should be identified with $v_K$, with $v_Z$, or even with some combination of them. We stress that this is an exquisitely relativistic question, since it arises from the a priori equivalence between reference frames; it can be relevant only in the non-Newtonian realisation of GR described above, with $v_D \sim v_K, v_Z$, which returns a non-negligible difference $(v_K - v_Z)/v_K \sim 10\%$ between the measured speeds.

We note that, in the literature (see [13,15,22,27]), it is often claimed that the "correct speed" we should use is the one measured by the inertial frame at infinity. However, here, we have to stress that such a property, i.e., to "be an inertial reference frame at spatial infinity", does not uniquely define a general–relativistic reference frame. Indeed, let us impose on the line element in Equation (1) the boundary condition of asymptotic Minkowskianity, as is it natural to do for an isolated system. Moreover, let us assume that $\Omega$ tends to zero at spatial infinity. Then, it is easily recognised that both SO and ZAMO define asymptotically inertial reference frames at $r, z \to \infty$, since both their time-like coordinate vectors are asymptotic to $\partial_t$. This arbitrariness is due to the nature of general–relativistic reference frames, which are point-wise defined, in contrast to special–relativistic ones. If we want to correctly identify our empirically inferred speed within the mathematically defined quantities, we cannot seek for an absolute observer. Instead, we have to exactly formalise the measuring procedure we are employing to infer the matter speed, and then obtain through the mathematical framework its correct relation to $v_K$, $v_Z$, $v_D$, and the other relevant physical quantities.

As a first example, we will consider here the conventionally considered matter speed $v_{\text{obs}}$ for a far away disc galaxy, i.e., the speed displayed in disc galaxy rotation curves plots. This is inferred via measurements of the redshift $z$ of the light emitted by stars or by the 21 cm line of gas particles. After the subtraction of the cosmological redshift given by the expansion of the universe, the remaining redshift is commonly interpreted with the special–relativistic Doppler shift formula $1 + z = \gamma_{obs}(1 + \cos\theta v_{\text{obs}}/c)$, where $\theta$ is the angle between the geodesic of the photon and the Killing vector $\partial_\phi$. On the other hand, we can calculate the same frequency shift in terms of our solution ((1)–(3)). For the idealized case of a detector placed at the Minkowskian infinity of our metric, and still with respect to the centre of the galaxy, the full frequency shift is given by [17–19,23]

$$\begin{aligned} 1 + z_{\text{drag}} &= (-H)^{-1/2}\left[1 + e^{-2\Phi/c^2}\frac{\Omega}{c}\left(W\cos\theta + \frac{g_{\phi\phi}\chi}{c}\right)\right] = \\ &= (-H)^{-1/2}\left[1 + e^{-\Phi/c^2}\gamma_K\frac{v_K}{c}\left(\cos\theta + \frac{v_D}{c}\right)\right]. \end{aligned} \tag{9}$$

For the energy regime of a typical galaxy, it has been shown that $v_{\text{obs}} \approx v_K \approx r\Omega$. That is, the inferred matter speed corresponds to the one measured by SO, and not with the speed $v_Z$ measured by ZAMO in the case of non-negligible frame dragging. Indeed, Equation (9) returns, at the first order, for the case of an edge-on galaxy

$$1 + \frac{v_{\text{obs}}}{c} \approx 1 + z_{\text{drag}} \approx 1 + \frac{v_K}{c}. \tag{10}$$

## 3. CMB in a Strong Frame-Dragging Vortex

We will now consider a completely new speed measurement procedure within our theoretical framework, i.e., the measure of our own speed inside our galaxy, performed by looking at the apparent kinetic dipole of the CMB distribution in the sky. This is indeed, a priori, a completely different technique for a speed measure. Moreover, we want to study the consequences of the presence of a strong dragging $v_D \sim v_K$ on the CMB multipole coefficients of higher order and ultimately answer to the question: *would the presence of a strong dragging vortex be detectable via CMB data analysis?*

In Equation (9), we wrote the frequency shift experienced by a photon emitted by a fluid element sourcing a dragging metric of the kind of Equation (1) and detected by an observer positioned at the Minkowskian infinity and still with respect to the galactic centre. The very same formula gives us the frequency shift for the opposite case, i.e, a photon emitted by a still, infinitely distant light source and detected by a matter particle within a dragging solution. Essentially, this represents the formula for a CMB photon, in the ideal case in which the galaxy is still in the CMB system of reference (CMB SoR), and in which the detecting satellite has exactly the four-velocity **U** of Equation (3). Indeed, given a random direction $\hat{n}$ of arrival of the photon, we have just to substitute $\cos\theta = \hat{n} \cdot \partial_\phi$ in Equation (9) to calculate the resulting redshift. In this case, we can recognize a distortion, with respect to the usual special–relativistic formula, due to the general–relativistic parameters $\Phi$ and $\chi$. However, both the general–relativistic terms will affect the frequency shift only from the second order of expansion onward.

To have a realistic prediction, we must complicate the above ideal description. We have to take into account the galactic proper motion with respect to the CMB SoR; the satellite's motion with respect to the self-gravitation matter **U**; and the CMB intrinsic noise. In principle, this would mean studying the dragging metric $\mathbf{g}_{\text{drag+FLRW}}$ for a galaxy surrounded by an expanding universe, i.e., a time-dependent generalization of Equation (1) that is spatially asymptotic to FLRW, rather than Minkowski. This would be a formidable mathematical task, which we can, however, simplify for our purposes.

Let us consider an ensemble of CMB photons, emitted at the surface of last scattering, i.e., $z \cong 1100$ in the standard $\Lambda$CDM cosmology, with average temperature $T_e$. For billions of years they travel on the geometry defined by the metric $\mathbf{g}_{\text{drag+FLRW}}$, but for most of this time they are so far from the dragging vortex that they do not feel its effects. Essentially, they travel in the FLRW asymptotic region, and they experience only the usual cosmological redshift, $z_{\text{FLRW}}$. Once they arrive at the periphery of the MW, say, crossing the galactic radius $r \cong r_G$, the photons acquired the average temperature $T_i(\hat{n}) \cong T_e/(1 + z_{\text{FLRW}})$; with the intrinsic noise, dependent on the direction of arrival $\hat{n}$, that is the anisotropy usually provided by $\Lambda$CDM.

We can now perform a boost from the CMB SoR to the reference frame of the centre of the MW. This means a special–relativistic Doppler shift $1 + z_{b1} = \gamma_1(1 + \vec{v}_1 \cdot \hat{n}/c)$, if we call $\vec{v}_1$ the relative speed between the MW's centre and the CMB SoR. Once this boost has been performed, the photons have to complete their journey from the galactic periphery $r \cong R_G$ up to the satellite that detects them. However, such a short route is completed in a very short time, when compared to the cosmological scales, so that the time dependence of $g_{\text{drag+FLRW}}$ is negligible, and the metric can be approximated by the geometry depicted in Equation (1). The contribution to the frequency shift is thus given by Equation (9).

Finally, we should also take into account the satellite's motion, which has, in general, some velocity $\vec{v}_2$ with respect to the average matter four-velocity **U**. This means a second special–relativistic boost, which carries an ulterior Doppler shift $z_{b2}$. Overall, our approximation allows us to trivially multiply all these frequency shifts contributions, i.e.,

$$z_{\text{TOT}} \cong (1 + z_{\text{FLRW}})(1 + z_{b1})(1 + z_{\text{drag}})(1 + z_{b2}) \,, \tag{11}$$

so that

$$T_d(\hat{n}) \cong \frac{T_i(\hat{n})}{(1 + z_{b1})(1 + z_{\text{drag}})(1 + z_{b2})} \,. \tag{12}$$

Furthermore, since the contributions of the two boosts are just multiplied, Special Relativity (SR) allows us to dial the velocities as $\vec{v}_b = \vec{v}_1 \oplus \vec{v}_2$, where we denote with $\oplus$ the velocity-addition formula of SR, so that we can consider a single boost

$$[1 + z_b(\vec{v}_1)][1 + z_b(\vec{v}_2)] =: 1 + z_b(\vec{v}_b) \approx 1 + \frac{\vec{v}_b}{c} \cdot \hat{n} + \frac{v_b^2}{2c^2}\,, \tag{13}$$

between the CMB and the satellite's SoR. Moreover, we can write in the LEL, up to the second order, Equation (9) as

$$\begin{aligned}1 + z_{\rm drag} &= \left[1 + \frac{v_K^2}{2c^2} - \frac{v_K v_D}{c^2} - \frac{\Phi}{c^2} + \mathcal{O}\left(v^3/c^3\right)\right]\left[1 + \frac{v_K}{c}\cos\theta + \frac{v_K v_D}{c^2} + \mathcal{O}\left(v^3/c^3\right)\right] \\ &= 1 + \frac{\vec{v}_K \cdot \hat{n}}{c} + \frac{v_K^2}{2c^2} - \frac{\Phi}{c^2} \approx [1 + z_b(\vec{v}_K)]\left[1 - \frac{\Phi}{c^2}\right] + \mathcal{O}\left(v^3/c^3\right).\end{aligned} \tag{14}$$

Here, we define the vector observed velocity as $\vec{v}_K = v_K \partial_\phi$, and analogously the vector dragging velocity is $\vec{v}_D = v_D \partial_\phi$. In this way, we can appreciate how this formula for the frequency shift is analogous to the special–relativistic one, with some general–relativistic corrections. Interestingly, even at the second order in the expansion, we have no direct contribution from the off-diagonal terms (the dragging), but only from the on-diagonal ones. Furthermore, in analogy with what has been done before, we can define the total velocity by dialing $\vec{v}_{\rm TOT} = \vec{v}_b \oplus \vec{v}_K$. Given our approximation, and using Equations (14) and (12), we are now able to express how the intrinsic CMB anisotropies are distorted in the system of reference of the detecting satellites if these are embedded in a GR dragging metric

$$\begin{aligned}T_d(\hat{n}) &\cong \frac{T_i(\hat{n})}{[1 + z_b(\vec{v}_b)][1 + z_{\rm drag}]} = \frac{T_i(\hat{n})}{[1 + z_b(\vec{v}_{\rm TOT})][1 - \Phi/c^2]} + \mathcal{O}\left(v^3/c^3\right) \\ &= T_i(\hat{n})\left[1 - \frac{\vec{v}_{\rm TOT} \cdot \hat{n}}{c} + \frac{(\vec{v}_{\rm TOT} \cdot \hat{n})^2}{c^2} - \frac{v_{\rm TOT}^2}{2c^2} + \frac{\Phi}{c^2}\right] + \mathcal{O}\left(v^3/c^3\right).\end{aligned} \tag{15}$$

Here, we stress again that $\vec{v}_{\rm TOT}$ is the total velocity of the satellite in the CMB SoR. Again, we do not see any contribution from the dragging $v_D$, up to the second order. Such terms can be found only from the third order onwards. Indeed, a more detailed calculation (see Appendix A) gives us the following formula:

$$\begin{aligned}\frac{T_d(\hat{n})}{T_i(\hat{n})} =& 1 - \frac{\vec{v}_{\rm TOT} \cdot \hat{n}}{c} + \frac{1}{c^2}\left[\Phi - \frac{v_{\rm TOT}^2}{2} + (\vec{v}_{\rm TOT} \cdot \hat{n})^2\right] \\ &+ \frac{1}{c^3}\left[(\vec{v}_{\rm TOT} \cdot \hat{n})\left(\frac{v_{\rm TOT}^2}{2} - \Phi\right) + (\vec{v}_K \cdot \hat{n})\left(\Phi - \frac{v_K^2}{2} + v_K v_D\right) - (\vec{v}_{\rm TOT} \cdot \hat{n})^3\right] + \mathcal{O}\left(v^4/c^4\right).\end{aligned} \tag{16}$$

From (16), we find that it is possible to detect a strong dragging $v_D \sim 10^{-4}c$ from the third order term $(\vec{v}_K \cdot \hat{n})v_K v_D/c^3$, or even from a comparison between the other third order terms in $v_{\rm TOT}$ and those in $v_K \approx v_{\rm TOT} - v_b$. However, the practical application of this idea would require an implausibly exquisite measure of the CMB dipole, since the crucial term $(\vec{v}_K \cdot \hat{n})v_K v_D/c^3$ has a magnitude of $\sim 10^{-7}$ times the magnitude of the dominant dipole term $-\vec{v}_{\rm TOT} \cdot \hat{n}/c$ (see the discussion in Section 4.2).

## 4. CMB Multipole Analysis and Frame-Dragging Estimation

Now we will apply the above results to the study of the CMB landscape across the sky, studying the consequences on its stochastic and systematic multipole coefficients. The CMB anisotropies are conventionally [32,33] expanded in multipoles, i.e.,

$$T(\hat{n}) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{T,lm} Y_l^m(\hat{n}) \,, \tag{17}$$

where

$$a_{T,lm} = \int_{S^2} T(\hat{n}) Y_l^m(\hat{n})^* d^2\hat{n} \,. \tag{18}$$

Such a decomposition can be performed both for the intrinsic and the detected temperature. In particular, we can immediately recognise that $a^d_{T,00} = a^i_{T,00}(1 + \mathcal{O}(v^2/c^2))$, since

$$\int_{S^2} T_d(\hat{n}) d^2\hat{n} = \int_{S^2} T_i(\hat{n}) \left[1 + \frac{(\vec{v}_{\rm TOT} \cdot \hat{n})^2}{c^2} - \frac{v^2_{\rm TOT}}{2c^2} + \frac{\Phi}{c^2} + \mathcal{O}\left(v^3/c^3\right)\right] d^2\hat{n} = \int_{S^2} T_d(\hat{n})(1 + \mathcal{O}\left(v^2/c^2\right)) d^2\hat{n} \tag{19}$$

Thus, the average CMB temperature suffers from a systematic correction of only $\sim 1$ μK. Moreover, we can see from (15) that the kinematic second-order corrections affect only the CMB coefficients starting from the quadrupole term $\propto \cos^2\theta$, as there is no effect on the dipole. Thus, we are interested in the multipole expansion up to the quadrupole. The relative spherical harmonics can be made explicit, as

$$2\sqrt{\pi} Y_0^0(\hat{n}) \equiv 1 \,, \tag{20}$$

$$2\sqrt{\frac{\pi}{3}} Y_1^0(\hat{n}) = \hat{n} \cdot \hat{e}_3 \,, \tag{21}$$

$$\pm 2\sqrt{\frac{2}{3}\pi} Y_1^{\pm 1}(\hat{n}) = -\hat{n} \cdot (\hat{e}_1 \pm i\hat{e}_2) \,, \tag{22}$$

$$4\sqrt{\frac{\pi}{5}} Y_2^0(\hat{n}) = 3(\hat{n} \cdot \hat{e}_3)^2 - 1 \,, \tag{23}$$

$$\pm 2\sqrt{\frac{2\pi}{15}} Y_2^{\pm 1} = -(\hat{n} \cdot \hat{e}_3)[\hat{n} \cdot (\hat{e}_1 \pm i\hat{e}_2)] \,, \tag{24}$$

$$4\sqrt{\frac{2\pi}{15}} Y_2^{\pm 2} = [\hat{n} \cdot (\hat{e}_1 \pm i\hat{e}_2)]^2 \,. \tag{25}$$

Here, we expressed these harmonics with respect to a given coordinate frame $\{\hat{e}_j\}_{j=1}^3 \in SO(3)$. The multipole expansion depends on such an arbitrary choice. Therefore, we should more rigorously write $a_{T,lm} = a_{T,lm}(\hat{e}_j)$ and $Y_l^m(\hat{n}) = Y_l^m(\hat{n}|\hat{e}_j)$. We recall that the intrinsic CMB anisotropy is expected to be purely stochastic within the ΛCDM framework, so that, by taking the average of any multipole quantity with respect to the frame choice, we should have

$$\langle a^i_{T,lm} \rangle = \frac{1}{8\pi^2} \int_{SO(3)} a^i_{T,lm}(\hat{e}_j) d^3\hat{e}_j = 0 \,, \tag{26}$$

as well as

$$\langle a^i_{T,lm} (a^i_{T,l'm'})^* \rangle = C^i_{TT,l} \delta_{ll'} \delta_{mm'} \,. \tag{27}$$

The only exception to (26) should be given by the $l = 0$ case, which gives the average temperature of the CMB, i.e., $\langle T_i \rangle = a^i_{T,00}/2\sqrt{\pi} \cong 2.7$ K. Moreover, it is the custom to normalise the variances $C$ as

$$D_l^{TT,i} = \frac{l(l+1)}{2\pi} C^i_{TT,l} \,. \tag{28}$$

We note that the empirical measures of the $D_l$s are mostly in agreement with the predictions of the $\Lambda$CDM cosmological model. The only relevant tension is related to the quadrupole variance, which should be given by $D_2^{TT,i} \cong 1170\,\mu\text{K}^2$ according to $\Lambda$CDM, but it is found to be $D_2^{TT,d} \cong 200\,\mu\text{K}^2$ [36–38].

However, from Equation (15), we see how the detector's SoR confers systematic multipole coefficients to the temperature profile, so that we can expect $\langle a^d_{T,lm}\rangle \neq 0$, with an order of magnitude of $\sim \langle T\rangle v^l/c^l$ in general. It is then useful to study such systematic multipoles in a coordinate frame that is aligned to the detector's kinematic, e.g., $\hat{v}_{\text{TOT}} := \hat{e}_3$ and $\hat{v}_K = \hat{v}_D = \partial_\phi := \sin\alpha\hat{e}_1 + \cos\alpha\hat{e}_3$, where $\alpha$ is the angle between $\vec{v}_{\text{TOT}}$ and $\partial_\phi$, could be a good choice. For such coordinates, Equation (15) becomes

$$\begin{aligned} T_d(\hat{n}) &= \left[\sum_{l,m} a^i_{T,lm} Y^m_l(\hat{n})\right]\left[\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{2c^2}\right) - \frac{v_{\text{TOT}}}{c} n_3 + \frac{v^2_{\text{TOT}}}{c^2} n_3^2\right] + \mathcal{O}\left(v^3/c^3\right) \\ &= \sum_{l=1}^{2}\sum_{m=-l}^{l} a^i_{T,lm}\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{2c^2}\right) Y^m_l - \frac{a^i_{T,00}}{2\sqrt{\pi}}\frac{v_{\text{TOT}}}{c} 2\sqrt{\frac{\pi}{3}} Y^0_1 - a^i_{T,10}\frac{v_{\text{TOT}}}{c}\left(\frac{1}{\sqrt{3}}Y^0_0 + \frac{2}{\sqrt{3}}Y^0_2\right) \\ &\quad - \sum_{\pm} a^i_{T,1,\pm 1}\frac{v_{\text{TOT}}}{c}\frac{Y^{\pm 1}_2}{\sqrt{5}} + \frac{a^i_{T,00}}{2\sqrt{\pi}}\frac{v^2_{\text{TOT}}}{c^2}\left(\frac{2}{3}\sqrt{\pi}Y^0_0 + \frac{4}{3}\sqrt{\frac{\pi}{5}}Y^0_2\right) + \mathcal{O}\left(v^3/c^3\right), \end{aligned} \tag{29}$$

where $n_3 = \hat{n}\cdot\hat{e}_3$. From here, we can calculate the multipole coefficients via Equation (18). Thus, up to the dipole, we have

$$a^d_{T,00} = a^i_{T,00}\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{2c^2}\right) - a^i_{T,10}\frac{v_{\text{TOT}}}{c}\frac{1}{\sqrt{3}} + \frac{a^i_{T,00}}{2\sqrt{\pi}}\frac{v^2_{\text{TOT}}}{c^2}\frac{2}{3}\sqrt{\pi} + \mathcal{O}\left(v^3/c^3\right), \tag{30}$$

$$a^d_{T,10} = a^i_{T,10}\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{2c^2}\right) - \frac{a^i_{T,00}}{2\sqrt{\pi}}\frac{v_{\text{TOT}}}{c} 2\sqrt{\frac{\pi}{3}} + \mathcal{O}\left(v^3/c^3\right), \tag{31}$$

$$a^d_{T,1,\pm 1} = a^i_{T,1,\pm 1}\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{2c^2}\right) + \mathcal{O}\left(v^3/c^3\right). \tag{32}$$

Hence, it follows that

$$\begin{aligned} 2\sqrt{\pi}\langle T_d\rangle &= \langle a^d_{T,00}\rangle = \left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{2c^2} + \frac{v^2_{\text{TOT}}}{3c^2}\right)\langle a^i_{T,00}\rangle - \frac{v_{\text{TOT}}}{\sqrt{3}c}\langle a^i_{T,10}\rangle + \mathcal{O}\left(v^3/c^3\right) \\ &= 2\sqrt{\pi}\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{6c^2}\right)\langle T_i\rangle + \mathcal{O}\left(v^3/c^3\right), \end{aligned} \tag{33}$$

$$\begin{aligned} \langle a^d_{T,10}\rangle &= \left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{2c^2}\right)\langle a^i_{T,10}\rangle - \frac{v_{\text{TOT}}}{\sqrt{3}c}\langle a^i_{T,00}\rangle + \mathcal{O}\left(v^3/c^3\right) = -2\sqrt{\frac{\pi}{3}}\langle T_i\rangle\frac{v_{\text{TOT}}}{c} + \mathcal{O}\left(v^3/c^3\right) \\ &= -2\sqrt{\frac{\pi}{3}}\left(1 - \frac{\Phi}{c^2} + \frac{v^2_{\text{TOT}}}{6c^2}\right)\langle T_d\rangle\frac{v_{\text{TOT}}}{c} + \mathcal{O}\left(v^3/c^3\right) = -2\sqrt{\frac{\pi}{3}}\langle T_d\rangle\frac{v_{\text{TOT}}}{c} + \mathcal{O}\left(v^3/c^3\right), \end{aligned} \tag{34}$$

$$\langle a^d_{T,1,\pm 1}\rangle = \left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\text{TOT}}}{2c^2}\right)\langle a^i_{T,1,\pm 1}\rangle + \mathcal{O}\left(v^3/c^3\right) = \mathcal{O}\left(v^3/c^3\right). \tag{35}$$

### *4.1. Measure of Galaxy Rotation from Systematic Dipole*

In SR, the dipole has already an $m = 0$ coefficient with non-zero average, which describes the motion $\vec{v}_{\text{TOT}}$ in the CMB SoR. Indeed, in this way it is possible to measure the direction, $\hat{e}_3$, and the intensity of such velocity, i.e., $|v_{\text{TOT}}| \approx \sqrt{3}\left(\langle a^d_{T,10}\rangle/\langle a^d_{T,00}\rangle\right)c$. Here, we recall that the current best fit to CMB data gives $|v_{\text{TOT}}| = (1.231 \pm 0.003)10^{-3}c$ [44].

The very same relation holds for the magnitude of the velocity up to the first order also in our GR calculations. Since the current measure of this quantity has a precision of $\approx 0.3\% \gg 10^{-6}$, the higher-order terms are not appreciable with the current data. Hence, here we have a measure of the satellite's speed $\vec{v}_{\mathrm{TOT}} = \vec{v}_b \oplus \vec{v}_K$ in the SoR where, again, $v_K$ is the galaxy rotation speed as described by SO. This is the first noteworthy result we anticipated: for both systematic CMB dipole measurements (observed in the MW) and of rotation curves inferred via frequency shift data (observed for a far galaxy), what one measures can be always identified to the matter's speed $v_K$ according to the SO, and not the matter's speed $v_Z$ according to ZAMO.

*4.2. Measure of Dragging from Systematic Multipole Coefficients*

In the special–relativistic case, we recall that $v_D = 0$, so that the $m = \pm 1$ dipole coefficients are expected to have a zero average, if it is chosen that $\hat{e}_3 = \hat{v}_{\mathrm{TOT}}$, as we did. In GR, we see a non-zero average emerge for both $a^d_{T,1,\pm 1}$, although this is only a third-order quantity. We can estimate it by repeating the previous multipole expansion with the more exact formula given in Equation (16). The calculation proceeds analogously, and the only terms in $Y_1^{\pm 1}$ found in $T_d(\hat{n})$ are

$$\cdots + \frac{a^i_{T,00}}{2\sqrt{\pi}} \frac{v_K \sin\alpha}{c^3} \left( \Phi - \frac{v_K^2}{2} + v_K v_D \right) \sqrt{\frac{2}{3}} \pi (Y_1^{-1} - Y_1^1) + \dots, \quad (36)$$

which derive from the novel term $(\vec{v}_K \cdot \hat{n})(\Phi + v_K v_D - v_K^2/2)/c^3$ in (16). We thus find the dipole coefficient of misalignment to be

$$\langle a^d_{T,1,\pm 1} \rangle \approx 2\sqrt{\frac{2}{3}} \pi \sin\alpha \langle T \rangle \frac{v_K}{c} \left( \frac{\Phi}{c^2} - \frac{v_K^2}{2c^2} + \frac{v_K v_D}{c^2} \right). \quad (37)$$

In principle, it would be possible to measure such a coefficient for the CMB, checking if a dragging speed $v_D$ is needed to explain the data. However, we believe such a test to be ultimately infeasible.

Indeed, it would be extremely challenging to separate the impact of $v_D$ on the data from the contributions of $v_K$ and $\Phi$. Both $v_K$ and $\Phi$ should be obtained from some other relations between multipole coefficients, which will in turn have to involve higher-order quantities given by coefficients with $l > 1$. The precision required for all the needed measurements is currently unachievable. Additionally, we stress that these higher-order contributions will also be masked by the cosmic variance intrinsic in the CMB observations [30,45], further complicating their possible detection. Moreover, we note that the exact direction of the coordinates $\hat{e}_3, \hat{e}_1$ are subject to some errors, which would then impact any dragging measure via Equation (37). We can formalize this statement by saying that, in fact, $\hat{v}_{\mathrm{TOT}}$ is not exactly pointed towards $\hat{e}_3$, but it is instead given by $\cos\varepsilon \hat{e}_3 + \sin\varepsilon \hat{e}_1$, for some small angle $\varepsilon$. In the same way, $\partial_\phi = \cos(\alpha - \varepsilon)\hat{e}_3 + \sin(\alpha - \varepsilon)\hat{e}_1$, where $\alpha \cos^{-1}(\hat{v}_{\mathrm{TOT}} \cdot \vec{v}_K)$ is fixed. Under these assumptions, the systematic dipole coefficients become

$$\langle a^d_{T,10} \rangle \approx -2\sqrt{\frac{\pi}{3}} \cos\varepsilon \langle T \rangle \frac{v_{\mathrm{TOT}}}{c}, \quad (38)$$

$$\langle a^d_{T,1,\pm 1} \rangle \approx \mp 2\sqrt{\frac{2}{3}} \pi \langle T \rangle \left[ \frac{v_K}{c} \left( \frac{\Phi}{c^2} - \frac{v_K^2}{2c^2} + \frac{v_K v_D}{c^2} \right) \sin(\alpha - \varepsilon) - \frac{v_{\mathrm{TOT}}}{c} \left( 1 - \frac{v_{\mathrm{TOT}}^2}{2c^2} + \frac{\Phi}{c^2} \right) \sin\varepsilon \right]. \quad (39)$$

If the intensity and direction of the velocity $\vec{v}_{\rm TOT}$ is determined from the CMB data, the parameter $\varepsilon$ is fixed in order to make the misalignment disappear, so that $\langle a^d_{T,1,\pm1}\rangle \equiv 0$. This can be done for both $m = \pm 1$ with the choice

$$\varepsilon \approx \sin\varepsilon \approx \sin\alpha \frac{v_K}{v_{\rm TOT}}\left(\frac{\Phi}{c^2} - \frac{v_K^2}{2c^2} + \frac{v_K v_D}{c^2}\right). \tag{40}$$

We stress that this is the natural choice and there is no way to check it, except by considering even higher-order coefficients. The latter is a consequence of the dominant role of the special–relativistic kinematic term $-(\vec{v}_{\rm TOT}\cdot\hat{n})/c$ in determining the CMB dipole, masking any higher-order effects. The other $l-$pole coefficients are also dominated by the analogous SR kinematical term $(-\vec{v}_{\rm TOT}\cdot\hat{n}/c)^l$. We show this explicitly in Appendix A. An intuitive understanding of this result for all the $l$ comes by considering the dragging frequency shift (see Equation (9)): the dependence on the direction (i.e., the angle $\theta$) is related only to $v_K$ and it does not appear in a denominator; hence, in the coefficient of any $l$-pole, namely $\cos^l\theta$, the matter speed $v_K$ (or $v_{\rm TOT}$) will dominate with respect to $v_D$. The conclusion is that any measure of the dragging $v_D$ from the systematic CMB multipole coefficients $\langle a^d_{lm}\rangle$ is currently unattainable, requiring a precision of at least $10^{-6}$ on the data.

### *4.3. Quadrupole Tension in the Dragging Framework*

Let us now calculate the quadrupole coefficients.

$$a^d_{T,20} = a^i_{T,20}\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\rm TOT}}{2c^2}\right) - a^i_{T,10}\frac{v_{\rm TOT}}{c}\frac{2}{\sqrt{3}} + \frac{a^i_{T,00}}{2\sqrt{\pi}}\frac{v^2_{\rm TOT}}{c^2}\frac{4}{3}\sqrt{\frac{\pi}{5}} + \mathcal{O}\left(v^3/c^3\right), \tag{41}$$

$$a^d_{T,2,\pm1} = a^i_{T,2,\pm1}\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\rm TOT}}{2c^2}\right) - a^i_{T,1,\pm1}\frac{v_{\rm TOT}}{c}\frac{1}{\sqrt{5}} + \mathcal{O}\left(v^3/c^3\right), \tag{42}$$

$$a^d_{T,2,\pm2} = a^i_{T,2,\pm2}\left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\rm TOT}}{2c^2}\right) + \mathcal{O}\left(v^3/c^3\right). \tag{43}$$

Hence, it follows that

$$\begin{aligned}\langle a^d_{T,20}\rangle &= \left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\rm TOT}}{2c^2}\right)\langle a^i_{T,20}\rangle - \frac{2v_{\rm TOT}}{\sqrt{3}c}\langle a^i_{T,10}\rangle + \frac{2v^2_{\rm TOT}}{3\sqrt{5}c^2}\langle a^i_{T,00}\rangle + \mathcal{O}\left(v^3/c^3\right) \\ &= \frac{4}{3}\sqrt{\frac{\pi}{5}}\langle T\rangle\frac{v^2_{\rm TOT}}{c^2} + \mathcal{O}\left(v^3/c^3\right),\end{aligned} \tag{44}$$

$$\langle a^d_{T,2,\pm1}\rangle = \left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\rm TOT}}{2c^2}\right)\langle a^i_{T,2,\pm1}\rangle - \frac{v_{\rm TOT}}{\sqrt{5}c}\langle a^i_{T,1,\pm1}\rangle + \mathcal{O}\left(v^3/c^3\right) = \mathcal{O}\left(v^3/c^3\right), \tag{45}$$

$$\langle a^d_{T,2,\pm2}\rangle = \left(1 + \frac{\Phi}{c^2} - \frac{v^2_{\rm TOT}}{2c^2}\right)\langle a^i_{T,2,\pm2}\rangle + \mathcal{O}\left(v^3/c^3\right) = \mathcal{O}\left(v^3/c^3\right). \tag{46}$$

As it is for the dipole, the quadrupole also has negligible systematic misalignment coefficients $\langle a^d_{T,2,\pm1}\rangle$, $\langle a^d_{T,2,\pm2}\rangle$, once the coordinates are chosen aligned to $\hat{e}_3||\vec{v}_{\rm TOT}$. At this order, any possible misalignment can be nullified by choosing a suitable angle $\varepsilon$, as in Section 4.2. We note that, because of the relativistic frequency shift, the detected CMB landscape $T_d(\hat{n})$ is no more Gaussian, i.e., it does not possess multipole variances with form (27). Indeed, by substituting, we find

$$\langle a^d_{T,20}(a^d_{T,20})^*\rangle \approx \left(1+\frac{\Phi}{c^2}-\frac{v^2_{\rm TOT}}{2c^2}\right)^2\langle a^i_{T,20}(a^i_{T,20})^*\rangle + \frac{4}{3}\frac{v^2_{\rm TOT}}{c^2}\langle a^i_{T,10}(a^i_{T,10})^*\rangle + \frac{4}{45}\frac{v^4_{\rm TOT}}{c^4}\langle a^i_{T,00}(a^i_{T,00})^*\rangle$$
$$-\frac{4}{\sqrt{3}}\frac{v_{\rm TOT}}{c}\left(1+\frac{\Phi}{c^2}-\frac{v^2_{\rm TOT}}{2c^2}\right)\langle a^i_{T,20}(a^i_{T,10})^*\rangle + \frac{4}{3\sqrt{5}}\frac{v^2_{\rm TOT}}{c^2}\left(1+\frac{\Phi}{c^2}-\frac{v^2_{\rm TOT}}{2c^2}\right)\langle a^i_{T,20}(a^i_{T,00})^*\rangle$$
$$-\frac{8}{3\sqrt{15}}\frac{v^3_{\rm TOT}}{c^3}\langle a^i_{T,10}(a^i_{T,00})^*\rangle$$
$$=\left(1+\frac{\Phi}{c^2}-\frac{v^2_{\rm TOT}}{2c^2}\right)^2 C^i_{TT,2} + \frac{4}{3}\frac{v^2_{\rm TOT}}{c^2}C^i_{TT,1} + \frac{16\pi}{45}\frac{v^4_{\rm TOT}}{c^4}\langle T\rangle^2\,, \tag{47}$$

$$\langle a^d_{T,2,\pm1}(a^d_{T,2,\pm1})^*\rangle \approx \left(1+\frac{\Phi}{c^2}-\frac{v^2_{\rm TOT}}{2c^2}\right)^2 C^i_{TT,2} + \frac{1}{5}\frac{v^2_{\rm TOT}}{c^2}C^i_{TT,1}\,, \tag{48}$$

$$\langle a^d_{T,2,\pm2}(a^d_{T,2,\pm2})^*\rangle \approx \left(1+\frac{\Phi}{c^2}-\frac{v^2_{\rm TOT}}{2c^2}\right)^2 C^i_{TT,2}\,, \tag{49}$$

while $\langle a^d_{T,2,m}(a^d_{T,2,m'})^*\rangle = \mathcal{O}(v^6/c^6)$ for all the other cases $m \neq m'$. Because of this non-Gaussianity, the "detected quadrupole", $C^d_{TT,2}$, is not uniquely defined, and each of these three quantities could be legitimately identified with it. Moreover, under the hypothesis that the observed quadrupole tension is only determined by the systematic frequency shift—either in SR or in GR—the intrinsic quantities $C^i_{TT,l}$ should coincide with the ΛCDM predictions for the early growth of structures. In particular, $C^i_{TT,2} \cong (3/\pi)\cdot 1170\,\mu\mathrm{K}^2 \cong 1.225\cdot 10^{-9}\,\mathrm{K}^2$. On the other hand, since $v_{\rm TOT} \cong 1.231\cdot 10^{-3}c$, the last term $(16\pi/45)\cdot(v^4_{\rm TOT}/c^4)\langle T\rangle \cong 1.87\cdot 10^{-11}\,\mathrm{K}^2$ would be negligible. We note that the GR corrections in the factor $1+(\Phi/c^2)-(v^2_{\rm TOT}/2c^2)$ would be negligible as well, as $v^2_{\rm TOT}/c^2 \sim 10^{-6}$ and, according to the LEL, $\Phi/c^2$ has the same order of magnitude. Indeed, it would require a huge Newtonian potential, namely $\Phi \cong -0.59c^2$, to justify the observed tension $200\,\mu\mathrm{K}^2 \cong D^{TT,d}_2 \approx (1+\Phi/c^2)D^{TT,i}_2$. However, this cannot be the case, as such a non-negligible GR correction would give $1+(\Phi/c^2)-(v^2_{\rm TOT}/2c^2) \cong 0.41$, which would also affect the higher multipoles $D^{TT,d}_{l>2}$, that, in turn, are coherent with the ΛCDM previsions $D^{TT,i}_{l>2}$. Thus, the only non-negligible correction can come from the terms $\propto (v^2_{\rm TOT}/c^2)C^i_{TT,1}$. However, these are necessarily *positive*, being unable to justify a *reduction* of $D^{TT}_2$, as instead it is observed. Therefore, we can conclude that the systematic kinematic corrections from SR and GR cannot explain the tension on the CMB quadrupole.

Finally, we point out that our conclusions remain true even within a more general framework. Let us suppose that a different relativistic model returns a correction $a^d_{T,20} := K_2 a^i_{T,20} + K_1 a^i_{T,10} + K_0 a^i_{T,00}$ on the quadrupole coefficient. The detected quadrupole would then be given by $C^d_{TT,0} = K^2_2 C^i_{TT,2} + K^2_1 C^i_{TT,1} + 4\pi K^2_0\langle T\rangle$. It must be $K_2 \cong 1$, following the previous argument on the factor $1+(\Phi/c^2)-(v^2_{\rm TOT}/2c^2)$. Then, the two remaining terms in $K^2_1$, $K^2_0$ can only *increase* the quadrupole, since they cannot be negative. The negative tension of the observed to predicted quadrupole $\cong 200(\mu\mathrm{K})^2 - 1170(\mu\mathrm{K})^2$ cannot be explained in any way via peculiar GR corrections associated only to the Galaxy patched on a ΛCDM background. We note that in this work we assumed the local cosmological structure to determine only peculiar velocities within the bulk flow; if this were not to be the case, then further calculations would be required.

## 5. Conclusions

In this paper, we studied the consequences on the CMB temperature landscape, as it would be detected from realistic measurement procedures by a satellite, of a peculiar GR effect without Newtonian analogue, i.e., a strong off-diagonal component of the galactic metric. We thus provide novel insight into such an unexpected, purely GR phenomenon that has been discussed in the recent literature [10,12–27,46,47].

We have described the frequency shift experienced by a CMB photon travelling through a spacetime described patch-wise by an FLRW and dragging metric. We derived the corrections to its temperature, depending on the direction of arrival, up to the fourth order for relative subrelativistic speeds between emitter and observer, and weak gravitational potentials—including the dragging term. We then performed a multipole analysis and found that the speed measured from the systematic CMB dipole is obtained by dialing the Milky Way velocity with respect to the CMB system of reference and the rotation speed of the satellite around the galaxy, where the latter is identified with $v_K$.

We point out that the result mentioned above is noteworthy, for in a non-diagonal metric, as that in Equation (1), there is no unique natural reference frame. Indeed, in the current literature, it is debated whether the speed of the galactic matter should be described according to the zero angular momentum observer or the stationary observer in such metrics [13,15,22,23,27,48]. In general, this depends on the exact technique one uses to infer, or measure, the speed. We have found that, for what regards the technique exploiting the CMB dipole, the identified speed is the one described by the stationary observer. This is the same result that was already found for the technique that makes use of the frequency shift while looking to far away galaxies to infer their rotation curves [17–19,23].

Continuing with the multipole analysis, we showed how it would be extremely difficult to detect the presence of the hypothetical dragging vortex around the Milky Way from the CMB multipole coefficients of the temperature anisotropies. Indeed, at each multipole order, the corrections from a dragging speed $v_D \sim 10^{-4} c$ would be too small with respect to the dominant terms and ultimately masked by the cosmic variance [30,45]. Therefore, we conclude that the presence of a galactic vortex of the type currently being considered in the literature cannot impact in any meaningful way the standard CMB analysis, unless the polarization landscape is taken in consideration. The study of the apparent polarization of the CMB photons from the point of view of an observer surrounded by a dragging metric as (1) is beyond the scopes of the present work. It is, however, worthy of future investigations, since the non-negligible off-diagonal terms of the spacetime metric might affect the polarization of a traveling photon in a detectable way.

Finally, we checked if such general–relativistic phenomena could give an explanation to the tension on the CMB quadrupole. We showed that such tension cannot be justified by the dragging hypothesis, nor by any other analogous relativistic framework at the galactic scale.

**Author Contributions:** F.R.: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Resources, Data curation, Writing—Original draft, Writing—review & editing, Visualization, Supervision, Project administration. M.G.: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Resources, Data curation, Writing—Original draft, Writing—review & editing, Visualization, Supervision, Project administration, Funding acquisition. M.D.: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Resources, Data curation, Writing—Original draft, Writing—review & editing, Visualization, Supervision, Project administration, Funding acquisition. All authors have read and agreed to the published version of the manuscript.

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## Appendix A. Expansions at the Fourth Order

In order to expand (9) to higher orders in $v_K$ and $v_D$, we need exact formulae for $H$ and $\gamma_K$ in terms of $v_K$ and $v_D$. We start by considering again the normalisation of the four-velocity, i.e., $U_\mu U^\mu = -c^2$, which gives (as previously discussed)

$$-H = e^{2\Phi/c^2}(1+A\Omega)^2 - e^{-2\Phi/c^2}\frac{W^2\Omega^2}{c^2}\,. \tag{A1}$$

From Equation (4), we can substitute $\Omega = e^{\Phi/c^2}\gamma_K v_K/W$. We can also express the field $A(r,z)$ in terms of $v_D$ by making use of the relation $c^2e^{2\Phi/c^2}A = -g_{t\phi} = g_{\phi\phi}\chi = Wv_D$, according to Equation (6). Equation (A1) thus becomes

$$-H = \left(e^{\Phi/c^2} + \gamma_K\frac{v_K v_D}{c^2}\right)^2 - \gamma_K^2\frac{v_K^2}{c^2}\,. \tag{A2}$$

On the other hand, the Lorentz factor as measured by SO, which has $e_S^\mu = e^{-\Phi/c^2}\partial_t$ as the time-like component of its tetrad, can be exactly expressed as

$$\gamma_K := e_\mu^S U^\mu = e^{\Phi/c^2}\frac{1+A\Omega}{\sqrt{-H}} = \frac{e^{\Phi/c^2}}{\sqrt{-H} - v_K v_D/c^2}\,. \tag{A3}$$

In principle, we could solve the two Equations (A2) and (A3) as an algebraic system, hence finding exact formulas for $H$, and $\gamma_K$. However, this would involve a four degree algebraic equation that, although analytically solvable, has a uselessly complicated solving formula. Hence, we will already expand at this stage our quantities up to the fourth order.

Let us start from the second-order expansion of Equation (7). Equation (A3) returns us

$$\gamma_K = 1 + \frac{v_K^2}{2c^2} + \mathcal{O}\left(v^4/c^4\right), \tag{A4}$$

which is identical to the special–relativistic second-order expansion. We can substitute our result in Equation (A2), so that at the fourth order we have

$$-H = 1 + \frac{1}{c^2}\left(2\Phi + 2v_K v_D - v_K^2\right) + \frac{1}{c^4}\left(2\Phi^2 + 2\Phi v_K v_D + v_K^2 v_D^2 + v_K^3 v_D - v_K^4\right) + \mathcal{O}\left(v^6/c^6\right). \tag{A5}$$

This is indeed the fourth-order expansion of Equation (7). However, both in Equations (A3) and (9), we require its square root, i.e,

$$\sqrt{-H} = 1 + \frac{1}{c^2}\left(\Phi + v_K v_D - \frac{v_K^2}{2}\right) + \frac{1}{c^4}\left(\frac{\Phi^2}{2} + \frac{\Phi v_K^2}{2} + v_K^3 v_D - \frac{5}{8}v_K^4\right) + \mathcal{O}\left(v^6/c^6\right). \tag{A6}$$

We can now substitute Equation (A6) in Equation (A3), finding

$$\gamma_K = 1 + \frac{v_K^2}{2c^2} + \frac{1}{c^4}\left(\frac{7}{8}v_K^4 - \Phi v_K^2 - v_K^3 v_D\right) + \mathcal{O}\left(v^6/c^6\right). \tag{A7}$$

Substituting now in (9), we get

$$
\begin{aligned}
1+z =& \frac{1}{\sqrt{-H}}\left[1+e^{-\Phi/c^2}\gamma_K\left(\cos\theta\frac{v_K}{c}+\frac{v_K v_D}{c^2}\right)\right] = \\
=& 1+\cos\theta\frac{v_K}{c}+\frac{1}{c^2}\left(\frac{v_K^2}{2}-\Phi\right)+\cos\theta\frac{v_K}{c^3}\left(v_K^2-2\Phi-v_Kv_D\right)+\frac{1}{c^4}\left(\frac{7}{8}v_K^4-\frac{3}{2}\Phi v_K^2+\frac{\Phi^2}{2}-\frac{3}{2}v_K^3v_D\right) \\
&+\cos\theta\frac{v_K}{c^5}\left(2v_K^4-3\Phi v_K^2+2\Phi^2-\frac{7}{2}v_K^3v_D+v_K^2v_D^2\right)+\mathcal{O}\left(v^6/c^6\right).
\end{aligned} \tag{A8}
$$

Equation (A8) thus describes $z_{dr}$ expanded to the fourth order in the kinematic quantities. Nonetheless, what we require for the CMB analysis is the factor $1+z_{TOT}=[1+z_{dr}][1+z_b(\vec{v}_b)]$, where

$$
\begin{aligned}
1+z_b(\vec{v}_b) &= \left(1+\frac{\vec{v}_b\cdot\hat{n}}{c}\right)\left(1-\frac{v_b^2}{c^2}\right)^{-1/2} \\
&= 1+\frac{\vec{v}_b\cdot\hat{n}}{c}+\frac{v_b^2}{2c^2}+\frac{v_b^2}{2c^3}\vec{v}_b\cdot\hat{n}-\frac{v_b^4}{8c^4}-\frac{v_b^4}{8c^5}\vec{v}_b\cdot\hat{n}+\mathcal{O}\left(v^6/c^6\right).
\end{aligned} \tag{A9}
$$

In order to write Equation (A9) in terms of $\vec{v}_{\mathrm{TOT}} := \vec{v}_b \oplus \vec{v}_K$, it is useful to decompose $1+z_{dr}=[1+z_b(\vec{v}_K)][1+f/c^2]$, for some expression $f$, as to get $1+z_T=[1+z_b(\vec{v}_{\mathrm{TOT}})][1+f/c^2]$. Let us hence calculate

$$
\begin{aligned}
\frac{1}{1+z_b(\vec{v}_b)} =& \left(1+\frac{\vec{v}_b\cdot\hat{n}}{c}\right)^{-1}\left(1-\frac{v_b^2}{c^2}\right)^{1/2} \\
=& 1-\frac{\vec{v}_b\cdot\hat{n}}{c}+\frac{1}{c^2}\left((\vec{v}_b\cdot\hat{n})^2-\frac{v_b^2}{2}\right)+\frac{\vec{v}_b\cdot\hat{n}}{c^3}\left(\frac{v_b^2}{2}-(\vec{v}_b\cdot\hat{n})^2\right)+\frac{1}{c^4}\left(\frac{v_b^4}{8}-\frac{v_b^2}{2}(\vec{v}_b\cdot\hat{n})^2+(\vec{v}_b\cdot\hat{n})^4\right) \\
&-\frac{\vec{v}_b\cdot\hat{n}}{c^5}\left(\frac{v_b^4}{8}-\frac{v_b^2}{2}(\vec{v}_b\cdot\hat{n})^2+(\vec{v}_b\cdot\hat{n})^4\right)+\mathcal{O}\left(v^6/c^6\right).
\end{aligned} \tag{A10}
$$

Using $v_K\cos\theta=\vec{v}_K\cdot\hat{n}$, we have

$$
\begin{aligned}
1+\frac{f}{c^2} :=& \frac{1+z_{dr}}{1+z_b(\vec{v}_K)} \\
=& 1-\frac{\Phi}{c^2}+\frac{\vec{v}_K\cdot\hat{n}}{c^3}\left(\frac{v_K^2}{2}-\Phi-v_Kv_D\right) \\
&+\frac{1}{c^4}\left(\frac{3}{4}v_K^4-\Phi v_K^2+\frac{\Phi^2}{2}-\frac{3}{2}v_K^3v_D-\frac{v_K^2}{2}(\vec{v}_K\cdot\hat{n})^2+\Phi(\vec{v}_K\cdot\hat{n})^2+v_Kv_D(\vec{v}_K\cdot\hat{n})^2\right)+\mathcal{O}\left(v^5/c^5\right).
\end{aligned} \tag{A11}
$$

The total frequency shift is thus expanded as

$$
\begin{aligned}
1+z_T =& 1+\frac{\vec{v}_{\mathrm{TOT}}\cdot\hat{n}}{c}+\frac{1}{c^2}\left(\frac{v_{\mathrm{TOT}}^2}{2}-\Phi\right)+\frac{\vec{v}_{\mathrm{TOT}}\cdot\hat{n}}{c^3}\left(\frac{v_{\mathrm{TOT}}^2}{2}-\Phi\right)+\frac{\vec{v}_K\cdot\hat{n}}{c^3}\left(\frac{v_K^2}{2}-\Phi-v_Kv_D\right) \\
&+\frac{(\vec{v}_K\cdot\hat{n})^2}{c^4}\left(\Phi+v_kv_D-\frac{v_K^2}{2}\right)+\frac{(\vec{v}_{\mathrm{TOT}}\cdot\hat{n})(\vec{v}_K\cdot\hat{n})}{c^4}\left(\frac{v_K^2}{2}-\Phi-v_Kv_D\right) \\
&+\frac{1}{c^4}\left(\frac{3}{4}v_K^4-\Phi v_K^2+\frac{\Phi^2}{2}-\frac{3}{2}v_K^3v_D-\frac{v_{\mathrm{TOT}}^4}{8}-\frac{\Phi v_{\mathrm{TOT}}^2}{2}\right)+\mathcal{O}\left(v^5/c^5\right).
\end{aligned} \tag{A12}
$$

The CMB's systematic anisotropy is, hence, finally given by

$$\frac{T_d(\hat{n})}{T_i(\hat{n})} = \left[1 + \frac{1}{c^2}\left(\Phi - \frac{v_{\rm TOT}^2}{2}\right) + \frac{1}{c^4}\left(\Phi v_K^2 - \frac{3}{4}v_K^4 + \frac{\Phi^2}{2} + \frac{3}{2}v_K^3 v_D + \frac{3v_{\rm TOT}^4}{8} - \frac{\Phi v_{\rm TOT}^2}{2}\right)\right] - \frac{(\vec{v}_{\rm TOT}\cdot\hat{n})^3}{c^3} \tag{A13}$$
$$- \frac{1}{c}\left[(\vec{v}_{\rm TOT}\cdot\hat{n})\left(1 + \frac{\Phi}{c^2} - \frac{v_{\rm TOT}^2}{2c^2}\right) + (\vec{v}_K\cdot\hat{n})\left(\frac{v_K^2}{2c^2} - \frac{\Phi}{c^2} - \frac{v_K v_D}{c^2}\right)\right] + \frac{(\vec{v}_{\rm TOT}\cdot\hat{n})^4}{c^4}$$
$$+ \frac{1}{c^2}\left[(\vec{v}_{\rm TOT}\cdot\hat{n})^2\left(1 - \frac{v_{\rm TOT}^2}{2c^2} + \frac{\Phi}{c^2}\right) + (\vec{v}_{\rm TOT}\cdot\hat{n} + \vec{v}_K\cdot\hat{n})(\vec{v}_K\cdot\hat{n})\left(\frac{v_K^2}{2c^2} - \frac{\Phi}{c^2} - \frac{v_K v_D}{c^2}\right)\right] + \mathcal{O}\left(v^5/c^5\right).$$

## Notes

1. We employ the term "strong dragging vortex" to identify the nonnegligible, purely geometrical, frame-dragging effect possibly present on galactic scales.
2. Here, we note that, by employing the geometry of Equation (1), we neglect non-axisymmetric morphological features within the disc galaxy. However, the change in the local spacetime geometry with respect to the presence of such features can be seen as a perturbation of the line element in Equation (1). Therefore, their impact on our results would give only higher-order corrections.
3. We will use the symbol $\approx$ to say that two quantities are equal up to higher-order corrections. Approximations of another nature will be indicated by the symbol $\cong$. If just the order of magnitude is given, one will find $\sim$.

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