

Horofunctions of infinite Sierpiński polygon graphs

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Abstract

Generalizing works of D’Angeli and Donno, we describe, given an infinite sequence over the alphabet $\{0, \dots, r - 1\}$, with r not divisible by 4, a sequence of pointed finite graphs. We study the pointed Gromov–Hausdorff limit graphs giving a description of isomorphism classes in terms of dihedral groups and we provide a description of the horofunction boundary, distinguishing between Busemann and non-Busemann points.

Introduction

The horofunction boundary is a powerful compactification of a metric space based on analytic tools ([Gro81]). Its interest lies not only in its generality, but also in the fact that horofunctions admit a geometric description as *weakly-geodesic rays* ([Rie02]). Considerable effort has been devoted to understanding the subset of Busemann points ([Gro81]), that is, horofunctions which can be described via geodesic rays. Among many different aspects, there have been investigations regarding its connections to dynamical systems ([CP93; Paw15; Per23]), group theory ([BT24; TY16]) and graph theory ([WW06]).

The aim of this paper is to generalize and complete the work of the first author ([DAn17]) on Sierpiński triangle graphs. In particular, we consider Sierpiński polygon graphs where the perimeter of the polygon is not a multiple of 4. They are Gromov-Hausdorff limits of recursively generated finite graphs inspired by the Sierpiński polygon fractals. The techniques reflect elements from the first author and Donno ([DD14],[DD16]), where Sierpiński gasket graphs were taken into account. Notice that in the case of the Sierpiński carpet ($r = 4$ in the notation that we will use later), the corresponding fractal

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is not finitely ramified (see [Tep08]), and that is why these strategies cannot be directly applied: it will become evident in our study that the analysis relies on the local cut points of the fractal. The same holds for any $r = 4i$ and $i \in \mathbb{N}$.

In Section 1 we give a proper definition of the graphs $\Gamma_\xi^{(r)}$, where ξ is an infinite sequence over a finite alphabet of r letters. The sequence helps describing the recursion process which defines the graph. We then address the isomorphism problem in Section 2.

Theorem 2.4. *Let D_r be dihedral group of the r -polygon. Two graphs $\Gamma_\xi^{(r)}$ and $\Gamma_\eta^{(r)}$ are isomorphic if and only if there exists $\sigma \in D_r$ such that η and $\sigma(\xi)$ are cofinal.*

In order to prove it, we exploit the aforementioned Gromov-Hausdorff convergence, together with some considerations which highlight a connection with Schreier and tile graphs of self-similar groups ([BDN17]).

In Section 3 we analyze the structure of cut points in the graphs and give a countable collection of non-Busemann points and a finite collection of Busemann points when $\Gamma_\xi^{(r)}$ is eventually constant, resulting in the following:

Theorem 3.5. *The horofunction boundary of $\Gamma_\xi^{(r)}$ with $\xi = wj^\infty$ and w a finite prefix, contains two Busemann points and countably many non-Busemann points.*

1 Sierpiński polygon graphs

Let r be a positive integer that is not a multiple of 4. Consider the cyclic graph P_r with r vertices and denote them $V_r = \{0, 1, \dots, r-1\}$. Denote by V_r^∞ the set of infinite sequences over V_r , define $f(r) := \min\{i \mid 4i > r\}$ and $\tilde{f}(r) := 2f(r)$.

The **Sierpiński graphs** represent a sequence of graphs inspired by the Sierpiński fractals. The r -th **Sierpiński sequence** $\{\Gamma_k^{(r)} : k \geq 1\}$ is recursively constructed as follows:

1. $\Gamma_1^{(r)}$ coincides with P_r whose vertex set has labels $\{0, 1, \dots, r-1\}$;
2. $\Gamma_k^{(r)}$ is obtained by taking r copies of $\Gamma_{k-1}^{(r)}$ numbered from 0 to $r-1$. A vertex labeled by v in $\Gamma_{k-1}^{(r)}$ and belonging to the i -th copy is re-labeled vi . Identify, by gluing the copies, exactly two vertices for each $\Gamma_{k-1}^{(r)}$. One will be identified with a vertex of the $(i+1)$ -th copy and the other with a vertex in the $(i-1)$ -th copy. In particular,

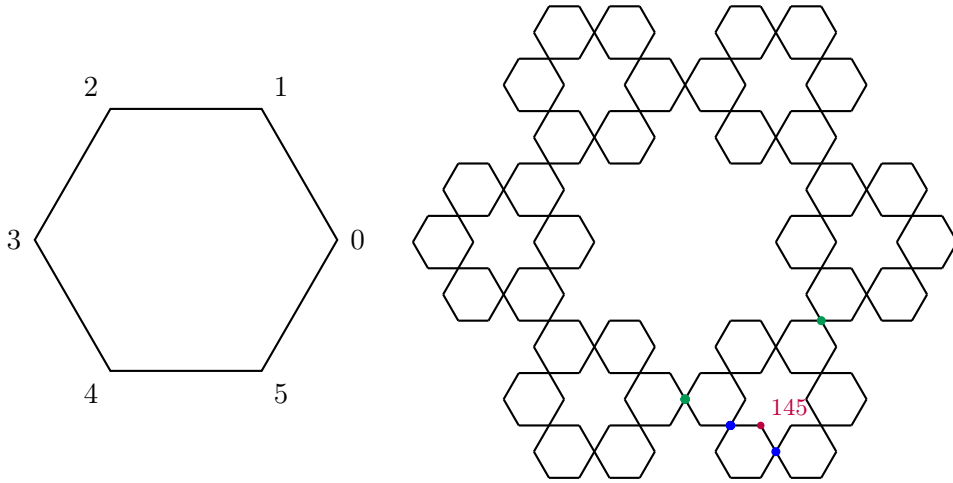


Figure 1: The cyclic graph P_6 and the corresponding pointed graph $(\Gamma_3^{(6)}, 145)$ with some gluing vertices highlighted: in blue $535 = 245$ and $045 = 355$, while in green $004 = 335$ and $440 = 115$.

$$(i+f(r))^{k-1}i \pmod r \text{ identifies with } ((i+1)+\tilde{f}(r))^{k-1}(i+1) \pmod r.$$

For example if $r = 6$, then $f(6) = 2$ and $\tilde{f}(6) = 4$ and hence 004 and 335 identify in $\Gamma_3^{(6)}$, as well as 115 and 440 (see green vertices in Figure 1).

Before going on, let us note that such a type of recursive constructions are already present in literature. For example, in theory of automata groups, in particular when describing Schreier graphs (see e.g. [BDN17]), and in rational dynamical systems where such graphs are associated to iterations of the system (see [HP24]). In the first case, these analogies persist (see further details below), whereas in the second case they cease, as the graphs are intended to approximate the standard Sierpiński fractals.

The graph $\Gamma_k^{(r)}$ can be seen as a pointed graph $(\Gamma_k^{(r)}, w)$, once we choose a sequence w of length k over V_r (see e.g. Figure 1). The choice of the notation for the sequences and the fractal construction growing instead of remaining of the same size is justified by the next definitions.

Definition 1.1. A sequence of pointed graphs $\{(X_m, p_m)\}_{m=1}^\infty$ **strongly converges in the Gromov-Hausdorff sense** to a pointed graph (X, p) if for every $r > 0$ there exists $M(r) > 0$ such that for every $m > M(r)$ there is an isometry between the two balls $B_{X_m}(p_m, r)$ and $B_X(p, r)$ sending p_m to p .

Note that this convergence implies the classic Gromov-Hausdorff convergence for pointed metric spaces (see e.g. [BBI01, Definition 8.1.1]).

In our case, we can consider an infinite sequence $\xi \in V_r^\infty$ and denote by $\xi[k]$ the prefix of length k of ξ . In this way, we define $(\Gamma_\xi^{(r)}, \xi)$ to be the limit

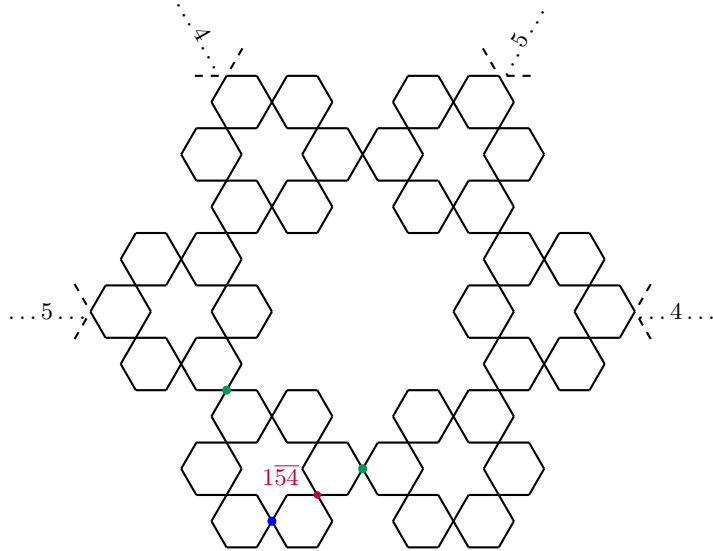


Figure 2: A portion of the graph $\Gamma_{154}^{(6)}$. Highlighted in blue the gluing vertex $044\bar{5} = 3\bar{5}4$ relative to the copy $(\Gamma_1^{(6)}, 15)$, the other one is $\bar{15}4$ itself, while in green $224\bar{5} = 553\bar{5}4$ and $004\bar{5} = 335\bar{5}4$ relative to the copy $(\Gamma_2^{(6)}, 154)$. The dotted arrows represent how the next copies are attached after reading the corresponding digit.

of the strong Gromov-Hausdorff convergence

$$(\Gamma_k^{(r)}, \xi[k]) \rightarrow (\Gamma_\xi^{(r)}, \xi).$$

See Figure 2 for an example with $r = 6$ and $\xi = \bar{15}4$. The intuition of the Gromov-Hausdorff convergence is that, once we have fixed a sequence ξ , our infinite graph is growing around it and the k -digit of the sequence is telling us the position of the k -step finite graph inside Γ_{k+1} . An analog can be witnessed in the theory of infinite Schreier graphs of bounded automata groups, where the vertices are infinite sequences and the neighborhood of a vertex is determined by a finite Schreier graph of a prefix (see e.g. [BDN17]). Following this point of view, another resemblance between the two cases can be spotted in Lemma 2.2. Note that, depending on the infinite sequence we choose, we might end up with different limit graphs $(\Gamma_\xi^{(r)}, \xi)$. To avoid cumbersome notation, we will simply refer to Γ_k and Γ_ξ , dropping the r . Also, we will write $B_k(\xi[k], \ell) = B_{\Gamma_k}(\xi[k], \ell)$ and $B(\xi, \ell) = B_{\Gamma_\xi}(\xi, \ell)$

We will refer to vertices of Γ_ξ belonging to two different copies of Γ_k as **gluing vertices**.

2 Isomorphism classification

In this section, we aim to determine when two infinite graphs are isomorphic. We begin by considering the pointed cases (Γ_ξ, ξ) and (Γ_η, η) , and then proceed to the unpointed setting.

Consider the r -cycle graph P_r and its isometry group given by the dihedral group D_r seen as a subgroup of $\text{Sym}(V_r)$. Given an infinite sequence $\xi = x_1x_2 \cdots$ in V_r^∞ , we define the action of $\sigma \in D_r$ on V_r^∞ by

$$\sigma(\xi) = \sigma(x_1)\sigma(x_2) \cdots .$$

Remark 2.1. It is easy to see that the map σ induces an isomorphism $(\Gamma_k^{(r)}, \xi[k]) \rightarrow (\Gamma_k^{(r)}, \sigma(\xi[k]))$ of pointed graphs and, in fact, the group of all isomorphisms of $(\Gamma_k^{(r)}, \xi[k])$ is isomorphic to D_r . To better visualize this, one can think of the permutations of the r copies of Γ_{k-1} , which would provide exactly D_r and then easily verify that, for compatibility, copies of $\Gamma_{k'}$ with $k' < k - 1$ do not play a role in the definition of further isomorphisms.

Two infinite sequences $\xi = x_1x_2 \cdots$ and $\eta = y_1y_2 \cdots$ are **cofinal** if there exists an $N \in \mathbb{N}$ such that $x_i = y_i$ for any $i \geq N$. Being cofinal defines an equivalence relation and we write $\xi \sim \eta$. With a slight abuse of notation let us regard an infinite sequence as a vertex of a limit graph Γ_ξ .

As already mentioned above, the following result is shared with infinite Schreier graphs of bounded automata groups.

Lemma 2.2. *The sequence η represents a vertex in Γ_ξ if and only if $\eta \sim \xi$. Therefore, in this case, we have $\Gamma_\xi = \Gamma_\eta$.*

Proof. Suppose that $\eta \sim \xi$ and $\xi = x_1x_2 \dots$, that is for some \tilde{k} we have $\eta = y_1 \dots y_{\tilde{k}} x_{\tilde{k}+1} \dots$. By definition, this means that, after a finite amount of steps, we are always picking the same copy where to be. Hence, by construction of the graphs, $\eta[k]$ is contained in a ball of $\Gamma_{\xi[k]}$ centered in $\xi[k]$ with radius $\text{diam}(\Gamma_{\tilde{k}})$ for all $k \geq \tilde{k}$. Hence η belongs to Γ_ξ by Gromov-Hausdorff convergence. \square

We have remarked that to define the limit graph we must consider it in the space of pointed graphs. Once we have the infinite (pointed) graph we might forget the distinguished vertex and look at the structure of the graph.

We are interested in the isomorphism problem of such infinite graphs.

Lemma 2.3. *The map $\sigma : (\Gamma_\xi, \xi) \rightarrow (\Gamma_{\sigma(\xi)}, \sigma(\xi))$ is an isomorphism of pointed graphs. Moreover, if ϕ is an isomorphism between two pointed graphs, then $\phi = \sigma$ for some $\sigma \in D_r$.*

Proof. The first claim follows directly by definition, $\phi = \sigma$ is the limit of $\{\phi_k\}_{k=1}^\infty$ where $\phi_k : (\Gamma_k, \xi[k]) \rightarrow (\Gamma_k, \sigma(\xi)[k])$ is the pointed isometry defined by σ . By definition of Gromov-Hausdorff convergence, we conclude that ϕ is an isometry of the limits.

For the second claim, we want to prove that each isomorphism of $(\Gamma_k, \xi[k])$ is induced by a $\sigma \in D_r$. Let $\phi : (\Gamma_\xi, \xi) \rightarrow (\Gamma_\eta, \eta)$ be a pointed graph isomorphism. This means that $B(\xi, \ell)$ is isomorphic to $B(\eta, \ell)$ for all $\ell > 0$. Fix an $\ell > 0$ then, by the Gromov-Hausdorff convergence there exists L such that for all $m \geq L$ we have $B_m(\xi[m], \ell) \simeq B(\xi, \ell)$ and $B_m(\eta[m], \ell) \simeq B(\eta, \ell)$. Composing all the pointed isomorphisms, we have an isomorphism $\phi_{\ell, m} : B_m(\xi[m], \ell) \rightarrow B_m(\eta[m], \ell)$. Note that, by construction, the restriction of $\phi_{\ell, m}$ to the ball $B_m(\xi[m], \ell - 1)$ is equivalent to $\phi_{\ell-1, m}$ and the restriction of $\phi_{\ell, m}$ to the isomorphic copy of $B_{m-1}(\xi[m-1], \ell)$ embedded in $B_m(\xi[m], \ell)$ is equivalent to $\phi_{\ell, m-1}$. This implies that these pointed isomorphisms are each compatible to one another. Now an isomorphic copy of $(\Gamma_k, \xi[k])$ is contained in $B_m(\xi[m], \ell)$ for some suitable k and the family of pointed isomorphisms $\widehat{\phi}_{\ell, m}$ restricted to each $(\Gamma_k, \xi[k])$ provides a family of pointed isomorphisms $\widehat{\phi}_k : (\Gamma_k, \xi[k]) \rightarrow (\Gamma_k, \eta[k])$, so by Remark 2.1 we have $\widehat{\phi}_k = \sigma$ for some $\sigma \in D_r$. The compatibility stated above ensures that the isomorphism $\widehat{\phi}_k$ is induced by the same σ for all k . Arguing as in the first part, we get that $\phi = \sigma$ as requested. \square

Theorem 2.4. *Two graphs Γ_ξ and Γ_η are isomorphic if and only if there exists $\sigma \in D_r$ such that $\eta \sim \sigma(\xi)$.*

Proof. By Lemma 2.3, the isomorphism between Γ_ξ and Γ_η is described by some $\sigma \in D_r$. Hence $\sigma(\xi) \in \Gamma_\eta$ and by Lemma 2.2 we conclude that $\sigma(\xi) \sim \eta$. On the other hand, suppose that σ is such that $\sigma(\xi) \sim \eta$. Again by Lemma 2.3, the element σ induces an isomorphism between Γ_ξ and $\Gamma_{\sigma(\xi)}$. Finally, we have $\Gamma_{\sigma(\xi)} = \Gamma_\eta$ by Lemma 2.2. \square

3 Horofunction boundary

We need to recall some definitions and basic results. We fix a **locally finite graph** X , that is, each vertex is incident to only finitely many edges. A **path** in X is a sequence of vertices such that each consecutive pair of vertices in the sequence is connected by an edge. Given two paths where the last vertex of the first is the first vertex of the second, their **concatenation** is the sequence of vertices of the first followed by the sequence of vertices of the second (the common vertex is counted once). We endow X with the standard path metric d , where the distance between two vertices is given by the length (i.e., the number of edges) of a shortest path (or a **geodesic**) connecting them.

Let $C(X, \mathbb{Z}) = \mathbb{Z}^X$ be the abelian group of all integer-valued functions on the vertex set of X , which we endow with the product topology and where \mathbb{Z} has the discrete topology. Let us fix a base vertex $x_0 \in X$. For each vertex $v \in X$, we define

$$\bar{d}_v(x) := d(x, v) - d(x_0, v), \quad \forall x \in X.$$

There is a well-defined embedding $\iota : X \hookrightarrow C(X, \mathbb{Z})$, defined as $\iota(v) := \bar{d}_v$. We denote by $\overline{\iota(X)}$ the closure of $\iota(X)$ in $C(X, \mathbb{Z})$.

Definition 3.1. The space $\overline{\iota(X)}$ is called the **horofunction compactification** of X . The **horofunction boundary** of X is the set $\overline{\iota(X)} \setminus \iota(X)$ and it is denoted by $\partial_h X$.

We will refer to elements of $\partial_h X$ simply as horofunctions.

We want a geometric description of the horofunctions of the graph. We need the following definition (see [Rie02] for the general definition in metric spaces).

Definition 3.2. Let X be a locally finite graph, and let T be an unbounded subset of \mathbb{N} containing 0. Let $\gamma : T \rightarrow X$ such that $\gamma(t)$ is a vertex for each $t \in T$ and such that $d(\gamma(t), \gamma(0)) = t$. Then

1. γ is a **geodesic ray** if $d(\gamma(t), \gamma(s)) = |t - s|$, for all $t, s \in T$;
2. γ is an **almost-geodesic ray** if there exists an integer N such that, for every $t, s \in T$ with $t \geq s \geq N$, one has:

$$|d(\gamma(t), \gamma(s)) + d(\gamma(s), \gamma(0)) - t| = 0;$$

3. γ is a **weakly-geodesic ray** if, for every $y \in X$, there exists an integer N such that, for every $s, t \geq N$, one has:

$$|d(\gamma(t), y) - d(\gamma(s), y) - (t - s)| = 0.$$

It is easy to see that the following implications hold

$$\text{geodesic ray} \implies \text{almost-geodesic ray} \implies \text{weakly-geodesic ray}.$$

The following result is a reformulation of [Rie02, Theorem 4.7] in the context of locally finite graphs.

Theorem 3.3. *Let X be a locally finite graph and let γ be a weakly-geodesic ray in X . Then $\lim_{t \rightarrow +\infty} \bar{d}_{\gamma(t)}(y)$ pointwise converges to a horofunction. Conversely, every horofunction is a pointwise limit of a weakly-geodesic ray.*

The previous theorem helps us describing horofunctions from a more geometric point of view. We are now ready to introduce a specific type of horofunctions.

Definition 3.4. A horofunction defined by an almost-geodesic ray is called a **Busemann point**.

The aim of this section is to prove the following result.

Theorem 3.5. *The horofunction boundary of Γ_ξ with $\xi = wj^\infty$ has exactly two Busemann points and contains countably many non-Busemann points.*

In order to study horofunctions in our recursively defined graphs, we will need to consider *holes* and *cut points*. We start with the following.

Definition 3.6. Let $\xi \in V_r^\infty$. We say that Γ_ξ **grows away from** $j \in \{1, \dots, r-1\}$ $j \in \{0, \dots, r-1\}$ if j appears infinitely many times in the infinite sequence ξ .

Note that this notion is well-defined for a graph. In fact, due to Lemma 2.2, it is independent of the choice of ξ .

Definition 3.7. A **k -hole** is the subgraph of Γ_k isomorphic to the k -iteration of a generalized Koch snowflake defining the central hole of the graph. A bit more formally, such a subgraph is obtained by joining the gluing vertices with suitable curves made by replacing the edges of the $(k-1)$ -hole with geodesics between their endpoints in the graph Γ_{k-1} .

A formal and explicit definition of such a curve exists, but it is rather technical and not relevant for our purposes (see Figure 3 for graphical aid). A copy of Γ_{k-1} is said to **facing** a k -hole if its intersection with the hole is non trivial. A **hole** H in Γ_ξ is the k -hole of $\Gamma_{\xi[k]}$ for some $k \in \mathbb{N}$.

Recall that Γ_k contains r copies of Γ_{k-1} numbered from 0 to $r-1$. If n is even, we say that **antipodal copy** of the i -th copy is the $(i+r/2)$ -th copy where $i+r/2$ is taken modulo r . If r is odd, we say that there are two antipodal copies, the $(i+\lfloor r/2 \rfloor)$ -th and the $(i+\lceil r/2 \rceil)$ -th.

Definition 3.8. If r is even, an **antipodal point** p of a hole H in Γ_ξ with respect to ξ is a vertex of H such that ξ and p belong to the antipodal copies of Γ_m facing the hole and such that $d(\bar{b}_m, p) = d(\underline{b}_m, p)$ where \bar{b}_m and \underline{b}_m are the two gluing vertices for the copy of Γ_m containing ξ . More precisely if Γ_m is in position i , then \bar{b}_m glues Γ_m to the $(i+1)$ -th copy and \underline{b}_m to the $(i-1)$ -th copy (see again Figure 3 for graphical aid).

If r is odd, an antipodal point p of a hole H in Γ_ξ with respect to ξ is the unique vertex gluing the two antipodal copies of the copy of Γ_m containing ξ .

Note that, in the latter case, the condition $d(\bar{b}_m, p) = d(\underline{b}_m, p)$ still holds since they are the two gluing vertices (not equal to the antipodal point) of the two copies of Γ_m .

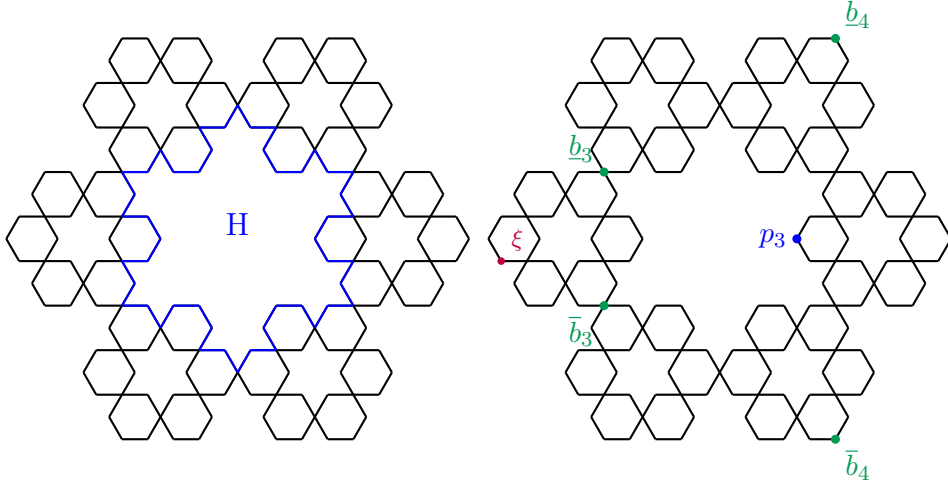


Figure 3: On the left side, a 3-hole with six facing copies of $\Gamma_3^{(6)}$. On the right side, the points \bar{b}_m and \underline{b}_m for $\xi = 433$ and $i = 3$, together with the antipodal point of the hole H in the left figure with respect to ξ .

Remark 3.9. If Γ_ξ grows away from j , then there exists an unbounded subset M of \mathbb{N} such that ξ belongs to the j -th copy of $\Gamma_{\xi[m]}$ for all $m \in M$. Call H_m the copy of the m -hole in $\Gamma_{\xi[m]}$, so that $\text{diam}(H_{m_1}) < \text{diam}(H_{m_2})$ with $m_1 < m_2$ in M . Now, set p_m to be the antipodal point of H_m with respect to ξ , in particular ξ and p_m belong to antipodal copies of $\Gamma_{\xi[m]}$ and $d(\xi, p_m) \rightarrow \infty$ as $m \rightarrow \infty$.

Exploiting the remark, we provide the following.

Definition 3.10. We call the sequence $\{p_m\}_{m \in M}$ obtained from Remark 3.9, an **antipodal sequence**.

Before working with antipodal sequences, we first examine when certain sequences cannot be horofunctions, making use of holes. Such sequences are those for which there exists a copy Γ_m containing the base vertex ξ , for which reaching the base vertex requires passing infinitely many times through each of the two gluing vertices of Γ_m . An example is shown in Figure 4: in this case it is easy to see that

$$\begin{aligned} |d(q_{m+1}, \xi) - d(q_{m+1}, \bar{b}_1) - (d(q_m, \xi) - d(q_m, \bar{b}_1))| = \\ |4 \cdot 3^m - (4 \cdot 3^m - 2) - (4 \cdot 3^{m-1} - 4 \cdot 3^{m-1})| = 2. \end{aligned}$$

Therefore, it follows that the sequence fails to be weakly geodesic.

Lemma 3.11. *Let ξ be such that Γ_ξ grows away from j . Suppose that $\{q_m\}_{m \in M}$ is a sequence of vertices such that $d(\xi, q_m) \rightarrow \infty$ as $m \rightarrow \infty$ and*

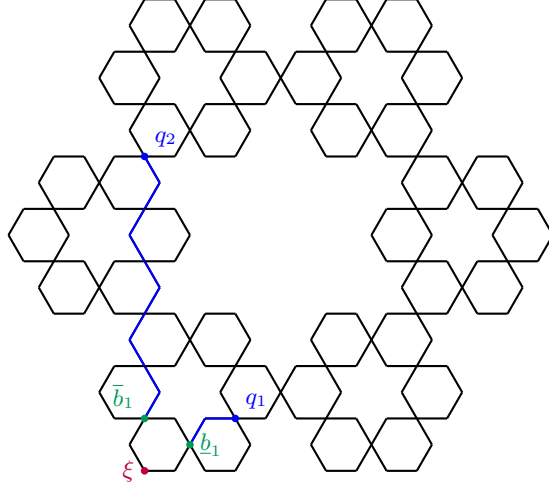


Figure 4: A portion of the graph Γ_ξ with $\xi = \bar{4}$ and with the first two vertices $q_1 = 40\bar{4}$ and $q_2 = 442\bar{4}$ of a sequence $\{q_m\}_{m \in M}$ as in Lemma 3.11 with $\{q_m\} = \{4^m 0\bar{4} \mid m \text{ odd}\}$ and $\{q_{\bar{m}}\} = \{4^m 2\bar{4} \mid m \text{ even}\}$. Here $N = 1$, with geodesics highlighted in blue.

suppose there exist two \bar{b}_N and \underline{b}_N related to a j copy $\Gamma_{\xi[N]}$ such that there are two infinite subsequences $\{q_{\bar{m}}\}$ and $\{q_{\underline{m}}\}$ with $q_{\bar{m}}$ (resp. $q_{\underline{m}}$) such that a geodesic connecting $q_{\bar{m}}$ (resp. $q_{\underline{m}}$) to any vertex in $\Gamma_{\xi[N]}$ passes through \bar{b}_N (resp. \underline{b}_N). Then $\{q_m\}_{m \in M}$ does not represent a horofunction.

Proof. Let us prove that $\{q_m\}_{m \in M}$ is not a weakly-geodesic ray. We fix $y = \bar{b}_N$ and compute

$$|d(q_{\bar{m}}, \xi) - d(q_{\bar{m}}, y) - (d(q_{\underline{m}}, \xi) - d(q_{\underline{m}}, y))|.$$

By hypothesis,

$$d(q_{\bar{m}}, \xi) = d(q_{\bar{m}}, \bar{b}_N) + d(\bar{b}_N, \xi) \text{ and } d(q_{\underline{m}}, \bar{b}_N) = d(q_{\underline{m}}, \underline{b}_N) + d(\underline{b}_N, \bar{b}_N).$$

Hence,

$$|d(q_{\bar{m}}, \xi) - d(q_{\bar{m}}, \bar{b}_N) - (d(q_{\underline{m}}, \xi) - d(q_{\underline{m}}, \bar{b}_N))| = |d(\bar{b}_N, \xi) - d(\underline{b}_N, \xi) + d(\underline{b}_N, \bar{b}_N)|$$

and possibly choosing another vertex inside the smallest subgraph containing the three vertices $\xi, \underline{b}_N, \bar{b}_N$ (see Lemma 2.2), one can get $d(\underline{b}_N, \xi) < d(\underline{b}_N, \bar{b}_N) + d(\bar{b}_N, \xi)$. This means that

$$|d(q_{\bar{m}}, \xi) - d(q_{\bar{m}}, \bar{b}_N) - (d(q_{\underline{m}}, \xi) - d(q_{\underline{m}}, \bar{b}_N))| > 0$$

and $\{q_m\}_{m \in M}$ is not a weakly-geodesic ray. By virtue of Theorem 3.3 the proof is complete. \square

This means that we can distinguish three possible main types of weakly-geodesic rays: the ones that almost always pass through \bar{b}_m , the ones through \underline{b}_m and the ones that almost always are at the same distance from \bar{b}_m and \underline{b}_m . In particular, for $\xi = wj^\infty$, we want to show that in each of the first two types, which are symmetric due to the geometry of the limit graphs, there exists a unique Busemann point (see Proposition 3.12), while the third type provides a non-Busemann point (see Lemma 3.13) together with countably many other non-Busemann points (see Proposition 3.14).

Proposition 3.12. *If Γ_ξ is such that $\xi = wj^\infty$, then the boundary has exactly two Busemann points.*

Proof. By Lemma 2.2, we can assume $\xi = j^\infty$. Consider the antipodal sequence $\{p_m\}_{m \in M}$ and its associated gluing vertices sequences $\{\bar{b}_m\}_{m \in M}$ and $\{\underline{b}_m\}_{m \in M}$. We claim that these two sequences lie on two different geodesic rays. With a slight abuse of notation, we set $d(\bar{b}_m, \xi) = d(\underline{b}_m, \xi) = m$. We consider the path $\bar{\gamma}$ such that $\bar{\gamma}(m) = \bar{b}_m$ and $\bar{\gamma}([m_1, m_2])$ is a geodesic going from \bar{b}_{m_1} to \bar{b}_{m_2} where m_1 and m_2 are two consecutive indices in M . In a similar way, we construct the path $\underline{\gamma}$ passing through the points of the sequence $\{\underline{b}_m\}_{m \in M}$. We have to show that the concatenation of $\bar{\gamma}|_{[m_1, m_2]}$ followed by $\bar{\gamma}|_{[m_2, m_3]}$ is a geodesic. By construction, $\bar{\gamma}([m_2, m_3])$ lies entirely in $\Gamma_{\xi[m_3]}$ and $\bar{\gamma}([m_2, m_3]) \cap \Gamma_{\xi[m_2]} = \{\bar{\gamma}(m_2)\}$. On the other hand, removing $\bar{\gamma}(m_2)$ and $\underline{\gamma}(m_2)$ from $\Gamma_{\xi[m_3]}$ disconnects $\Gamma_{\xi[m_2]}$ and this implies that a geodesic from $\bar{\gamma}(m_1)$ and $\bar{\gamma}(m_3)$ must pass through one of these points and its length must be the sum of the length of two geodesics, one from $\bar{\gamma}(m_1)$ to p and the other from p to $\bar{\gamma}(m_3)$, where $p \in \{\bar{b}_{m_2}, \underline{b}_{m_2}\}$. We can exclude $p = \underline{b}_{m_2}$ because, by construction,

$$d(\bar{b}_{m_t}, \bar{b}_{m_2}) < d(\bar{b}_{m_t}, \underline{b}_{m_2}) \quad \text{for } t = 1, 3,$$

and we now explain why. Suppose that \bar{b}_{m_1} is gluing two copies of Γ_k for some k . Then:

1. To reach \bar{b}_{m_2} , one must pass through $f(r)$ copies of Γ_k ;
2. To reach \underline{b}_{m_2} , one must either:
 - pass through \underline{b}_{m_1} and then through $f(r)$ copies of Γ_k ; or
 - pass through more than $f(r) + 1$ copies of Γ_k .

In both cases under point (2), the path requires at least $f(r) + 1$ copies of Γ_k , hence

$$d(\bar{b}_{m_t}, \bar{b}_{m_2}) < d(\bar{b}_{m_t}, \underline{b}_{m_2}),$$

and the required geodesic does not pass through \underline{b}_{m_2} . Note that all the previous considerations on coarse distances are true because we are implicitly exploiting the symmetries of the case $\xi = j^\infty$. To finish the proof, let us note that, since $\{\bar{b}_m\}_{m \in M}$ and $\{\underline{b}_m\}_{m \in M}$ are sequences of cut points, another infinite geodesic starting from ξ should pass through infinitely many $\{\bar{b}_m\}_{m \in M}$ or infinitely many $\{\underline{b}_m\}_{m \in M}$. Hence, it must define a horofunction that corresponds to one of the two previously constructed Busemann points. \square

Lemma 3.13. *Let Γ_ξ be such that $\xi = wj^\infty$. Then the antipodal sequence describes a horofunction, which is not a Busemann point.*

Proof. First, we need to prove that $P : T \rightarrow \Gamma_\xi$ where $P(d(\xi, p_m)) := p_m$ is a weakly-geodesic ray, that is, if we fix a vertex $y \in \Gamma_\xi$, eventually

$$|d(p_m, y) - d(p_n, y) - (d(\xi, p_m) - d(\xi, p_n))| = 0.$$

Let N be the smallest n such that ξ and y belong to $\Gamma_{\xi[n]}$, and $\Gamma_{\xi[n]}$ is the j copy in $\Gamma_{\xi[n+1]}$. From now on, we consider $n, m \in T$ such that $n, m \geq N$. Let b_m (resp. b_n) be (possibly the same) gluing vertex of $\Gamma_{\xi[N]}$ to the rest of $\Gamma_{\xi[N+1]}$ such that a geodesic connecting ξ and p_m (resp. p_n) passes through b_m (resp. b_n), so that

$$d(\xi, p_m) = d(\xi, b_m) + d(b_m, p_m) \text{ and } d(\xi, p_n) = d(\xi, b_n) + d(b_n, p_n). \quad (3.13a)$$

Analogously, let \tilde{b}_m and \tilde{b}_n be the gluing vertices such that a geodesic connecting y and p_m (resp. p_n) passes through \tilde{b}_m (resp. \tilde{b}_n), again we have

$$d(y, p_m) = d(y, \tilde{b}_m) + d(\tilde{b}_m, p_m) \text{ and } d(y, p_n) = d(y, \tilde{b}_n) + d(\tilde{b}_n, p_n). \quad (3.13b)$$

The definition of antipodal point implies that $d(p_N, b_N) = d(p_N, \tilde{b}_N)$ with \tilde{b}_N as in the previous sentence lying on a geodesic between y and p_N . We claim that $d(p_m, b_m) = d(p_m, \tilde{b}_m)$ for all $m > N$. If $b_m = \tilde{b}_m$ we are done, otherwise without loss of generality suppose that b_m is closer to \tilde{b}_m and \tilde{b}_m is closer to \underline{b}_m . This means that

$$d(p_m, b_m) = d(p_m, \tilde{b}_m) + d(\tilde{b}_m, b_m) = d(p_m, \underline{b}_m) + d(\underline{b}_m, \tilde{b}_m) = d(p_m, \tilde{b}_m),$$

where $d(p_m, \tilde{b}_m) = d(p_m, \underline{b}_m)$ follows again by definition of antipodal point, while $d(\tilde{b}_m, b_m) = d(\underline{b}_m, \tilde{b}_m)$ follows from the assumption and the geometry of Γ_k (see Figure 5, these are the finite geodesics described in Proposition 3.12). Again, here we are exploiting the symmetries of the case $\xi = wj^\infty$.

Combining the fact that $d(p_m, b_m) = d(p_m, \tilde{b}_m)$ with the equations 3.13a and 3.13b, we have

$$\begin{aligned} |d(p_m, y) - d(\xi, p_m) - (d(p_n, y) - d(\xi, p_n))| = \\ |d(\tilde{b}_m, y) - d(\xi, b_m) - (d(\tilde{b}_n, y) - d(\xi, b_n))|. \end{aligned}$$

By reasoning as before and recalling the symmetries of the case $\xi = wj^\infty$, we have $d(p_m, \underline{b}_N) = d(p_m, \tilde{b}_N)$. This means that b_m is the closest vertex to ξ among \underline{b}_N and \tilde{b}_N , since this holds for any m , we have $b_m = b_n$. In the same way, $\tilde{b}_m = \tilde{b}_n$. Hence

$$\begin{aligned} |d(\tilde{b}_m, y) - d(\xi, b_m) - (d(\tilde{b}_n, y) - d(\xi, b_n))| = \\ |d(\tilde{b}_m, y) - d(\xi, b_m) - (d(\tilde{b}_m, y) - d(\xi, b_m))| = 0 \end{aligned}$$

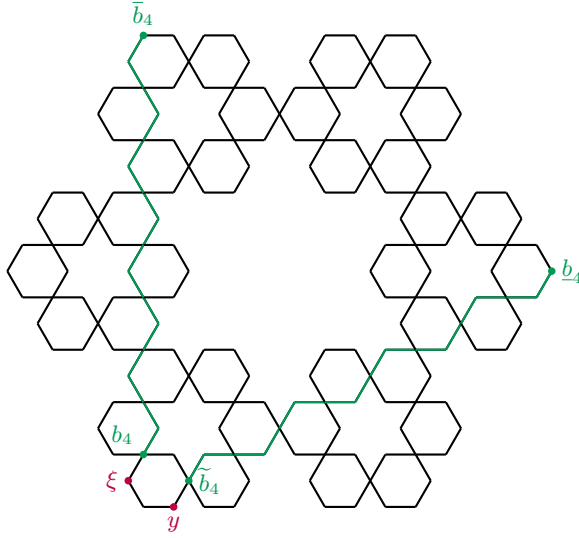


Figure 5: Two geodesics connecting the gluing vertices as in the proof of Lemma 3.13: they have the same length.

To conclude, we should prove that $P : T \rightarrow \Gamma_\xi$ is not a Busemann point. But this follows directly, almost verbatim, from the proof of the [DD16, Proposition 4.2], taking v_1 and v_2 as \underline{b}_N and \bar{b}_N and noticing that $\varphi_y(x) = \bar{d}_x(y)$. \square

Before continuing, it is worth mentioning that the horofunction described in the previous lemma can be seen as sequence of atoms (see [BBM21; Per23] for an introduction to the formalism) which are totally disconnected. It is straightforward to verify this, for example, when $\xi = \bar{4}$, $r = 6$ and one computes the first level atoms (i.e. the atoms related to $B(\xi, 1)$). The existence of disconnected atoms is yet to be understood in the hyperbolic and CAT(0) cases.

Proposition 3.14. *If Γ_ξ is such that $\xi = wj^\infty$, then $\partial_h \Gamma_\xi$ contains countably many non-Busemann points.*

Proof. The strategy of the proof is to consider the antipodal sequence constructed in [Proposition 3.14](#) [Lemma 3.13](#) and shift it along a suitable geodesic, in such a way that the resulting sequences yield distinct horofunctions. Finally, the previous argument shows that each of these horofunctions is non-Busemann.

Let y be any vertex in the graph Γ_ξ and let N be as in Lemma 3.13. Consider the antipodal sequence $\{p_m\}_{m \in M}$. Let A_m and B_m be the vertices gluing the antipodal copy (or copies) of p_m to the other copies in Γ_m . For clarity, note that A_m and B_m are gluing vertices associated with the antipodal

copy, and should not be confused with the gluing vertices $b_m, \tilde{b}_m, \underline{b}_m$ and \bar{b}_m used elsewhere in the text. Then by the symmetry of the space we have that $d(A_m, B_m)$ is even. Hence, one can consider \hat{p}_m the midpoint of a geodesic γ_m realizing such distance and applying the same technique as in Lemma 3.13, one can show that $\{\hat{p}_m\}_{m \in M}$ describes the same horofunction of the antipodal sequence.

To retrieve a countable number of horofunctions, let us “shift” the sequence we have just constructed. We simply mean that if $\gamma_m(\hat{t}) = \hat{p}_m$, then we define $p_{m,t} := \gamma_m(\hat{t} + t)$ with $t \in \mathbb{Z}$ such that $0 < \hat{t} + t < d(A_m, B_m)$. Moreover, we denote by $b_{m,t}$ (resp. $\tilde{b}_{m,t}$) the vertex in $\{\bar{b}_N, \underline{b}_N\}$ lying on a geodesic connecting $p_{m,t}$ to ξ (resp. y). Note that $b_{m,t} = b_{n,t}$ and similarly one sees that $\tilde{b}_{m,t} = \tilde{b}_{n,t}$, as we are about to show. Indeed, suppose that $d(A_m, p_{m,t}) < d(B_m, p_{m,t})$, the other case being analogous, and note that then the same holds for n , that is, we must also have $d(A_n, p_{n,t}) < d(B_n, p_{n,t})$. If $d(\xi, \bar{b}_N) \leq d(\xi, \underline{b}_N)$, then there is nothing to show. Conversely, the choice of one of the two vertices depend only on t . In fact, it is easy to see that $d(\underline{b}_N, p_{m,t}) = d(\bar{b}_N, p_{m,t}) + 2|t|$ for all $m > N$.

In the same fashion as in Lemma 3.13, we get the following simplification

$$\begin{aligned} & |d(p_{m,t}, y) - d(p_{n,t}, y) - (d(\xi, p_{m,t}) - d(\xi, p_{n,t}))| = \\ & \quad |d(b_{m,t}, p_{m,t}) - d(\tilde{b}_{m,t}, p_{m,t}) - (d(b_{n,t}, p_{n,t}) - d(\tilde{b}_{n,t}, p_{n,t}))|. \end{aligned}$$

If $b_{m,t} = \tilde{b}_{m,t}$ there is nothing to show. Otherwise, suppose without loss of generality that $d(b_{m,t}, A_m) < d(b_{m,t}, B_m)$, so that we must also have $d(\tilde{b}_{m,t}, A_m) > d(\tilde{b}_{m,t}, B_m)$. Now, we can simplify once more by noticing that $d(b_{m,t}, A_m) = d(\tilde{b}_{m,t}, B_m)$. Namely, we have

$$\begin{aligned} & |d(b_{m,t}, p_{m,t}) - d(\tilde{b}_{m,t}, p_{m,t}) - (d(b_{n,t}, p_{n,t}) - d(\tilde{b}_{n,t}, p_{n,t}))| \\ & \quad |d(A_m, p_{m,t}) - d(B_m, p_{m,t}) - (d(A_n, p_{n,t}) - d(B_n, p_{n,t}))| \end{aligned}$$

Again, suppose without loss of generality that $\gamma_m(0) = A_m$ and $t > 0$, then $d(A_m, p_{m,t}) = d(B_m, p_{m,t}) + 2t$ and the same holds for n . In the end the previous equality becomes

$$|d(A_m, p_{m,t}) - d(B_m, p_{m,t}) - (d(A_n, p_{n,t}) - d(B_n, p_{n,t}))| = |2t - 2t| = 0.$$

As a second step, we need to show that the sequences describe different horofunctions.

If $t_1 t_2 < 0$, then we can assume that p_{m,t_1} (resp. p_{m,t_2}) is nearer to A_m (resp. B_m). Fix $y = A_m$ and determine the possible values of t_1 and t_2 such that

$$d(A_m, p_{m,t_1}) - d(\xi, p_{m,t_1}) = d(A_m, p_{m,t_2}) - d(\xi, p_{m,t_2}).$$

We thus suppose that t_1, t_2 satisfy the equation above. We observe that $d(A_m, p_{m,t_1}) - d(\xi, p_{m,t_1}) = -d(\xi, A_m)$ and $d(\xi, p_{m,t_2}) = d(\xi, A_m) + d(B_m, p_{m,t_2})$, which we substitute in the equation above to get that

$$d(A_m, p_{m,t_2}) = d(B_m, p_{m,t_2}),$$

and so, by definition, this means that $p_{m,t_2} = \widehat{p}_m$. Fixing $y = B_m$ gives $p_{m,t_1} = \widehat{p}_m$.

Now suppose again without loss of generality that $\gamma_m(0) = b_{m,t}$ and that $t_1 \leq t_2 < 0$. Fix y to be the other gluing vertex different from $b_{m,t}$. We want to determine when

$$d(b_{m,t}, p_{m,t_1}) - d(\xi, p_{m,t_1}) = d(b_{m,t}, p_{m,t_2}) - d(\xi, p_{m,t_2}).$$

By construction $d(\xi, p_{m,t_2}) = d(\xi, p_{m,t_1}) + |t_1| - |t_2|$, while

$$d(y, p_{m,t_1}) = d(y, \widehat{p}_m) + |t_1| \text{ and } d(y, p_{m,t_2}) = d(y, \widehat{p}_m) + |t_2|.$$

Combining all together, we end up with

$$|t_1| = |t_2| - (|t_1| - |t_2|)$$

which is true if and only if $t_1 = t_2$.

All is left to do is to prove that they are not Busemann points, but, as in Lemma 3.13, this follows from [DD16, Proposition 4.2]. \square

Remark 3.15. More precisely, the antipodal point p_m and the point \widehat{p}_m considered in the proof are in the same atom (as in the notation mentioned immediately before Proposition 3.14).

Combining together Proposition 3.14 and Proposition 3.12, we get the proof of Theorem 3.5.

4 Final remarks

In general, for sequences that are not eventually constant, the situation is more involved. Indeed, it may happen that ξ must pass through distinct cut points \underline{b}_m and \bar{b}_m in order to reach the points of the weakly geodesic sequence $\{p_n\}$. This would imply that we cannot apply the cancellation argument used in Proposition 3.14. The reason is that subsequences of the form sj in ξ , with $s \neq j$, may introduce distortions in the geometry of the graph Γ_ξ .

By examining this distortion and the symmetries of the graph in the different cases, we strongly believe that our result, with the addition of further technical details, could be generalized to a generic Γ_ξ , henceforth we make the following.

Conjecture 4.1. *Let Γ_ξ be such that it grows away from j and define $S := \{s \in V_r \mid s \neq j \text{ and } sj \text{ occurs infinitely many times in } \xi\}$.*

1. *If S is empty, we are in the conditions of Theorem 3.5.*
2. *For every $s \in S$ there are two Busemann points $\lambda_\alpha(s) \neq \lambda_\beta(s)$ and countably many non-Busemann points $\Lambda(s) = \{\lambda_n(s)\}_{n \in \mathbb{N}}$. All such points are pairwise distinct depending on whether:*

$$(a) \ s \in S_I := \left[j + \frac{r - f(r)}{2}, j + \frac{r + f(r)}{2} \right],$$

$$(b) \ s \in S_L := \left(0, j + \frac{r - f(r)}{2} \right),$$

$$(c) \ s \in S_R := \left(j + \frac{r + f(r)}{2}, r - 1 \right).$$

More precisely, $\lambda_\alpha(s_1) = \lambda_\alpha(s_2)$, $\lambda_\beta(s_1) = \lambda_\beta(s_2)$, $\Lambda(s_1) = \Lambda(s_2)$ if and only if both s_1, s_2 belong to the same of the three subsets S_I, S_L, S_R . Moreover, if s_1, s_2 belong to distinct subsets, then

$$[\{\lambda_\alpha(s_1), \lambda_\beta(s_1)\} \cup \Lambda(s_1)] \cap [\{\lambda_\alpha(s_2), \lambda_\beta(s_2)\} \cup \Lambda(s_2)] = \emptyset.$$

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