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Research paper

Equivariant optimisation for the gravitational n-body problem: A computational factory of symmetric orbits

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ABSTRACT

In this paper we present SymOrb.jl (2024), a software which combines group representation theory and variational methods to provide numerical solutions of singular dynamical systems of paramount relevance in Celestial Mechanics and other interacting particles models. Among all, it prepares for large-scale search of symmetric periodic orbits for the classical n-body problem and their classification, paving the way towards a computational validation of Poincaré conjecture about the density of periodic orbits. Through the accessible language of Julia, Symorb.jl offers a unified implementation of an earlier version (dlfer – symorb, 2017). This paper provides theoretical and practical guidelines for the specific approach we adopt, complemented with examples.

Introduction

Among natural dynamical systems arising in classical mechanics, the gravitational n-body problem has become a paradigmatic example of a singular system, both for its simple and clean formulation, and the notable complexity of its orbits' structure. The question is the following: what are the trajectories of n heavy bodies that move in the Euclidean space under their mutual gravitational attractions, and how do we describe their motions? For $x_1, \dots, x_n \in \mathbb{R}^3$ heavy bodies with masses $m_1, \dots, m_n > 0$, classical Newton laws provide the equations of motion, which

$$m_k \ddot{x}_k(t) = -\sum_{j \neq k} m_k m_j \frac{x_k(t) - x_j(t)}{\|x_k(t) - x_j(t)\|^3}, \quad \text{for each } k \in \{1, \dots, n\},$$

where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^3$, so that a solution of the n-body problem is the list of trajectories of the bodies. At first glance, the set of singularities of this system of equations might be large and difficult to manage: whenever a body x_j collapses on another body x_i, the motion equations are not defined. When $n = 2$, the picture is quite clear: each body follows a conic section, which can be easily classified in terms of the mechanical energy. Nonetheless, the 3-body problem is already a non-integrable system, and small perturbations on initial conditions can cause a dramatic displacement of nearby orbits. This sensitivity of the initial data flow

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has even caught the attention of contemporary folklore, raising the interest of this elegant old problem. The theoretical foundations which motivate the research fervour around this problem can be traced back to the famous Poincaré conjecture [1]. A possibly chaotic dynamics is strictly related to the existence of a dense family of periodic solutions:

"D'ailleurs, ce qui nous rend ces solutions périodiques si précieuses, c'est qu'elles sont, pour ainsi dire, la seule brèche par où nous puissons essayer de pénétrer dans une place jusqu'ici réputée inabordable...".

Indeed, according to Poincaré, all trajectories can be approximated by periodic ones:

"...voici un fait que je n'ai pu démontrer rigoureusement, mais qui me parait pourtant très vraisemblable. Étant données des équations de la forme définie dans le n. 13,[1] *et une solution particulière quelconque de ces équations, one peut toujours trouver une solution périodique (dont la période peut, il est vrai, être très longue), telle que la différence entre les deux solutions soit aussi petite qu'on le veut, pendant un temps aussi long qu'on le veut".*

As a result, the presence of rich sets of periodic solutions is a suitable indicator of complexity for dynamical systems. Moreover, the knowledge of classical periodic orbits has profound implications for the associated quantum problem. Periodic orbits form the backbone of semiclassical approximations, such as the Gutzwiller trace formula, which connects the quantum energy levels with the classical periodic trajectories. This connection is pivotal for interpreting spectral properties and predicting quantum states.

Since the 90s of the last century, variational methods have shown to be successful in producing selected solutions of singular systems (among others, see [2–7]) and eventually such techniques have found a powerful application in search of periodic orbits for the n-body problem (see in particular the celebrated papers [8,9]). Nowadays, it is widely known that one could look for collision-less periodic solutions among critical points of the associated Lagrange-action functional

$$\mathcal{A}(x) = \int_0^T \left[\frac{1}{2} \sum_{i=1}^n m_i \|\dot{x}(t)\|^2 + \sum_{i<j} \frac{m_i m_j}{\|x_i(t) - x_j(t)\|} \right] dt. \tag{1}$$

over the space of T-periodic loops: a peculiar guide on this approach can be found in the monograph [10]. Within the class of all critical points, it seems more natural to look for minimisers of $\mathcal{A}$, especially if it is feasible to recover coercivity and to work with *closed* subsets of the paths space.

To a wider extent, an essential guide in the hunting of solutions via variational methods can be summarised in these steps:

- introduce a functional which is differentiable over a space of candidate solutions. Such a functional can have a geometrical or dynamical connection with the problem and naturally arises from the weak formulation of the motion equations. This naturally leads to work with the Sobolev space of H^1 functions (see e.g. [11]). Usually, Jacobi-length and Maupertuis functionals reflect the Riemannian structure of the problem, while Energy and Lagrange-action functionals are more related to the dynamical nature of the problem;
- state and prove a variational principle which guarantees that non-trivial collision-less critical points of this functional, or at least some re-parametrisations of them, solve the motion equations. Depending on the functional involved, one can have a *Maupertuis principle* or a *Least action principle*;
- look for critical points of the chosen functional. For instance, if coercivity and compactness of sub-levels for the functional can be proved, direct methods in calculus of variations apply (e.g., see again [11]) and one naturally looks for minimisers.

Concerning the n-body problem, the first obstruction is the lack of compactness in the open set of collision-less paths and hence of the sub-levels of $\mathcal{A}$. Furthermore, an elementary remark excludes coercivity of the action functional: any sequence of constant loops with diverging mutual distances between the bodies has a diverging H^1-norm, but $\mathcal{A}$ remains bounded on such sequence. Nonetheless, different types of reduction of the n-body problem to more affordable models allow to recover coercivity and to succeed in the minimisation process (for the symmetric n-body problem see [8,9,12–19]. We also refer to the n-centre problem [20–26]). This paper is concerned with a particular reduction case, requiring the candidate paths to satisfy some chosen symmetries in time, space, and body labels.

In this direction, a very successful approach has been presented in [9], where the authors recover coercivity by symmetrising the space of paths through the action of a finite group G. Under major hypotheses on the action of G, the authors have shown that minimisers of $\mathcal{A}$ are collision-less and so they actually solve a *symmetrical* n-body problem (see also [27–31]). This rigorous argument has been supported by a numerical implementation of a constrained optimisation algorithm, which allows us to find a finite group G verifying some chosen symmetries and to check if a G-equivariant optimisation is achievable. Once a G-equivariant space of closed paths is generated, an optimisation routine provides a candidate numerical solution of the n-body problem, which displays the symmetries fixed by G.

The first version of this algorithm, named *Symorb* by its creator Davide Ferrario, is available in [32]. Symorb is based on a combination of Python, Fortran, and the computer algebra symbolic language GAP (see [33]). The present paper introduces *SymOrb.jl* [34], a renovation of Symorb's Fortran and Python routines in a unified language, Julia [35]. It offers theoretical and practical guidelines for the variational approach we are adopting. We provide instructions for generating symmetric, periodic solutions to the n-body problem and subsequently computing them numerically.

[1] Formula 13 mentioned by Poincaré, is the Hamilton equation.

We emphasise that this work goes far beyond a mere reimplementation of Symorb in a different language, but it introduces a modular and extensible refactoring of the original software, which lays the groundwork for future developments and experimental directions. Thanks to this flexible framework, the core operations of *SymOrb.jl* can be easily extended to support a wider range of applications, including both quantitative and qualitative analyses of symmetric critical orbits. For instance, trajectory features such as stability and topological indexes (e.g., the Morse index, the Floquet exponents, and other topological indicators) can be easily incorporated. To explore bifurcation phenomena, natural parameters like the angular velocity with respect to an intertial frame can be effortlessly introduced. On the performance side, we introduce a new approach that reduces the number of parameters involved in each optimisation step, which allows to obtain solutions with the same accuracy in a shorter time (see Remark 3.4). Notice that this idea could not be developed starting from the original Symorb implementation without rewriting the whole optimisation routines from scratch. Finally, the enhanced adaptability of the new framework enables the organisation of critical orbits into databases, the improvement of numerical integration methods, and the exploration of the efficiency of non-classical optimisation methods (see also [36]).

Conclusions and perspectives. The variational-computational approach to the search for symmetric solutions in the gravitational n-body problem represents a novel and flexible methodology. Through a combination of analytical-functional methods, it stands out from past research based on solving initial data problems. It offers unprecedented efficiency in identifying stable and unstable periodic orbits and understanding the intricate dynamics of multi-body celestial systems. It can easily be used for large-scale searches of equivariant trajectories, which we intend to classify with respect to their symmetry properties, variational characterisation and stability. We can collect symmetric orbits in an accessible database and we plan to use them as building blocks in the search of complex periodic trajectories, which do not necessarily satisfy a symmetry constraint. Furthermore, our computational exploration can be complemented with computer-assisted proofs, for rigorously proving the existence of new periodic solutions (see [37–41]).

On the computational side, Julia provides first-class support for linear and nonlinear operations on mutidimensional arrays with an accessible, concise syntax. Julia's Just In Time (JIT) compilation model enables execution speed on par with statically compiled languages while allowing for an efficient and interactive developer experience. This is particularly crucial in our scenario, as `SymOrb.jl` needs to run complex custom code in the hot loop of an optimisation routine, thus not only library but also user-defined code needs to be highly performant. Additionally, the Julia ecosystem encompasses support for GPU programming [42] and modern optimisation methods [43].

Notations. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space and $m_1, \dots, m_n > 0$ be positive masses, with $n \geq 2$. We introduce the configuration space of n heavy particles with centre of mass in the origin:

$$\mathcal{X} \doteq \left\{ x = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n \;\middle|\; \sum_{i=1}^n m_i x_i = 0 \right\}.$$

Concerning collisions, we define the following singular sets

$$\Delta_{ij} \doteq \left\{ x \in \mathcal{X} \;\middle|\; x_i = x_j \right\}, \quad \Delta \doteq \bigcup_{i,j} \Delta_{ij}$$

so that we can also introduce the set of collision-free configurations $\hat{\mathcal{X}} \doteq \mathcal{X} \setminus \Delta$. The potential function associated with the classical n-body problem is usually introduced as

$$U(x) \doteq \sum_{i<j} \frac{m_i m_j}{\|x_i - x_j\|}, \quad \text{for } x \in \mathcal{X},$$

while the kinetic energy function is

$$K(\dot{x}) \doteq \frac{1}{2} \sum_{i=1}^n m_i \|\dot{x}_i\|^2, \quad \text{for } \dot{x} \in T_x\mathcal{X};$$

finally, the Lagrangian function reads

$$L(x, \dot{x}) \doteq K(\dot{x}) + U(x), \quad \text{for } x \in \mathcal{X},\ \dot{x} \in T_x\mathcal{X}.$$

With these notations, the motion equations of n bodies $x_1(t), \dots, x_n(t)$ in the configuration space $\mathcal{X}$ read:

$$m_i \ddot{x}_i = \frac{\partial U}{\partial x_i}, \qquad i = 1, \dots, n, \tag{2}$$

which are nothing but the Euler–Lagrange equations of the above Lagrange function L.

Remark 0.1. The theoretical results recalled in this paper and proved in [9], together with the numerical procedures summarised in the last section, adapt to a wider class of potential functions U. In particular, we refer to the minimal hypotheses required on U introduced in [44], in order to guarantee the existence of collision-less critical points of $\mathcal{A}$. For instance, the α-homogenous n-body problem can be considered, but also quasi-homogenous and logaritmic potentials are allowed in this analysis (for completeness, weparticularly refer to [44, Section 7]). As will be clear in Section 3, the numerical optimisation routine allows us to choose a generic function $f = f(\|x_i - x_j\|)$ instead of the singular quotient $\frac{1}{\|x_i-x_j\|}$. Clearly, the convergence to a collision-less critical point is not always guaranteed.

When dealing with the n-body problem, it is natural to rephrase this dynamical system in a uniform rotating frame. As a matter of fact, the results proved in [9] are valid in this more general situation, where the kinetic energy of the system clearly depends on x (see also Section 3). For the sake of simplicity, we prefer to present the theoretical statements of this paper in the interial frame.

As already remarked in the Introduction, there is a major interest in looking for periodic solutions of (2). A periodic solution of (2) has the property that every trajectory $x_i(t)$ of the ith body is a closed curve in $\mathbb{R}^d$, usually referred as a *loop* in the configuration space. As a short-hand notation, for $T > 0$ we consider the time circle $\mathbb{T} \simeq \mathbb{R}/T\mathbb{Z}$, so that the space of H^1-periodic loops in $\mathcal{X}$ can be defined as

$$\Lambda \doteq H^1(\mathbb{T};\mathcal{X}) = \{x \in L^2([0,T];\mathcal{X}) \,:\, \dot{x} \in L^2([0,T];\mathcal{X}),\ x(0) = x(T)\},$$

where $\dot{x}$ denotes the weak derivative of x and the boundary condition makes sense since H^1 functions are indeed continuous by classical Sobolev embedding. Recalling that $\hat{\mathcal{X}}$ is the set of non-collision configurations, we introduce also the non-collision counterpart of Λ

$$\hat{\Lambda} \doteq H^1(\mathbb{T};\hat{\mathcal{X}})$$

and, for $x = x(t) \in \Lambda$, we define the set of *collision times* of x as the set $x^{-1}\Delta \subseteq \mathbb{T}$. We finally consider the Lagrange action functional $\mathcal{A} : \Lambda \to \mathbb{R} \cup \{+\infty\}$ defined as

$$\mathcal{A}(x) \doteq \int_0^T L(x(t), \dot{x}(t))\,dt, \quad \text{for } x \in \Lambda.$$

Note that, since $\mathcal{A} \in C^1(\hat{\Lambda})$, it is well-known that if $x \in \hat{\Lambda}$ is a critical point of $\mathcal{A}$, then x is a T-periodic solution of (2) (see for instance Lemma 19.1 in [10]).

Outline of the paper. The paper is organised as follows. In Section 1 we briefly recall the concept of *group action* and its connection with symmetries. In Section 2 we will see how the machinery of group actions can be very useful to symmetrise the loop space Λ, we will state some ad-hoc variational principles for equivariant loops and we will collect some useful properties and classifications of groups acting on Λ. Finally, Section 3 is devoted to the search of equivariant critical points of $\mathcal{A}$, together with a survey on the numerical implementation of this procedure. The appendix sections contains some supplementary resources on the topics of this paper.

1. Symmetries as group actions

Representation theory offers the perspective of thinking of groups as structures which *act* on mathematical objects, such as sets or spaces. Our first aim is to endow the space of H^1-periodic loops Λ with a prescribed symmetry constraint. A powerful way to describe such a symmetry is through the *action* of a finite group G on Λ. A rigorous definition of *group action* is given in the following:

Definition 1.1 (*Group Action*)**.** We say that a finite group G acts (on the left) on a set X if there exists a map $\phi : G \times X \to X$ such that

- $\phi(1, x) = x$ for any $x \in X$;
- $\phi(g, \phi(h, x)) = \phi(gh, x)$ for any $g, h \in G$ and $x \in X$,

i.e., the map ϕ commutes with the group operation. We will often write gx instead of $\phi(g, x)$. We say that the action of G on a set X is *transitive* if for any $x, y \in X$ there exists $g \in G$ such that $gx = y$.

Therefore, a group action is essentially a way for G to interact with the set X. Given $x \in X$, it could be useful to regroup those elements of G which fix x; on the other hand, we could need to gather those elements of X which are invariant through the action of the whole G or of a subgroup H of G. For instance, if $X = \mathbb{R}^2$ and $g \in G$ is a reflection with respect to the line $y = x$, we see that all the points (x, x) are fixed by g. In the following, we will use the notations $H < G$ and $H \lhd G$ to denote respectively the subgroup and normal subgroup relations.

Definition 1.2 (*Isotropy Groups and Projectors*)**.** For an element $x \in X$ we define the *isotropy group* of x in G as

$$G_x \doteq \{g \in G \,:\, gx = x\},$$

and $G_x < G$. Moreover, if $H < G$, we define the set of *points fixed by* H as the set

$$X^H \doteq \{x \in X \,:\, G_x \supseteq H\} = \{x \in X \,:\, hx = x,\ \forall\, h \in H\}.$$

We define also the *projector onto* X^H as the map $\pi_H : X \to X^H$ which acts in this way:

$$\pi_H\, x \doteq \frac{1}{|H|} \sum_{h \in H} hx. \tag{3}$$

Remark 1.3. We recall that if $H < G$, then the normaliser $N_G H$ of H in G is the smaller subgroup of G in which H is normal. Moreover, the *Weyl group* of H in G is the quotient group $W_G H = N_G H/H$. If G acts on X, then the Weyl group $W_G H$ acts on X^H. Indeed, since and $H \triangleleft N_G H$, then the cosets of H naturally act on X^H.

From a geometrical point of view groups can be considered as *sets of symmetries* of a mathematical object. For instance, the symmetries of a square are exactly the elements of the dihedral group D_8 (see Definition A.1).

In this context, given a set X and a group G acting on it, the action of G on X induces a group homomorphism between G and Σ_X (the group of all permutations of elements of X)

$$a : G \to \Sigma_X, \quad a(g) : X \to X, \quad a(g)x = gx.$$

Indeed, for any $g \in G$, the group action implies that the map $a(g)$ is a bijection with inverse $a(g^{-1})$; for the same reason, the set $\{a(g) : g \in G\}$ is a subgroup of Σ_X. The homomorphism a is an example of *representation* of G, which is obtained by starting from its action on X. Below, we list some examples of representations of G.

Example 1.4 (X *Finite, Action of* D_{2n}). Take $X = \{1, \dots, n\}$ so that $\Sigma_X = \Sigma_n$, the *symmetric group* of n elements. We can label the vertices of a regular n-gon with elements of X and take $G = D_{2n}$. If g is the rotation of $\frac{2\pi}{n}$, then g corresponds to the cycle $(1, \dots, n)$. On the other hand, if n is even and g is the reflection with respect to the symmetry axis through the vertices 1 and $n/2 + 1$, then g is represented in Σ_n as the permutation $(2, n)(3, n-1) \cdots (n/2, n/2 + 2)$. If n is odd, the reflection is represented as the permutation $(2, n)(3, n-1) \cdots ((n + 1)/2, (n + 3)/2)$. We then understand that the way a group G acts on a space determines a representation of G, and viceversa.

Example 1.5 (X *Linear Space*). Assume now that X is a linear space over $\mathbb{R}$. It is natural to require that the action of G on X preserves the linear structure of the vector space, that is, for any $g \in G$, $x_1, x_2 \in X$ and $\lambda, \mu \in \mathbb{R}$

$$a(g)(\lambda x_1 + \mu x_2) = \lambda a(g)x_1 + \mu a(g)x_2.$$

This means that, for any $g \in G$, the map $a(g)$ is a linear transformation from X to itself and thus $a(g) \in GL(X)$. In this case we refer to $a : G \to GL(X)$ as a *linear representation of* G. On the other hand, when $X = \mathbb{R}^n$ then the whole $GL(n)$ acts on $\mathbb{R}^n$ in the natural way. A group G acting on $\mathbb{R}^n$ can be represented as a subgroup of $GL(n)$; in particular, one could require not only the linearity of the group action, but also to be angle-preserving. In the latter case, G is represented into $O(n)$ and the representation is usually called *orthogonal*. Note that such representation always exists (the trivial action is always admitted).

2. Action of finite groups on Λ

In this section we deepen the main ideas of [9] and we define group actions on the loop space $\Lambda = H^1(\mathbb{T}; \mathcal{X})$ in order to introduce a symmetry constraint. As a preliminary remark, since Λ is infinite-dimensional, it is clear that we have to look for finite dimensional objects on which a group can effectively act.

Recall that $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$ and that, for any $x = x(t) \in \Lambda$, we can write

$$x(t) = (x_1(t), \dots, x_n(t)) \in \mathcal{X} \subseteq (\mathbb{R}^d)^n, \quad \text{for any } t \in \mathbb{T}.$$

We can identify three finite dimensional objects in Λ:

- the *space* $\mathbb{R}^d$ in which every component $x_i(t)$ lies;
- the *time circle* $\mathbb{T} \subset \mathbb{R}^2$ which represents the period of a trajectory;
- the *index set* $\{1, \dots, n\}$ on which the n-bodies are labelled.

A way to define the action of G on Λ is then through three different representations of G as a group action respectively on $\mathbb{R}^d$, $\mathbb{R}^2$ and $\{1, \dots, n\}$. More precisely, G is represented as a subgroup of $O(d)$, of $O(2)$ and of the symmetric group Σ_n (see Examples 1.4 and 1.5) via the following homomorphisms

$$\rho : G \to O(d), \quad \tau : G \to O(2), \quad \sigma : G \to \Sigma_n.$$

As an intermediate step, we use ρ and σ to define the action of G on $\mathcal{X}$

$$\phi : G \times \mathcal{X} \to \mathcal{X}, \quad \phi(g, x) = gx,$$

where

$$gx = g(x_1, \dots, x_n) = \left(\rho(g)x_{\sigma(g^{-1})1}, \dots, \rho(g)x_{\sigma(g^{-1})n}\right),$$

or, shortly

$$(gx)_i = gx_{g^{-1}i}.$$

Let us observe that this action is orthogonal, in the sense that it preserves the inner product of $(\mathbb{R}^d)^n$; this is easy to verify since $\rho(g) \in O(d)$ and the permutation $\sigma(g^{-1})$ does not affect the inner product. For reasons that will be clear later, it makes sense to require that $\sigma : G \to \Sigma_n$ satisfies the following property

$$\forall\, g \in G,\ \text{if } \sigma(g^{-1})i = j \text{ for some } i \neq j,\ \text{then } m_i = m_j, \tag{4}$$

meaning that only bodies with equal masses can be *interchanged* by σ. For instance, if the action of G is transitive, then we will work with n equal masses.

Since the time circle $\mathbb{T}$ is a 1-dimensional object immersed in $\mathbb{R}^2$, the action of G on $\mathbb{T}$ can be defined through the representation $\tau : G \to O(2)$ in this way

$$gt = \tau(g^{-1})t,\ \text{for } t \in \mathbb{T},\ g \in G.$$

Remark 2.1. Since $\tau : G \to O(2)$ is a homomorphism, then:

- $\ker \tau \lhd G$;
- $G/\ker \tau \simeq \tau(G)$ and so it is a finite subgroup of $O(2)$.

Thanks to Proposition A.2, we deduce that $\tau(G)$ is either a cyclic or dihedral subgroup of $O(2)$. Since τ models an action of G on $\mathbb{T}$, then any $\tau(g)$ is either a rotation or a reflection in the plane. With a slight abuse of notations, or assuming that θ and t belong to the same circle, for any $t \in \mathbb{T}$ we have:

- if $\tau(g)$ is a counter-clockwise rotation of ϑ, then:
$$\tau(g)t = \vartheta + t;$$
- if $\tau(g)$ is a reflection with respect to a line through the origin which forms an angle of $\vartheta/2$ with the horizontal axis, then:
$$\tau(g)t = \vartheta - t.$$

We have finally obtained a way to represent the action of G on the loop space Λ by means of the homomorphisms ρ, σ and τ, which is the following

$$(gx)(t) = (\rho(g)x_{\sigma(g^{-1})1}(\tau(g^{-1})t), \dots, \rho(g)x_{\sigma(g^{-1})n}(\tau(g^{-1})t))$$

which in short we will write as

$$(gx)(t) = (gx_{g^{-1}i}(g^{-1}t))_i,$$

for any $g \in G$, $t \in \mathbb{T}$ and $x \in \Lambda$. We also define the set of loops in Λ fixed by G, or *G-equivariant loops*, as the set

$$\Lambda^G \doteq \{x \in \Lambda : (gx)(t) = x(t),\ \forall\, t \in \mathbb{T},\ g \in G\}$$

and its collision-less subset

$$\hat{\Lambda}^G \doteq \{x \in \hat{\Lambda} : (gx)(t) = x(t),\ \forall\, t \in \mathbb{T},\ g \in G\}.$$

Example 2.2 ($G = D_6$)**.** We consider the dihedral group

$$D_6 = \langle s, r\, :\, s^2 = r^3 = (sr)^2 = 1\rangle,$$

which is the group of symmetries of a regular triangle in the (x, y)-plane. For simplicity, assume that $T = 2\pi$ and one of the vertices of the triangle lies on the line $y = x$. According to Definition A.1, we can define the representations τ, ρ and σ through their values on the generators s and r

$$\tau(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau(r) = \begin{pmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{pmatrix},$$

$$\rho(s) = -\mathrm{Id}_2, \quad \rho(r) = \mathrm{Id}_2,$$

$$\sigma(s) = (1, 2), \quad \sigma(r) = (1, 2, 3).$$

Note that, using these homomorphisms, we are modelling a 3-body problem in the plane. It is clear that these representations depends on the geometrical interpretation of the elements of D_6 as transformations which fix a regular triangle. Let us observe that

$$\ker(\tau) = \ker(\sigma) = 1, \quad \ker(\rho) = \langle r\rangle,$$

and also that, by assumption (4), we have $m_1 = m_2 = m_3$. Let us remark that $x \in \Lambda^G$ if and only if for every $t \in \mathbb{T}$

$$(sx)(t) = x(t) \quad \text{and} \quad (rx)(t) = x(t).$$

The first relation can be rewritten in this way

$$\rho(s)x_{\sigma(s^{-1})j}(\tau(s^{-1})t) = x_j(t), \quad \forall\, j \in \{1, 2, 3\}$$

and $\tau(s)$ is a reflection with respect to the line $y = x$. By means of the notations introduced in Remark 2.1, $\tau(s)$ acts as the map $t \to \frac{2\pi}{4} - t$ and so the equivariance reads:

$$\begin{cases} x_1(t) = -x_2\left(\frac{2\pi}{4} - t\right) \\ x_2(t) = -x_1\left(\frac{2\pi}{4} - t\right) \\ x_3(t) = -x_3\left(\frac{2\pi}{4} - t\right). \end{cases}$$

Now, $\rho(r)$ is the identity matrix and $\tau(r)$ is a counter-clockwise rotation of $\frac{2\pi}{3}$; therefore $\tau(r^{-1})$ is a clockwise rotation of the same angle, that in $\mathbb{T}$ corresponds to a time-shift of $-\frac{2\pi}{3}$. Hence the equivariance with respect to r reads

$$\begin{cases} x_1(t) = x_2\left(t - \frac{2\pi}{3}\right) \\ x_2(t) = x_3\left(t - \frac{2\pi}{3}\right) \\ x_3(t) = x_1\left(t - \frac{2\pi}{3}\right) \end{cases}$$

which means that every D_6-equivariant loop is in particular a *choreography*.

2.1. Equivariant variational principles

The next result guarantees that Λ^G is a loop space which fits a variational argument to find periodic solutions as critical points of the action functional. We will not give the details of the proof since they are quite standard and follow from the definition of group action on Λ. In particular, the equivariance of $\mathcal{A}$ follows from assumption (4).

Proposition 2.3. *Let G be a finite group, with orthogonal representations $\rho : G \to O(d)$, $\tau : G \to O(2)$ and $\sigma : G \to \Sigma_n$ satisfying (4). Then the following assertions hold true:*

(i) Λ^G is a closed linear subspace of Λ;
(ii) the Lagrange action functional $\mathcal{A}$ is G-equivariant, i.e.,

$$\mathcal{A}(gx) = \mathcal{A}(x),\ \forall\, g \in G,\ \forall\, x \in \Lambda;$$

(iii) the colliding set Δ is G-equivariant, i.e.,

$$x \in \Delta \implies gx \in \Delta,\ \forall\, g \in G.$$

As a consequence of these facts, we deduce this version of the *Palais principle of symmetric criticality*, whose proof is a slight modification of the classical one given in [45].

Lemma 2.4. *Let $\mathcal{A}|_{\Lambda^G}$ be the restriction of the Lagrange action functional $\mathcal{A}$ to the G-equivariant loop space Λ^G. Then, a collision-less critical point $\bar{x} \in \hat{\Lambda}^G$ of $\mathcal{A}|_{\Lambda^G}$ is also a (collision-less) critical point of $\mathcal{A}$ over the whole Λ.*

Now, we introduce the set of equivariant configurations

$$\mathcal{X}^G \doteq \left\{x \in \mathcal{X} \,:\, \rho(g)x_{\sigma(g^{-1})i} = x_i,\ \forall\, i = 1, \dots, n\right\}$$

and recall this crucial result contained in [9, Proposition 4.1]

Proposition 2.5. *Assume that $\mathcal{A}|_{\Lambda^G}$ is not identically $+\infty$. Then*

$$\mathcal{A}|_{\Lambda^G} \text{ is coercive} \iff \mathcal{X}^G = \{0\}.$$

As a consequence, if $\mathcal{X}^G = \{0\}$, there exists at least a minimiser of $\mathcal{A}$ in Λ^G.

2.2. Reducible, bound to collisions and admissible actions on Λ

It is convenient to refine the choice of a finite group G with respect to its action on the periodic loops in Λ, in order to rule out those representations of G which lead to *reducible* group actions. This process, described in the present section, will yield a definition of *admissible* group action on Λ. All the results and definitions of this paragraph have been introduced in [9], or follows as direct consequences of such results. We will add some proofs for the reader's convenience.

Definition 2.6. Consider a finite group G with orthogonal representations $\rho : G \to O(d)$, $\tau : G \to O(2)$ and $\sigma : G \to \Sigma_n$ satisfying (4). We say that the action of G on Λ is *non-reducible* if the following assumptions are satisfied:

(i) $\ker \tau \cap \ker \rho \cap \ker \sigma = 1$;

(ii) there is no proper linear subspace V of $\mathbb{R}^d$ such that

$$x_i(t) \in V, \quad \forall\, i \in \{1,\dots,n\},\ \forall\, x \in \Lambda^G,\ \forall\, t \in \mathbb{T};$$

(iii) there is no integer $k \neq \pm 1$ such that

$$\forall\, x \in \Lambda^G\ \exists\, y \in \Lambda \text{ such that } x(t) = y(kt),\ \forall\, t \in \mathbb{T}.$$

Remark 2.7. We can motivate the choices in the previous definition in this way:

(i) If $K = \ker \tau \cap \ker \rho \cap \ker \sigma \neq 1$ then it is enough to consider $G' = G/K$ instead of G, since in this case $\Lambda^{G'} = \Lambda^G$ up to homeomorphisms.
(ii) If V is a proper subspace of $\mathbb{R}^d$ then $\dim V < d$ and one could change the representation ρ of G.
(iii) This assumption simply means that T is the minimal period for equivariant loops.

We can actually determine some situations in which the action of a group G is reducible.

Proposition 2.8. *Let G be a finite group acting on Λ. If one of the following holds*

(a) $\ker \tau \cap \ker \sigma \neq 1$
(b) $|\ker \rho \cap \ker \sigma| > 2$

then the action of G is reducible.

Proof. Assume (a). Let $g \in (\ker \tau \cap \ker \sigma) \setminus \{1\}$. If $g \in \ker \rho$ then the action is reducible since (i) in Definition 2.6 fails. Otherwise, for $x \in \Lambda^G$ we have

$$\rho(g) x_i(t) = x_i(t),\ \forall\, t \in \mathbb{T},\ \forall\, i \in \{1,\dots,n\}$$

and so $x_i(t)$ belongs to the proper subspace $V^g = \{v \in \mathbb{R}^d\ :\ \rho(g)v = v\} = \ker \rho(g)$; this is against (ii) in Definition 2.6.

Assume (b). Let $g_0, g_1 \in (\ker \rho \cap \ker \sigma) \setminus \ker \tau$. Since $\tau(G)$ is a finite subgroup of $O(2)$, then g_0 and g_1 act as rotations or reflections on $\mathbb{T}$ (see Remark 2.1). If both act as reflections, their composition $\hat{g} \in \ker \rho \cap \ker \sigma$ and acts as a rotation. Then, if we consider $x \in \Lambda^G$, the equivariance with respect to $\hat{g}$ reads

$$x_i(t) = x_i(t + \hat{\vartheta}), \quad \forall\, t \in \mathbb{T},\ i = 1,\dots,n,$$

for some $\hat{\vartheta} \in \mathbb{R}/T\mathbb{Z}$. Since x is a T-periodic loop, the previous relation implies that $T = k\hat{\vartheta}$ for some $k \in \mathbb{Z} \setminus \{0, \pm 1\}$. In this case, T would not be the minimal period for $x \in \Lambda^G$, and this would lead again to a reducible action of G for condition (iii) of Definition 2.6. Clearly □

Remark 2.9. Note that if $|\ker \rho \cap \ker \sigma| = 2$ the action of G is not necessarily reducible. Indeed, the element $g \in (\ker \rho \cap \ker \sigma) \setminus \{1\}$ may act as a reflection, which would not imply property (iii) of Definition 2.6.

Since we are looking for classical solutions of the n-body problem, it is important to know that the action of some groups G can make $\hat{\Lambda}^G$ an empty set, and *forcing* the bodies to collide.

Definition 2.10. We say that the action of G on Λ is *bound to collisions* if

$$\forall\, x \in \Lambda^G \text{ it holds } x^{-1}\Delta \neq \emptyset,$$

or, equivalently, if

$$\hat{\Lambda}^G = \emptyset.$$

The next easy result provides a necessary condition for $\hat{\Lambda}^G$ to be non empty.

Proposition 2.11. *Let G be a finite group with non-reducible action on Λ. Then, if $\ker \tau \cap \ker \rho \neq 1$, the action of G is bound to collisions.*

Proof. If $g \in (\ker \tau \cap \ker \rho) \setminus \{1\}$ then $\sigma(g)$ has to be non-trivial for condition (i) of Definition 2.6 and thus, if $x \in \Lambda^G$, we have that

$$x_{\sigma(g^{-1})i}(t) = x_i(t),\ \forall\, t \in \mathbb{T},\ i = 1,\dots,n,$$

which proves that any G-equivariant loop is a collision loop. □

To conclude, in order to exclude those finite groups which *surely* induce a reducible or bound to collisions action, we may further assume the following hypotheses on G, motivated by Propositions 2.8–2.11 and give a definition of *admissible* group action on Λ. We notice that an admissible action is not necessarily non reducible, nor not bound to collisions. Nonetheless, the properties required in the admissibility are easy to check and refine the choice of the groups G we are interested in.

Definition 2.12 (*Admissible Action*)**.** We say that the action of a finite group G on Λ is *admissible* if:

- $\ker\tau \cap \ker\sigma = 1$. This shows in particular that G is a subgroup of $O(2)\times\Sigma_n$ through canonical isomorphism considering the product homomorphism (τ,σ);
- $|\ker\rho\cap\ker\sigma| \le 2$ and the non-trivial element in $\ker\rho\cap\ker\sigma$ acts as a reflection in time;
- $\ker\tau\cap\ker\rho = 1$. Again this shows that G is a subgroup of $O(2)\times O(d)$.

2.3. Classification of the group action on Λ and fundamental domain

From now on, we assume that G generates an admissible action on Λ as described in Definition 2.12. At this point we aim to give a classification of finite groups which is based on the *type of action* they provide on Λ. Actually, since $\tau(G)$ is a finite subgroup of $O(2)$, Proposition A.2 and Remark 2.1 suggest that the easiest and most significant classification has to take into account the action that G yields on $\mathbb{T}$. In particular, $G/\ker\tau$ is itself a finite subgroup of $O(2)$ and it is useful to see how the action of G on $\mathcal{X}$ is related to the one of $G/\ker\tau$ on $\mathcal{X}^{\ker\tau}$. To start with, let us define

$$\bar{G} \doteq G/\ker\tau.$$

We recall that G acts on the configuration space $\mathcal{X}$ through the representations $\rho: G\to O(d)$ and $\sigma: G\to\Sigma_n$. Since $\ker\tau\triangleleft G$, by Remark 1.3 we see that $\bar{G}$ acts on the fixed space $\mathcal{X}^{\ker\tau}$. The quotient $\bar{G}$ is a useful refinement of the whole group G, mostly because it is a subgroup of $O(2)$. This makes more convenient to work with $\bar{G}$-*equivariant* loops which live in $\mathcal{X}^{\ker\tau}$ and this turns out to be also reasonable thanks to the next result.

Proposition 2.13. *The H^1 loop spaces*

$$\Lambda^G = H^1(\mathbb{T};\mathcal{X})^G, \quad \Omega^{\bar{G}} \doteq H^1(\mathbb{T};\mathcal{X}^{\ker\tau})^{\bar{G}}$$

coincide pointwise.

Proof. As a first remark, if $x\in\Lambda^G$ then $x(t)\in\mathcal{X}^{\ker\tau}$ for any $t\in\mathbb{T}$. Indeed, the G-equivariance of an element $x\in\Lambda^G$ implies in particular that

$$hx_{h^{-1}i}(h^{-1}t) = x_i(t), \qquad \forall\, i\in\{1,\dots,N\},\ \forall\, t\in\mathbb{T},$$

and for any $h\in\ker\tau$, but obviously $h^{-1}t = t$. Moreover, the G-equivariance of x clearly implies the $\bar{G}$-equivariance, so $\Lambda^G\subseteq\Omega^{\bar{G}}$.

Assume now that $x\in\Omega^{\bar{G}}$; then, for any $g\in G$, $h\in\ker\tau$ we have that

$$\rho(gh)x_{\sigma((gh)^{-1})}(\tau((gh)^{-1})t) = x_i(t) \tag{5}$$

or

$$\rho(g)\rho(h)x_{\sigma(h^{-1})\sigma(g^{-1})}(\tau(h^{-1})\tau(g^{-1})t) = x_i(t).$$

Now, if we call $s=\tau(g^{-1})t$ and $j=\sigma(g^{-1})i$ we can rewrite the left hand side as

$$\rho(g)\rho(h)x_{\sigma(h^{-1})j}(\tau(h^{-1})s).$$

Now, recall that $x(t)\in\mathcal{X}^{\ker\tau}$ for any $t\in\mathbb{T}$ and that $h\in\ker\tau$, so that $\tau(h^{-1})s = s$. Therefore, we can rewrite the previous expression as

$$\rho(g)x_j(s)$$

and now, since $j=\sigma(g^{-1})i$ and $s=\tau(g^{-1})t$ we have actually proved that (5) holds if and only if

$$\rho(g)x_{\sigma(g^{-1})i}(\tau(g^{-1})t) = x_i(t), \quad \text{for any } g\in G,$$

and so $x\in\Lambda^G$. $\square$

Note that, since $\bar{G}$ is a finite subgroup of $O(2)$, then Proposition A.2 guarantees that $\bar{G}$ is either *cyclic* or *dihedral*. This motivates the following definition.

Definition 2.14. We classify the action of a finite group G on Λ in this way:

- if the quotient $\bar{G}$ acts trivially on the orientation of $\mathbb{T}$, then $\bar{G}$ is cyclic and we say that the action of G on Λ is of *cyclic type*. In this case, $\bar{G}$ is a subgroup of $SO(2)$ and only contains rotations;
- if $\bar{G}$ is made by a single reflection on $\mathbb{T}$, then we say that the action of G on Λ is of *brake type*. In particular, $\bar{G}$ is a cyclic group of order 2;
- if none of the previous is satisfied we say that the action of G on Λ is of *dihedral type* and $\bar{G}$ is a dihedral subgroup of $O(2)$.

The peculiarity of symmetric loops is that, once the group G is fixed, it is possible to detect a sub-interval of $\mathbb{T}$ which characterises the entire loop. This is a fundamental step when one looks for critical points of $\mathcal{A}$, since it allows to switch from the analysis of periodic loops to the one of fixed-ends paths which share initial and final conditions fixed on the extrema of such subinterval. For this reason, we firstly give the following definition.

Definition 2.15. The isotropy subgroups of the action of G on $\mathbb{T}$ are the sets

$$G_t = \{g \in G \,:\, \tau(g)t = t\}, \quad \text{for } t \in \mathbb{T},$$

and are called the $\mathbb{T}$-*isotropy subgroups* of G.

From the previous definition, the $\mathbb{T}$-isotropy subgroups of G correspond to the isotropy subgroups of the action of $\bar{G}$ on $\mathbb{T}$. This information is very useful, since we can classify again the $\mathbb{T}$-isotropy subgroups of G as subgroups of $O(2)$.

Remark 2.16. The number of distinct $\mathbb{T}$-isotropy subgroups completely determines the action type of G. At first, notice that $\ker\tau$ is always a $\mathbb{T}$-isotropy subgroup of G. Moving to the action of $\bar{G}$, observe that any $\bar{G}_t$ cannot contain rotations. Denoting by l the number of distinct $\mathbb{T}$-isotropy subgroups, we obtain this characterisation:

- $l = 1$, i.e., the maximal $\mathbb{T}$-isotropy subgroup is $\ker\tau \iff$ the action is *cyclic*;
- $l = 2$, i.e., the $\mathbb{T}$-isotropy subgroups are $\ker\tau$ and a maximal one $\iff$ the action is of *brake type*. In this case, the maximal $\mathbb{T}$-isotropy subgroup coincides with G;
- $l \geq 3$, i.e., the $\mathbb{T}$-isotropy subgroups are either maximal or $\ker\tau \iff$ the action is of *dihedral type*. In this case, every maximal $\mathbb{T}$-isotropy subgroup is cyclic of order 2. This can be seen by working with the isotropy subgroups of the action of $\bar{G}$ and observing that no rotations can belong to any $\bar{G}_t$. Note that the product of two reflections is a rotation.

We now define the already mentioned sub-interval of $\mathbb{T}$.

Definition 2.17 (*Fundamental Domain*)**.** The *fundamental domain* $\mathbb{I} \subset \mathbb{T}$ of the action of G on $\mathbb{T}$ is defined as follows:

- if the action of G is cyclic, then $\mathbb{I}$ is the closed interval which connects the instant $t = 0$ with its image $\tau(g^{-1})0$, where g is the cyclic generator of $\bar{G}$;
- if the action of G is brake or dihedral, then $\mathbb{I}$ is a closed interval whose extrema are two distinct elements of $\mathbb{T}$ generating two distinct maximal $\mathbb{T}$-isotropy subgroups, such that no other instants related to maximal $\mathbb{T}$-isotropy subgroups are included in $\mathbb{I}$. Note that, if the action is of brake type, the unique maximal $\mathbb{T}$-isotropy subgroup is G itself.

In particular, there are $|\bar{G}|$ of such domains and we have

$$\mathbb{T} = \bigcup_{[g]\in\bar{G}} \tau(g^{-1})\mathbb{I} \quad \text{and} \quad |\mathbb{I}| = \frac{|\mathbb{T}|}{|\bar{G}|}.$$

The previous definition may seem rather technical and difficult to picture, but the following example should clarify its important meaning.

Example 2.18 (*Fundamental Domain Of D_6*)**.** Recalling that D_6 is generated by a rotation r and a reflection s (see Example 2.2), we know that

$$D_6 = \left\{1, s, r, r^2, sr, sr^2\right\}.$$

In order to determine a fundamental domain $\mathbb{I}$, we need to detect first the $\mathbb{T}$-isotropy subgroups of D_6. We observe that the rotations r, r^2 do not fix any element of $\mathbb{T}$. Direct computations show that within a period of 2π:

- s fixes the instants $t = \frac{\pi}{4}, \frac{5\pi}{4}$;
- sr fixes the instants $t = \frac{11\pi}{12}, \frac{23\pi}{12}$;
- sr^2 fixes the instants $t = \frac{7\pi}{12}, \frac{19\pi}{12}$.

We conclude that D_6 has three distinct non-trivial $\mathbb{T}$-isotropy subgroups

- $G_{\frac{\pi}{4}} = G_{\frac{5\pi}{4}} = \langle s \rangle$
- $G_{\frac{11\pi}{12}} = G_{\frac{23\pi}{12}} = \langle sr \rangle$
- $G_{\frac{7\pi}{12}} = G_{\frac{19\pi}{12}} = \langle sr^2 \rangle$

which are cyclic of order 2. Following Definition 2.17, we see that for instance

$$\mathbb{I} = \left[\frac{\pi}{4}, \frac{7\pi}{12}\right], \quad |\mathbb{I}| = \frac{2\pi}{6}.$$

3. *G*-Equivariant critical points of the action functional

This final section explains the numerical optimisation process which has been prepared until now. We recall that we want to optimise the action functional

$$\mathcal{A}(x) = \frac{1}{2}\int_0^T \sum_{i=1}^n m_i \|\dot{x}_i(t)\|^2\, dt + \int_0^T \sum_{i<j} \frac{m_i m_j}{\|x_i(t) - x_j(t)\|}\, dt$$

over the symmetric loop space Λ^G. In particular, we firstly present the theoretical approach which permits to switch from the optimisation of $\mathcal{A}$ on Λ^G to a suitable *fixed-ends* problem depending on the symmetry. Secondly, we will deepen the numerical procedures which produce G-equivariant periodic orbits for the symmetrical n-body problem. We start with this first result.

Proposition 3.1. *Consider a finite group G and the usual representations ρ, τ and σ. Let $\mathbb{I} = [t_0, t_1]$ be the fundamental domain of the action of G.*

(i) If the action of G is brake or dihedral, let H_0 and H_1 be the two maximal $\mathbb{T}$-isotropy subgroups associated respectively with t_0 and t_1. Then, the space $\Omega^{\bar{G}} = H^1(\mathbb{T}; \mathcal{X}^{\ker\tau})^{\bar{G}}$ is homeomorphic to the space of fixed-end paths

$$Y = \{x \in H^1(\mathbb{I}; \mathcal{X}^{\ker\tau}) \,:\, x(t_0) \in \mathcal{X}^{H_0},\ x(t_1) \in \mathcal{X}^{H_1}\}.$$

(ii) If the action of G is cyclic, then $\Omega^{\bar{G}}$ is homeomorphic to the space

$$Y = \{x \in H^1(\mathbb{I}; \mathcal{X}^{\ker\tau}) \,:\, gx(t_0) = x(t_1)\},$$

where g is the generator of $\bar{G}$.

Proof. We start by defining the map $\pi : \Omega^{\bar{G}} \to Y$ which associates to any loop $x \in \Omega^{\bar{G}}$ its restriction to the fundamental domain $\pi(x) \doteq x|_{\mathbb{I}}$. The map π is clearly well-defined: the boundary conditions are satisfied in both cases, since x is $\bar{G}$-equivariant. Note that the boundary conditions are different since when $\bar{G}$ is cyclic the unique maximal $\mathbb{T}$-isotropy subgroup is $\ker\tau$

On the other hand, for any $y \in Y$ we can construct a unique path in $\Omega^{\bar{G}}$ by concatenating the symmetrised paths gy, for any $g \in \bar{G}$. Depending on the action type, anyone of these paths is defined on a reflection or rotation $g\mathbb{I}$ of the fundamental domain. Recalling that

$$\mathbb{T} = \bigcup_{[g]\in\bar{G}} g\mathbb{I}$$

the proof is complete. □

At this point, for any action type, we can define the restriction of the action functional to the fundamental domain $\mathbb{I}$ on Y in this way

$$\begin{aligned} &\mathcal{A}_{\mathbb{I}} : Y \longrightarrow \mathbb{R} \cup \{+\infty\} \\ &\quad y \mapsto \mathcal{A}_{\mathbb{I}}(y) \doteq \int_{\mathbb{I}} L(y(t), \dot{y}(t))\, dt. \end{aligned}$$

Arguing as in the proof of Proposition 3.1, if $y \in Y$ and $y = \pi(x)$, where x is the concatenation of symmetrised paths gy, using also the equivariance of the action functional proved in Proposition 2.3, it is easy to check that

$$\mathcal{A}(x) = |\bar{G}|\mathcal{A}_{\mathbb{I}}(y). \tag{6}$$

The following result establishes a relation between collision-less critical points of $\mathcal{A}_{\mathbb{I}}$ and symmetric solutions of the n-body problem.

Theorem 3.2. *Let G be a finite group acting on Λ and let $\mathbb{I}$ be the foundamental domain of its action. Assume that there exists a collision-less critical point $y \in Y$ of the restricted action functional $\mathcal{A}_{\mathbb{I}}$. Then, the symmetrised path $\pi^{-1}y$ is a solution of the G-equivariant n-body problem.*

Proof. Consider a critical point $y \in Y$ and the symmetrised path $x = \pi^{-1}y$, which is the concatenation of the fixed-end paths gy, for any $g \in G/\ker\tau$. This is clearly a G-equivariant path, which is also collision-less by assumption. Since y is a critical point of the restricted action functional on Y, by straightforward computations and Proposition 3.1 we see that the symmetrised path x is a critical point for $\mathcal{A}$ on Λ^G, and actually a collision-less free critical point on the whole Λ (see Lemma 2.4). We are only left to prove that x is C^1 on the whole $\mathbb{T}$, since singularities may appear on the boundary of $g\mathbb{I}$ when we symmetrise y. We split the proof in two cases:

- brake or dihedral action;
- cyclic action.

When the action is brake or dihedral, we exploit the criticality of y in Y, which imposes some boundary conditions: not only that $y(t_0) \in \mathcal{X}^{H_0}$ and $y(t_1) \in \mathcal{X}^{H_1}$, but also that $M\dot{y}(t_0) \perp \mathcal{X}^{H_0}$ and $M\dot{y}(t_1) \perp \mathcal{X}^{H_1}$, where $M \in \mathbb{R}^{nd} \times \mathbb{R}^{nd}$ denotes the diagonal matrix of the masses of the bodies. To see that, it is enough to expand the equation $d\mathcal{A}_{\mathbb{I}}(gy)[\varphi] = 0$, for any $\varphi \in Y$.

Now assume that $G/\ker\tau = \langle g \,:\, g^k = 1\rangle$, for some $k \in \mathbb{N}$. Following Definition 2.17, in this case we have $\mathbb{I} = [0, T/k]$ and $H_0 = H_1 = \ker\tau$. This means that the path $y(t) \in \mathcal{X}^{\ker\tau}$ for any $t \in [0, T/k]$ and the only condition to be satisfied in order to have a C^1 junction is that

$$\dot{y}(T/k) = g\dot{y}(0).$$

Again, this follows from the criticality condition on $\mathcal{A}_{\mathbb{I}}$. □

As already remarked in the introduction, the search for critical points of the action functional $\mathcal{A}$ is also motivated by a simple condition which guarantees the coercivity of $\mathcal{A}$ (see Proposition 2.5). There exists also a *local* version of this result, which establishes the coercivity of $\mathcal{A}_{\mathbb{I}}$ on fixed-ends paths under the same condition on G.

Lemma 3.3. *The functional $\mathcal{A}_{\mathbb{I}}$ is coercive on Y if and only if $\mathcal{X}^G = \{0\}$.*

Proof. By Propositions 2.5–2.13, we prove that $\mathcal{A}_{\mathbb{I}}$ is coercive if and only if $\mathcal{A}|_{\Omega^{\bar{G}}}$ is. Moreover, Proposition 3.1 guarantees that, for any $x \in \Omega^{\bar{G}}$ there exists a constant $C > 0$, depending exclusively on $|\bar{G}|$, such that

$$\|\pi(x)\|_{H^1} = C\|x\|_{H^1}.$$

This clearly provides the assertion. □

This quite clean condition motivates the search for *local minimisers* of $\mathcal{A}_{\mathbb{I}}$, and it is straightforward to show that the periodic symmetrisation of these paths minimises $\mathcal{A}$ on Λ. In addition, any candidate minimiser of $\mathcal{A}_{\mathbb{I}}$ belongs to the space Y introduced in Proposition 3.1 and so the implementantion of the numerical procedures requires to deal with a boundary condition problem. Nevertheless, a suitable definition of *local minimiser* has to be given and major assumptions on G are required in order to prove the absence of collisions. We prefer to postpone a survey on such results to Appendix B.

3.1. Approximating critical points of $\mathcal{A}$

At this point, we present the numerical procedure which succeeds in finding some examples of such solutions as critical points of the action functional. The first version of this optimisation routine was introduced in [32] and now it is also implemented in the Julia package `SymOrb.jl`.

3.1.1. Configuration space and projectors

As a first step, we need to define a finite-dimensional approximation of the space Y introduced in Proposition 3.1. The space Y already contains two finite-dimensional spaces, the starting and final manifolds $\mathcal{X}^{H_0}$ and $\mathcal{X}^{H_1}$ (which both coincide with $\mathcal{X}^{\ker\tau}$ when the action is cyclic), and thus it is trivially diffeomorphic to the space

$$H^1(\mathbb{I}; \mathcal{X}^{\ker\tau}) \times \mathcal{X}^{H_0} \times \mathcal{X}^{H_1}.$$

A natural way to approximate elements in $H^1(\mathbb{I}; \mathcal{X}^{\ker\tau})$ is by using truncated Fourier series, while the manifolds $\mathcal{X}^{H_i}$ are already embedded in $(\mathbb{R}^d)^n$. As a result, our optimisation procedure will take place in the linear space

$$((\mathbb{R}^d)^n)^F \times (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n,$$

where $F \in \mathbb{N}$ stands for the length of Fourier polynomials. From this point, we assume that $\mathbb{I} = [0, \pi]$. Consider $A_1, \dots, A_F \in \left(\mathbb{R}^d\right)^n$ as the Fourier coefficients and $A_0 = x_0, A_{F+1} = x_1 \in (\mathbb{R}^d)^n$ as the initial and final configurations. In this way, an approximating path will be defined as

$$y(t) \doteq x(t) + s(t), \tag{7}$$

where $x(t)$ stands for the segment between x_0 and x_1

$$x(t) = x_0 + \frac{t}{\pi}(x_1 - x_0), \quad \text{for } t \in [0, \pi],$$

and $s(t)$ is the truncated Fourier series

$$s(t) \doteq \sum_{k=1}^{F} A_k \sin(kt), \quad \text{for } t \in [0, \pi].$$

Consider a discretisation $(t_h)_{h=0\dots S}$ of the interval $[0, \pi]$, where $t_0 = 0$ and $t_S = \pi$. Then $\Delta t = \frac{\pi}{S}$ and $t_h = h\Delta t$, so that

$$x(t_h) = x_0 + \frac{h}{S}(x_1 - x_0)$$

and

$$s(t_h) = \sum_{k=1}^{F} A_k \sin\left(\frac{\pi}{S} kh\right).$$

The functions we want to approximate pointwisely belong to the space $\mathcal{X}^{\ker\tau}$ and also the boundary conditions have to be imposed (note again that they change according with the action type). Therefore, recalling the definition of projector on fixed spaces given in (3), we consider the maps

$$\pi_{\ker\tau} : (\mathbb{R}^d)^n \to \mathcal{X}^{\ker\tau}$$
$$\pi_{H_i} : (\mathbb{R}^d)^n \to \mathcal{X}^{H_i}, \quad i = 0, 1,$$

which project any configuration in the correct equivariant space. Looking at the definition of Y in Proposition 3.1, the cyclic case deserves a different treatment. In that case, the boundary conditions requires an equivariance property, which can be translated by using the following projector

$$\begin{aligned}\pi_g : (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n &\longrightarrow V_g \\ (v, w) &\longmapsto \pi_g(v, w) \doteq \left(\frac{1}{2}(v + g^{-1}w), \frac{1}{2}(gv + w)\right)\end{aligned}$$

where g is the cyclic generator of $\bar{G}$ and $V_g \doteq \{(v, w) \in (\mathbb{R}^d)^n \times (\mathbb{R}^d)^n \,:\, gv = w\}$. With these definitions, the projector of a pair of points on the right boundary condition space is the following

$$\pi_{bc}(v, w) = \begin{cases} (\pi_{H_0} v, \pi_{H_1} w) & \text{if the action is brake/dihedral} \\ \pi_g(v, w) & \text{if the action is cyclic} \end{cases}$$

Remark 3.4 (*Fixing the centre of mass*)**.** We recall that the centre of mass in $\mathcal{X}$ is fixed at the origin. In `SymOrb` [32], to accomplish this requirement, at every step of the optimisation process the whole configuration is suitably translated. In `SymOrb.jl` we propose a different approach: only $n-1$ configurations are actually free variables, while the nth is a function of the former so that the centre of mass remains at the origin. Beside avoiding a projection at each step, this method considerably reduces the number of parameters to be optimised, hence shortening the computational time, especially for low number of bodies.

3.1.2. The action functional

Each optimisation step requires the evaluation of the action functional $\mathcal{A}$ introduced in (1) on a given path, together with its gradient (when we choose a first-order optimisation method) and its Hessian (when we choose a second order method). It is possible to compute them without approximating the velocities $\dot{y}(t)$ and the accelerations $\ddot{y}(t)$ of the path $y(t)$ in a (possibly) rotating reference frame with infinitesimal rotation matrix $\Omega \in \mathfrak{so}(d)$, the Lie algebra of $SO(d)$.

We introduce the kinetic and the potential components of the action functional on $[0, \pi]$

$$\mathcal{A}(y) = \mathcal{K}(y, \dot{y}) + \mathcal{U}(y) = \int_0^{\pi} K(y, \dot{y})\, dt + \int_0^{\pi} U(y)\, dt;$$

note that the kinetic part here depends not only on $\dot{y}$, but also on y, because we can also consider the n-body problem in a rotating frame (see also Remark 0.1). In the following, we are going to compute the gradient and the Hessian of $\mathcal{A}$ and, since we are working with approximated paths, such operators will be computed by differentiating with respect to the coefficients matrices $A_0, A_1, \ldots, A_F, A_{F+1} \in (\mathbb{R}^d)^n$. For this reason, we will often write $y(A)$ and $\mathcal{A}(A)$ to point out the dependence on such coefficients matrices.

The kinetic part. Consider, in a possibly rotating reference frame, a path $y(t)$ as in (7) and its velocity $\dot{y}(t)$. Let $\tilde{y}(t)$ be the same path in the inertial frame, so that $\dot{\tilde{y}}(t) = \dot{y}(t) - \Omega y$. The kinetic energy is then

$$\begin{aligned} K(y, \dot{y}) &= \frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{\tilde{y}}_i(t)\|^2 = \frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{y}_i(t) - \Omega y(t)\|^2 \\ &= \frac{1}{2} \sum_{i=1}^{n} m_i \Big[\underbrace{\dot{y}_i^t \dot{y}_i}_{\text{linear}} - \underbrace{y_i^t \Omega^2 y_i}_{\text{centrifugal}} + \underbrace{\left(\dot{y}_i^t \Omega y_i - y_i^t \Omega \dot{y}_i\right)}_{\text{Coriolis}} \Big] \\ &= K_{\text{lin}}(\dot{y}) + K_{\text{centr}}(y) + K_{\text{Cor}}(y, \dot{y}). \end{aligned}$$

Integrating from 0 to π, since the dependence of y on the coefficients A_k is well understood, the kinetic part of the action can be written as a quadratic form $\mathcal{K} : ((\mathbb{R}^n)^d)^{F+2} \to \mathbb{R}$, such that $\mathcal{K}(A) = \frac{1}{2} y^t(A)\, \mathbf{K}\, y(A)$ and $\mathbf{K} = \mathbf{K}_{\text{lin}} + \mathbf{K}_{\text{centr}} + \mathbf{K}_{\text{Cor}}$. In particular, it is possible to rewrite the three components as

$$\mathbf{K}_{\text{lin}} = (K_{jk}^{\text{lin}}) \otimes (\mathbf{M} \otimes \mathbf{Id}_d),$$
$$\mathbf{K}_{\text{centr}} = (K_{jk}^{\text{centr}}) \otimes (\mathbf{M} \otimes \Omega^2),$$
$$\mathbf{K}_{\text{Cor}} = (K_{jk}^{\text{Cor}}) \otimes (\mathbf{M} \otimes \Omega),$$

where $K^{\mathrm{lin}}_{jk}, K^{\mathrm{centr}}_{jk}, K^{\mathrm{Cor}}_{jk}$ are $(F+2)\times(F+2)$ matrices, $\mathbf{M} = \mathrm{diag}(m_1,\dots,m_n)$ is the diagonal matrix of masses and $\otimes$ is the Kronecker product.

Hence, the gradient of the kinetic part of the action functional is given by $\nabla\mathcal{K}(A) = \mathbf{K}\, y(A)$ and the Hessian is $\nabla^2\mathcal{K}(A) = \mathbf{K}$.

The potential part. Consider the potential function $U(y)$. It can be written as

$$U(y) = \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{m_i m_j}{f\left(\|y_{ij}\|\right)} = \sum_{i=1}^{n}\sum_{j=i+1}^{n} U_{ij}(y_{ij})$$

where $y_{ij} = y_i - y_j$, $f : \mathbb{R}^+ \to \mathbb{R}$ is a function of the norm $\|y_{ij}\|$. Formally, the components of the gradient and of the Hessian $\frac{\partial U_{ij}(y)}{\partial y_i} \in \mathbb{R}^d$ and $\frac{\partial^2 U_{ij}(y)}{\partial y_i \partial y_j} \in \mathbb{R}^{d\times d}$ are given by

$$\frac{\partial U_{ij}(y)}{\partial y_i} = -m_i m_j \frac{f'}{f^2}\frac{y_{ij}}{\|y_{ij}\|}$$

$$\frac{\partial^2 U_{ij}(y)}{\partial y_i \partial y_j} = \frac{m_i m_j}{f^2\cdot\|y_{ij}\|^2}\left[(y_{ij}\otimes y_{ij})\left(f'' - \frac{f'}{\|y_{ij}\|} - \frac{2f'^2}{f}\right) + (f'\cdot\|y_{ij}\|)\mathrm{Id}_d\right]$$

where the dependence of f on y_{ij} is understood. Hence, for $i = 1,\dots,n$ we have

$$\frac{\partial U(y)}{\partial y_i} = \sum_{j\neq i}\frac{\partial U_{ij}(y)}{\partial y_i} \qquad \frac{\partial^2 U(y)}{\partial y_i^2} = \sum_{j\neq i}\frac{\partial^2 U_{ij}(y)}{\partial y_i^2},$$

while, for $i, j \in \{1,\dots,n\}$, $i \neq j$ we obtain

$$\frac{\partial^2 U(y)}{\partial y_i \partial y_j} = \frac{\partial}{\partial y_j}\left(\frac{\partial U(y)}{\partial y_i}\right) = \frac{\partial^2 U_{ij}(y)}{\partial y_i \partial y_j}.$$

Therefore, integrating in $[0,\pi]$, for $k, h = 0,\dots,F+1$ we finally have

$$\begin{aligned}
\mathcal{U}(A) &= \int_0^\pi U(y)dt \\
\mathcal{U}(A)_k &= \int_0^\pi \sum_{i=1\dots n}\frac{\partial U(y)}{\partial y_i}\frac{\partial y_i}{\partial A_k}dt \\
\mathrm{Hess}\,\mathcal{U}(A)_{kh} &= \int_0^\pi \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\frac{\partial^2 U(y)}{\partial y_i \partial y_j}\frac{\partial y_j}{\partial A_h}\right)^t \frac{\partial y_i}{\partial A_k}dt
\end{aligned}$$

where the integration is performed numerically.[2] Note that $\frac{\partial y_i}{\partial A_k}$ is independent of A_k and can be precomputed.

Remark 3.5. As explained in Remark 3.4, analysing the code it turns out that each matrix involved in the computation actually refers to $n-1$ bodies and already encodes the dependence of the last one from the others. For the sake of clarity and consistency with the theoretical framework, we prefer to not change the dimensions neither in the next section, understanding that the nth body is a function of the others $n-1$.

3.2. Numerical algorithm

Below we collect the main steps involved in the numerical optimisation routine:

- Consider an initial sequence

  $$A_0, A_1, \dots, A_F, A_{F+1} \in (\mathbb{R}^d)^n,$$

  where A_0 and A_{F+1} correspond to the initial and final configurations, while $A_1,\dots,A_F$ represent Fourier coefficients. This sequence can be random or chosen ad-hoc to improve the convergence of the algorithm.
- Project A_0 and A_{F+1} onto the boundary manifolds $\mathcal{X}^{H_0}, \mathcal{X}^{H_1}$ using the map π_{bc} and thus obtaining the initial and final configurations

  $$(x_0, x_1) = \pi_{bc}(A_0, A_{F+1}).$$

  Then project $x_0, x_1, A_1, \dots, A_F$ onto $\mathcal{X}^{\ker\tau}$ using the map $\pi_{\ker\tau}$. Note that, since the path depends linearly on the Fourier coefficients, it is possible to project the latter instead of the former.

[2] We use the trapezoidal integration rule.

- Optimise the action functional, written in terms of Fourier coefficients on the fundamental domain $\mathbb{I} = [0, \pi]$, using a suitable numerical method that stops when a convergence conditions is fulfilled (see in particular Section 4). At each optimisation step, before computing the action, its gradient and its Hessian, project $x_0, x_1, A_1, \dots, A_F$ as before. Moreover, project also the gradient and the Hessian of the action, to ensure that the optimisation is performed in the correct space. The gradient is projected using the same projectors used for the Fourier coefficients. The Hessian is projected using the same projectors used for the Fourier coefficients applied to the rows and columns of the Hessian.

3.2.1. Extension to the whole period

The optimised path is defined on the fundamental domain $\mathbb{I} = [0, \pi]$ via Fourier coefficients and can be extended to the whole period $\mathbb{I} = [0, m\pi]$, where $m = |\bar{G}|$, using the symmetries of the group and an inverse Fourier transform. In fact, consider $y(t)$ as in (7) where the coefficients $A_0, \dots, A_{F+1}$ are the optimised ones. Let $y_h \doteq y(t_h), 0 \le h \le S$ be its discretisation.

Cyclic action. In the cyclic case, the group $\bar{G}$ is generated by $g \in G$. The path can be extended up to y_{mS} by

$$y_{jS+k} = \rho(g^j) y_k \qquad \text{for } 1 \le j \le m, 0 \le k < S.$$

Dihedral or brake action. In the dihedral or brake case, the group $\bar{G}$ is generated by $g_0 \in H_0$ and $g_1 \in H_1$. The path can be extended up to y_{2S} by

$$y_{S+k} = \rho(g_1) y_{S-k} \qquad \text{for } 0 \le k < S.$$

Then it can be extended to the whole period, up to mS, analogously to the cyclic case, replacing S with $2S$. Clearly, if the action is of brake type, this last step is not needed.

4. Algorithm overview and numerical results

This short final section contains an outline of the algorithm which stands behind the package `SymOrb.jl`. A practical guide, complemented with sessions examples and useful tips, can be reached at https://github.com/DipMathUniTO/SymOrb.jl. Here, we present the algorithm in two steps: the first one consists in specifying the symmetry group G which is acting, while the second one describes the optimisation routine and the re-construction of the whole symmetric periodic orbit.

Recalling Section 2, the action of a symmetry group G is based on the representations $\rho : G \to O(d), \tau : G \to O(2)$ and $\sigma : G \to \Sigma_n$, and the *action type* depends on the quotient $G/\ker\tau$, which is a finite subgroup of $O(2)$. On the other hand, in Section 3 we explained how the reduction from the search for periodic solutions to a fixed-end optimisation is based on the construction of the boundary manifolds $\mathcal{X}^{H_0}$ and $\mathcal{X}^{H_1}$, and the connection manifold $\mathcal{X}^{\ker\tau}$. Actually, `SymOrb.jl` returns a *configuration problem* `P`, which contains the above information and prepares for the optimisation routine. In particular, the whole group G is reconstructed using an ad-hoc `GAP` library, whose original version `SymOrb.g` has been created by Davide Ferrario (see [32]). Below, we sum up the configuration algorithm:

Algorithm 1 Symmetric problem configuration

Require:

- d (dimension of configuration space), n (number of bodies)
- Action type: 'cyclic', 'dihedral', or 'brake'
- Generators of $G/\ker\tau$ (matrices and permutations)
- Generators of $\ker\tau$ (matrices and permutations)
- Masses: list of positive real numbers
- Generator of the angular velocity $\Omega \in \mathfrak{so}(d)$ ($d \times d$ anti-symmetric matrix)
- F (number of Fourier coefficients)
- **(optional)** A function $f : \mathbb{R}^+ \to \mathbb{R}$ (denominator of the potential function U)

Specify the action type:
if cyclic **then**
    Use one generator of $G/\ker\tau$ (rotation)
else if dihedral or brake **then**
    Use two generators of $G/\ker\tau$ (rotation and reflection)
end if
Reconstruct $G/\ker\tau$
Determine optimisation spaces:
    Compute boundary manifolds $\mathcal{X}^{H_0}$ and $\mathcal{X}^{H_1}$
    Compute connection manifold $\mathcal{X}^{\ker\tau}$
return Configuration problem `P` needed to start the optimisation routine

---

The next step requires to parse the symmetry information contained in `P` and to choose a starting path, from which the optimisation routine begins. The software allows to proceed with the natural choice of random coefficients, but also to exploit user-provided initial paths, such as a circular guess, in which the bodies are equally spaced. At this point, the optimisation routine can finally start.

Algorithm 2 Optimisation Routine

Require:

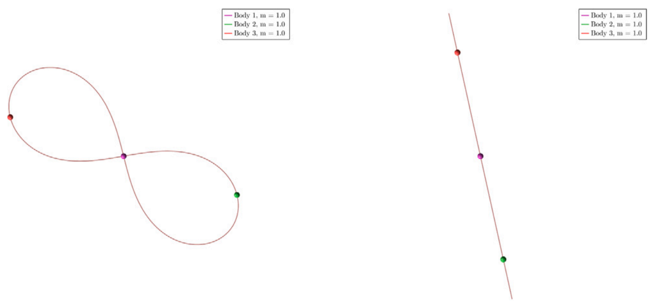
- Configuration problem P
- Optimisation methods
- **(optional)** user-provided paths

Choose starting path:
if user provides initial path **then**
    Use user-defined initial path (e.g., circular guess)
else
    Generate random Fourier coefficients
end if
Select optimisation method:
if First-order method **then**
    Choose from Conjugate-Gradient, Gradient Descent, or BFGS (`Optim.jl`)
else if Second-order method **then**
    Use Newton method (line search or trust region via `NLSolve.jl`)
end if
Set stopping criteria (max iterations, convergence condition)
Run optimisation routine
Optionally combine methods to improve convergence
if optimisation converges **then**
    Reconstruct periodic orbit using symmetries
    return symmetric periodic orbit
else
    return optimisation failed
end if
Visualise periodic orbits (`Makie.jl`)

4.1. Some numerical results

This final section contains examples of symmetry groups, with pictures of some numerical results in which the optimisation process succeeds.

Example 1 ($d = 2,\ m_1 = m_2 = m_3 = 1$)**.** Consider the dihedral symmetry group D_6 introduced in Examples 2.2 and 2.18. Note that the trajectory in panel (b) has double collisions.



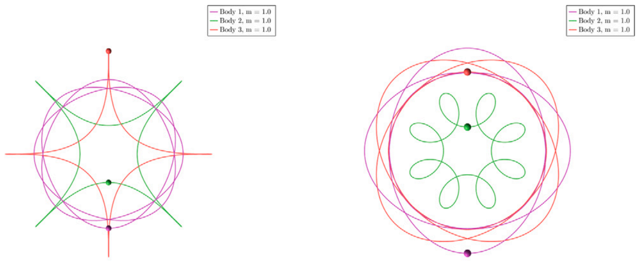
(a) $\mathcal{A} = 5.8584$
$\operatorname{grad}\mathcal{A} = 2.45 \times 10^{-8}$

(b) $\mathcal{A} = 7.5833$
$\operatorname{grad}\mathcal{A} = 5.31 \times 10^{-10}$

Example 2 ($d = 2,\ m_1 = m_2 = m_3 = 1$)**.** Take the symmetry group with generators r and s such that

$$\rho(r) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \sigma(r) = (), \qquad \rho(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma(s) = (),$$

where () is the identity permutation map.

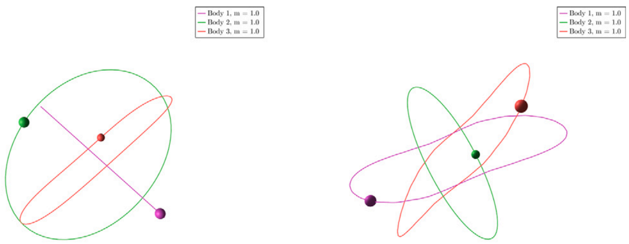


(a) $\mathcal{A} = 7.2653$
$\operatorname{grad} \mathcal{A} = 7.83 \times 10^{-9}$

(b) $\mathcal{A} = 8.9040$
$\operatorname{grad} \mathcal{A} = 1.16 \times 10^{-8}$

Example 3 ($d = 3,\ m_1 = m_2 = m_3 = 1$)**.**
Consider the cyclic symmetry group with generator r such that

$$\rho(r) = -\mathrm{Id}_3, \qquad \sigma(r) = ().$$

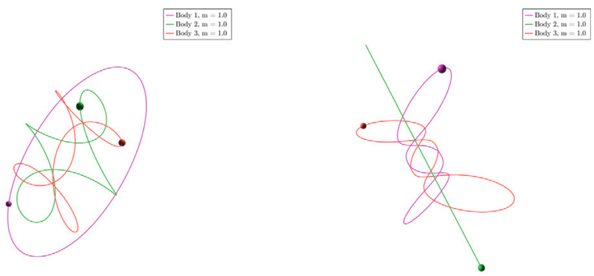


(a) $\mathcal{A} = 10.1572$
$\operatorname{grad} \mathcal{A} = 4.7 \times 10^{-7}$

(b) $\mathcal{A} = 11.3125$
$\operatorname{grad} \mathcal{A} = 2.25 \times 10^{-12}$

Example 4 ($d = 3,\ m_1 = m_2 = m_3 = 1$)**.** Choose the same r as in Example 3 and add the reflection generator s such that

$$\rho(s) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \sigma(s) = ().$$

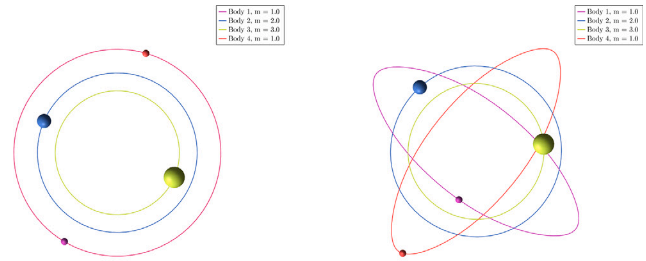


(a) $\mathcal{A} = 9.0005$
$\operatorname{grad} \mathcal{A} = 2.36 \times 10^{-9}$

(b) $\mathcal{A} = 10.1224$
$\operatorname{grad} \mathcal{A} = 3.96 \times 10^{-8}$

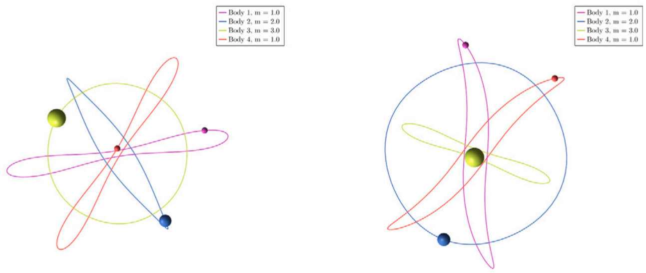
Example 5 ($d = 2,\ m_1 = m_4 = 1,\ m_2 = 2,\ m_3 = 3$)**.** For a planar 4-body system with different masses, consider the cyclic symmetry group generated by r such that

$$\rho(r) = -\mathrm{Id}_2 \qquad \sigma(r) = ().$$



(a) $\mathcal{A} = 42.5707$
grad $\mathcal{A} = 8.26 \times 10^{-9}$

(b) $\mathcal{A} = 43.5231$
grad $\mathcal{A} = 7.24 \times 10^{-7}$



(c) $\mathcal{A} = 45.2366$
grad $\mathcal{A} = 1.08 \times 10^{-6}$

(d) $\mathcal{A} = 45.4953$
grad $\mathcal{A} = 1.65 \times 10^{-9}$

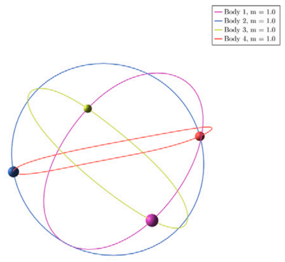
Example 6 ($d = 3,\ m_1 = m_2 = m_3 = m_4 = 1$)**.** Lastly, we consider a spatial 4-body system under the action of a symmetry group G with non-trivial $\ker\tau$. Choose κ as generator of $\ker\tau$ represented as

$$\rho(\kappa) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \sigma(\kappa) = (1, 2, 3, 4)$$

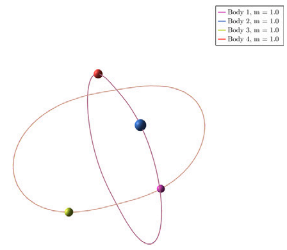
and take $\bar{G} = G/\ker\tau$ as the cyclic group generated by the element g such that

$$\rho(g) = -\mathrm{Id}_3 \qquad \sigma(g) = ().$$

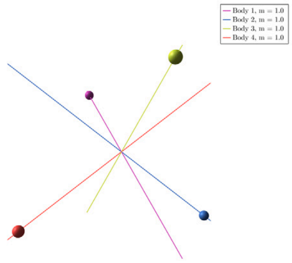
Note that the trajectory in panel (c) exhibits a total collision.



(a) $\mathcal{A} = 17.9761$
$\operatorname{grad} \mathcal{A} = 1.02 \times 10^{-8}$



(b) $\mathcal{A} = 20.7687$
$\operatorname{grad} \mathcal{A} = 1.99 \times 10^{-8}$



(c) $\mathcal{A} = 25.0553$
$\operatorname{grad} \mathcal{A} = 2.1 \times 10^{-8}$

CRediT authorship contribution statement

Vivina Barutello: Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization, Writing – review & editing. **Gian Marco Canneori:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Roberto Ciccarelli:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Susanna Terracini:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Mattia G. Bergomi:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization. **Pietro Vertechi:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization. **Davide L. Ferrario:** Writing – review & editing, Writing – original draft, Visualization, Validation, Supervision, Software, Resources, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Data curation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Finite subgroups of $O(2)$

Our major interest resides on some particular finite groups, which turn out to be very useful to describe the symmetries acting on the n-body problem. In Section 1 we have recalled the concept of *group action*, which is a very smart way of describing the symmetry of an object and allows to store in a group all the necessary information to describe these symmetries. There are some particular classes of groups whose action models some reductions by symmetry of the n-body problem and we collect them in this short section.

Definition A.1 (*Dihedral Groups*)**.** A *dihedral group* is the group of symmetries of a regular polygon, i.e., the group of *reflections* and *rotations* which fix the polygon. If we consider a polygon with n edges, also named n*-gon*, we define D_{2n} as its dihedral group. The elements of D_{2n} will be rotational symmetries, reflectional symmetries and their compositions. Note that rotations which fix a polygon with n edges are those of $\frac{2\pi}{n}$ and their multiples. The order of the dihedral group D_{2n} is exactly $2n$, since it contains n rotational symmetries and n reflectional symmetries. The dihedral group D_{2n} can be generated by a unique rotation r and a unique reflection s, so that it is usually defined through the following presentation:

$$\langle s, r\, :\ s^2 = r^n = (sr)^2 = 1\rangle.$$

It is quite clear that the 2-dimensional *orthogonal group* $O(2)$ plays a foundamental role in the context of this paper. We briefly recall that $O(2)$ is the matrix group of 2×2 real orthogonal matrices, and it contains the *special orthogonal group* $SO(2) = \{M \in O(2)\, :\ \det(M) = 1\}$. Therefore, the elements act either as a rotation about the origin or a reflection with respect to a line passing through the origin. For instance, we can represent a rotation of angle ϑ in this way:

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

and a reflection with respect to a line which forms an angle ϑ with the horizontal axis in this way:

$$\begin{pmatrix} \cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & -\cos 2\vartheta \end{pmatrix}.$$

Thus, we can expect that the classification of the finite subgroups of $O(2)$ is quite precise. We have indeed the following classical result.

Proposition A.2 (*Finite subgroups of $O(2)$*)**.** *Any finite subgroup H of $O(2)$ is either cyclic or dihedral. In particular:*

- *if $H \subseteq SO(2)$ then H is cyclic and only contains rotations;*
- *if $H = \{1, S\}$, with $S \in O(2) \setminus SO(2)$ a reflection, then H is cyclic of order 2;*
- *if H contains both elements of $SO(2)$ and $O(2) \setminus SO(2)$ then it is dihedral.*

Appendix B. Absence of collisions for local minimisers

As announced in Section 3, we collect here a survey on the arguments which guarantee the absence of collisions for local minimisers of the action functional $\mathcal{A}$ (we mainly refer to Sections 9–10 of [9]). For a finite group G, we recall that $\mathbb{I}$ is the fundamental of the action of G (see Definition 2.17) and the restriction of $\mathcal{A}$ to $\mathbb{I}$ is the functional $\mathcal{A}_{\mathbb{I}} : Y \to \mathbb{R} \cup \{+\infty\}$ which reads

$$\mathcal{A}_{\mathbb{I}}(y) = \int_{\mathbb{I}} \left[\frac{1}{2} \sum_{i=1}^{n} m_i \|\dot{y}(t)\|^2 + \sum_{i<j} \frac{m_i m_j}{\|y_i(t) - y_j(t)\|} \right] dt,$$

where Y is defined in Proposition 3.1. Lemma 3.3 inspires the search for minimisers for $\mathcal{A}_{\mathbb{I}}$ in Y. The following definition specifies the nature of minimisers in this context, with respect to local deformations in Y.

Definition B.1. We say that $y \in Y$ is a *local minimiser* of $\mathcal{A}_{\mathbb{I}}$ if, for any $\varphi \in Y$ and for any ε sufficiently small we have

$$\mathcal{A}_{\mathbb{I}}(x) \leq \mathcal{A}_{\mathbb{I}}(x + \varepsilon\varphi).$$

Assuming $\mathcal{X}^G = \{0\}$, the existence of at least a local minimiser is guaranteed by Lemma 3.3 (see also [9, Proposition 4.12]), which once symmetrised is a minimiser of $\mathcal{A}$ in Λ^G, defined on the whole period. Since Λ^G is closed, the minimiser could lie in the boundary and collisions between the bodies may occur. The following definitions prepare the reader to a stronger condition on G – known as the *rotating circle property* – which forces the minimisers to avoid collisions.

Definition B.2. Let G be a finite group acting orthogonally on $\mathbb{R}^d$ through the representation ρ and $\mathbb{S} \subseteq \mathbb{R}^d$ be a circle centred at the origin. We say that $\mathbb{S}$ is *rotating under a subgroup H* of G if the restriction $h|_{\mathbb{S}}$ of the action of every element $h \in H$ is a rotation of an angle ϑ_h. In particular, in this case $\mathbb{S}$ is invariant under the action of H.

Definition B.3. Let G be a finite group acting on the index set $\{1,\dots,n\}$ through the representation σ. Fix $i \in \{1,\dots,n\}$ and let H be a subgroup of G, let K_i denote the isotropy subgroup of the index i in H, i.e.,

$$K_i \doteq \{h \in K \,:\, hi = i\}.$$

A circle $\mathbb{S} \subseteq \mathbb{R}^d$ centred at the origin is termed *rotating for i under* H if

- $\mathbb{S}$ is rotating under H;
- $S \subseteq (\mathbb{R}^d)^{K_i}$.

Definition B.4. We say that a group G acts on Λ with the rotating circle property if, for every $\mathbb{T}$-isotropy subgroup G_t and for at least $n-1$ indexes $i \in \{1,\dots,n\}$, there exists a circle $\mathbb{S} \subseteq \mathbb{R}^d$ centred at the origin rotating for i under G_t.

The following result establishes in two steps the absence of collisions for local minimisers.

Lemma B.5 (*Theorem 10.3, [9]*)**.** *Assume that* $\ker\tau$ *acts on* Λ *with the rotating circle property. Then, any local minimiser* $y \in Y$ *of* $\mathcal{A}_{\mathbb{I}}$ *is free of interior collisions, i.e.,* $y(t) \notin \Delta$*, for any* $t \in (t_0, t_1)$*. Moreover, if also* H_0 *and* H_1 *act on* Λ *with the rotating circle property,* $y(t_0), y(t_1) \notin \Delta$ *and the minimiser is always collision-less.*

Note that, in the previous lemma, we have made no distinction between the cyclic and not cyclic action cases. The reason is again that, when $\bar{G}$ is cyclic, then $H_0 = H_1 = \ker\tau$ and the absence of collisions in the whole period follows from the fact that $\ker\tau$ acts with the rotating circle property. Moreover, note that if $\ker\tau = 1$, the rotating circle property holds trivially, so that any cyclic group G generates G-equivariant collision-less solutions as action-minimisers.

To summarise, the main result proved in [9] is the following

Theorem B.6. *Assume that* $\mathcal{X}^G = \{0\}$ *and that* $\ker\tau$*,* H_0 *and* H_1 *act on* Λ *with the rotating circle property. Then, there exists a classical solution of the* G*-equivariant* n*-body problem.*

Data availability

No data was used for the research described in the article.

References

[1] Poincaré H. Les méthodes nouvelles de la mécanique céleste. 1892.
[2] Bahri A, Rabinowitz PH. A minimax method for a class of Hamiltonian systems with singular potentials. J Funct Anal 1989;82(2):412–28.
[3] Bahri A, Rabinowitz PH. Periodic solutions of Hamiltonian systems of 3-body type. Ann Inst H. PoincarÉ C Anal Non Linéaire 1991;8(6):561–649.
[4] Serra E, Terracini S. Collisionless periodic solutions to some three-body problems. Arch Ration Mech Anal 1992;120(4):305–25.
[5] Majer P, Terracini S. Periodic solutions to some problems of n-body type. Arch Ration Mech Anal 1993;124(4):381–404.
[6] Serra E, Terracini S. Noncollision solutions to some singular minimization problems with Keplerian-like potentials. Nonlinear Anal 1994;22(1):45–62.
[7] Majer P, Terracini S. On the existence of infinitely many periodic solutions to some problems of n-body type. Comm Pure Appl Math 1995;48(4):449–70.
[8] Chenciner A, Montgomery R. A remarkable periodic solution of the three-body problem in the case of equal masses. Ann of Math (2) 2000;152(3):881–901.
[9] Ferrario DL, Terracini S. On the existence of collisionless equivariant minimizers for the classical n-body problem. Invent Math 2004;155(2):305–62.
[10] Ambrosetti A, Coti Zelati V. Periodic solutions of singular Lagrangian systems. In: Progress in nonlinear differential equations and their applications, vol. 10, Boston, MA: Birkhäuser Boston Inc.; 1993.
[11] Struwe M. Variational methods, In: applications to nonlinear partial differential equations and Hamiltonian systems, fourth ed.. vol. 34, Berlin: Springer-Verlag; 2008.
[12] Bessi U, Coti Zelati V. Symmetries and noncollision closed orbits for planar N-body-type problems. Nonlinear Anal 1991;16(6):587–98.
[13] Chen K-C. Action-minimizing orbits in the parallelogram four-body problem with equal masses. Arch Ration Mech Anal 2001;158(4):293–318.
[14] Ramos M, Terracini S. Noncollision periodic solutions to some singular dynamical systems with very weak forces. J Differential Equations 1995;118(1):121–52.
[15] Chenciner A, Venturelli A. Minima de l'intégrale d'action du problème newtonien de 4 corps de masses égales dans $\mathbf{R}^3$: orbites hip-hop. Celestial Mech Dynam Astronom 2001) 2000;77(2):139–52.
[16] Barutello V, Terracini S. Action minimizing orbits in the n-body problem with simple choreography constraint. Nonlinearity 2004;17(6):2015–39.
[17] Fusco G, Gronchi GF, Negrini. Platonic polyhedra P. Topological constraints and periodic solutions of the classical N-body problem. Invent Math 2011;185(2):283–332.
[18] Montaldi J, Steckles K. Classification of symmetry groups for planar n-body choreographies. Forum Math Sigma 2013;1(55). Paper No. e5.
[19] Simó C. New families of solutions in N-body problems. In: European congress of mathematics, vol. I (Barcelona, 2000). Progr. math., vol. 201, Basel: Birkhäuser; 2001, p. 101–15.
[20] Bolotin S V. Nonintegrability of the problem of n centers for $n > 2$. Vestnik Moskov. Univ Ser I Mat. Mekh. 1984;3:65–8.
[21] Klein M, Knauf A. Chaotic motion in Coulombic potentials. In: Mathematical physics x (Leipzig, 1991). Berlin: Springer; 1992, p. 308–12.
[22] Knauf A. The n-centre problem of celestial mechanics for large energies. J Eur Math Soc (JEMS) 2002;4(1):1–114.
[23] Soave N, Terracini S. Symbolic dynamics for the N-centre problem at negative energies. Discrete Contin Dyn Syst 2012;32(9):3245–301.
[24] Bolotin SV, Kozlov VV. Topological approach to the generalized n-centre problem. Uspekhi Mat. Nauk 2017;72((3(435)):65–96.
[25] Barutello V, Canneori GM, Terracini S. Symbolic dynamics for the anisotropic N-centre problem at negative energies. Arch Ration Mech Anal 2021;242(3):1749–834.
[26] Baranzini S, Canneori GM. Chaotic phenomena for generalised N-centre problems. Arch Ration Mech Anal 2024;248(3). Paper No 39.
[27] Ferrario DL. Symmetries and periodic orbits for the n-body problem: about the computational approach. 2024, Preprint, arXiv:2405.07737 . [math.CA].

[28] Ferrario DL. Symmetry groups and non-planar collisionless action-minimizing solutions of the three-body problem in three-dimensional space. Arch Ration Mech Anal 2006;179:389–412.
[29] Ferrario DL. Transitive decomposition of symmetry groups for the n-body problem. Adv Math 2007;213(2):763–84.
[30] Barutello V, Ferrario DL, Terracini S. Symmetry groups of the planar three-body problem and action-minimizing trajectories. Arch Ration Mech Anal 2008;190(2):189–226.
[31] Xia Z, Zhou T. Applying the symmetry groups to study the n body problem. J Differential Equations 2022;310:302–26.
[32] dlfer. symorb. 2017, https://github.com/dlfer/symorb.
[33] The GAP Group. GAP – groups, algorithms, and programming. 2024, Version 4.13.1.
[34] DipMathUniTO. Symorb.jl. 2024, https://github.com/DipMathUniTO/SymOrb.jl.
[35] Bezanson J, Edelman A, Karpinski S, Shah VB. Julia: A fresh approach to numerical computing. SIAM Rev 2017;59(1):65–98.
[36] Ciccarelli R, Introna M, Terracini S, Vasile M. New solutions for the symmetrical n-body problem through variational approach and optimisation techniques. Aerotec Missili & Spaz 2025.
[37] Kapela T, Zgliczyński P. The existence of simple choreographies for the N-body problem—a computer-assisted proof. Nonlinearity 2003;16(6):1899–918.
[38] Arioli G, Barutello V, Terracini S. A new branch of mountain pass solutions for the choreographical 3-body problem. Comm Math Phys 2006;268(2):439–63.
[39] Kapela T, Simó C. Computer assisted proofs for nonsymmetric planar choreographies and for stability of the eight. Nonlinearity 2007;20(5):1241–55, With multimedia enhancements available from the abstract page in the online journal.
[40] Calleja R, García-Azpeitia C, Lessard J-P, Mireles James JD. Torus knot choreographies in the n-body problem. Nonlinearity 2021;34(1):313–49.
[41] Capiński MJ, Kepley S, Mireles James JD. Computer assisted proofs for transverse collision and near collision orbits in the restricted three body problem. J Differential Equations 2023;366:132–91.
[42] Besard T, Churavy V, Edelman A, De Sutter B. Rapid software prototyping for heterogeneous and distributed platforms. Adv Eng Softw 2019;132:29–46.
[43] Mogensen PK, Riseth AN. Optim: A mathematical optimization package for Julia. J Open Source Softw 2018;3(24):615.
[44] Barutello V, Ferrario DL, Terracini S. On the singularities of generalized solutions to n-body-type problems. Int Math Res Not IMRN 2008;78, pages Art. ID rnn 069.
[45] Palais RS. The principle of symmetric criticality. Comm Math Phys 1979;69(1):19–30.