Journal of Functional Analysis 290 (2026) 111225

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Regular Article

Quantization and reduction for torsion free CR manifolds

Andrea Galasso[a,*,1], Chin-Yu Hsiao[b]

[a] *Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi 55, 20125 Milano, Italy*
[b] *Department of Mathematics, National Taiwan University, Taiwan*

A R T I C L E I N F O

Article history:
Received 7 November 2024
Accepted 27 September 2025
Available online 9 October 2025
Communicated by Valentino Tosatti

MSC:
32Vxx
32A25
53D50

Keywords:
CR manifolds
Quantization commutes with reduction

A B S T R A C T

Consider a compact torsion free CR manifold X and assume that X admits a compact CR Lie group action G. Let L be a G-equivariant rigid CR line bundle over X. It seems natural to consider the space of G-invariant CR sections in the high tensor powers as quantization space, on which a certain weighted G-invariant Fourier–Szegő operator projects. Under certain natural assumptions, we show that the group invariant Fourier–Szegő projector admits a full asymptotic expansion. As an application, if the tensor power of the line bundle is large enough, we prove that quantization commutes with reduction.

Contents

* Corresponding author.
E-mail addresses: andrea.galasso@unimib.it, andrea.galasso.91@gmail.com (A. Galasso), chinyuhsiao@ntu.edu.tw, chinyu.hsiao@gmail.com (C.-Y. Hsiao).

1 **Scholarship:** Titolare di una borsa per l'estero dell'Istituto Nazionale di Alta Matematica.

https://doi.org/10.1016/j.jfa.2025.111225

1. Introduction and motivation

The study of quantization commutes with reduction in various geometric settings plays an important role in analysis, geometry, and mathematical physics. The famous geometric quantization conjecture of Guillemin and Sternberg [8] states that for a compact prequantizable symplectic manifold admitting a Hamiltonian action of a compact connected Lie group, the principle of "quantization commutes with reduction" holds.

This conjecture was first proved independently by Meinrenken [28] and Vergne [34] in the case where the Lie group is abelian, and later by Meinrenken [29] in the general case. Tian and Zhang [33] subsequently provided a purely analytic proof, with various generalizations. In the case of a non-compact symplectic manifold M with a compact connected Lie group action G, the conjecture was resolved by the foundational work of Ma and Zhang [26], as a solution to a conjecture posed by Vergne in her ICM 2006 plenary lecture [35]; see also [24] for a survey. In [9] the first author studied this principle in the pseudo-Kähler setting.

While most results in geometric quantization concern the symplectic case, the situation is different when the underlying manifold is contact or CR: the literature is significantly more limited. Nevertheless, the study of quantization on CR, contact, and Sasakian manifolds is deeply connected to several central problems in both CR and contact geometry (see, e.g., [17], [22], [7]). Theorem 1.4 in this paper can be regarded as a first step toward a geometric quantization theory for CR line bundles L, in analogy with the Kostant–Souriau framework.

More specifically, when some of our main results are specialized to the case where X is a circle bundle, they recover known results in the literature. We refer to Section 1.2 for a comparison. In fact, in certain geometric situations, CR manifolds reduce to the setting of circle bundles; see, for instance, [31, Theorems 1.11 and 5.1].

A key aspect of our analysis is the presence of an $\mathbb{R}$-action on X, which lifts naturally to the CR line bundle L. This lifting property plays a crucial role in the construction of the quantization space. The reason this lift exists is that the given $\mathbb{R}$-action on X arises from a torus action. To clarify this point, let us recall a result from [14, Corollary 3.8]: let X be a connected, strongly pseudoconvex CR manifold equipped with a transversal CR $\mathbb{R}$-action. Then the $\mathbb{R}$-action is induced by a CR torus action if and only if at least one of the following conditions holds: (a) there exists an orbit which is not closed; (b) there exists a compact orbit. In particular, if X is compact, then (b) is automatically satisfied, and hence the $\mathbb{R}$-action must arise from a CR torus action. Therefore, in our setting—since X is compact and the $\mathbb{R}$-action is CR and transversal—the action is indeed induced by a CR torus action on X. A key consequence for our purposes is [14, Theorem 3.12] (see also Theorem 2.1 below), which ensures that the $\mathbb{R}$-action on X can be lifted to the CR line bundle L.

This structure plays an important role when comparing our setting with that of [22], where the authors study CR geometric quantization. However, their framework does not assume the existence of a CR line bundle. Our work generalizes their perspective by incorporating the line bundle structure. Furthermore, the main results of this article intersect with the classical geometric quantization conjecture in the orbifold setting when the S^1-action is locally free. We refer again to Section 1.2 for a comparison, and to [10] for a treatment of quantization in the orbifold context.

Let us now provide a more precise discussion and offer an additional motivation. Let $(X, T^{1,0}X)$ be a compact CR manifold equipped with a Reeb one-form ω_0. Assume that X admits a CR action of a compact Lie group G, and that the action of G preserves ω_0. The form ω_0 induces a CR moment map μ. Under the assumptions that 0 is a regular value of μ, and that X is strongly pseudoconvex near $\mu^{-1}(0)$, it was shown in [22] that—up to some finite-dimensional subspaces of L^2 G-invariant CR functions and L^2 CR functions on the reduced space—quantization commutes with reduction. In [22], the regularity of 0 as a value of μ is a standing assumption.

This raises a natural question: if 0 is not a regular value of μ, can one still perform geometric quantization?

In this paper, we introduce the concept of a mixed moment map $\hat{\mu}_t$. We always assume that 0, in the Lie algebra of the group, is a regular value for $\hat{\mu}_t$. This allows us to treat cases in which 0 is not a regular value of the standard CR moment map μ. A special but important case occurs when the G-action is horizontal on X; in that case, 0 is not a regular value of μ but it is a regular valued of the mixed moment map $\hat{\mu}_t$. Consider, for example, the product $\hat{X} := M \times X$, where M is a complex manifold. The Reeb one-form on X lifts to a Reeb one-form on $\hat{X}$. If G acts only on M, then it is horizontal on $\hat{X}$. We observe that if M admits a positive line bundle, one can study geometric quantization by using the curvature of the line bundle and the Reeb one-form on X. Therefore, even when 0 is not a regular value of the moment map induced by the one-form on X, it is still possible to study geometric quantization by using the curvature of the CR line bundle and the Reeb one-form on the base manifold X.

It is also worth mentioning that several technical objects used in the proof were previously introduced in earlier works. More specifically, the operator σ_k appearing in the proof of Theorem 1.4 is quite similar in spirit to [25, Equation (4.108)]. As kindly pointed out by a referee, the following observation sheds light on connections between our construction and previous works in the context of holomorphic extension as developed in [3], [4], [5], and [6]. In particular, the operator σ_k appearing in the proof of Theorem 1.4 shares structural similarities with the family of operators $A_p^{X|Y}$ studied by S. Finski in [6, Section 4.2]. These operators, introduced in the context of the Ohsawa–Takegoshi extension problem, quantify the defect between the optimal extension and its factorization through an intermediate submanifold; see, for instance, equations (4.27) and (4.29), and Theorem 5.1 in [6], as well as Theorems 4.1 and 5.7, and equation (5.40) in [5] (see also the discussion after Theorem 1.4).

Although our setting is different since in our case the submanifold Y corresponds to the CR moment level set $\hat{\mu}_t^{-1}(0)$, which is generally not a complex submanifold, the philosophy is similar: we start with a G-invariant CR section on the ambient manifold X, restrict it to $\hat{\mu}_t^{-1}(0)$, and then consider the induced section on the quotient. The interplay between restriction and the group-invariant Szegő projector resembles the composition of extension and restriction operators in the holomorphic context. In particular, our analysis of the asymptotics of σ_k captures, in a different geometric framework, effects analogous to those described by $A_p^{X|Y}$.

We believe that further investigation of this analogy, especially in the direction of understanding the quantization–reduction process through the lens of optimal L^2 extension, may provide valuable insight into both CR and complex geometric quantization, especially in the setting of manifolds of bounded geometry.

All of the considerations in the previous paragraphs serve as the starting point of our project. In this work, we consider a compact torsion-free CR manifold with a CR action of a compact Lie group G. Let $L \to X$ be a G-equivariant rigid CR line bundle. We define the quantization space as the space of G-invariant CR sections of high tensor powers of L, onto which a certain weighted G-invariant Fourier–Szegő projector acts. Under natural assumptions on the curvature of the CR line bundle, the Reeb one-form, and the Lie group action, we prove that the group-invariant Fourier–Szegő projector admits a full asymptotic expansion, and that for sufficiently large tensor powers, quantization commutes with reduction.

1.1. Statements of the main results

We now formulate our results. We refer the reader to Section 2 for the terminology and notations used here. Let $(X, T^{1,0}X)$ be a compact orientable CR manifold of dimension $2n+1$, $n \geq 1$ with a transversal and CR Reeb vector field $T \in \mathcal{C}^\infty(X, TX)$ (see (30)). Let $\eta : \mathbb{R} \times X \to X$, $(\eta, x) \to \eta \cdot x$, be the $\mathbb{R}$-action induced by the flow of T (see (31)) and let $\omega_0 \in \mathcal{C}^\infty(X, T^*X)$ be the Reeb one form given by (32) below. Let $HX := \operatorname{Re} T^{1,0}X$ and let $J : HX \to HX$ be the complex structure map given by $J(u + \overline{u}) = iu - i\overline{u}$, $u \in T^{1,0}X$. Assume that X admits an action of a compact Lie group G of dimension d. We assume that

Assumption 1.1 *(Group action assumption). G commutes with the $\mathbb{R}$-action η, preserves the CR structure, $g^*\omega_0 = \omega_0$ on X and $g_*J = Jg_*$ on HX, for every $g \in G$, where g^* and g_* denote the pull-back map and push-forward map of G, respectively.*

Let $(L, h^L) \to X$ be a G-equivariant rigid CR line bundle (see Definition 2.6, Definition 2.10), where h^L is a $G \times \mathbb{R}$-invariant Hermitian metric on L (see Definition 2.7). Let R^L be the curvature of L induced by h^L (see Definition 2.8). Let $\omega_0 \in \mathcal{C}^\infty(X, T^*X)$ be the Reeb one form given by (32) and let $\mathcal{L}_x$ be the Levi form of X at $x \in X$ given by (34). In this work, we assume that

Assumption 1.2 *(Curvature assumption). There exists a bounded open interval $I \subset \mathbb{R}$ such that $R^L_x - 2s\mathcal{L}_x$ is positive definite on $T^{1,0}_xX$ at every $x \in X$, for every $s \in I$.*

From now on, we assume that the Hermitian metric $\langle \cdot \,|\, \cdot \rangle$ on $\mathbb{C}TX$ and h^L are $G \times \mathbb{R}$-invariant. Note that $\langle \cdot \,|\, \cdot \rangle$ satisfies the following: $T^{1,0}X$ is orthogonal to $T^{0,1}X$, $\langle u \,|\, v \rangle$ is real if u, v are real tangent vectors, $\langle T \,|\, T \rangle = 1$ and T is orthogonal to $T^{1,0}X \oplus T^{0,1}X$.

For every $\xi \in \mathfrak{g}$, we write ξ_X to denote the infinitesimal vector field on X induced by ξ. Put

$$\underline{\mathfrak{g}} := \{\xi_X \in \mathcal{C}^\infty(X, TX);\, \xi \in \mathfrak{g}\}.$$

Let

$$\gamma : X \to \mathfrak{g}^*$$

be the moment map induced by h^L (see Definition 2.11 and Lemma 2.2). Let

$$\mu : X \to \mathfrak{g}^*$$

be the moment map induced by ω_0 (see Definition 2.12). For every $t \in I$, let

$$\hat{\mu}_t := \gamma - 2t\mu : X \to \mathfrak{g}^*. \tag{1}$$

Recall that I is the open bounded interval as in Assumption 1.2. In this work, we assume that

Assumption 1.3. *$\hat{\mu}_t^{-1}(0) = \gamma^{-1}(0) \cap \mu^{-1}(0)$, zero is a regular value of $\hat{\mu}_t$, for all $t \in I$ and the action G is free near $\hat{\mu}_t^{-1}(0)$.*

Remark 1.1. (i) If the action G is horizontal, that is, $\omega_0(\xi_X) = 0$, for every $\xi_X \in \underline{\mathfrak{g}}$, then $\hat{\mu} = \gamma$ and $\hat{\mu}_t^{-1}(0)$ is independent of $t \in I$.

(ii) If $\gamma = \omega_0$, then $\hat{\mu}_t^{-1}(0)$ is independent of $t \in I$.

We refer the reader to Section 1.2 for more examples.

We now introduce our result about G-invariant weighted Fourier-Szegő projection. Let $(\cdot\,|\,\cdot)_k$ be the L^2 inner product on $\mathcal{C}^\infty(X,L^k)$ induced by $\langle\cdot\,|\,\cdot\rangle$ and h^{L^k}. Let

$$\mathcal{H}^0_b(X,L^k)^G := \left\{u\in L^2(X,L^k);\, \overline{\partial}_b u = 0, g^*u = u, \quad \forall g\in G\right\},$$

where $\overline{\partial}_b$ is the tangential Cauchy-Riemann operator with values in L^k. Let Π^G_k : $L^2(X,L^k)\to\mathcal{H}^0_b(X,L^k)^G$ be the orthogonal projection (G-invariant Szegő projection). We extend $-iT$ to L^2 space by

$$\begin{aligned}&-iT : \mathrm{Dom}\,(-iT)\subset L^2(X,L^k)\to L^2(X,L^k),\\ &\mathrm{Dom}\,(-iT)=\left\{u\in L^2(X,L^k);\, -iTu\in L^2(X,L^k)\right\}.\end{aligned}$$

From [14, Theorems 4.1, 4.5], $-iT$ is self-adjoint with respect to $(\cdot\,|\,\cdot)_k$, $\mathrm{Spec}\,(-iT)$ is countable and every element in $\mathrm{Spec}\,(-iT)$ is an eigenvalue of $-iT$, where $\mathrm{Spec}\,(-iT)$ denotes the spectrum of $-iT$. Let $\tau\in\mathcal{C}^\infty_c(I,\mathbb{R}_+)$, $\tau_k(t):=\tau(\frac{t}{k})$. Let $\tau_k(-iT)$ denote the operator defined by applying the functional calculus to $-iT$ using the function τ_k. Since T preserves CR structure, commutes with the action G and L is rigid, $\tau_k(-iT)$ commutes with the G-invariant Szegő projection Π^G_k. Let

$$P^G_{k,\tau} := \Pi^G_k\circ\tau_k(-iT) : L^2(X,L^k)\to\mathcal{H}^0_b(X,L^k)^G. \tag{2}$$

Let s be a local $G\times\mathbb{R}$-invariant CR trivializing section defined on an open set $D\subset X$, $|s|^2_{h^L}=e^{-2\Phi}$. The localized operator of $P^G_{k,\tau}$ is given by

$$P^G_{k,\tau,s} := s^{-k}e^{-k\Phi}P^G_{k,\tau}s^k e^{k\Phi} : \mathcal{C}^\infty_c(D)\to\mathcal{C}^\infty(D). \tag{3}$$

Let $P^G_{k,\tau,s}(x,y)\in\mathcal{C}^\infty(D\times D)$ be the distribution kernel of $P^G_{k,\tau,s}$. Let $Y:=\hat{\mu}_t^{-1}(0)$. The first main result of this work is the following

Theorem 1.1 *(Semi-classical G-invariant Fourier Szegő kernel). With the notations and assumptions above, let $\chi\in\mathcal{C}^\infty(X)$ with $\mathrm{supp}\,\chi\cap Y=\emptyset$. Then,*

$$\chi P^G_{k,\tau^2} = O(k^{-\infty}) \quad \text{on } X. \tag{4}$$

Let $p\in Y$ and let s be a local $G\times\mathbb{R}$-invariant CR trivializing section defined on an open set $D\subset X$, $p\in D$, $|s|^2_{h^L}=e^{-2\Phi}$. Then

$$P^G_{k,\tau^2,s}(x,y) = \int_{\mathbb{R}} e^{ikA(x,y,t)}g(x,y,t,k)dt + O(k^{-\infty}) \tag{5}$$

on $D\times D$, where

$$\begin{aligned}&g\in S^{n+1-d/2}_{\mathrm{loc}}(1;D\times D\times I),\\ &\mathrm{supp}_{\,t}\, g(x,y,t,k)\subset I,\end{aligned} \tag{6}$$

is a symbol with expansion

$$\begin{aligned}
&g(x,y,t,k) \sim \sum_{j=0}^{+\infty} g_j(x,y,t)k^{n+1-d/2-j} \quad \text{in } S^{n+1-d/2}_{\mathrm{loc}}(1; D\times D\times I),\\
&g_j(x,y,t) \in \mathcal{C}^\infty(D\times D\times I), \quad j=0,1,\dots,\\
&\operatorname{supp}_t g_j(x,y,t) \subset I, \quad j=0,1,\dots.
\end{aligned} \tag{7}$$

Furthermore $A \in \mathcal{C}^\infty(D\times D\times I)$ *is a complex phase function with* $\operatorname{Im} A \geq 0$*, and*

$$\mathrm{d}_x A(x,\,y,\,t)_{|x=y} = -\mathrm{d}_y A(x,\,y,\,t)_{|x=y} = -2\operatorname{Im}\overline{\partial}_b\Phi(x) + t\,\omega_0 \tag{8}$$

for every $x\in Y$*,* $A(x,\,y,\,t)=0$ *if and only if* $x=y\in Y$ *and*

$$\operatorname{Im} A(x,y,t) \geq C\Big(d^2(x,Y)+d^2(x,Y)\Big), \tag{9}$$

$x,y\in D$*, where* $C>0$ *is a constant. For a local description of the phase* A *in terms of local coordinates defined in Proposition 3.1, we refer to equation (117) and we refer the reader to Theorem 3.3 for more properties of* $\operatorname{Im} A$*.*

We refer the reader to Section 2.2 and the discussion after (49) for the semi-classical notations used in Theorem 1.1. In particular, the precise meaning of the space $S^{n+1-d/2}_{\mathrm{loc}}$ is given by Definition 2.1.

We now give a formula for the leading term of $g(x,y,t,k)$ in (7). We need to recall one more piece of notation. Fix Y, for every $t\in I$, consider the linear map

$$\begin{aligned}
R_x(t)\ :\ &\underline{\mathfrak{g}}_x \to \underline{\mathfrak{g}}_x\\
&u \mapsto R_x(t)u,
\end{aligned} \tag{10}$$

where

$$\langle R_x(t)u|v\rangle = \langle -i(R^L(x)-2t\mathcal{L}_x),\, Ju\wedge v\rangle\,.$$

Let $\det R_x(t) = \mu_1(x,t)\cdots\mu_d(x,t)$, where $\mu_j(x,t)$, $j=1,\dots,d$ are the eigenvalues of $R_x(t)$. Furthermore, put $Y_x=\{g\cdot x;\, g\in G\}$. Then Y_x is a d-dimensional submanifold of X. The G-invariant Hermitian metric induces a volume form dV_{Y_x} on Y_x. Put

$$V_{\mathrm{eff}}(x) := \int_{Y_x} \mathrm{dV}_{Y_x}.$$

Theorem 1.2. *In the same setting of Theorem 1.1, for any* $x\in Y\cap D$ *and* $t\in I$*, we have*
In the same setting of Theorem 1.1, for any $x\in Y\cap D$ *and* $t\in I$*, we have*

$$g_0(x,x,t) = 2^{-n-1+d}\,\frac{1}{V_{\text{eff}}(x)}\,|\det R_x(t)|^{-1/2}\pi^{-n-1+d/2}\,|\det(R_x^L - 2t\mathcal{L}_x)|\cdot\tau^2(t), \quad (11)$$

where $\det(R_x^L - 2t\mathcal{L}_x) = \lambda_1(x,t)\cdots\lambda_n(x,t)$, $\lambda_j(x,t)$, $j = 1,\ldots,n$, *are the eigenvalues of* $R_x^L - 2t\mathcal{L}_x$ *with respect to* $\langle\cdot|\cdot\rangle$.

Let $X_G := Y/G$. By Assumptions 1.3, we have (see Proposition 2.1)

Theorem 1.3. *X_G is a torsion free CR manifold of dimension $2n-2d+1$ and $L_G := L/G$ is a rigid CR line bundle over X_G.*

Theorem 1.3 says that $X_G := Y/G$ is endowed by a CR structure in a natural way. We say that X_G is the CR reduction with respect to curvature data. From Theorem 1.3, we can study quantization commutes with reduction.

For every $\lambda \in \operatorname{Spec}(-iT)$, put

$$\mathcal{H}^0_{b,\lambda}(X, L^k)^G := \left\{u \in \mathcal{H}^0_b(X, L^k)^G;\ -iTu = \lambda u\right\}. \quad (12)$$

It is not difficult to see that $\dim \mathcal{H}^0_{b,\lambda}(X, L^k)^G < +\infty$. In the discussion before (42), we explain why $\dim \mathcal{H}^0_{b,\lambda}(X, L^k)^G$ is finite. Let T_{X_G} be the vector field on X_G induced by the $\mathbb{R}$-action on X_G. For every $\lambda \in \operatorname{Spec}(-iT_{X_G})$, put

$$\mathcal{H}^0_{b,\lambda}(X_G, L^k_G) := \left\{u \in \mathcal{H}^0_b(X_G, L^k_G);\ -iT_{X_G}u = \lambda u\right\}. \quad (13)$$

The following is the quantization commutes with reduction result obtained in this work

Theorem 1.4 *(Quantization commutes with reduction). With the same notations and assumptions used above, suppose that $I = (a,b)$, $a < b < +\infty$. There is a $k_0 \in N$ such that for all $\lambda \in (ka, kb)$, $k \geq k_0$, $\lambda \in \operatorname{Spec}(-iT) \cap \operatorname{Spec}(-iT_{X_G})$, we have*

$$\dim \mathcal{H}^0_{b,\lambda}(X, L^k)^G = \dim \mathcal{H}^0_{b,\lambda}(X_G, L^k_G). \quad (14)$$

Moreover, if $\lambda \in \operatorname{Spec}(-iT)$ and $\lambda \notin \operatorname{Spec}(-iT_{X_G})$, $\lambda \in (ka, kb)$, $k \geq k_0$, we have that $\dim \mathcal{H}^0_{b,\lambda}(X, L^k)^G = 0$. Similarly, if $\lambda \in \operatorname{Spec}(-iT_{X_G})$ and $\lambda \notin \operatorname{Spec}(-iT)$, $\lambda \in (ka, kb)$, $k \geq k_0$, then $\dim \mathcal{H}^0_{b,\lambda}(X_G, L^k_G) = 0$.

Theorem 1.4 can be seen as an application of Theorems 1.1 and 1.2. Let

$$\sigma_k : \mathcal{C}^\infty(X, L^k) \to \mathcal{H}^0_b(X_G, L^k_G)$$

be the operator given by (146) below. We actually prove that for $k \gg 1$, the map

$$\sigma_k : \mathcal{H}^0_{b,\lambda}(X, L^k)^G \to \mathcal{H}^0_{b,\lambda}(X_G, L^k_G)$$

is an isomorphism, for all $\lambda \in (ka, kb)$. The operator σ_k constructed in (146) was inspired by [25, Equation (4.108)].] Moreover, several technique objects used in the proof (see Section 4) were introduced in Ma-Zhang [25]. It is worth reminding readers that, in this work, we use the method of complex Fourier integral operators, whereas Ma-Zhang's book uses the technique from local index theory.

Another point worth bringing to the reader's attention is that, in these important papers [3], [4], [5], [6], Finski used asymptotic behaviors similar to those of $\sigma_k^*\sigma_k$ to study significant holomorphic extension problems in complex geometry. More precisely, in [6], Finski considered complex manifolds X, Y with a complex embedding $\iota : Y \to X$, a positive line bundle (L, h^L) over X. In [6, Theorem 5.1, equations (4.27), (4.29)], Finski studied the asymptotic behavior of the operator

$$A_p^{X|Y} = \mathrm{Res}_Y \circ (\mathrm{Res}_Y \circ B_p^X)^*,$$

where B_p^X is the Bergman projection with values in L^p and Res_Y is the operator of the restriction on Y (see also [5, Theorems 4.1, 5.7, equation (5.40)]). The operator $A_p^{X|Y}$ is, in spirit, analogous to $\sigma_k\sigma_k^*$, and the asymptotic behavior of $A_p^{X|Y}$ is similar to that of $\sigma_k\sigma_k^*$ (see Theorem 4.6). It is worth noting that, Finski computed the first two coefficients in the expansion of $A_p^{X|Y}$ (see [6, Theorem 5.1]) and we compute the leading term of $\sigma_k\sigma_k^*$.

The technique of the proof of Theorem 1.4 comes from [17], [22]. Furthermore, we recall that the way to establish the isomorphism from kernel expansion for k large comes from [25]. Moreover, they show that this isomorphism becomes an asymptotic isometry when both spaces are equipped with their natural L^2-metrics (see Theorem 0.10 in [25]). Note that the L^2-metric for sections on the symplectic reduction is defined in [25, equation (0.28)].

We also point out that related notions of asymptotic isometries in geometric quantization have been investigated in [30] and in [11], in a symplectic and almost-complex framework. In [30] and [11], the authors also study the asymptotic isometry of the Guillemin and Sternberg map. They show that this map is not asymptotically isometric unless the effective potential of the action is constant. In [25], the effective potential is explicitly included in the definition [25, equation (0.28)], which is why the resulting isomorphism is indeed an isometry. The arguments in [25] could be readily adapted to our setting. Moreover, from Theorem 4.7, we see that there is a sequence $\delta_k > 0$ with $\lim_{k\to+\infty} \delta_k = 0$, such that for all $\lambda \in (ka, kb)$, $\lambda \in \mathrm{Spec}\,(-iT) \cap \mathrm{Spec}\,(-iT_{X_G})$, if $\{u_1, \ldots, u_{d_k}\}$ is an orthonormal basis of $\mathcal{H}^0_{b,\lambda}(X, L^k)^G$, then

$$(\,\sigma_k u_j \,|\, \sigma_k u_\ell\,)_{X_G,k} = \delta_{j,\ell} + \varepsilon_k, \tag{15}$$

for all $j, \ell = 1, \ldots, d_k$, where $|\varepsilon_k| \leq \delta_k$, for all $k \gg 1$. Note that in Ma-Zhang's book, they have asymptotic isometry as (15) (see [25, Theorem 0.10]) but in Ma-Zhang's work, they can take δ_k to be $\frac{1}{k}$. We believe that δ_k will have the form $\frac{1}{k}$ and $\frac{1}{\sqrt{k}}$, but this requires a more delicate analysis, and due to length limitations, we will not pursue it further.

In the end of this section, in order to help readers gain a clearer understanding of our results, we describe our results in the Sasakian case. We say that X is a Sasakian manifold if X is strongly pseudoconvex. Recall that T is a transversal and CR vector field. We will use the same notations as before. For the relation between Sasaki geometry and CR geometry, we refer to [31, Theorems 1.11, 5.1]. We now assume that X is a Sasakian manifold. We have

Lemma 1.1. *We can find a rigid positive CR line bundle (L, h^L) over X such that $R^L_x = \mathcal{L}_x$, for all $x \in X$.*

Proof. Let $x = (z, \theta)$ be a BRT chart of X defined in an open set D of X (see discussion before Example 2.11 of [18] for the meaning of BRT charts). Then

$$T^{1,0}_x X = \operatorname{span}\left\{\frac{\partial}{\partial z_j} - i\frac{\partial\phi}{\partial z_j}\frac{\partial}{\partial\theta}, j = 1, \cdots, n\right\}, \quad \phi(z) \in C^\infty(D),$$
$$T = -\frac{\partial}{\partial\theta}.$$

Also, $\mathcal{L}_x = \sum_{j,\ell=1}^n \frac{\partial^2\phi}{\partial z_j \partial\overline{z}_\ell} dz_j \wedge d\overline{z}_\ell\Big|_{T^{1,0}_x X}$. Let s_ϕ be a local CR trivialization on D with $|s_\phi|^2_{h^L} = e^{-2\phi}$, $\partial\overline{\partial}\phi > 0$. Let (w, η) be another BRT chart of X on D, then

$$T^{1,0}_x X = \operatorname{span}\left\{\frac{\partial}{\partial w_j} - i\frac{\partial\widetilde{\phi}}{\partial w_j}\frac{\partial}{\partial\eta}, j = 1, \cdots, n\right\}.$$

We have

$$\begin{cases} w = (w_1, \cdots, w_n) = H(z) = (H_1(z), \cdots, H_n(z)), \\ \overline{\partial}H = 0, \quad \widetilde{\phi}(H(z)) = \phi(z) + \log|g(z)|, \\ \eta = \theta + \operatorname{Im} g(z), \quad g \in \mathcal{C}^\infty(D), \quad g \neq 0, \quad \overline{\partial}g = 0. \end{cases} \tag{16}$$

From (16), we have

$$s_{\widetilde{\phi}} = g^{-1} s_\phi \qquad \text{on } D.$$

Thus, $\{s_\phi\}$ defines a $\mathbb{R}$-invariant CR line bundle L. $\square$

Let (L, h^L) be a rigid positive CR line bundle over X such that $R^L_x = \mathcal{L}_x$, for all $x \in X$. From Lemma 1.1, we see that this is always possible. Here $\gamma = \omega_0$ and Assumption 1.3 holds. Let $I = (-\varepsilon, \varepsilon)$, $\varepsilon > 0$, $\varepsilon \ll 1$. Since $R^L_x = \mathcal{L}_x$, for all $x \in X$, $R^L_x - 2s\mathcal{L}_x$ is positive definite, for all $s \in I$, if $\varepsilon > 0$ is small enough. Theorem 1.4 tells us that there is a $k_0 \gg 1$ such that for all $k \geq k_0$, we have

$$\mathcal{H}^0_{b,\lambda}(X, L^k)^G \cong \mathcal{H}^0_{b,\lambda}(X_G, L^k_G), \tag{17}$$

for all $\lambda \in (-k\varepsilon, k\varepsilon)$. (17) can be seen as "quantization commutes with reduction" on Sasakian manifolds. It should be mention that in [22], the authors also established "quantization commutes with reduction" on Sasakian manifolds. But in [22], they don't consider CR line bundle; they consider CR G-invariant CR functions. We notice that $X/\mathbb{R}$ is a complex manifold with singularities. Moreover, when the $\mathbb{R}$-action is actually s S^1-action, X/S^1 is a complex orbifold with cyclic singularities. Therefore (17) intersect with the classical geometric quantization conjecture in the orbifold/singular setting.

We refer the reader to Example 1.2 below when the $\mathbb{R}$-action is actually a S^1-action.

1.2. Examples

In this subsection, we will give two very simple examples and we verify that the assumptions of Section 1.1 hold.

Example 1.1. Let M be a compact orbifold with cyclic singularities and let $(L, h^L) \to M$ be an orbifold line bundle, where h^L is a Hermitian metric of L. Assume that the curvature of L induced by h^L is positive definite. Suppose that M admits a compact holomorphic Lie group action G and the action G can be lifted to L. Assume further that L and h^L are G-invariant. Consider the circle bundle

$$X := \{v \in L^*;\, |v|_{h^{L^*}} = 1\}.$$

Since the singularities of M are cyclic, X is a smooth CR manifold. Actually, X is a quasi-regular Sasakian manifold. The line bundle L can be considered as a CR line bundle over X (we still denote by L). X is a torsion free CR manifold. We will use the same notations as in Section 1. The $\mathbb{R}$-action on X is the S^1-action on X acting on the fiber of X. For $m \in \mathrm{Spec}\,(-iT)$, we have

$$\mathcal{H}^0_{b,m}(X, L^k)^G = \mathcal{H}^0(M, L^m \otimes L^k)^G, \tag{18}$$

where $\mathcal{H}^0(M, L^k \otimes L^m)^G$ denotes the space of all G-invariant holomorphic sections of M with values in $L^k \otimes L^m$.

Take any G-invariant volume form dV_M on M. For every $k, m \in \mathbb{Z}$, let $\left\{f^{k,m}_1, \ldots, f^{k,m}_{d_{k,m}}\right\}$ be an orthonormal basis for $\mathcal{H}^0(M, L^k \otimes L^m)^G$ with respect to the L^2 inner product induced by h^L and dV_M. Let

$$B^G_{k,m}(x) := \sum_{j=1}^{d_{k,m}} |f^{k,m}_j(x)|^2_{h^{L^k}} \in \mathcal{C}^\infty(M).$$

For every $k, m \in \mathbb{Z}$, let $\left\{g_1^{k,m}, \ldots, g_{d_{k,m}}^{k,m}\right\}$ be an orthonormal basis for $\mathcal{H}^0_{b,m}(X, L^k)^G$ with respect to the L^2 inner product induced by h^L and dV_M. We have

$$\sum_{j=1}^{d_{k,m}} |f_j^{k,m}(\pi(x))|^2_{h^{L^k}} = \sum_{j=1}^{d_{k,m}} |g_j^{k,m}(x)|^2_{h^{L^k}},$$

where $\pi : X \to M$ is the natural projection. Let s be a local rigid G-invariant CR trivializing section of L defined on an open set D of X, $|s|^2_{h^L} = e^{-2\Phi}$. On D, write $g_j^{k,m} = s^k \otimes \tilde{g}_j^{k,m}$, $\tilde{g}_j^{k,m} \in \mathcal{C}^\infty(D)$, $j = 1, \ldots, d_{k,m}$. Then,

$$P^G_{k,\tau^2,s}(x, y) = \sum_{j=1}^{d_{k,m}} \tau^2(\frac{m}{k})e^{-k\Phi(x)}\tilde{g}_j^{k,m}(x)\overline{\tilde{g}_j^{k,m}}(y)e^{-k\Phi(y)},$$

where $P^G_{k,\tau^2,s}(x, y)$ is as in (3). In particular,

$$\begin{aligned} &P^G_{k,\tau^2,s}(x, x) \\ &= \sum_{j=1}^{d_{k,m}} \tau^2(\frac{m}{k})e^{-2k\Phi(x)}\left|\tilde{g}_j^{k,m}(x)\right|^2 \\ &= \sum_{m\in\mathbb{Z}} \tau^2\left(\frac{m}{k}\right) B^G_{k,m}(\pi(x)). \end{aligned} \tag{19}$$

Theorem 1.1 tells us that the weighted Bergman kernel (19) admits a full asymptotic expansion. It should be notice that when G is identity, Ross and Thomas [32] used different weighted Bergman kernel and established a full asymptotic expansion for their weighted Bergman kernel (see also [2]).

We have

Lemma 1.2. *$\gamma = 2\mu$.*

Proof. Let $x_0 \in X$ and let $p := \pi(x_0)$. Let s be a holomorphic trivializing section of L defined on an open set U of p in M, $|s|^2_{h^L} = e^{-2\phi}$. Let $z = (z_1, \ldots, z_n)$ be holomorphic coordinates of M defined on U (we may take U small enough) such that $z(p) = 0$. Consider the following map:

$$x = (x_1, \ldots, x_{2n+1}) = (z, x_{2n+1}) \in U \times (-\pi, \pi) \to e^{-\phi(z)}s^*(z)e^{ix_{2n+1}} \in X.$$

Then, $x = (x_1, \ldots, x_{2n+1})$ are local coordinates of X defined on $D = U \times (-\pi, \pi)$. On D, we have

$$\begin{aligned} &T^{1,0}_x X = \operatorname{span}\left\{\frac{\partial}{\partial z_j} + i\frac{\partial\phi}{\partial z_j}\frac{\partial}{\partial x_{2n+1}};\, j=1,\dots,n\right\},\\ &T = \frac{\partial}{\partial x_{2n+1}}. \end{aligned} \tag{20}$$

From (20), it is straightforward to check that on D,

$$\omega_0 = -dx_{2n+1} + \sum_{j=1}^{n}\Big(i\frac{\partial\phi}{\partial z_j}dz_j - i\frac{\partial\phi}{\partial\overline{z}_j}d\overline{z}_j\Big). \tag{21}$$

From (21), we see that

$$\langle\, \mu(x)\,,\, \xi\,\rangle = \langle\, i\partial\phi - i\overline{\partial}\phi\,,\, \xi_X\,\rangle, \tag{22}$$

where $\xi\in\mathfrak{g}$, ξ_X is the vector field on X induced by ξ. From Definition 2.11, we see that

$$\langle\, \gamma\,,\, \xi\,\rangle = (-2i)\langle\, \overline{\partial}\phi - \partial\phi\,,\, \xi_X\,\rangle. \tag{23}$$

From (22) and (23), the lemma follows. □

From Lemma 1.2, we see that $\hat{\mu}_t^{-1}(0)$ is independent of $t\in I$. It should be noticed that since R^L is positive, we can take I to be any small open interval of $0\in\mathbb{R}$ and hence Assumption 1.2 holds. From Theorem 1.1, (19), we deduce that if zero is a regular value of $\hat{\mu}_t$, for some $t\in I$ and the action G is free near $\hat{\mu}_t^{-1}(0)$, then

$$\sum_{m\in\mathbb{Z}}\tau^2\left(\frac{m}{k}\right)\, B^G_{k,m}(\pi(x))$$

admits a full asymptotic expansion.

Example 1.2. Eventually we give another simple example and we verify that the assumptions of Section 1.1 hold. Let

$$X := \left\{z\in\mathbb{C}^{n+1};\, |z_1|^{2\alpha_1} + (|z_2|^{2\alpha_2}\cdots + |z_{n+1}|^{2\alpha_{n+1}})^m = 1\right\},$$

where $\alpha_1,\dots,\alpha_{n+1},m\in\mathbb{N}$. Then X is a compact CR manifold. X admits a transversal and CR $\mathbb{R}$-action:

$$\eta\cdot(z_1,\dots,z_{n+1}) = (e^{i\beta_1\eta}z_1,\dots,e^{i\beta_{n+1}\eta}z_{n+1}),$$

where $(\beta_1,\dots,\beta_{n+1})\in\mathbb{R}^{n+1}_+$. Then, X is a torsion free CR manifold. Let L be the trivial line bundle with non-trivial Hermitian metric $|1|^2_{h^L} = e^{-2|z|^2}$. The CR manifold X admits a S^1-action:

$$G = S^1 : e^{i\theta}\cdot z = (e^{-i\theta}z_1, e^{i\theta}z_2,\dots,e^{i\theta}z_{n+1}).$$

We can calculate

$$\hat{\mu}_t^{-1}\left(\frac{\partial}{\partial\theta}\right) = \Big(|z_1|^2 - \sum_{j=2}^{n+1}|z_j|^2\Big) - 2t\Big(\alpha_1|z_1|^{2\alpha_1} - m(\sum_{j=2}^{n+1}|z_j|^{2\alpha_j})^{m-1}(\sum_{j=2}^{n+1}\alpha_j|z_j|^{2\alpha_j})\Big). \tag{24}$$

From (24), we see that if $\alpha_1 = m$, $\alpha_2 = \cdots = \alpha_{n+1} = 1$, then $\hat{\mu}_t^{-1}(0) = \mu^{-1}(0) = \gamma^{-1}(0)$. Moreover, we can check that zero is a regular value of $\hat{\mu}_t$ at $t = 0$ and we can take I to be any small open interval of $0 \in \mathbb{R}$ such that Assumption 1.2 holds.

In [25, Section 3.4], Ma-Zhang calculated explicitly the Schwartz kernel of G-invariant Bergman kernel on some model case. Example 1.2, can be seen as a model case of our study. However, in this situation, it is still very difficult to obtain an explicit formula for P^G_{k,τ^2}. The main difficulty lies in the fact that even on the sphere, the Szegő kernel does not have a nice expression. In general, to obtain a good expression for the Szegő kernel on the sphere, one needs to extend CR functions into the domain and then represent them using complex coordinates. To help readers understand Theorem 1.1, we will, in the case of a circle bundle, express P^G_{k,τ^2} in terms of the G-invariant Bergman kernel of the complex manifold.

We will use the same notations as Example 1.1. It was proved in [17, Theorem 1.5], [25, Theorem 0.2] that for $m \in \mathbb{Z}$, $m \in \operatorname{supp}\chi_k$, $\chi_k(x) ;= \chi(\frac{x}{k})$, we have

$$B^G_{k,m}(x) = e^{i(k+m)\varphi(x)}b(x,k,m) + O(k^{-\infty}), \tag{25}$$

where $\varphi(x) \in \mathcal{C}^\infty(M)$, $\operatorname{Im}\varphi \geq 0$, $\operatorname{Im}\varphi(x) \sim d(x,\mu^{-1}(0))^2$ and

$$b(x,k,m) \sim \sum_{j=0}^{+\infty}(k+m)^{n+\frac{d}{2}-j}b_j(x), \quad b_j(x) \in \mathcal{C}^\infty(M).$$

From (19) and (25), we have

$$P^G_{k,\tau^2,s}(y,y) = \sum_{m\in\mathbb{Z}}\tau^2(\frac{m}{k})e^{i(k+m)\varphi(x)}b(x,k,m) + O(k^{-\infty}), \tag{26}$$

where $\pi(y) = x$, $\pi : X \to M$ is the natural projection. Let's recall Poisson summation formula (see [13, Theorem 7.2.1]):

$$\sum_{m\in\mathbb{Z}}\hat{g}(m) = (2\pi)\sum_{m\in\mathbb{Z}}g(2\pi m), \tag{27}$$

for all $g \in \mathcal{C}^\infty_c(\mathbb{R})$, where $\hat{g}(t) = \int e^{-itx}g(x)dx$. Fox fix $x \in M$, let

$$g(t) := \tau^2(\frac{t}{2\pi k})e^{i(k+\frac{t}{2\pi})\varphi(x)}b(x,k,\frac{t}{2\pi}).$$

From (26) and (27), we have

$$\begin{aligned} P^G_{k,\tau^2,s}(y,y) &= \sum_{m\in\mathbb{Z}} g(2\pi m) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \hat{g}(m) \\ &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \int \tau^2(\frac{t}{2\pi k})e^{i(k+\frac{t}{2\pi})\varphi(x)-itm} b(x,k,\frac{t}{2\pi})dt \\ &= \sum_{m\in\mathbb{Z}} \int \tau^2(t)e^{ik(1+t)\varphi(x)-i(2\pi k)tm} b(x,k,kt)kdt \\ &= \int \tau^2(t)e^{ik(1+t)\varphi(x)} b(x,k,kt)kdt \\ &\quad + \sum_{m\in\mathbb{Z},m\neq 0} \int \tau^2(t)e^{ik(1+t)\varphi(x)-i(2\pi k)tm} b(x,k,kt)kdt. \end{aligned} \tag{28}$$

We can integrate by parts in t several times and deduce that

$$\frac{1}{2\pi}\sum_{m\in\mathbb{Z},m\neq 0} \int \tau^2(t)e^{ik(1+t)\varphi(x)-i(2\pi k)tm} b(x,k,kt)kdt = O(k^{-\infty}). \tag{29}$$

From (28) and (29), we get Theorem 1.1 in the circle bundle cases.

2. Preliminaries

2.1. Standard notation and symbols

We use the following notations: $\mathbb{N} = \{1, 2, \ldots\}$ is the set of natural numbers excluding 0 and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = \{x \in \mathbb{R}; x > 0\}$, $\overline{\mathbb{R}}_+ = \{x \in \mathbb{R}; x \geq 0\}$. Furthermore we adopt the standard multi-index notation: we write $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$ if $\alpha_j \in \mathbb{N}_0$, $j = 1, \ldots, n$.

Let M be a smooth paracompact manifold. We let TM and T^*M denote respectively the tangent bundle of M and the cotangent bundle of M. The complexified tangent bundle $TM \otimes \mathbb{C}$ of M will be denoted by $\mathbb{C}TM$, similarly we write $\mathbb{C}T^*M$ for the complexified cotangent bundle of M. Consider $\langle \cdot, \cdot \rangle$ to denote the pointwise duality between TM and T^*M; we extend $\langle \cdot, \cdot \rangle$ bi-linearly to $\mathbb{C}TM \times \mathbb{C}T^*M$. Let B be a smooth vector bundle over M. The fiber of B at $x \in M$ will be denoted by B_x. Let E be a vector bundle over a smooth paracompact manifold N. We write $B \boxtimes E^*$ to denote the vector bundle over $M \times N$ with fiber over $(x, y) \in M \times N$ consisting of the linear maps from E_y to B_x.

Let $Y \subset M$ be an open set. From now on, the spaces of distribution sections of B over Y and smooth sections of B over Y will be denoted by $\mathcal{D}'(Y, B)$ and $\mathcal{C}^\infty(Y, B)$, respectively. Let $\mathcal{E}'(Y, B)$ be the subspace of $\mathcal{D}'(Y, B)$ whose elements have compact support in Y. Let $\mathcal{C}_c^\infty(Y,B) := \mathcal{C}^\infty(Y,B) \cap \mathcal{E}'(Y,B)$. For $m \in \mathbb{R}$, let $H^m(Y,B)$ denote the Sobolev space of order m of sections of B over Y. Let us denote

$$H^m_{\mathrm{loc}}(Y,B) = \left\{u \in \mathcal{D}'(Y,B);\, \varphi u \in H^m(Y,B),\, \forall \varphi \in \mathcal{C}_c^\infty(Y)\right\},$$

and

$$H^m_{\mathrm{comp}}(Y,B) = H^m_{\mathrm{loc}}(Y,B) \cap \mathcal{E}'(Y,B)\,.$$

Let B and E be smooth vector bundles over paracompact orientable manifolds M and M_1, respectively, equipped with smooth densities of integration. We now recall the Schwartz kernel theorem. Let $A : \mathcal{C}_c^\infty(N,E) \to \mathcal{D}'(M,B)$ be a continuous operator. By Schwartz kernel theorem (see [13, Theorem 5.2.1]), we can find a distribution $A(x,y) \in \mathcal{D}'(M \times N, B \boxtimes E^*)$ such that

$$\langle\, Au\,, v\,\rangle = \langle\, A(x,y)\,,\, v \otimes u\,\rangle,$$

for all $u \in \mathcal{C}_c^\infty(N,E)$, $v \in \mathcal{C}_c^\infty(M,B)$. We call $A(x,y)$ the distribution kernel of A. The following two statements are equivalent

1. A is continuous: $\mathcal{E}'(N,E) \to \mathcal{C}^\infty(M,B)$,
2. $A(x,y) \in \mathcal{C}^\infty(M \times N, B \boxtimes E^*)$.

If A satisfies (1) or (2), we say that A is smoothing on $M \times N$. We say that A is properly supported if the restrictions of the two projections $(x,y) \mapsto x$, $(x,y) \mapsto y$ to $\mathrm{supp}\,(A(x,y))$ are proper.

Let $H(x,y) \in \mathcal{D}'(M \times N, B \boxtimes E^*)$. We write H to denote the unique continuous operator $\mathcal{C}_c^\infty(N,E) \to \mathcal{D}'(M,B)$ with distribution kernel $H(x,y)$. In this work, we identify H with $H(x,y)$.

2.2. Some more notations in semi-classical analysis

Let W_1 be an open set in $\mathbb{R}^{N_1}$ and let W_2 be an open set in $\mathbb{R}^{N_2}$. Let E and F be vector bundles over W_1 and W_2, respectively. A k-dependent continuous operator $A_k : \mathcal{C}_c^\infty(W_2,F) \to \mathcal{D}'(W_1,E)$ is called k-negligible on $W_1 \times W_2$ if, for k large enough, A_k is smoothing and, for any $K \Subset W_1 \times W_2$, any multi-indices α, β and any $N \in \mathbb{N}$, there exists $C_{K,\alpha,\beta,N} > 0$ such that

$$\left|\partial_x^\alpha \partial_y^\beta A_k(x,y)\right| \le C_{K,\alpha,\beta,N} k^{-N} \text{ on } K, \quad \forall k \gg 1.$$

In that case we write

$$A_k(x,y) = O(k^{-\infty}) \text{ on } W_1 \times W_2, \quad \text{or} \quad A_k = O(k^{-\infty}) \text{ on } W_1 \times W_2.$$

If $A_k, B_k : \mathcal{C}_c^\infty(W_2, F) \to \mathcal{D}'(W_1, E)$ are k-dependent continuous operators, we write $A_k = B_k + O(k^{-\infty})$ on $W_1 \times W_2$ or $A_k(x,y) = B_k(x,y) + O(k^{-\infty})$ on $W_1 \times W_2$ if $A_k - B_k = O(k^{-\infty})$ on $W_1 \times W_2$. When $W = W_1 = W_2$, we sometime write "on W".

Let X and M be smooth manifolds and let E and F be vector bundles over X and M, respectively. Let $A_k, B_k : \mathcal{C}^\infty(M, F) \to \mathcal{C}^\infty(X, E)$ be k-dependent smoothing operators. We write $A_k = B_k + O(k^{-\infty})$ on $X \times M$ if on every local coordinate patch D of X and local coordinate patch D_1 of M, $A_k = B_k + O(k^{-\infty})$ on $D \times D_1$. When $X = M$, we sometime write on X.

We recall the definition of the semi-classical symbol spaces

Definition 2.1. Let W be an open set in $\mathbb{R}^N$. Let

$$S(1) = S(1; W) := \Big\{ a \in \mathcal{C}^\infty(W);\ \forall \alpha \in \mathbb{N}_0^N : \sup_{x \in W} |\partial^\alpha a(x)| < \infty \Big\},$$

$$S^0_{\mathrm{loc}}(1; W) := \Big\{ (a(\cdot, m))_{m \in \mathbb{R}};\ \forall \alpha \in \mathbb{N}_0^N, \forall \chi \in \mathcal{C}_c^\infty(W) : \sup_{m \in \mathbb{R}, m \geq 1} \sup_{x \in W} |\partial^\alpha (\chi a(x,m))| < \infty \Big\}.$$

We define $a(\cdot, k) \in S^\ell_{\mathrm{loc}}(1; W)$ if for every $\alpha \in \mathbb{N}_0^N$ and $\chi \in \mathcal{C}_c^\infty(W)$, there exists $C_\alpha > 0$ independent of k, such that $|\partial^\alpha(\chi a(\cdot, k))| \leq C_\alpha k^\ell$ holds on W.

Consider a sequence $a_j \in S^{\ell_j}_{\mathrm{loc}}(1)$, $j \in \mathbb{N}_0$, where $\ell_j \searrow -\infty$, and let $a \in S^{\ell_0}_{\mathrm{loc}}(1)$. We say

$$a(\cdot, k) \sim \sum_{j=0}^\infty a_j(\cdot, k) \text{ in } S^{\ell_0}_{\mathrm{loc}}(1),$$

if, for every $N \in \mathbb{N}_0$, we have $a - \sum_{j=0}^N a_j \in S^{\ell_{N+1}}_{\mathrm{loc}}(1)$. For a given sequence a_j as above, we can always find such an asymptotic sum a, which is unique up to an element in $S^{-\infty}_{\mathrm{loc}}(1) = S^{-\infty}_{\mathrm{loc}}(1; W) := \cap_\ell S^\ell_{\mathrm{loc}}(1)$.

Let $\ell \in \mathbb{R}$ and let

$$S^\ell_{\mathrm{loc,cl}}(1) := S^\ell_{\mathrm{loc,cl}}(1; W)$$

be the set of all $a \in S^\ell_{\mathrm{loc}}(1; W)$ such that we can find $a_j \in \mathcal{C}^\infty(W)$ independent of k, $j = 0, 1, \ldots$, such that

$$a(\cdot, k) \sim \sum_{j=0}^\infty k^{\ell - j} a_j(\cdot) \text{ in } S^{\ell_0}_{\mathrm{loc}}(1).$$

Similarly, we can define $S^\ell_{\mathrm{loc}}(1; Y, E)$, $S^\ell_{\mathrm{loc\,,cl}}(1; Y, E)$ in the standard way, where Y is a smooth manifold and E is a vector bundle over Y.

2.3. Preliminaries on CR geometry and CR line bundles

We recall some notations concerning CR geometry. Let $(X, T^{1,0}X)$ be a compact and orientable CR manifold of dimension $2n+1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of X. There is a unique sub-bundle HX of TX such that $\mathbb{C}\,HX = T^{1,0}X \oplus T^{0,1}X$, $T^{0,1}X = \overline{T^{1,0}X}$. Let $J : HX \to HX$ be the complex structure map given by $J(u + \overline{u}) = iu - i\overline{u}$, for every $u \in T^{1,0}X$. By complex linear extension of J to $\mathbb{C}\,TX$, the i-eigenspace of J is $T^{1,0}X$. We shall also write (X, HX, J) to denote a CR manifold. In this work, we assume that

Assumption 2.1. *There is a global vector field $T \in \mathcal{C}^\infty(X, TX)$ such that*

$$\begin{aligned} &T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T(x) = \mathbb{C}T_xX, \quad \text{for all } x \in X, \\ &[T, \mathcal{C}^\infty(X, T^{1,0}X)] \subset \mathcal{C}^\infty(X, T^{1,0}X). \end{aligned} \tag{30}$$

We say that T is a transversal CR vector field.

From now on, we fix $T \in \mathcal{C}^\infty(X, TX)$ such that (30) hold. Let

$$\begin{aligned} \eta : \mathbb{R} \times X &\to X, \\ (\eta, x) &\to \eta \cdot x, \end{aligned} \tag{31}$$

be the $\mathbb{R}$-action induced by the flow of T, that is,

$$(Tu)(x) = \frac{\partial}{\partial \eta}(u(\eta \cdot x))|_{\eta=0}, \quad \text{for all } u \in \mathcal{C}^\infty(X).$$

Since X is orientable and T is CR, the characteristic bundle is a trivial real line sub-bundle and thus there exists a real non-vanishing global 1-form $\omega_0 \in \mathcal{C}^\infty(X, T^*X)$ such that

$$\begin{aligned} &\langle\, \omega_0(x)\,,\, u\, \rangle = 0, \text{ for every } u \in H_xX, \\ &\omega_0(T) \equiv -1, \quad \mathrm{d}\omega_0(T, \cdot) \equiv 0 \ \text{ on } TX. \end{aligned} \tag{32}$$

We give an argument that why ω_0 in (32) exists. Let $p \in X$. From [1, section 1], we can find local coordinates $x = (x_1, \ldots, x_{2n+1}) = (z, x_{2n+1}) = (z_1, \ldots, z_n, x_{2n+1})$ defined on an open set D of p in X such that on D, we have

$$\begin{aligned} &T^{1,0}_xX = \text{span}\left\{ \frac{\partial}{\partial z_j} + i\frac{\partial \phi}{\partial z_j}(z)\frac{\partial}{\partial x_{2n+1}};\, j = 1, \ldots, n \right\}, \quad x \in D, \\ &T = \frac{\partial}{\partial x_{2n+1}} \ \text{ on } D, \end{aligned} \tag{33}$$

where $\phi \in \mathcal{C}^\infty(D, \mathbb{R})$, ϕ is independent of x_{2n+1}. Let

$$\omega_0 := -dx_{2n+1} + \sum_{j=1}^{n}(i\frac{\partial\phi}{\partial z_j}(z)dz_j - i\frac{\partial\phi}{\partial\overline{z}_j}(z)d\overline{z}_j) \in \mathcal{C}^\infty(D, T^*X).$$

Then, ω_0 is a real one form on D. We can check that on D, ω_0 satisfies (32). Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T^{*1,0}X$ on D. Let $\hat{\omega}_0 \in \mathcal{C}^\infty(D, T^*X)$ be another one form satisfying (32). We have

$$\hat{\omega}_0 = \lambda\omega_0 + \sum_{j=1}^{n}(\mu_j e_j + \overline{\mu}_j\overline{e}_j) \ \text{ on } D,$$

where $\lambda \in \mathbb{R}$, $\mu_j \in \mathbb{C}$, $j = 1, \ldots, n$. From (32), we can check that $\lambda = 1$ and $\mu_j = 0$, for all $j = 1, \ldots, n$. Thus, we can take ω_0 to be a global one form.

For each $x \in X$, we define a Hermitian quadratic form $\mathcal{L}_x$ on $T^{1,0}_x X$ as follows: for $U, V \in T^{1,0}_x X$,

$$\mathcal{L}_x(U, \overline{V}) = \frac{1}{2}\,\mathrm{d}\omega_0(JU, \overline{V}) = -\frac{1}{2i}\,\mathrm{d}\omega_0(U, \overline{V}). \tag{34}$$

The Hermitian quadratic form $\mathcal{L}_x$ on $T^{1,0}_x X$ is called Levi form at x.

Fix a smooth Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$ so that $T^{1,0}X$ is orthogonal to $T^{0,1}X$, $\langle\, u\,|\,v\,\rangle$ is real if u, v are real tangent vectors, $\langle\, T\,|\,T\,\rangle = 1$ and T is orthogonal to $T^{1,0}X \oplus T^{0,1}X$. For $u \in \mathbb{C}TX$, we write $|u|^2 := \langle\, u\,|\,u\,\rangle$. Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. They can be identified with sub-bundles of the complexified cotangent bundle $\mathbb{C}T^*X$. For $q = 0, 1, \ldots, n$, let $T^{*0,q}X := \Lambda^q(T^{*0,1}X)$. The Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\mathbb{C}TX$ induces by duality a Hermitian metric $\langle\,\cdot\,|\,\cdot\,\rangle$ on $\oplus_{q=1}^{n} T^{*0,q}X$.

Let $\Omega^{0,q}(X) := \mathcal{C}^\infty(X, T^{*0,q}X)$ and for an open set $D \subset X$, let $\Omega^{0,q}_c(D) := \mathcal{C}^\infty_c(D, T^{*0,q}X)$.

Let $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operator. Since the $\mathbb{R}$-action is CR, it is straightforward to see that

$$T\overline{\partial}_b = \overline{\partial}_b T$$

on $\Omega^{0,q}(X)$.

Definition 2.2. Let D be a sufficiently small open set. We say that a function $u \in \mathcal{C}^\infty(D)$ is rigid if $Tu = 0$. We say that a function $u \in \mathcal{C}^\infty(D)$ is CR if $\overline{\partial}_b u = 0$. We say that $u \in \mathcal{C}^\infty(D)$ is rigid CR if $\overline{\partial}_b u = 0$ and $Tu = 0$.

The following definitions for CR vector bundles can be found in [12].

Definition 2.3. A complex line bundle $\pi : L \to X$ is called a CR line bundle if

(i) L is a CR manifold of codimension 2,

(ii) $\pi: L \to X$ is a CR submersion,
(iii) $L \oplus L \ni (\xi_1, \xi_2) \to \xi_1 + \xi_2 \in L$ and $\mathbb{C} \times E \ni (\lambda, \xi) \to \lambda\xi \in L$ are CR maps.

A smooth section $s \in \mathcal{C}^\infty(U, L)$ defined on an open set $U \subset X$ is called CR section if the map $s: U \to L$ is CR.

Remark 2.1. Each condition in Definition 2.3 ensures that the complex line bundle structure is compatible with the CR geometry of both the base and the total space, and is crucial for our construction of CR geometric quantization. The first assumption is natural since we are dealing with a complex line bundle over a CR base. The second condition guarantees that the CR structure of the total space L projects nicely to the CR structure on X. It ensures that the fibers are transversal to the CR directions, which is necessary to define CR sections. Eventually, the third condition ensures that the vector space operations (addition and scalar multiplication) are compatible with the CR structure, so that L behaves as a genuine CR vector bundle. This is essential to define CR analogues of geometric objects such as CR connections, CR curvature, and, ultimately, to carry out the quantization process in the CR category.

These conditions together define what can be thought of as a “holomorphic line bundle” in the CR category, allowing us to extend classical notions of geometric quantization to the CR setting. For example, the existence of enough CR sections plays a role analogous to that of holomorphic sections in the Kähler case.

Let L be a CR line bundle over X. The tangential Cauchy-Riemann operator can be defined on sections of L:

$$\overline{\partial}_b : \Omega^{0,q}(X, L) \to \Omega^{0,q+1}(X, L^k),$$

where $\Omega^{0,q}(X, L) := \mathcal{C}^\infty(X, L \otimes T^{*0,q}X)$.

Definition 2.4. A CR line bundle $L \to X$ is called locally CR trivializable if for any point $p \in X$ there exists an open neighborhood $U \subset X$ such that $L|_U$ is CR line bundle isomorphic to the trivial CR vector bundle $U \times \mathbb{C}$.

Definition 2.5. Let L be a CR line bundle over X. A CR bundle lift of T to L is a smooth linear partial differential operator

$$T = T^L : \mathcal{C}^\infty(X, L) \to \mathcal{C}^\infty(X, L)$$

such that

(i) $T^L(f \cdot s) = T(f) \cdot s + fT^L(s)$ for all $f \in \mathcal{C}^\infty(X)$ and $s \in \mathcal{C}^\infty(X, L)$,
(ii) $[T^L, \overline{\partial}_b] = 0$.

In order to define $[T^L, \overline{\partial}_b]$ we need to define T^L on $(0,1)$ forms with values in L first. But this definition follows immediately from the fact that any $w \in \Omega^{0,1}(X, L)$ locally can be written $w = s \otimes g$ where g is a $(0,1)$-form and s is a local frame of L and that T is defined also for $(0,q)$-forms using the Lie derivative.

Definition 2.6. A CR line bundle $L \to X$ with a CR bundle lift T^L of T is called rigid CR (with respect to T^L) if for every point $p \in X$ there exists an open neighborhood U around p and a CR frame s of $L|_U$ with $T^L(s) = 0$.

A section $s \in \mathcal{C}^\infty(X, L)$ is called a rigid CR section if $T^L s = 0$ and $\overline{\partial}_b s = 0$. The frame s in Definition 2.6 is called a rigid CR frame of $L|_U$. Note that it follows from [14, Lemma 2.6] that any rigid CR line bundle is locally CR trivializable. The following is well-known [14, Lemma 2.10]

Lemma 2.1. *Let $L \to X$ be a CR line bundle over X and recall that $T \in \mathcal{C}^\infty(X, TX)$ is a CR vector field. The following are equivalent:*

(i) *T has a CR bundle lift T^L such that $L \to X$ is rigid CR with respect to T^L.*
(ii) *There exist an open cover $\{U_j\}_{j\in\mathbb{N}}$ of X and CR frames $\{s_j\}$ for $E|_{U_j}$, $j \in \mathbb{N}$, such that the corresponding transition functions are rigid CR.*

From now on, we assume that $L \to X$ is a rigid CR line bundle (with respect to the lifting T^L). To simplify the notations, we also write T to denote the lifting T^L.

Definition 2.7. Let $L \to X$ be a rigid CR line bundle (with respect to the lifting T^L). Let h^L be a Hermitian metric on L. We say that h^L is a rigid Hermitian metric or $\mathbb{R}$-invariant Hermitian metric if for every local rigid frame s of L, we have $T|s|^2_{h^L} = 0$.

By Lemma 2.1, there exists an open covering $(U_j)_{j=1}^N$ and a family of rigid CR trivializing frames $\{s_j\}_{j=1}^N$ with each s_j defined on U_j and the transition functions between different rigid CR frames are rigid CR functions. Let L^k be the k-th tensor power of L. Then $\{s_j^k\}_{j=1}^N$ is a family of rigid CR trivializing frames on each U_j. Let $\overline{\partial}_b : \Omega^{0,q}(X, L^k) \to \Omega^{0,q+1}(X, L^k)$ be the tangential Cauchy-Riemann operator. Since L^k is rigid CR, we have $\overline{\partial}_b f = \overline{\partial}_b f_j \otimes s_j^k$, $Tf = (Tf_j) \otimes s_j^k$ for any $f = f_j \otimes s_j^k \in \Omega^{0,q}(X, L^k)$ and

$$T\overline{\partial}_b = \overline{\partial}_b T \quad \text{on } \Omega^{0,q}(X, L^k). \tag{35}$$

Let h^L be a rigid Hermitian fiber metric on L. The local weight of h^L with respect to a local rigid CR trivializing section s of L over an open subset $D \subset X$ is the function $\Phi \in \mathcal{C}^\infty(D, \mathbb{R})$ for which

$$|s(x)|^2_{h^L} = e^{-2\Phi(x)}, x \in D. \tag{36}$$

We denote by Φ_j the weight of h^L with respect to s_j.

Definition 2.8. Let h^L be a rigid Hermitian metric on L. The curvature of (L, h^L) is the Hermitian quadratic form $R^L = R^{(L,h^L)}$ on $T^{1,0}X$ defined by

$$R^L_p(U, V) = \left\langle d(\overline{\partial}_b\Phi_j - \partial_b\Phi_j)(p), U \wedge \overline{V} \right\rangle,\ U, V \in T^{1,0}_p X,\ p \in U_j. \tag{37}$$

Due to [19, Proposition 4.2], R^L is a well-defined global Hermitian form, since the transition functions between different frames s_j are annihilated by T.

Definition 2.9. We say that (L, h^L) is positive if there is an interval $I \subset \mathbb{R}$ such that the associated curvature $R^L_x - 2s\mathcal{L}_x$ is positive definite at every $x \in X$, for every $s \in I$.

In this work, we assume that

Assumption 2.2. *There is a positive rigid Hermitian metric h^L on L.*

From now on, we fix a positive rigid Hermitian metric h^L on L and we have

$$R^L_x - 2s\mathcal{L}_x \text{ is positive definite, for every } x \in X,\ s \in I, \tag{38}$$

where $I \subset \mathbb{R}$ is a bounded open interval. As we mentioned in the introduction, from [14, Corollary 3.8], we see that the $\mathbb{R}$-action on X comes form a torus action on X. The following is well-known [14, Theorem 3.12],

Theorem 2.1. *With the assumptions and notations above, we can find local CR rigid trivializations of L defined on open sets D_j, $j = 1, \ldots, N$, such that $X = \bigcup_{j=1}^N D_j$, and*

$$D_j = \bigcup_{\eta\in\mathbb{R}} \{\eta \cdot x;\ x \in D_j\} \tag{39}$$

for every $j = 1, \ldots, N$.

Theorem 2.1 tells us that the $\mathbb{R}$-action on X can be lifted to L. From now on, for any local CR rigid trivialization s defined on an open set D of X, we will always assume that $D = \bigcup_{\eta\in\mathbb{R}}\{\eta \cdot x;\ x \in D\}$ and s is $\mathbb{R}$-invariant.

2.4. The weighted Fourier-Szegő operator

In this section we study the microlocal properties of the weighted Fourier-Szegő operator P_{k,τ^2} introduced in Section 1.1. As pointed out to us by a referee, in [20] the authors consider a Toeplitz operator of the form $A := \Pi(-iT)\Pi$, where Π is the Bergman projection associated with a strictly pseudoconvex domain, and T is as in Assumption 2.1.

They then define the operator $\chi_k(A)$ via functional calculus, for suitable functions χ_k. This construction allows them to obtain semi-classical asymptotics for $\chi_k(A)$.

In our setting, we define the Fourier–Szegő kernel using a spectral cut-off for the operator $-iT$ with respect to the natural L^2-structure on the CR manifold. This operator acts on the space of CR functions (or sections) and is a smoothing operator that approximates the projection onto the low-frequency part of the spectrum of $-iT$. Thus, both constructions are based on spectral calculus applied to a Toeplitz-type operator of the form $\Pi(-iT)\Pi$.

We will use the same notations and assumptions as before. Let L^k be the k-th power of L. The Hermitian metric on L^k induced by h^L is denoted by h^{L^k}. We assume that the Hermitian metric $\langle\cdot|\cdot\rangle$ is $\mathbb{R}$-invariant. We denote by dV_X the volume form on X induced by $\langle\cdot|\cdot\rangle$. Let $q\in\{0,1,\ldots,n\}$. Let $(\cdot|\cdot)_k$ be the L^2 inner product on $\Omega^{0,q}(X,L^k)$ induced by h^{L^k} and dV_X. Let $L^2_{(0,q)}(X,L^k)$ be the completion of $\Omega^{0,q}(X,L^k)$ with respect to $(\cdot|\cdot)_k$. We extend $(\cdot|\cdot)_k$ to $L^2_{(0,q)}(X,L^k)$. For $q=0$, we write $L^2(X,L^k):=L^2_{(0,0)}(X,L^k)$.

Consider the operator

$$-iT:\mathcal{C}^\infty(X,L^k)\to\mathcal{C}^\infty(X,L^k)$$

and we extend $-iT$ to L^2 space by

$$\begin{aligned}&-iT:\mathrm{Dom}\,(-iT)\subset L^2(X,L^k)\to L^2(X,L^k),\\&\mathrm{Dom}\,(-iT)=\left\{u\in L^2(X,L^k);\ -iTu\in L^2(X,L^k)\right\}.\end{aligned}$$

From [14, Theorems 4.1, 4.5], $-iT$ is self-adjoint with respect to $(\cdot|\cdot)_k$, $\mathrm{Spec}\,(-iT)$ is countable and every element in $\mathrm{Spec}\,(-iT)$ is an eigenvalue of $-iT$, where $\mathrm{Spec}\,(-iT)$ denotes the spectrum of $-iT$.

Let $\overline{\partial}_b:\Omega^{0,q}(X,L^k)\to\Omega^{0,q+1}(X,L^k)$ be the tangential Cauchy-Riemann operator with values in L^k. For every $\alpha\in\mathrm{Spec}\,(-iT)$, put

$$\mathcal{C}^\infty_\alpha(X,L^k):=\left\{u\in\mathcal{C}^\infty(X,L^k);\ -iTu=\alpha u\right\},\tag{40}$$

and

$$\mathcal{H}^0_{b,\alpha}(X,L^k):=\left\{u\in\mathcal{C}^\infty_\alpha(X,L^k);\ \overline{\partial}_bu=0\right\}.\tag{41}$$

Let $\overline{\partial}^*_b:\Omega^{0,1}(X,L^k)\to\mathcal{C}^\infty(X,L^k)$ be the formal adjoint of $\overline{\partial}_b$ with respect to $(\cdot|\cdot)_k$. Let

$$\Box_b:=\overline{\partial}^*_b\,\overline{\partial}_b:\mathcal{C}^\infty(X,L^k)\to\mathcal{C}^\infty(X,L^k).$$

Let

$$\triangle_b:=\Box_b-T^2:\mathrm{Dom}\,\triangle_b\subset L^2(X,L^k)\to L^2(X,L^k),$$

where $\operatorname{Dom}\triangle_b := \{u \in L^2(X, L^k); \triangle_b u \in L^2(X, L^k)\}$. Note that for every $\alpha \in \operatorname{Spec}(-iT)$, $\mathcal{H}^0_{b,\alpha}(X, L^k)$ is in the eigenspace of $\triangle_b$. From this observation and note that $\triangle_b$ is elliptic, we deduce that

$$\dim \mathcal{H}^0_{b,\alpha}(X, L^k) < \infty, \tag{42}$$

for every $\alpha \in \operatorname{Spec}(-iT)$.

For any interval $J \subset \mathbb{R}$, put

$$\mathcal{H}^0_{b,J}(X, L^k) := \bigoplus_{\alpha\in\operatorname{Spec}(-iT),\alpha\in J} \mathcal{H}^0_{b,\alpha}(X, L^k). \tag{43}$$

For every $\alpha \in \operatorname{Spec}(-iT)$, let $L^2_\alpha(X, L^k) \subset L^2(X, L^k)$ be the eigenspace of $-iT$ with eigenvalue α. It is easy to see that $L^2_\alpha(X, L^k)$ is the completion of $C^\infty_\alpha(X, L^k)$ with respect to $(\cdot\,|\,\cdot)_k$. Let

$$Q_{\alpha,k} : L^2(X, L^k) \to L^2_\alpha(X, L^k) \tag{44}$$

be the orthogonal projection with respect to $(\cdot\,|\,\cdot)_k$. We have the Fourier decomposition

$$L^2(X, L^k) = \bigoplus_{\alpha\in\operatorname{Spec}(-iT)} L^2_\alpha(X, L^k).$$

We first construct a bounded operator on $L^2(X, L^k)$ by putting a weight on the components of the Fourier decomposition with the help of a cut-off function. Let

$$\tau \in \mathcal{C}^\infty_c(I), \quad \tau \geq 0, \tag{45}$$

where I is the interval as in (38). Let $F_{k,\tau} : L^2(X, L^k) \to L^2(X, L^k)$ be the bounded operator given by

$$\begin{aligned} F_{k,\tau} : L^2(X, L^k) &\to L^2(X, L^k), \\ u &\mapsto \sum_{\alpha\in\operatorname{Spec}(-iT)} \tau\left(\frac{\alpha}{k}\right) Q_{\alpha,k}(u). \end{aligned} \tag{46}$$

We consider the partial Szegő projector

$$\Pi_{k,I} : L^2(X, L^k) \to \mathcal{H}^0_{b,I}(X, L^k) \tag{47}$$

which is the orthogonal projection on the space of $\mathbb{R}$-equivariant CR functions in $\mathcal{H}^0_{b,I}(X, L^k)$. Finally, we consider the weighted Fourier-Szegő operator

$$P_{k,\tau^2} := F_{k,\tau} \circ \Pi_{k,I} \circ F_{k,\tau} : L^2(X, L^k) \to \mathcal{H}^0_{b,I}(X, L^k). \tag{48}$$

The Schwartz kernel of P_{k,τ^2} with respect to dV_X is the smooth section $(x, y) \mapsto P_{k,\tau^2}(x, y) \in L^k_x \otimes (L^k_y)^*$ satisfying

$$(P_{k,\tau^2}u)(x) = \int_X P_{k,\tau^2}(x, y)u(y)\, dV_X(y)\,, \;\; u \in L^2(X, L^k). \tag{49}$$

We pause and introduce some notations. Let $A_k : \mathcal{C}^\infty_c(X, L^k) \to \mathcal{C}^\infty(X, L^k)$ be a continuous operator with distribution kernel $A_k(x, y) \in \mathcal{D}'(X \times X, L^k \boxtimes (L^k)^*)$. Let s_1, s_2 be local trivializing CR rigid sections of L defined on $\mathbb{R}$-invariant open sets $D_1 \subset X$, $D_2 \subset X$, respectively, $|s_j(x)|^2_{h^L} = e^{-2\Phi_j(x)}$, $\Phi_j \in \mathcal{C}^\infty(D_j)$, $j = 1, 2$. The localization of A_k with respect to s_1 and s_2 are given by

$$\begin{aligned}&A_{k,s_1,s_2} : \mathcal{C}^\infty_c(D_1) \to \mathcal{C}^\infty(D_2),\\ &A_{k,s_1,s_2}(u) := e^{-k\Phi_2} s_2^{-k} A_k(s_1^k e^{k\Phi_1} u), \forall u \in \mathcal{C}^\infty_c(D_1).\end{aligned}$$

When $s = s_1 = s_2$, $D_1 = D_2$, we write $A_{k,s} := A_{k,s_1,s_2}$. We write $A_k = O(k^{-\infty})$ on X or $A_k(x, y) = O(k^{-\infty})$ on $X \times X$ if for any local trivializing CR rigid sections s_1, s_2 of L defined on $\mathbb{R}$-invariant open sets $D_1 \subset X$, $D_2 \subset X$, respectively, we have

$$A_{k,s_1,s_2} = O(k^{-\infty}) \;\text{ on } D_1 \times D_2.$$

Let s be a local trivializing CR rigid section of L defined on a $\mathbb{R}$-invariant open set D, $|s(x)|^2_{h^L} = e^{-2\Phi(x)}$, $x \in D$. As before, let $P_{k,\tau^2,s} : \mathcal{C}^\infty_c(D) \to \mathcal{C}^\infty(D)$ be the localization of P_{k,τ^2} with respect to s. Let $P_{k,\tau^2,s}(x, y) \in \mathcal{C}^\infty(D \times D)$ be the Schwartz kernel of $P_{k,\tau^2,s}$ with respect to dV_X. We have the following [14, Theorem 1.1]

Theorem 2.2. *With the notations and assumptions above, consider a point $p \in X$ and a coordinates neighborhood $(D, x = (x_1, \ldots, x_{2n+1}))$ centered at $p = 0$ with $T = -\frac{\partial}{\partial x_{2n+1}}$. Let s be a local rigid CR trivializing section of L on D and set $|s|^2_h = e^{-2\Phi}$. With the notations used above,*

$$P_{k,\tau^2,s}(x, y) = \int_{\mathbb{R}} e^{ik\varphi(x,y,t)} a(x, y, t, k)dt + O(k^{-\infty}) \;\; on\ D \times D, \tag{50}$$

where $\varphi \in \mathcal{C}^\infty(D \times D \times I)$ is a phase function such that for some constant $c > 0$ we have

$$\begin{aligned}&d_x\varphi(x,y,t)|_{x=y}=-2\mathrm{Im}\,\overline{\partial}_b\Phi(x)+t\omega_0(x),\\&d_y\varphi(x,y,t)|_{x=y}=2\mathrm{Im}\,\overline{\partial}_b\Phi(x)-t\omega_0(x),\\&\mathrm{Im}\,\varphi(x,y,t)\geq c|z-w|^2,\\&(x,y,t)\in D\times D\times I, x=(z,x_{2n+1}), y=(w,y_{2n+1}),\\&\mathrm{Im}\,\varphi(x,y,t)+\left|\frac{\partial\varphi}{\partial t}(x,y,t)\right|^2\geq c\,|x-y|^2,\ (x,y,t)\in D\times D\times I,\\&\varphi(x,y,t)=0 \text{ and } \frac{\partial\varphi}{\partial t}(x,y,t)=0 \text{ if and only if } x=y,\end{aligned}\tag{51}$$

and $a(x,y,t,k)\in S^{n+1}_{\mathrm{loc}}(1;D\times D\times I)\cap\mathcal{C}^\infty_c(D\times D\times I)$ *is a symbol with expansion*

$$a(x,y,t,k)\sim\sum_{j=0}^{\infty}a_j(x,y,t)k^{n+1-j}\ \text{ in } S^{n+1}_{\mathrm{loc}}(1;D\times D\times I),\tag{52}$$

and for $x\in D_0$ *and* $t\in I$ *we have*

$$a_0(x,x,t)=(2\pi)^{-n-1}\left|\det\!\left(R^L_x-2t\mathcal{L}_x\right)\right|\tau^2(t),\tag{53}$$

where $\omega_0\in\mathcal{C}^\infty(X,T^*X)$ *is the global real* 1*-form of unit length orthogonal to* $T^{*1,0}X\oplus T^{*0,1}X$*, see* (32)*,* $\left|\det\left(R^L_x-2t\mathcal{L}_x\right)\right|=|\lambda_1(x,t)\cdots\lambda_n(x,t)|$*, where* $\lambda_j(x,t)$*,* $j=1,\ldots,n$*, are the eigenvalues of the Hermitian quadratic form* $R^L_x-2t\mathcal{L}_x$ *with respect to* $\langle\cdot|\cdot\rangle$*,* R^L_x *and* $\mathcal{L}_x$ *denote the curvature two form of* L *and the Levi form of* X *respectively (see Definition 2.8 and* (34)*).*

Now, we shall introduce local coordinates and give a local expression for the phase function. In Section 4.4 of [16] the author determined the tangential Hessian of the phase function. We recall [21, Theorem 3.13], which is an improvement of the result in [16] for the case of CR manifolds with transversal CR $\mathbb{R}$-action.

Theorem 2.3. *Fix* $(p,p,t_0)\in D\times D\times I$*, and let* $\overline{W}_{1,t_0},\ldots,\overline{W}_{n,t_0}$ *be an orthonormal rigid frame of* $T^{1,0}_xX$ *varying smoothly with* x *in a neighborhood of* p*, for which the Hermitian quadratic form* $R^L_x-2t_0\mathcal{L}_x$ *is diagonal at* p*, that is,*

$$(R^L_p-2\,t_0\mathcal{L}_p)(\overline{W}_{j,t_0}(p),\,W_{k,t_0}(p))=\lambda_j(t_0)\,\delta_{j,k}\,,$$

for $j,\ k=1,\ldots,n$*. Let* s *be a local rigid CR frame of* L *defined on* D*,* $|s|^2_{h^L}=e^{-2\Phi}$*,* $\Phi\in\mathcal{C}^\infty(D)$*. Let* $x=(x_1,\ldots,x_{2n+1})$ *be local coordinates defined in some small neighborhood of* p *with* $x(p)=0$ *such that*

$$
\begin{aligned}
&\omega_0(p) = \mathrm{d}x_{2n+1}\,, \qquad T = -\frac{\partial}{\partial x_{2n+1}}, \\
&\overline{W}_{j,\,t_0}(p) = \frac{\partial}{\partial z_j} + i\sum_{t=1}^{n} \tau_{j,\,t}\overline{z}_t \frac{\partial}{\partial x_{2n+1}} + O(|z|^2) \quad \text{for } j = 1, \ldots, n\,, \\
&\left\langle \frac{\partial}{\partial x_j}(p) \middle| \frac{\partial}{\partial x_\ell}(p) \right\rangle = 2\,\delta_{j,\ell} \quad \text{for } j, \ell = 1, \ldots, 2n\,, \\
&z_j = x_j + i x_{d+j}, \quad j = 1, \ldots, d, \\
&z_j = x_{2j-1} + i x_{2j}, \quad j = d+1, \ldots, n,
\end{aligned}
\tag{54}
$$

and

$$
\Phi(x) = \frac{1}{2}\sum_{l,\,t=1}^{n} \mu_{t,l} z_t \overline{z}_l + \sum_{l,\,t=1}^{n} (a_{l,t} z_l z_t + \overline{a}_{l,t}\overline{z}_l \overline{z}_t) + O(|z|^3)\,, \tag{55}
$$

where $\tau_{t,l}$, $\mu_{t,l}$, $a_{t,l} \in \mathbb{C}$, $\mu_{t,l} = \overline{\mu}_{l,t}$, $l, t = 1, \ldots, n$. *Then, there exists a neighborhood of* $(p,\, p)$ *such that*

$$
\begin{aligned}
\varphi(x, y, t_0) =& t_0(x_{2n+1} - y_{2n+1}) - \frac{i}{2}\sum_{j,l=1}^{n} (a_{l,j} + a_{j,l})(z_j z_l - w_j w_l) \\
&+ \frac{i}{2}\sum_{j,l=1}^{n} (\overline{a_{l,j}} + \overline{a_{j,l}})(\overline{z}_j \overline{z}_l - \overline{w}_j \overline{w}_l) + \frac{i\,t_0}{2}\sum_{j,l=1}^{n} (\overline{\tau_{l,j}} - \tau_{j,l})(z_j \overline{z}_l - w_j \overline{w}_l) \\
&- \frac{i}{2}\sum_{j=1}^{n} \lambda_j(t_0)(z_j \overline{w}_j - \overline{z}_j w_j) \\
&+ \frac{i}{2}\sum_{j=1}^{n} \lambda_j(t_0)|z_j - w_j|^2 + O(|(z, w)|^3).
\end{aligned}
\tag{56}
$$

2.5. Preliminaries on CR manifolds and group actions

From now on, we assume that X admits a d-dimensional compact Lie group action G with Lie algebra $\mathfrak{g}$. As before, let us denote by $\underline{\mathfrak{g}}$ the space of vector field on X induced by the Lie algebra $\mathfrak{g}$ of G. Furthermore for every $\xi \in \mathfrak{g}$, we write ξ_X to denote the infinitesimal vector field on X induced by ξ. Recall that we work with Assumption 1.1.

As Definition 2.6, we can define G-equivariant rigid CR line bundle.

Definition 2.10. A rigid CR line bundle $L \to X$ is called G-equivariant rigid CR if the action G can be lifted to L, for every $\xi_X \in \underline{\mathfrak{g}}$, ξ_X can be lifted to L and for every point $p \in X$ there exists an open neighborhood U around p and a CR frame s of $L|_U$ with $T(s) = 0$ and $\xi_X s = 0$, for every $\xi_X \in \underline{\mathfrak{g}}$.

From now on, we assume that L is a G-equivariant rigid CR line bundle.

The proof of the following is similar to the proof of [14, Theorem 3.12].

Theorem 2.4. *With the assumptions and notations above, we can find local CR rigid G-invariant trivializing section s_j defined on an open subset D_j of X, $j = 1, \dots, N$, $N \in \mathbb{N}$, such that $X = \cup_{j=1}^N D_j$ and $D_j = \cup_{(g,\eta)\in G\times\mathbb{R}}\{g \cdot \eta \cdot x;\, x \in D_j\}$, for every $j = 1, \dots, N$.*

Proof. From the discussion after Assumption 2.2, we see that the $\mathbb{R}$-action on X comes from a torus action T^r on X and L is T^r-equivariant. Let $\hat{G} := G \times T^r$. Then, $\hat{G}$ is a compact Lie group action acting on X and L is $\hat{G}$-equivariant. Fix $x_0 \in X$. Let

$$\begin{aligned} A_{x_0} := \{&U \subset \hat{G};\, U \text{ is an open set of } \hat{G} \text{ and we can find a local CR trivializing}\\ &\text{section } s \text{ of } L,\, Ts = 0,\, \xi_X s = 0, \text{ for all } \xi \in \mathfrak{g}, \text{ defined on } \bigcup\nolimits_{g\in U} gD,\\ &D \text{ is an open set of } x_0\}, \end{aligned}$$

where $gD := \{g \cdot x;\, x \in D\}$. Let $U_1, U_2 \in A_{x_0}$. We claim that

$$U_1 \bigcup U_2 \in A_{x_0}. \tag{57}$$

By definition, we can find local CR trivializing sections s_1, s_2 of L, $Ts_1 = 0$, $Ts_2 = 0$, $\xi_X s_1 = 0$, $\xi_X s_2 = 0$, for all $\xi \in \mathfrak{g}$, defined on $\Omega_1 = \bigcup_{g\in U_1} gD_1$, $\Omega_2 = \bigcup_{g\in U_2} gD_2$, respectively, where D_1, D_2 are open sets of x_0. If $\Omega_1 \bigcap \Omega_2 = \emptyset$. Let s be the function on $\Omega = \bigcup_{g\in U_1\cup U_2} g(D_1 \cap D_2)$ given by $s = s_1$ on $\cup_{g\in U_1} g(D_1 \cap D_2)$, $s = s_2$ on $\cup_{g\in U_2} g(D_1 \cap D_2)$. Then s is a local CR trivializing section of L, $Ts = 0$, $\xi_X s = 0$, for all $\xi \in \mathfrak{g}$, defined on Ω. Hence, $U_1 \bigcup U_2 \in A_{x_0}$.

If $\Omega_1 \bigcap \Omega_2 \neq \emptyset$. We have $s_1 = f s_2$ on $\Omega_1 \cap \Omega_2$, f is a CR function on $\Omega_1 \bigcap \Omega_2$, $Tf = 0$, $\xi_X f = 0$, for all $\xi \in \mathfrak{g}$, and $f(x) \neq 0$, for all $x \in \Omega_1 \bigcap \Omega_2$. Thus, f is a CR function on $\bigcup_{g\in U_1\cap U_2} g(D_1 \cap D_2)$. Fix $h \in U_1 \bigcap U_2$. Let $\widetilde{f}$ be the CR function on $\bigcup_{g\in U_1\cup U_2} g(D_1 \cap D_2)$ given by $\widetilde{f}(g \cdot x) := f(h \cdot x)$, $x \in D_1 \cap D_2$. We check now that $\widetilde{f}$ is well-defined. If $g \cdot x = \hat{g} \cdot y \in \bigcup_{g\in U_1\cup U_2} g(D_1 \cap D_2)$, $x, y \in D_1 \cap D_2$, $g, \hat{g} \in U_1 \cup U_2$. Then, $y = (\hat{g})^{-1} g \cdot x \in D_1 \cap D_2$. Since $Tf = 0$, $\xi_X f = 0$, for all $\xi \in \mathfrak{g}$, $f(h \cdot x) = f(h(\hat{g})^{-1} g \cdot x) = f(h \cdot y)$. Thus, $\widetilde{f}$ is well-defined. Let s be the function on $\Omega = \bigcup_{g\in U_1\cup U_2} g(D_1 \cap D_2)$ given by $s = s_1$ on $\bigcup_{g\in U_1} g(D_1 \cap D_2)$ and $s = \widetilde{f} s_2$ on $\bigcup_{g\in U_2} g(D_1 \cap D_2)$. Then s is a well-defined local CR trivializing section of L, $Ts = 0$, $\xi_X s = 0$, for all $\xi \in \mathfrak{g}$, defined on Ω. Hence, $U_1 \bigcup U_2 \in A_{x_0}$. The claim (57) follows.

It is clear that we can find $U_j \in A_{x_0}$, $j = 1, 2, \dots, N$, such that $\hat{G} = \bigcup_{j=1}^N U_j$. From (57), we see that $\bigcup_{j=1}^N U_j \in A_{x_0}$. Thus, $\hat{G} \in A_{x_0}$. Hence, we can find a local CR trivializing section s of L, $Ts = 0$, $\xi_X s = 0$, for all $\xi \in \mathfrak{g}$, defined on $\bigcup_{g\in\hat{G}} gD$, D is an open set of x_0. The theorem follows. □

From now on, for any local CR rigid trivializing section s defined on an open set D of X, by Theorem 2.4, without loss of generality, we will always assume that D and s are $G \times \mathbb{R}$-invariant.

From now on, we assume that the Hermitian metrices $\langle \cdot | \cdot \rangle$ and h^L are $G\times\mathbb{R}$-invariant. We now introduce moment map associated to R^L and ω_0.

Definition 2.11. Let s be a CR rigid G-invariant trivializing section of L defined on an open subset $D \subset X$, $|s|^2_{h^L} = e^{-2\Phi}$, $\Phi \in \mathcal{C}^\infty(D)$. Let

$$\gamma_\Phi := 4\mathrm{Im}\,\overline{\partial}_b\Phi = (-2i)(\overline{\partial}_b\Phi - \partial_b\Phi).$$

The moment map on D associated to the local weight Φ is the map $\gamma_\Phi : D \to \mathfrak{g}^*$ such that for all $x \in D$ and $\xi \in \mathfrak{g}$, we have

$$\langle\, \gamma_\Phi(x)\,,\, \xi\,\rangle = \gamma_\Phi(\xi_X(x)), \tag{58}$$

where ξ_X is the vector field on X induced by ξ.

Lemma 2.2. *Definition 2.11 is global defined. More precisely, the moment map γ_Φ given by (58) is independent of the choices of Φ.*

Proof. Let s_1, s_2 be CR rigid G-invariant trivializing sections of L defined on an open subset $D \subset X$, $|s_j|^2_{h^L} = e^{2\Phi_j}$, $\Phi_j \in \mathcal{C}^\infty(D)$, $j = 1, 2$. We have $s_1 = fs_2$ on D, f is a rigid CR G-invariant function on D, $f(x) \neq 0$, for every $x \in D$. Thus,

$$|s_1|^2_{h^L} = e^{-2\Phi_1} = |f|^2 e^{-2\Phi_2} = e^{-2\Phi_2 + \log|f|^2}$$

on D. Hence,

$$\Phi_1 = \Phi_2 - \frac{1}{2}\log|f|^2.$$

Thus,

$$\overline{\partial}_b\Phi_1 = \overline{\partial}_b\Phi_2 - \frac{1}{2}\frac{\overline{\partial}_b\overline{f}}{\overline{f}}$$

on D and hence

$$\begin{aligned}\gamma_{\Phi_1} &= (-2i)(\overline{\partial}_b\Phi_1 - \partial_b\Phi_1)\\ &= (-2i)(\overline{\partial}_b\Phi_2 - \partial_b\Phi_2) + i(\frac{\overline{\partial}_b\overline{f}}{\overline{f}} - \frac{\partial_b f}{f})\\ &= \gamma_{\Phi_2} + i(\frac{\overline{\partial}_b\overline{f}}{\overline{f}} - \frac{\partial_b f}{f}).\end{aligned} \tag{59}$$

Since f is G-invariant, we have

$$\langle \frac{\overline{\partial}_b \overline{f}}{\overline{f}} - \frac{\partial_b f}{f}, \xi_X \rangle = 0 \quad \text{on } X, \tag{60}$$

for every $\xi \in \mathfrak{g}$. The lemma follows. □

From now on, we write

$$\gamma : X \to \mathfrak{g}^*$$

to denote the moment map given by (58).

Definition 2.12. The moment map associated to the form ω_0 is the map $\mu : X \to \mathfrak{g}^*$ such that, for all $x \in X$ and $\xi \in \mathfrak{g}$, we have

$$\langle \mu(x), \xi \rangle = \omega_0(\xi_X(x)), \tag{61}$$

$\xi \in \mathfrak{g}$, ξ_X: the vector field on X induced by ξ.

For every $t \in I$, let $\hat{\mu}_t : X \to \mathfrak{g}^*$ be as in (1). Recall that in this work, we work with Assumption 1.3.

2.6. *CR reduction with respect to curvature data*

Let $Y := \hat{\mu}_t^{-1}(0)$ and denote by $HY := \operatorname{Ker} \omega_0 \cap TY$. Let $X_G := Y/G$ and let $\pi : Y \to X_G$ be the natural projection. Let $HX_G := \pi_* HY$, π_* is the push-forward map of π. We are now going to prove that X_G is a CR manifold. Fix $t_0 \in I$. Recall that zero is a regular value of $\hat{\mu}_{t_0}$. Recall that I is the open interval as in (38). For every $x \in Y$, let

$$b_x(\cdot, \cdot) := ((-2i)R_x^L - 2t_0 d\omega_0(x))(\cdot, J\cdot)$$

be the bilinear form on $H_x X$. Let

$$\underline{\mathfrak{g}}^{\perp_b} := \{v \in HX;\, b(\xi_X, v) = 0, \quad \text{for all } \xi_X \in \underline{\mathfrak{g}}\}.$$

Since $R_x^L - 2t_0 \mathcal{L}_x$ is positive,

$$\underline{\mathfrak{g}} \cap \underline{\mathfrak{g}}^{\perp_b} = \{0\}. \tag{62}$$

Hence, b is a non-degenerate bilinear form and thus

$$\underline{\mathfrak{g}} \oplus \underline{\mathfrak{g}}^{\perp_b} = H_x X, \tag{63}$$

for every $x \in Y$. Let $H^H Y := \underline{\mathfrak{g}}^{\perp_b}|_Y \cap HY$. From (63), we have

$$HY = \underline{g}|_Y \oplus H^H Y. \tag{64}$$

Lemma 2.3. *We have*

$$\underline{\mathfrak{g}}^{\perp_b}|_Y = JHY, \tag{65}$$

and

$$HX|_Y = J\underline{\mathfrak{g}}|_Y \oplus HY = J\underline{\mathfrak{g}}|_Y \oplus \underline{\mathfrak{g}}|_Y \oplus H^H Y. \tag{66}$$

Proof. Fix $p \in Y$ and let s be a local CR rigid G-invariant trivializing section of L defined on a G-invariant open subset D of p in X, $|s|^2_{h^L} = e^{-2\Phi}$. For $V \in H_pX$ and $\xi \in \mathfrak{g}$, we have

$$b_p(\xi_X, JV) = d\gamma_\Phi(\xi_X, JV). \tag{67}$$

From (67), we see that $V \in \underline{\mathfrak{g}}_p^{\perp_b}$ if and only if $d\gamma_\Phi(\xi_X, JV) = 0$, for all $\xi_X \in \underline{\mathfrak{g}}_p$. Since 0 is a regular value of $\hat{\mu}_{t_0}$, $d\gamma_\Phi(\xi_X, JV) = 0$, for all $\xi_X \in \underline{\mathfrak{g}}_p$ if and only if $JV \in H_pY$. We get (65).

Now, for $V \in H_pY$ and $\xi \in \mathfrak{g}$, we have

$$b_p(J\xi_X, V) = d\gamma_\Phi(V, \xi_X) = 0. \tag{68}$$

From (68),

$$\dim H_pX = \dim H_pY + \dim J\underline{\mathfrak{g}}_p$$

and the fact that b_p is non-degenerate, we obtain (66). □

From (65), we have

$$H^H Y = JHY \cap HY. \tag{69}$$

From (64), we can identify HX_G with $H^H Y$ and from (69), we can define complex structure map J_G on HX_G: For $V \in HX_G$, we denote by V^H its lift in $H^H Y$, and we define J_G on X_G by

$$(J_G V)^H = J(V^H). \tag{70}$$

Hence, we have $J_G : HX_G \to HX_G$ such that $J_G^2 = -\,\mathrm{id}$, where id denotes the identity map $\mathrm{id} : HX_G \to HX_G$. By complex linear extension of J_G to $\mathbb{C}TX_G$, we can define the i-eigenspace of J_G is given by $T^{1,0}X_G = \left\{V \in \mathbb{C}HX_G\,;\, J_G V = \sqrt{-1}V\right\}$.

Proposition 2.1. *The subbundle $T^{1,0}X_G$ is a CR structure of X_G.*

Proof. Let $u, v \in \mathcal{C}^\infty(X_G, T^{1,0}X_G)$, then we can find $U, V \in \mathcal{C}^\infty(X_G, HX_G)$ such that

$$u = U - \sqrt{-1}J_G U, \qquad v = V - \sqrt{-1}J_G V.$$

By (70), we have

$$u^H = U^H - \sqrt{-1}JU^H, \quad v = V^H - \sqrt{-1}JV^H \in T^{1,0}X \cap \mathbb{C}HY.$$

Since $T^{1,0}X$ is a CR structure and it is clearly that $[u^H, v^H] \in \mathbb{C}HY$, we have $[u^H, v^H] \in T^{1,0}X \cap \mathbb{C}HY$. Hence, there is a $W \in \mathcal{C}^\infty(X, HX)$ such that

$$[u^H, v^H] = W - \sqrt{-1}JW.$$

In particular, $W, JW \in HY$. Thus, $W \in HY \cap JHY = H^H Y$. Let $X^H \in H^H Y$ be a lift of $X \in TX_G$ such that $X^H = W$. Then we have

$$[u, v] = \pi_*[u^H, v^H] = \pi_*(X^H - \sqrt{-1}JX^H) = X - \sqrt{-1}J_G X \in T^{1,0}X_G,$$

i.e. we have $[\mathcal{C}^\infty(X_G, T^{1,0}X_G), \mathcal{C}^\infty(X_G, T^{1,0}X_G)] \subset \mathcal{C}^\infty(X_G, T^{1,0}X_G)$. Therefore, $T^{1,0}X_G$ is a CR structure of X_G. □

3. Proofs of Theorem 1.1 and Theorem 1.2

This section is organized as follows. First, we shall introduce local coordinates compatible with the actions of G on X and the $\mathbb{R}$-action in Section 3.1. Then, in Section 3.2, we shall study the microlocal structure of the G-invariant Szegő kernel leading to the proof of Theorem 1.1 and Theorem 1.2.

3.1. Local coordinates

In this chapter we specialize the local coordinates introduced in Section 2.3 taking into account the action of the group G and $\mathbb{R}$. The main results are two: Proposition 3.1, which introduces the local coordinates, and Proposition 3.2, which expresses the phase function φ in terms of the coordinates introduced in Proposition 3.1.

Let $e_0 \in G$ be the identity element and let $v = (v_1, \ldots, v_d)$ be the local coordinates of G defined in a neighborhood V of e with $v(e_0) = (0, \ldots, 0)$. From now on, we will identify the element $g \in V$ with $v(g)$.

Proposition 3.1. *Fix $t_0 \in I$. Let $p \in Y$ and let s be a local CR rigid G-invariant trivializing section of L defined on a G-invariant open set D of p in X, $|s|^2_{h^L} = e^{-2\Phi}$. Then there exist local coordinates $x = (x_1, \ldots, x_{2n+1})$ on X defined in a neighborhood $U = U_1 \times U_2 \subset D$ of p with $p \equiv 0$, $U_1 \subset \mathbb{R}^d$ (resp. $U_2 \subset \mathbb{R}^{2n+1-d}$) is an open neighborhood* *of* $0 \in \mathbb{R}^d$ *(resp.* $0 \in \mathbb{R}^{2n+1-d}$*) and a smooth function* $\Gamma = (\Gamma_1, \ldots, \Gamma_d) \in \mathcal{C}^\infty(U_2, U_1)$ *with* $\Gamma(0) = 0 \in \mathbb{R}^d$*,* Γ *is independent of* x_{2n+1}*, such that*

$$\begin{aligned} &T = -\frac{\partial}{\partial x_{2n+1}} \ \text{on } U, \\ &(v_1, \ldots, v_d) \circ (\Gamma(x_{d+1}, \ldots, x_{2n}), x_{d+1}, \ldots, x_{2n+1}) \\ &\quad = (v_1 + \Gamma_1(x_{d+1}, \ldots, x_{2n}), \ldots, v_d + \Gamma_d(x_{d+1}, \ldots, x_{2n}), x_{d+1}, \ldots, x_{2n+1}) \end{aligned} \tag{71}$$

for each $(v_1, \ldots, v_d) \in V$ *and for each* $(x_{d+1}, \ldots, x_{2n+1}) \in U_2$*,*

$$\begin{aligned} &\underline{\mathfrak{g}} = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right\}, \\ &Y = \{x_{d+1} = \cdots = x_{2d} = 0\}, \\ &J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial x_{d+j}} \quad \text{on } Y \text{ for } j = 1, \ldots, d, \end{aligned} \tag{72}$$

$$\begin{aligned} &T^{1,0}X = \operatorname{span}\{Z_{1,t_0}, \ldots, Z_{n,t_0}\}, \\ &Z_{j,t_0}(p) = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_{d+j}}\right)(p), \quad j = 1, \ldots, d, \\ &Z_{j,t_0}(p) = \frac{1}{2}\left(\frac{\partial}{\partial x_{2j-1}} - i\frac{\partial}{\partial x_{2j}}\right)(p), \quad j = d+1, \ldots, d, \\ &\left(R^L_p - 2t_0\mathcal{L}_p\right)\left(Z_{j,t_0}(p), \overline{Z}_{\ell,t_0}(p)\right) = \lambda_j(t_0)\delta_{j,\ell}, \qquad j, \ell = 1, 2, \ldots, n, \\ &\langle Z_{j,t_0}(p) | Z_{\ell,t_0}(p)\rangle = \delta_{j,\ell}, \qquad j, \ell = 1, 2, \ldots, n, \end{aligned} \tag{73}$$

where $\{Z_{1,t_0}, \ldots, Z_{n,t_0}\}$ *is an orthonormal basis of* $T^{1,0}X$ *on* U *depending smoothly in* $x \in U$*.*

Moreover, let $\widetilde{x}_j$*,* $j = 1, \ldots, 2n+1$*, be the coordinates as in Proposition 2.3, we have*

$$\begin{aligned} &\widetilde{x}_j = x_j + O(|\mathring{x}^2|), \quad j = 1, \ldots, 2n, \\ &\widetilde{x}_{2n+1} = x_{2n+1} + \sum_{j,\ell=1}^{d} \frac{i}{2}(\tau_{j,\ell} - \overline{\tau_{j,\ell}})x_j x_\ell + \sum_{j,\ell=1}^{d} -(\tau_{j,\ell} + \overline{\tau_{j,\ell}})x_{d+j}x_\ell \\ &\quad + \sum_{j=d+1}^{n}\sum_{\ell=1}^{d} i(\tau_{j,\ell} - \overline{\tau_{j,\ell}})x_{2j-1}x_\ell + \sum_{j=d+1}^{n}\sum_{\ell=1}^{d} -(\tau_{j,\ell} + \overline{\tau_{j,\ell}})x_{2j}x_\ell + O(|\mathring{x}^3|), \end{aligned} \tag{74}$$

where $\mathring{x} = (x_1, \ldots, x_{2n})$*.*

Proof. From the standard proof of Frobenius Theorem, it is not difficult to see that there exist local coordinates $v = (v_1, \ldots, v_d)$ of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$ and local coordinates $x = (x_1, \ldots, x_{2n+1})$ of X defined in a neighborhood $U \subset D$ of p with $x(p) = 0$ such that

$$\begin{aligned}&(v_1,\ldots,v_d)\circ(0,\ldots,0,x_{d+1},\ldots,x_{2n+1})\\&=(v_1,\ldots,v_d,x_{d+1},\ldots,x_{2n+1}),\quad\forall(v_1,\ldots,v_d)\in V,\quad\forall(0,\ldots,0,x_{d+1},\ldots,x_{2n+1})\in U,\end{aligned}\tag{75}$$

and

$$\underline{\mathfrak{g}}=\operatorname{span}\left\{\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_d}\right\},\tag{76}$$

$$T=-\frac{\partial}{\partial x_{2n+1}}\quad\text{on } D.\tag{77}$$

Consider the linear map

$$\begin{aligned}&R:\underline{\mathfrak{g}}_p\to\underline{\mathfrak{g}}_p,\\&\qquad u\to Ru,\quad\langle\,Ru\,|\,v\,\rangle=\langle\,(-2i)R^L_p-2t_0d\omega_0(p))\,,\,Ju\wedge v\,\rangle.\end{aligned}$$

Since R is self-adjoint, by using linear transformation in $(x_1,\ldots,x_d)$, we can take $(x_1,\ldots,x_d)$ such that, for $j,\ell=1,2,\ldots,d$,

$$\langle\,R\frac{\partial}{\partial x_j}(p)\,|\,\frac{\partial}{\partial x_\ell}(p)\,\rangle=4\lambda_j(t_0)\delta_{j,\ell},\quad\langle\,\frac{\partial}{\partial x_j}(p)\,|\,\frac{\partial}{\partial x_\ell}(p)\,\rangle=2\delta_{j,\ell}.\tag{78}$$

By taking linear transformation in $(v_1,\ldots,v_d)$, (75) still hold.

Let $\hat{\mu}_{t_0}(\frac{\partial}{\partial x_j})=a_j(x)\in\mathcal{C}^\infty(U)$, $j=1,2,\ldots,d$. Since $a_j(x)$ is $G\times\mathbb{R}$-invariant, we have $\frac{\partial a_j(x)}{\partial x_s}=0$, $\frac{\partial a_j}{\partial x_{2n+1}}=0$, $j,s=1,2,\ldots,d$. By the definition of the moment map, we have

$$Y\bigcap U=\{x\in U;\,a_1(x)=\cdots=a_d(x)=0\}\,.$$

Since 0 is a regular value of the moment map $\hat{\mu}_{t_0}$, the matrix $\left(\frac{\partial a_j}{\partial x_s}(p)\right)_{1\leq j\leq d,d+1\leq s\leq 2n+1}$ is of rank d. We may assume that the matrix $\left(\frac{\partial a_j}{\partial x_s}(p)\right)_{1\leq j\leq d,d+1\leq s\leq 2d}$ is non-singular. Thus, $(x_1,\ldots,x_d,a_1,\ldots,a_d,x_{2d+1},\ldots,x_{2n+1})$ are also local coordinates of X. Hence, we can take $v=(v_1,\ldots,v_d)$ and $x=(x_1,\ldots,x_{2n+1})$ such that (75), (76), (77) and (78) hold and

$$Y\bigcap U=\{x=(x_1,\ldots,x_{2n+1})\in U;\,x_{d+1}=\cdots=x_{2d}=0\}\,.\tag{79}$$

On $Y\bigcap U$, let

$$J(\frac{\partial}{\partial x_j})=b_{j,1}(x)\frac{\partial}{\partial x_1}+\cdots+b_{j,2n+1}(x)\frac{\partial}{\partial x_{2n+1}},\quad j=1,2,\ldots,d.$$

Since we only work on Y, $b_{j,\ell}(x)$ is independent of $x_{d+1},\ldots,x_{2d}$, for all $j=1,\ldots,d$, $\ell=1,\ldots,2n+1$. Moreover, it is easy to see that $b_{j,\ell}(x)$ is also independent of

$x_1, \ldots, x_d$, x_{2n+1}, for all $j = 1, \ldots, d$, $k = 1, \ldots, 2n+1$. Let $\widetilde{x}'' = (x_{2d+1}, \ldots, x_{2n})$. Hence, $b_{j,\ell}(x) = b_{j,\ell}(\widetilde{x}'')$, $j = 1, \ldots, d$, $\ell = 1, \ldots, 2n+1$. We claim that the matrix $(b_{j,\ell}(\widetilde{x}''))_{1 \leq j \leq d, d+1 \leq k \leq 2d}$ is non-singular near p. If not, it is easy to see that there is a non-zero vector $u \in J\underline{\mathfrak{g}} \bigcap HY$. Let $u = Jv$, $v \in \underline{\mathfrak{g}}$. Then, $v \in \underline{\mathfrak{g}} \bigcap JHY = \underline{\mathfrak{g}} \bigcap \underline{\mathfrak{g}}^{\perp_b}$. Since $\underline{\mathfrak{g}} \cap \underline{\mathfrak{g}}^{\perp_b} = \{0\}$ on Y, we deduce that $v = 0$ and we get a contradiction. The claim follows. From the claim, we can use linear transformation in $(x_{d+1}, \ldots, x_{2d})$ (the linear transform depends smoothly on $\widetilde{x}''$) such that on Y,

$$J(\frac{\partial}{\partial x_j}) = b_{j,1}(\widetilde{x}'')\frac{\partial}{\partial x_1} + \cdots + b_{j,d}(\widetilde{x}'')\frac{\partial}{\partial x_d} + \frac{\partial}{\partial x_{d+j}} + b_{j,2d+1}(\widetilde{x}'')\frac{\partial}{\partial x_{2d+1}} + \cdots \\ + b_{j,2n+1}(\widetilde{x}'')\frac{\partial}{\partial x_{2n+1}},$$

where $j = 1, 2, \ldots, d$. Consider the coordinates change:

$$x = (x_1, \ldots, x_{2n+1}) \to u = (u_1, \ldots, u_{2n+1}),$$

$$(x_1, \ldots, x_{2n+1}) \to (x_1 - \sum_{j=1}^{d} b_{j,1}(\widetilde{x}'')x_{d+j}, \ldots, x_d - \sum_{j=1}^{d} b_{j,d}(\widetilde{x}'')x_{d+j}, \\ x_{d+1}, \ldots, x_{2d}, x_{2d+1} - \sum_{j=1}^{d} b_{j,2d+1}(\widetilde{x}'')x_{d+j}, \ldots, x_{2n+1} - \sum_{j=1}^{d} b_{j,2n+1}(\widetilde{x}'')x_{d+j}).$$

Then,

$$\frac{\partial}{\partial x_j} \to \frac{\partial}{\partial u_j}, \quad j = 1, \ldots, d, 2d+1, \ldots, 2n+1,$$

$$\frac{\partial}{\partial x_{d+j}} \to -b_{j,1}\frac{\partial}{\partial u_1} - \cdots - b_{j,d}\frac{\partial}{\partial u_d} + \frac{\partial}{\partial u_{d+j}} - b_{j,2d+1}\frac{\partial}{\partial u_{2d+1}} - \cdots - b_{j,2n+1}\frac{\partial}{\partial u_{2n+1}}, \\ j = 1, \ldots, d.$$

Hence, on $Y \bigcap U$, $J(\frac{\partial}{\partial x_j}) \to \frac{\partial}{\partial u_{d+j}}$, $j = 1, \ldots, d$. Thus, we can take $v = (v_1, \ldots, v_d)$ and $x = (x_1, \ldots, x_{2n+1})$ such that (71), (76), (78), (79) hold and on $Y \bigcap U$,

$$J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial x_{d+j}}, \quad j = 1, 2, \ldots, d.$$

Let $Z_j = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial x_{d+j}})(p) \in T^{1,0}_p X$, $j = 1, \ldots, d$. From (78), we can check that

$$(R^L_p - 2t_0\mathcal{L}_p)(Z_j, \overline{Z}_k) = \lambda_j(t_0)\delta_{j,k}, \quad \langle\, Z_j \,|\, Z_k \,\rangle = \delta_{j,k}, \quad j, k = 1, \ldots, d.$$

Since $\underline{\mathfrak{g}}_p$ is orthogonal to $H_pY \bigcap JH_pY$ and $H_pY \bigcap JH_pY \subset \underline{\mathfrak{g}}^{\perp_b}_p$, we can find an orthonormal frame $\{Z_1, \ldots, Z_d, V_1, \ldots, V_{n-d}\}$ for $T^{1,0}_p X$ such that $R^L_p - 2t_0\mathcal{L}_p$ is diagonalized with respect to $Z_1,\ldots,Z_d,V_1,\ldots,V_{n-d}$, where $V_1\in\mathbb{C}H_pY\bigcap J\mathbb{C}H_pY,\ldots,V_{n-d}\in\mathbb{C}H_pY\bigcap J\mathbb{C}H_pY$. Write

$$\operatorname{Re}V_j=\sum_{k=1}^{2n}\alpha_{j,k}\frac{\partial}{\partial x_k},\quad \operatorname{Im}V_j=\sum_{k=1}^{2n}\beta_{j,k}\frac{\partial}{\partial x_k},\quad j=1,\ldots,n-d.$$

We claim that $\alpha_{j,k}=\beta_{j,k}=0$, for all $k=d+1,\ldots,2d$, $j=1,\ldots,n-d$. Fix $j=1,\ldots,n-d$. Since $\operatorname{Re}V_j\in\underline{\mathfrak{g}}_p^{\perp_b}$ and span $\left\{\frac{\partial}{\partial x_{d+1}},\ldots,\frac{\partial}{\partial x_{2d}}\right\}\in\underline{\mathfrak{g}}_p^{\perp_b}$, we conclude that

$$\sum_{k=1}^{d}\alpha_{j,k}\frac{\partial}{\partial x_k}+\sum_{k=2d+1}^{2n}\alpha_{j,k}\frac{\partial}{\partial x_k}\in\underline{\mathfrak{g}}_p^{\perp_b}\bigcap H_pY. \tag{80}$$

From (65) and (80), we deduce that

$$\sum_{k=1}^{d}\alpha_{j,k}\frac{\partial}{\partial x_k}+\sum_{k=2d+1}^{2n}\alpha_{j,k}\frac{\partial}{\partial x_k}\in JH_pY\bigcap H_pY=\underline{\mathfrak{g}}_p^{\perp_b}\bigcap H_pY$$

and hence

$$J\Big(\sum_{k=1}^{d}\alpha_{j,k}\frac{\partial}{\partial x_k}+\sum_{k=2d+1}^{2n}\alpha_{j,k}\frac{\partial}{\partial x_k}\Big)\in\underline{\mathfrak{g}}_p^{\perp_b}\bigcap H_pY. \tag{81}$$

From (81) and notice that $J(\operatorname{Re}V_j)\in\underline{\mathfrak{g}}_p^{\perp_b}$, we deduce that

$$J(\sum_{k=d+1}^{2d}\alpha_{j,k}\frac{\partial}{\partial x_k})\in\underline{\mathfrak{g}}_p\bigcap\underline{\mathfrak{g}}_p^{\perp_b}=\{0\}\,.$$

Thus, $\alpha_{j,k}=0$, for all $k=d+1,\ldots,2d$, $j=1,\ldots,n-d$. Similarly, we can repeat the procedure above and deduce that $\beta_{j,k}=0$, for all $k=d+1,\ldots,2d$, $j=1,\ldots,n-d$.

Since span $\{\operatorname{Re}V_j,\operatorname{Im}V_j;\, j=1,\ldots,n-d\}$ is transversal to $\underline{\mathfrak{g}}_p\oplus J\underline{\mathfrak{g}}_p$, we can take linear transformation in $(x_{2d+1},\ldots,x_{2n})$ so that

$$\begin{aligned}\operatorname{Re}V_j&=\alpha_{j,1}\frac{\partial}{\partial x_1}+\cdots+\alpha_{j,d}\frac{\partial}{\partial x_d}+\frac{\partial}{\partial x_{2j-1+2d}},\quad j=1,2,\ldots,n-d,\\ \operatorname{Im}V_j&=\beta_{j,1}\frac{\partial}{\partial x_1}+\cdots+\beta_{j,d}\frac{\partial}{\partial x_d}+\frac{\partial}{\partial x_{2j+2d}},\quad j=1,2,\ldots,n-d.\end{aligned}$$

Consider the coordinates change:

$$x=(x_1,\ldots,x_{2n+1})\to u=(u_1,\ldots,u_{2n+1}),$$

$$(x_1,\ldots,x_{2n+1})\to(x_1-\sum_{j=1}^{d}\alpha_{j,1}x_{2j-1+2d}-\sum_{j=1}^{d}\beta_{j,1}x_{2j+2d},\ldots,x_d$$
$$-\sum_{j=1}^{d}\alpha_{j,d}x_{2j-1+2d}-\sum_{j=1}^{d}\beta_{j,d}x_{2j+2d},x_{d+1},\ldots,x_{2n+1})$$

Then,

$$\frac{\partial}{\partial x_j}\to\frac{\partial}{\partial u_j},\quad j=1,\ldots,2d,$$
$$\frac{\partial}{\partial x_{2j-1+2d}}\to-\alpha_{j,1}\frac{\partial}{\partial u_1}-\cdots-\alpha_{j,d}\frac{\partial}{\partial u_d}+\frac{\partial}{\partial u_{2j-1+2d}},\quad j=1,\ldots,n-d,$$
$$\frac{\partial}{\partial x_{2j+2d}}\to-\beta_{j,1}\frac{\partial}{\partial u_1}-\cdots-\beta_{j,d}\frac{\partial}{\partial u_d}+\frac{\partial}{\partial u_{2j+2d}},\quad j=1,\ldots,n-d.$$

Thus, we can take $v=(v_1,\ldots,v_d)$ and $x=(x_1,\ldots,x_{2n+1})$ such that (71), (72) and (73) hold.

Let $\widetilde{x}=(\widetilde{x}_1,\ldots,\widetilde{x}_{2n+1})$ be the coordinates as in Theorem 2.3. It is easy to see that

$$\begin{aligned}&\widetilde{x}_j=x_j+h_j(\mathring{x}),\quad h_j(\mathring{x})=O(|x|^2),\quad j=1,2,\ldots,2n,\\&\widetilde{x}_{2n+1}=x_{2n+1}+h_{2n+1}(\mathring{x}),\quad h_{2n+1}(\mathring{x})=O(|x|^2),\end{aligned}\tag{82}$$

where $\mathring{x}=(x_1,\ldots,x_{2n})$. We may change x_{2n+1} be $x_{2n+1}+h_{2n+1}(0,\ldots,0,x_{d+1},\ldots,x_{2n})$ and we have

$$\frac{\partial^2\widetilde{x}_{2n+1}}{\partial x_j\partial x_k}(p)=0,\quad j,k=\{d+1,\ldots,2n\}.\tag{83}$$

Note that when we change x_{2n+1} to $x_{2n+1}+h_{2n+1}(0,\ldots,0,x_{d+1},\ldots,x_{2n})$, $\frac{\partial}{\partial x_j}$ will change to $\frac{\partial}{\partial x_j}+\alpha_j(x)\frac{\partial}{\partial x_{2n+1}}$, $j=d+1,\ldots,2n$, where $\alpha_j(x)$ is a smooth function on $Y\bigcap U$, independent of $x_1,\ldots,x_d$, x_{2n+1} and $\alpha_j(0)=0$, $j=d+1,\ldots,2n$. Hence, on $Y\bigcap U$, we have $J(\frac{\partial}{\partial x_j})=\frac{\partial}{\partial x_{d+j}}+a_j(x)\frac{\partial}{\partial x_{2n+1}}$, $j=1,2,\ldots,d$, where $a_j(x)$ is a smooth function on $\mu^{-1}(0)\bigcap U$, independent of $x_1,\ldots,x_{2d}$, x_{2n+1} and $a_j(0)=0$, $j=1,\ldots,d$.

From (54) and (82), it is straightforward to see that

$$\begin{aligned}\omega_0(\widetilde{x})&=d\widetilde{x}_{2n+1}-i\sum_{j,t=1}^{n}\tau_{j,t}\overline{\widetilde{z}}_t d\widetilde{z}_j+i\sum_{j,t=1}^{n}\overline{\tau}_{j,t}\widetilde{z}_t d\overline{\widetilde{z}_j}+O(|\mathring{x}|^2)\\&=dx_{2n+1}-i\sum_{j,t=1}^{n}\tau_{j,t}\overline{z}_t dz_j+i\sum_{j,t=1}^{n}\overline{\tau}_{j,t}z_t d\overline{z_j}+\sum_{j=1}^{n}(\frac{\partial\widetilde{x}_{2n+1}}{\partial z_j}dz_j+\frac{\partial\widetilde{x}_{2n+1}}{\partial\overline{z}_j}d\overline{z}_j)\\&\quad+O(|\mathring{x}|^2).\end{aligned}\tag{84}$$

Note that ω_0 is G-invariant. From this observation and (84), we deduce that

$$\begin{aligned}
&\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial z_j \partial x_\ell}(p) = i\tau_{j,\ell}, \quad j \in \{1,\dots,n\}, \ell \in \{1,\dots,d\}, \\
&\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial \overline{z}_j \partial x_\ell}(p) = -i\overline{\tau_{j,\ell}}, \quad j \in \{1,\dots,n\}, \ell \in \{1,\dots,d\}.
\end{aligned} \tag{85}$$

From (85), we can compute that

$$\begin{aligned}
&\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial x_j \partial x_\ell}(p) = i\tau_{j,\ell} - i\overline{\tau_{j,\ell}}, \quad j,\ell \in \{1,\dots,d\}, \\
&\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial x_{d+j} \partial x_\ell}(p) = -(\tau_{j,\ell} + \overline{\tau_{j,\ell}}), \quad j,\ell \in \{1,\dots,d\}, \\
&\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial x_{2j-1} \partial x_\ell}(p) = i\tau_{j,\ell} - i\overline{\tau_{j,\ell}}, \quad j \in \{d+1,\dots,n\}, \quad \ell \in \{1,\dots,d\}, \\
&\frac{\partial^2 \widetilde{x}_{2n+1}}{\partial x_{2j} \partial x_\ell}(p) = -(\tau_{j,\ell} + \overline{\tau_{j,\ell}}), \quad j \in \{d+1,\dots,n\}, \quad \ell \in \{1,\dots,d\}.
\end{aligned} \tag{86}$$

From (83) and (86), we get (74). $\square$

Let us now write the phase function (56) in local coordinates defined above.

Proposition 3.2. *Let $p \in \mu^{-1}(0)$ and fix $t_0 \in I$. Let s be a local rigid CR frame of L defined on D, $|s|^2_{h^L} = e^{-2\Phi}$, $\Phi \in \mathcal{C}^\infty(D)$. Let $x = (x_1,\dots,x_{2n+1})$ be local coordinates as in Proposition 3.1. Then, there exists a neighborhood of (p, p) such that*

$$\begin{aligned}
\varphi(x,y,t_0) =& t_0(x_{2n+1} - y_{2n+1}) - \frac{i}{2}\sum_{j,l=1}^{n}(a_{l,j} + a_{j,l})(z_j z_l - w_j w_l) \\
&+ \frac{i}{2}\sum_{j,l=1}^{n}(\overline{a}_{l,j} + \overline{a}_{j,l})(\overline{z}_j\overline{z}_l - \overline{w}_j\overline{w}_l) + \frac{i\,t_0}{2}\sum_{j,l=1}^{n}(\overline{\tau}_{l,j} - \tau_{j,l})(z_j\overline{z}_l - w_j\overline{w}_l) \\
&- \frac{i}{2}\sum_{j=1}^{n}\lambda_j(t_0)(z_j\overline{w}_j - \overline{z}_j w_j) + \frac{i}{2}\sum_{j=1}^{n}\lambda_j(t_0)|z_j - w_j|^2 \\
&+ i\frac{t_0}{2}\sum_{j,l=1}^{d}(\tau_{j,l} - \overline{\tau_{j,l}})(x_j x_l - y_j y_l) + t_0\sum_{j,l=1}^{d}(\tau_{j,l} + \overline{\tau_{j,l}})(-x_{d+j}x_l + y_{d+j}y_l) \\
&+ t_0\sum_{j=d+1}^{n}\sum_{l=1}^{d} i(\tau_{j,l} - \overline{\tau_{j,l}})(x_{2j-1}x_l - y_{2j-1}y_l) \\
&+ t_0\sum_{j=d+1}^{n}\sum_{l=1}^{d} -(\tau_{j,l} + \overline{\tau_{j,l}})(x_{2j}x_l - y_{2j}y_l) + O(|(z,w)|^3),
\end{aligned} \tag{87}$$

where $\tau_{j,l}, a_{j,l} \in \mathbb{C}$, $j, l = 1, \ldots, n$, $\lambda_j(t_0)$, $j = 1, \ldots, n$, *are as in Theorem 2.3.*

Moreover, we have

$$\mu_{j,\ell} + t_0(\overline{\tau_{\ell,j}} + \tau_{j,\ell}) = \delta_{j,\ell}\lambda_j(t_0), \quad j, \ell = 1, \ldots, n, \tag{88}$$

and for all $j = 1, \ldots, d$, $\ell = 1, \ldots, n$

$$\begin{aligned} &\frac{1}{2}\mu_{j,\ell} + \overline{a_{j,\ell}} + \overline{a_{\ell,j}} = 0, \quad j = 1, \ldots, d, \quad \ell = 1, \ldots, n, \\ &\frac{1}{2}\mu_{\ell,j} + a_{j,\ell} + a_{\ell,j} = 0, \quad j = 1, \ldots, d, \quad \ell = 1, \ldots, n \end{aligned} \tag{89}$$

where $\mu_{j,\ell}$, $a_{j,ell}$, *are as in* (55)

Proof. From (56) and (74), we get the (87).

From (54), it is straightforward to check that

$$(R_p^L - 2t_0\mathcal{L}_p)(\overline{W}_{j,t_0}(p), W_{\ell,t}(p)) = \mu_{j,\ell} + t_0(\overline{\tau_{\ell,j}} + \tau_{j,\ell}) = \lambda_j(t_0)\delta_{j,\ell}, \tag{90}$$

for all $j, \ell = 1, \ldots, n$. From (90), we get (88).

Since Φ is G-invariant, we have

$$d\Phi(\xi_X) = 0, \quad \text{for all } \xi \in \mathfrak{g}. \tag{91}$$

From, (55), (72) and (74), we have

$$\begin{aligned} d\Phi &= \frac{1}{2}\sum_{\ell,t=1}^{n} \mu_{t,\ell}(\overline{z}_\ell dz_t + z_t d\overline{z}_\ell) \\ &+ \sum_{\ell,t=1}^{n}\Big(a_{\ell,t}z_\ell dz_t + a_{\ell,t}z_t dz_\ell\Big) + \sum_{\ell,t=1}^{n}\Big(\overline{a}_{\ell,t}\overline{z}_\ell d\overline{z}_t + \overline{a}_{\ell,t}\overline{z}_t d\overline{z}_\ell\Big) + O(|z|^2). \end{aligned} \tag{92}$$

From (72) and (91), we get for every $j = 1, \ldots, d$,

$$\begin{aligned} &d\Phi(\frac{\partial}{\partial x_j})(0) \\ &= \sum_{\ell=1}^{n}\frac{1}{2}\mu_{j,\ell}\overline{z}_\ell + \sum_{\ell=1}^{n}\frac{1}{2}\mu_{\ell,j}z_\ell + \sum_{\ell=1}^{n}a_{j,\ell}z_\ell + \sum_{\ell=1}^{n}a_{\ell,j}z_\ell + \sum_{\ell=1}^{n}\overline{a_{j,\ell}}\,\overline{z}_\ell + \sum_{\ell=1}^{n}\overline{a_{\ell,j}}\,\overline{z}_\ell = 0. \end{aligned}$$

Thus, for every $j = 1, \ldots, d$, we have

$$\sum_{\ell=1}^{n}\frac{1}{2}\mu_{j,\ell}\overline{z}_\ell + \sum_{\ell=1}^{n}\frac{1}{2}\mu_{\ell,j}z_\ell + \sum_{\ell=1}^{n}a_{j,\ell}z_\ell + \sum_{\ell=1}^{n}a_{\ell,j}z_\ell + \sum_{\ell=1}^{n}\overline{a_{j,\ell}}\,\overline{z}_\ell + \sum_{\ell=1}^{n}\overline{a_{\ell,j}}\,\overline{z}_\ell = 0. \tag{93}$$

From (93), we get (89). □

3.2. G-invariant Szegő kernels asymptotics near Y

The aim of this section is to prove Theorem 1.1, now we briefly explain the strategy of the proof. The G-invariant operator $P^G_{k,\tau^2,s}$ is obtained by taking the integral over G of the operator P_{k,τ^2}, which was studied in [23, Theorem 5.5]. The distributional kernel of the latter is an oscillatory integral, so the proof consists in applying the Stationary phase formula of Melin and Sjőstrand which leads to equation (102). The following Theorem 3.2, and in particular its consequence Theorem 3.3, concerning the imaginary part of the phase function A is standard in the analysis of complex Fourier integral operators. The next steps, after equation (102), aim to give a more precise description of the phase function A in local coordinates in Theorem 3.4. It remains to prove (4) which is done is Lemma 3.1 and Lemma 3.2. Eventually we prove Theorem 1.2 which gives a local description of the leading term in the asymptotic expansion of the symbol.

Now, let

$$\mathcal{H}^0_{b,I}(X,L^k)^G := \{u \in \mathcal{H}^0_{b,I}(X,L^k);\, g^*u = u, \quad \text{for all } g \in G\}.$$

Let

$$\Pi^G_{k,I} : L^2(X,L^k) \to \mathcal{H}^0_{b,I}(X,L^k)^G$$

be the orthogonal projection with respect to $(\cdot\,|\,\cdot)_k$. We consider the G-invariant weighted Fourier-Szegő operator

$$P^G_{k,\tau^2} := F_{k,\tau} \circ \Pi^G_{k,I} \circ F_{k,\tau} : L^2(X,L^k) \to \mathcal{H}^0_{b,I}(X,L^k), \tag{94}$$

where $F_{k,\tau}$ is as in (46). Let P_{k,τ^2} be as in (48). The following was proved in [23, Theorem 5.5].

Theorem 3.1. *Let $\chi, \hat{\chi} \in \mathcal{C}^\infty(X)$ with $\operatorname{supp}\chi \cap \operatorname{supp}\hat{\chi} = \emptyset$. Then,*

$$\chi P_{k,\tau^2}\hat{\chi} = O(k^{-\infty}) \quad \text{on } X \times X. \tag{95}$$

Now, recall that

$$P^G_{k,\tau^2}(x,y) = \int_G P_{k,\tau^2}(x, g\cdot y)\, d\mu(g)\,,$$

where $d\mu(g)$ is the Haar measure of G. Fix $p \in Y$ and let s be a local CR rigid trivializing section of L defined on an open set $D \subset X$. Let $d\mu(g)$ be the Haar measure on G with $\int_G d\mu(g) = 1$. Let V be an open neighborhood of $e_0 \in G$ as in Proposition 3.1. Note that

$$P^G_{k,\tau^2}(x,y) = \int_G P_{k,\tau^2}(x, g\cdot y) d\mu(g).$$

Let $\chi \in \mathcal{C}^\infty_c(V)$, $\chi = 1$ near e_0. We have

$$P^G_{k,\tau^2}(x,y) = \int_G \chi(g) P_{k,\tau^2}(x, g\cdot y) d\mu(g) + \int_G (1-\chi(g)) P_{k,\tau^2}(x, g\cdot y) d\mu(g).$$

Since G is freely on Y, if U and V are small, there is a constant $c > 0$ such that

$$d(x, g\cdot y) \geq c, \quad \forall x, y \in U, g \in \mathrm{Supp}\,(1-\chi), \tag{96}$$

where $U \subset D$ is an open set of $p \in Y$ as in Proposition 3.1. From now on, we take U and V small enough so that (96) holds. In view of Theorem 3.1, we see that $P_{k,\tau^2}(x,y)$ is k-negligible away from diagonal. From this observation and (96), we conclude that $\int_G (1-\chi(g)) P_{k,\tau^2}(x, g\cdot y) d\mu(g) = O(k^{-\infty})$ on $U \times U$ and hence

$$P^G_{k,\tau^2}(x,y) = \int_G \chi(g) P_{k,\tau^2}(x, g\cdot y) d\mu(g) + O(k^{-\infty}) \text{ on } U \times U. \tag{97}$$

From Theorem 2.2 and (97), we get

$$P^G_{k,\tau^2,s}(x,\, y) = \int_G \int_{\mathbb{R}} e^{ik\,\varphi(x, g\cdot y, t)} a(x, g\cdot y, t, k)\, \chi(g)\, \mathrm{d}t d\mu(g) + O(k^{-\infty}).$$

Recall that φ has the form

$$\varphi(x,y,t) = (x_{2n+1} - y_{2n+1})t + \varphi_0(\mathring{x}, \mathring{y}, t),$$

where $\mathring{x} = (x_1, \ldots, x_{2n})$, $\varphi(\mathring{x}, \mathring{y}, t) \in \mathcal{C}^\infty(D \times D \times I)$. Now, we use the coordinates as in Proposition 3.1. Put $x' = (x_1, \ldots, x_d)$, $x'' = (x_{d+1}, \ldots, x_{2n+1})$, $\mathring{x}'' = (x_{d+1}, \ldots, x_{2n})$, $x = (x',\, x'') = (x',\, \hat{x}'', \tilde{x}'')$ where $\hat{x}'' = (x_{d+1}, \ldots, x_{2d})$, and $\tilde{x}'' = (x_{2d+1}, \ldots, x_{2n+1})$. Since $P^G_{k,\tau^2}(x,y)$ is G-invariant we have

$$P^G_{k,\tau^2,s}(x,\, y) = P^G_{k,\tau^2,s}((0, x''),\, (\Gamma(\mathring{y}''), y''))$$

where Γ is as in Proposition 3.1.

Now, write $\mathring{x}'' = (x_{d+1}, \ldots, x_{2n})$. Assume that on V, we have

$$d\mu(g) = m(v)\, \mathrm{d}v = m(v_1, \ldots,\, v_d)\, \mathrm{d}v_1 \cdots \mathrm{d}v_d$$

on V and m is a real-valued smooth function on G. From Proposition 3.1, we have

$$\begin{aligned} &P^G_{k,\tau^2,s}((0,x''), (\Gamma(\mathring{y}''), y'')) \\ &\quad = \int_V \int_{\mathbb{R}} e^{ik\,\varphi((0,x''), (\Gamma(\mathring{y}'')+v, y''), t)} a((0,x''), (\Gamma(\mathring{y}'') + v, y''), t)\, \chi(v) m(v)\, \mathrm{d}t\, \mathrm{d}v \end{aligned}$$

$$+ O(k^{-\infty})$$

on $D \times D$, where V is a small open neighborhood of the identity e_0 of G. Let $v = (v_1, \ldots, v_d)$ be the local coordinates of G defined in a neighborhood V of e with $v(e) = (0, \ldots, 0)$. From now on, we will identify the element $g \in V$ with $v(g)$. By local coordinates of Proposition 3.1 and (87), it is easy to check that

$$\det\left(\left(\frac{\partial^2 \varphi}{\partial v_\ell \partial v_j}(p, p, t_0)\right)_{j,\,k=1}^{d}\right) = i^d\,|\lambda_1(t_0)| \cdots |\lambda_d(t_0)| \neq 0. \tag{98}$$

We aim to apply the stationary phase formula of Melin and Sjöstrand, see [27, Theorem 2.3], but we first need to introduce some notations.

Let W be an open set of $\mathbb{R}^N$, $N \in \mathbb{N}$. From now on, we write $W^{\mathbb{C}}$ to denote an open set in $\mathbb{C}^N$ with $W^{\mathbb{C}} \bigcap \mathbb{R}^N = W$ and for $f \in \mathcal{C}^\infty(W)$, from now on, we write $\widetilde{f} \in \mathcal{C}^\infty(W^{\mathbb{C}})$ to denote an almost analytic extension of f. For every $t \in I$, let $h(\mathring{x}'', \mathring{y}'', t) \in \mathcal{C}^\infty(U \times U, \mathbb{C}^d)$ be the solution of the system

$$\frac{\partial \widetilde{\varphi}_0}{\partial \tilde{y}_j}((0, \mathring{x}''), (h(\mathring{x}'', \mathring{y}'', t) + \Gamma(\mathring{y}''), \mathring{y}''), t) = 0 \tag{99}$$

for $j = 1, 2, \ldots, d$. Let us set

$$A(x'', y'', t) := (x_{2n+1} - y_{2n+1})t + \widetilde{\varphi}_0((0, \mathring{x}''), (h(\mathring{x}'', \mathring{y}'', t) + \Gamma(\mathring{y}''), \mathring{y}''), t)\,, \tag{100}$$

it is known that $\mathrm{Im} A \geq 0$, see [27, p. 147]. Furthermore, we note that

$$\begin{aligned} &\frac{\partial \varphi_0}{\partial v_j}(x' = v + \Gamma(y'') = 0,\, \hat{x}'' = \hat{y}'' = 0,\, \widetilde{x}'' = \widetilde{y}'',\, t) \\ &= \left\langle 2\,\mathrm{Im}\overline{\partial}_b \Phi(x) - t\,\omega_0(x),\, \frac{\partial}{\partial x_j} \right\rangle = 0, \end{aligned}$$

for every $j = 0, \ldots, d$ where $x = (0, (0, \widetilde{x}''))$. Hence, for every $t \in I$ the critical points are $\hat{x}'' = \hat{y}'' = 0$, $\widetilde{x}'' = \widetilde{y}''$, $x' = v + \Gamma(y'') = 0$ and we find that

$$\begin{aligned} &\mathrm{d}_x A(x'' = y'',\, \hat{x}'' = 0,\, t) = -2\,\mathrm{Im}\overline{\partial}_b \Phi(x) + t\,\omega_0(x), \\ &\mathrm{d}_y A(x'' = y'',\, \hat{x}'' = 0,\, t) = 2\,\mathrm{Im}\overline{\partial}_b \Phi(x) - t\,\omega_0(x), \\ &A(x'' = y'',\, \hat{x}'' = 0,\, t) = 0. \end{aligned} \tag{101}$$

Thus, we can now use the stationary phase formula of Melin and Sjöstrand, we can carry out the v integral and get

$$P^G_{k,\tau^2,s}(x, y) \equiv \int_{\mathbb{R}} e^{ikA(x'', y'', t)} b(x'', y'', t, k) dt + O(k^{-\infty}) \;\text{ on } U \times U, \tag{102}$$

$p \in U \subset D$, U is an open set as in Proposition 3.1,

$$\begin{aligned} &b(x'', y'', t, k) \sim \sum_{j=0}^{\infty} b_j(x'', y'', t) k^{n+1-\frac{d}{2}-j} \text{ in } S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\ &b_j(x'', y'', t) \in \mathcal{C}^\infty(U \times U \times I), \quad j = 0, 1, \dots, \\ &b(x'', y'', t, k) = b_j(x'', y'', t) = 0 \ \text{ if } t \notin I,\ j = 0, 1, \dots, \end{aligned} \tag{103}$$

and

$$b_0(p, p, t_0) = m(0)(2\pi)^{-n-1+\frac{d}{2}} |\lambda_1(t_0)|^{\frac{1}{2}} \cdots |\lambda_d(t_0)|^{\frac{1}{2}} |\lambda_{d+1}(t_0)| \cdots |\lambda_n(t_0)| \tau^2(t_0). \tag{104}$$

We now study the property of the phase $A(x'', y'', t)$. We need the following which is known (see Section 2 in [27])

Theorem 3.2. *There exist a constant $c > 0$ and an open set $\Omega \subset \mathbb{R}^d$, $0 \in \Omega$, such that*

$$\begin{aligned} \operatorname{Im} A(x'', y'', t) \geq c \inf_{v \in \Omega} \Big\{ &\operatorname{Im} \varphi_0((0, \mathring{x}''), (v + \Gamma(\mathring{y}''), \mathring{y}''), t) \\ &+ |d_v \varphi_0((0, \mathring{x}''), (v + \Gamma(\mathring{y}''), \mathring{y}''), t)|^2 \Big\}, \end{aligned} \tag{105}$$

for all $((0, x''), (0, y''), t) \in U \times U \times I$.

We can now prove

Theorem 3.3. *If U is small enough, then there is a constant $c > 0$ such that*

$$\operatorname{Im} A(x'', y'', t) \geq c\Big(|\hat{x}''|^2 + |\hat{y}''|^2 + |\mathring{x}'' - \mathring{y}''|^2 \Big), \quad \forall ((0, x''), (0, y'')) \in U \times U. \tag{106}$$

Proof. From (51), we see that there is a constant $c_1 > 0$ such that

$$\operatorname{Im} \varphi_0((0, \mathring{x}''), (v + \Gamma(\mathring{y}''), \mathring{y}''), t) \geq c_1(|v + \Gamma(\mathring{y}'')|^2 + |\mathring{x}'' - \mathring{y}''|^2), \quad \forall v \in \Omega, \tag{107}$$

where Ω is any open set of $0 \in \mathbb{R}^d$. From (105) and (107), we conclude that there is a constant $c_2 > 0$ such that

$$\operatorname{Im} A(x'', y'', t) \geq c_2(|\mathring{x}'' - \mathring{y}''|^2 + |d_{y'} \varphi_0((0, \mathring{x}''), (0, \mathring{x}''))|^2). \tag{108}$$

From (98), we see that the matrix

$$\left| \det \left(\frac{\partial^2 \varphi_0}{\partial x_j \partial x_\ell}(p, p, t) + \frac{\partial^2 \varphi_0}{\partial y_j \partial y_\ell}(p, p, t) \right)_{1 \leq \ell \leq d, d+1 \leq j \leq 2d} \right| \geq C,$$

for all $t \in I$, where $C > 0$ is independent of t. From this observation and notice that $d_{y'}\varphi_0((0,\mathring{x}''),(0,\mathring{x}''),t)|_{\mathring{x}''} = 0$, we deduce that if U is small enough then there is a constant $c_3 > 0$ such that

$$|d_{y'}\varphi_0((0,\mathring{x}''),(0,\mathring{x}''),t)| \geq c_3\,|\mathring{x}''|\,, \tag{109}$$

for all $t \in I$. From (108) and (109), the theorem follows. □

Now, we determine the Hessian of $A(x'', y'', t)$ at (p,p). Let

$$\hat{h}(\mathring{x}'',\mathring{y}'',t) := h(\mathring{x}'',\mathring{y}'',t) + \Gamma(\mathring{y}'') = (\hat{h}_1(\mathring{x}'',\mathring{y}'',t),\ldots,\hat{h}_d(\mathring{x}'',\mathring{y}'',t)).$$

By (99), we get

$$\begin{aligned}
&\frac{\partial^2\varphi_0}{\partial y_s\partial y_1}(p,p,t_0) + \sum_{j=1}^{d}\frac{\partial^2\varphi_0}{\partial y_1\,\partial y_j}(p,p,t_0)\,\frac{\partial\hat{h}_j}{\partial y_s}(p,p,t_0) = 0,\\
&\frac{\partial^2\varphi_0}{\partial x_s\partial y_1}(p,p,t_0) + \sum_{j=1}^{d}\frac{\partial^2\varphi_0}{\partial y_1\,\partial y_j}(p,p,t_0)\,\frac{\partial\hat{h}_j}{\partial x_s}(p,p,t_0) = 0,
\end{aligned} \tag{110}$$

$s = d+1,\ldots,2n$. Now, we would like to use (110) to get $\hat{h}$ up to second order. From (87) and by explicit computation, we get

$$\frac{\partial^2\varphi_0}{\partial y_1\partial x_s}(p,p,t_0) = \lambda_1(t_0)\delta_{s,d+1}, \tag{111}$$

$s = d+1,\ldots,2n$, and

$$\begin{aligned}
&\frac{\partial^2\varphi_0}{\partial y_1\partial y_j}(p,p,t_0) = i(a_{1,j}+a_{j,1}) - i(\overline{a_{1,j}}+\overline{a_{j,1}}) + \delta_{j,1}i\lambda_1(t_0), \quad j=1,\ldots,d,\\
&\frac{\partial^2\varphi_0}{\partial y_1\partial y_{d+s}}(p,p,t_0) = -(a_{1,s}+a_{s,1}) - (\overline{a_{1,s}}+\overline{a_{s,1}}) + \frac{t_0}{2}(\tau_{s,1}+\overline{\tau_{s,1}}) + \frac{t_0}{2}(\tau_{1,s}+\overline{\tau_{1,s}}),\\
&\quad s=1,\ldots,d,\\
&\frac{\partial^2\varphi_0}{\partial y_1\partial y_{2\ell-1}}(p,p,t_0) = i(a_{1,\ell}+a_{\ell,1}) - i(\overline{a_{1,\ell}}+\overline{a_{\ell,1}}) + \frac{i}{2}t_0\overline{\tau_{\ell,1}} - \frac{i}{2}t_0\tau_{\ell,1} + \frac{i}{2}t_0\tau_{1,\ell} - \frac{i}{2}t_0\overline{\tau_{1,\ell}},\\
&\quad \ell=d+1,\ldots,n,\\
&\frac{\partial^2\varphi_0}{\partial y_1\partial y_{2\ell}}(p,p,t_0) = -(a_{1,\ell}+a_{\ell,1}) - (\overline{a_{1,\ell}}+\overline{a_{\ell,1}}) + \frac{1}{2}t_0\overline{\tau_{\ell,1}} + \frac{1}{2}t_0\tau_{1,\ell} + \frac{1}{2}t_0\overline{\tau_{1,\ell}} + \frac{1}{2}t_0\tau_{\ell,1},\\
&\quad \ell=d+1,\ldots,n.
\end{aligned} \tag{112}$$

From (88) and (89), we have

$$\begin{aligned} &a_{1,j}+a_{j,1}=-\frac{1}{2}\mu_{1,j}=-\frac{1}{2}\mu_{j,1}=-\frac{1}{2}\overline{\mu_{1,j}},\quad j=1,\ldots,d,\\ &a_{1,\ell}+a_{\ell,1}=-\frac{1}{2}\mu_{\ell,1}=\frac{t_0}{2}(\overline{\tau_{1,\ell}}+\tau_{\ell,1}),\quad \ell=d+1,\ldots,n. \end{aligned} \tag{113}$$

From (88), (89), (112) and (113), it is straightforward to see that

$$\begin{aligned} &\frac{\partial^2\varphi_0}{\partial y_1\partial y_j}(p,p,t_0)=\delta_{j,1}i\lambda_1(t_0),\quad j=1,\ldots,d,\\ &\frac{\partial^2\varphi_0}{\partial y_1\partial y_{d+s}}(p,p,t_0)=\lambda_1(t_0)\delta_{1,s},\quad s=1,\ldots,d,\\ &\frac{\partial^2\varphi_0}{\partial y_1\partial y_{2\ell-1}}(p,p,t_0)=0,\quad \ell=d+1,\ldots,n,\\ &\frac{\partial^2\varphi_0}{\partial y_1\partial y_{2\ell}}(p,p,t_0)=0,\quad \ell=d+1,\ldots,n. \end{aligned} \tag{114}$$

From (110), (111) and (114), we get

$$\frac{\partial\hat{h}_1}{\partial y_s}(p,p,t_0)=\frac{\partial\hat{h}_1}{\partial x_s}(p,p,t_0)=i\delta_{s,d+1},\quad s=d+1,\ldots,2n. \tag{115}$$

In a similar way, we repeat the procedure above and get

$$\frac{\partial\hat{h}_j}{\partial x_s}(p,p,t_0)=\frac{\partial\hat{h}_j}{\partial y_s}(p,p,t_0)=i\delta_{s,d+j}, \tag{116}$$

$j=1,\ldots,d$, $s=1,\ldots,2n$.

By (87), (100) and (116), it is straightforward to check that

Theorem 3.4. *With the notations above, let $x=(x_1,\ldots,x_{2n+1})$ be the local coordinates as in Proposition 3.1. Then for $A(x'',y'',t)\in\mathcal{C}^\infty(U\times U\times I)$ in (100), we have*

$$\begin{aligned} A(x'',y'',t_0)=\,&t_0(x_{2n-1}-y_{2n-1})+\frac{i}{2}\sum_{j=1}^{d}\lambda_j(t_0)(x_{d+j}^2+y_{d+j}^2)\\ &+\frac{i}{2}\sum_{j,\ell=1}^{d}(a_{\ell,j}+a_{j,\ell})(x_{d+j}x_{d+\ell}-y_{d+j}y_{d+\ell})\\ &-\frac{i}{2}\sum_{j,\ell=1}^{d}(\overline{a_{\ell,j}}+\overline{a_{j,\ell}})(x_{d+j}x_{d+\ell}-y_{d+j}y_{d+\ell})\\ &+\frac{i}{2}t_0\sum_{j,\ell=1}^{d}(\overline{\tau_{\ell,j}}-\tau_{j,\ell})(x_{d+j}x_{d+\ell}-y_{d+j}y_{d+\ell}) \end{aligned}$$

$$
\begin{aligned}
&-\frac{i}{2}\sum_{j=1}^{d}\sum_{\ell=d+1}^{n}(a_{\ell,j}+a_{j,\ell})(ix_{d+j}z_\ell-iy_{d+j}w_\ell)\\
&-\frac{i}{2}\sum_{j=d+1}^{n}\sum_{\ell=1}^{d}(a_{\ell,j}+a_{j,\ell})(iz_jx_{d+j}-iw_jy_{d+\ell})\\
&-\frac{i}{2}\sum_{j,\ell=d+1}^{n}(a_{\ell,j}+a_{j,\ell})(z_jz_\ell-w_jw_\ell)\\
&+\frac{i}{2}\sum_{j=1}^{d}\sum_{\ell=d+1}^{n}(\overline{a_{\ell,j}}+\overline{a_{j,\ell}})(-ix_{d+j}\overline{z}_\ell+iy_{d+j}\overline{w}_\ell)\\
&+\frac{i}{2}\sum_{j=d+1}^{n}\sum_{\ell=1}^{d}(\overline{a_{\ell,j}}+\overline{a_{j,\ell}})(-i\overline{z}_jx_{d+j}+i\overline{w}_jy_{d+\ell})\\
&+\frac{i}{2}\sum_{j,\ell=d+1}^{n}(\overline{a}_{\ell,j}+\overline{a}_{j,\ell})(\overline{z}_j\overline{z}_\ell-\overline{w}_j\overline{w}_\ell)\\
&+\frac{it_0}{2}\sum_{j=1}^{d}\sum_{\ell=d+1}^{n}(\overline{\tau_{\ell,j}}-\tau_{j,\ell})(ix_{d+j}\overline{z}_\ell-iy_{d+j}\overline{w}_\ell)\\
&+\frac{it_0}{2}\sum_{j=d+1}^{n}\sum_{\ell=1}^{d}(\overline{\tau_{\ell,j}}-\tau_{j,\ell})(-ix_{d+\ell}z_j+iy_{d+\ell}w_j)\\
&+\frac{it_0}{2}\sum_{j,\ell=d+1}^{n}(\overline{\tau_{\ell,j}}-\tau_{j,\ell})(z_j\overline{z}_\ell-w_j\overline{w}_\ell)\\
&-\frac{i}{2}\sum_{j=d+1}^{n-1}\lambda_j(t_0)(z_j\overline{w}_j-\overline{z}_jw_j)\\
&+\frac{i}{2}\sum_{j=d+1}^{n}\lambda_j(t_0)|z_j-w_j|^2+O(|(\mathring{x}'',\mathring{y}'')|^3),
\end{aligned}
\tag{117}
$$

where $x''=(x_{d+1},\ldots,x_{2n+1})$, $\mathring{x}''=(x_{d+1},\ldots,x_{2n})$

From (102) and Theorem 3.4, we get (5).

Now, we prove (4). We need the following two lemmas.

Lemma 3.1. *Let* $p\notin\mu^{-1}(0)$. *Then, there are open sets* U *of* p *and* V *of* $e_0\in G$ *such that for any* $\chi\in\mathcal{C}^\infty_c(V)$, *we have*

$$\int_G P_{k,\tau^2}(x,g\cdot y)\chi(g)d\mu(g)=O(k^{-\infty})\quad\text{on } X\times U. \tag{118}$$

Proof. Take local coordinates $v = (v_1, \ldots, v_d)$ of G defined in a neighborhood V of e_0 with $v(e_0) = (0, \ldots, 0)$, local coordinates $x = (x_1, \ldots, x_{2n+1})$ of X defined in a neighborhood $U = U_1 \times U_2$ of p with $0 \leftrightarrow p$, where $U_1 \subset \mathbb{R}^d$ is an open set of $0 \in \mathbb{R}^d$, $U_2 \subset \mathbb{R}^{2n+1-d}$ is an open set of $0 \in \mathbb{R}^{2n+1-d}$, such that

$$
\begin{aligned}
&(v_1, \ldots, v_d) \circ (\gamma(x_{d+1}, \ldots, x_{2n}), x_{d+1}, \ldots, x_{2n+1}) \\
&= (v_1 + \gamma_1(x_{d+1}, \ldots, x_{2n}), \ldots, v_d + \gamma_d(x_{d+1}, \ldots, x_{2n}), x_{d+1}, \ldots, x_{2n+1}), \\
&\forall (v_1, \ldots, v_d) \in V, \quad \forall (x_{d+1}, \ldots, x_{2n+1}) \in U_2,
\end{aligned}
$$

and

$$
\underline{\mathfrak{g}} = \operatorname{span}\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right\},
$$

where $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathcal{C}^\infty(U_2, U_1)$ with $\gamma(0) = 0 \in \mathbb{R}^d$.

Let s be a local CR rigid trivializing section of L defined on an open set U of p. We first assume that $x, y \in U$. From Theorem 2.2, we have

$$
\begin{aligned}
&\int_G P_{k,\tau^2,s}(x, g \cdot y)\chi(g)d\mu(g) \\
&\quad \equiv \int e^{i(\varphi(x,(v+\gamma(y''),y''),t)k} a(x, (v + \gamma(y''), y''), t, k)\chi(v)m(v)dvdt,
\end{aligned}
$$

where $y'' = (y_{d+1}, \ldots, y_{2n+1})$, $m(v)dv = d\mu|_V$. Since $p \notin \mu^{-1}(0)$ and notice that $d_y\varphi(x, x, t) = 2\mathrm{Im}\,\overline{\partial}_b\Phi - t\omega_0(x)$, we deduce that if V and U are small then $d_v(\varphi(x, (v + \gamma(y''), y''), t)) \neq 0$, for every $v \in V$, $(x, y) \in U \times U$. Hence, by using integration by parts with respect to v, we get

$$
\int_G P_{k,\tau^2,s}(x, g \cdot y)\chi(g)d\mu(g) = O(k^{-\infty}) \text{ on } U. \tag{119}
$$

From (119), we get

$$
\int_G P_{k,\tau^2}(x, g \cdot y)\chi(g)d\mu(g) = O(k^{-\infty}) \text{ on } U \times U. \tag{120}
$$

Now fix $x_0 \in X$, $x_0 \notin U$. From Theorem 3.1, we can check that

$$
\int_G P_{k,\tau^2}(x, g \cdot y)\chi(g)d\mu(g) = O(k^{-\infty}) \text{ on } W \times U, \tag{121}
$$

where W is a small open neighborhood of x_0. From (120) and (121), the lemma follows. □

Lemma 3.2. *Let $p \notin \mu^{-1}(0)$ and let $h \in G$. We can find open sets U of p and V of h such that for every $\chi \in \mathcal{C}_c^\infty(V)$, we have*

$$\int_G P_{k,\tau^2}(x, g \cdot y)\chi(g)d\mu(g) = O(k^{-\infty}) \quad \textit{on } X \times U.$$

Proof. Let U and V be open sets as in Lemma 3.1. Let $\hat{V} = Vh$. Then, $\hat{V}$ is an open set of h. Let $\hat{\chi} \in \mathcal{C}_c^\infty(\hat{V})$. We have

$$\int_G P_{k,\tau^2}(x, g\cdot y)\hat{\chi}(g)d\mu(g) = \int_G P_{k,\tau^2}(x, g\cdot h\cdot y)\hat{\chi}(g\cdot h)d\mu(g) = \int_G P_{k,\tau^2}(x, g\cdot h\cdot y)\chi(g)d\mu(g), \tag{122}$$

where $\chi(g) := \hat{\chi}(g \cdot h) \in \mathcal{C}_c^\infty(V)$. From (122) and Lemma 3.1, we deduce that

$$\int_G P_{k,\tau^2}(x, g \cdot y)\hat{\chi}(g)d\mu(g) = O(k^{-\infty}) \quad \text{on } X \times U.$$

The lemma follows. □

Proof of (4). Fix $p \notin \mu^{-1}(0)$. Let $h \in G$. By Lemma 3.2, we can find open sets U_h of p and V_h of h such that for every $\chi \in \mathcal{C}_c^\infty(V_h)$, we have

$$\int_G P_{k,\tau^2}(x, g \cdot y)\chi(g)d\mu(g) = O(k^{-\infty}) \quad \text{on } X \times U_h. \tag{123}$$

Since G is compact, we can find open sets U_{h_j} and V_{h_j}, $j = 1, \ldots, N$, such that $G = \bigcup_{j=1}^N V_{h_j}$. Let $U = \bigcap_{j=1}^N U_{h_j}$ and let $\chi_j \in \mathcal{C}_c^\infty(V_{h_j})$, $j = 1, \ldots, N$, with $\sum_{j=1}^N \chi_j = 1$ on G. From (123), we have

$$\begin{aligned} P^G_{k,\tau^2}(x, y) &= \int_G P_{k,\tau^2}(x, g \cdot y)d\mu(g) \\ &= \sum_{j=1}^N \int_G P_{k,\tau^2}(x, g \cdot y)\chi_j(g)d\mu(g) = O(k^{-\infty}) \quad \text{on } X \times U. \quad \square \end{aligned}$$

Eventually we prove Theorem 1.2.

Proof of Theorem 1.2. We now determine the leading term $b_0(p, p, t_0)$. In view of (104), we only need to calculate $m(0)$. Put $Y_p = \{g \cdot p;\, g \in G\}$. Y_p is a d-dimensional submanifold of X. The G-invariant Hermitian metric $\langle \cdot | \cdot \rangle$ induces a volume form dV_{Y_p} on Y_p. Put

$$V_{\text{eff}}\,(p) := \int_{Y_p} dV_{Y_p}.$$

For $f(g) \in \mathcal{C}^\infty(G)$, let $\hat{f}(g\cdot p) := f(g)$, $\forall g \in G$. Then, $\hat{f} \in \mathcal{C}^\infty(Y_p)$. Let $d\hat{\mu}$ be the measure on G given by $\int_G f d\hat{\mu} := \int_{Y_p} \hat{f} dv_{Y_p}$, for all $f \in \mathcal{C}^\infty(G)$. It is not difficult to see that $d\hat{\mu}$ is a Haar measure and

$$\int_G d\hat{\mu} = V_{\text{eff}}\,(p). \tag{124}$$

In view of (73), we see that $\left\{\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}, \ldots, \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_d}\right\}$ is an orthonormal basis for $\underline{\mathfrak{g}}_p$. From this observation and (124), we deduce that

$$m(0) = 2^{\frac{d}{2}}\frac{1}{V_{\text{eff}}\,(p)}. \tag{125}$$

From (104) and (125), we get Theorem 1.2. $\square$

4. Proof of Theorem 1.4

Let

$$\sigma_k : \mathcal{C}^\infty(X, L^k) \to \mathcal{H}^0_b(X_G, L^k_G)$$

be the operator given by (146) below. The goal of this section, is to prove that for $k \gg 1$, $\sigma_k|_{\mathcal{H}^0_{b,\lambda}(X,L^k)^G} \to \mathcal{H}^0_{b,\lambda}(X_G, L^k_G)$ is an isomorphism, for all $\lambda \in (ka, kb)$, $\lambda \in \mathrm{Spec}\,(-iT) \cap \mathrm{Spec}\,(-iT_{X_G})$, $I = (a, b)$. To achieve this goal, we need to study the asymptotic behaviors of $F_k := \sigma_k^*\sigma_k$ and $\hat{F}_k := \sigma_k\sigma_k^*$.

In Section 4.1, the main objective is to prove Theorem 4.2 and Theorem 4.3, which are the key ingredients in the study of the compositions $F_k := \sigma_k^*\sigma_k$ and $\hat{F}_k := \sigma_k\sigma_k^*$. Both σ_k^* and σ_k are complex Fourier integral operators, and their compositions can be analyzed using the complex stationary phase formula of Melin and Sjöstrand.

In Section 4.2, the first goal is to provide semi-classical expressions for F_k and $\hat{F}_k$, thanks to Theorems 4.2 and 4.3 from the previous section. These expressions are given in Theorems 4.5 and 4.6.

From the semi-classical expression for F_k, we can show that for the operator

$$R_k := F_k - P^G_{k,\tau^8} : \mathcal{C}^\infty(X, L^k) \to \mathcal{H}^0_b(X, L^k)^G,$$

$I + R_k : \mathcal{C}^\infty(X, L^k) \to \mathcal{C}^\infty(X, L^k)$ is injective for k large (see Theorem 4.7). From the observation that

$$F_k|_{\mathcal{H}^0_{b,\lambda}(X,L^k)} = I + R_k,$$

for all $\lambda \in (ka, kb)$, we get that for $k \gg 1$, $\sigma_k|_{\mathcal{H}^0_{b,\lambda}(X,L^k)^G} \to \mathcal{H}^0_{b,\lambda}(X_G, L^k_G)$ is injective (see the discussion after Theorem 4.7). Similarly, we can repeat the same procedure, and deduce that for $k \gg 1$, $\sigma^*_k|_{\mathcal{H}^0_{b,\lambda}(X_G,L^k_G)^G}$ is injective and hence $\sigma_k|_{\mathcal{H}^0_{b,\lambda}(X,L^k)^G} \to \mathcal{H}^0_{b,\lambda}(X_G, L^k_G)$ is an isomorphism, for $k \gg 1$.

The advantage of working with F_k and $\hat{F}_k$ lies in the fact that F_k maps sections of L^k over X to sections of L^k over X, and $\hat{F}_k$ maps sections of L^k_G over X_G to sections of L^k_G over X_G. In contrast, $\sigma_k : \mathcal{H}^0_{b,\lambda}(X, L^k)^G \to \mathcal{H}^0_{b,\lambda}(X_G, L^k_G)$, which makes it more difficult to handle directly.

4.1. Preliminary operators for the proof

Fix $p \in Y$ and let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Proposition 3.1 defined in an open set U of p. We may assume that $U = \Omega_1 \times \Omega_2 \times \Omega_3 \times \Omega_4$, where $\Omega_1 \subset \mathbb{R}^d$, $\Omega_2 \subset \mathbb{R}^d$ are open sets of $0 \in \mathbb{R}^d$, $\Omega_3 \subset \mathbb{R}^{2n-2d}$ is an open set of $0 \in \mathbb{R}^{2n-2d}$ and Ω_4 is an open set of $0 \in \mathbb{R}$. From now on, we identify Ω_2 with

$$\{(0, \ldots, 0, x_{d+1}, \ldots, x_{2d}, 0, \ldots, 0) \in U;\ (x_{d+1}, \ldots, x_{2d}) \in \Omega_2\},$$

Ω_3 with $\{(0, \ldots, 0, x_{2d+1}, \ldots, x_{2n}, 0) \in U;\ (x_{d+1}, \ldots, x_{2n}) \in \Omega_3\}$, $\Omega_2 \times \Omega_3$ with

$$\{(0, \ldots, 0, x_{d+1}, \ldots, x_{2n}, 0) \in U;\ (x_{d+1}, \ldots, x_{2n}) \in \Omega_2 \times \Omega_3\}.$$

For $x = (x_1, \ldots, x_{2n+1})$, we write $x'' = (x_{d+1}, \ldots, x_{2n+1})$, $\mathring{x}'' = (x_{d+1}, \ldots, x_{2n})$, $\hat{x}'' = (x_{d+1}, \ldots, x_{2d})$,

$$\widetilde{x}'' = (x_{2d+1}, \ldots, x_{2n+1}), \quad \widetilde{\mathring{x}}'' = (x_{2d+1}, \ldots, x_{2n}).$$

From now on, we identify x'' with $(0, \ldots, 0, x_{d+1}, \ldots, x_{2n+1}) \in U$, $\mathring{x}'' = (x_{d+1}, \ldots, x_{2n})$ with $(0, \ldots, 0, x_{d+1}, \ldots, x_{2n}, 0) \in U$, $\hat{x}''$ with $(0, \ldots, 0, x_{d+1}, \ldots, x_{2d}, 0, \ldots, 0) \in U$, $\widetilde{x}''$ with $(0, \ldots, 0, x_{2d+1}, \ldots, x_{2n+1}) \in U$, $\widetilde{\mathring{x}}''$ with $(0, \ldots, 0, x_{2d+1}, \ldots, x_{2n}, 0)$. Since G acts freely on Y, we take Ω_2 and Ω_3 small enough so that if $x, x_1 \in \Omega_2 \times \Omega_3$ and $x \neq x_1$, then

$$g \cdot x \neq g_1 \cdot x_1, \quad \forall g, g_1 \in G. \tag{126}$$

Let $A(x, y, t) \in \mathcal{C}^\infty(U \times U \times I)$ be as in Theorem 3.4. From $\overline{\partial}_b P^G_{k,\tau^2} = 0$, we can check that

$$\overline{\partial}_b A(x, y, t) \text{ vanishes to infinite order at } \mathrm{diag}\Big((Y \bigcap U) \times (Y \bigcap U)\Big). \tag{127}$$

From (127) and notice that $\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial x_{d+j}} \in T^{0,1}_x X$, $j = 1, \ldots, d$, where $x \in Y$ and $\frac{\partial}{\partial x_j} A(x, y, t) = \frac{\partial}{\partial y_j} A(x, y, t) = 0$, $j = 1, \ldots, d$, we conclude that

$\frac{\partial}{\partial x_{d+j}}A(x,y,t)|_{x_{d+1}=\cdots=x_{2d}=0}$ and $\frac{\partial}{\partial y_{d+j}}A(x,y,t)|_{y_{d+1}=\cdots=y_{2d}=0}$ vanish to infinite order at diag $\Big(Y\bigcap U)\times Y\bigcap U)\Big)$.

Let

$$G_j(x,y,t):=\frac{\partial}{\partial y_{d+j}}A(x,y,t)|_{y_{d+1}=\cdots=y_{2d}=0}\,,$$

and

$$H_j(x,y,t):=\frac{\partial}{\partial x_{d+j}}A(x,y,t)|_{x_{d+1}=\cdots=x_{2d}=0}\,,$$

where $j=1,\ldots,d$. Put

$$\begin{aligned}A_1(x,y,t)&:=A(x,y,t)-\sum_{j=1}^{d}y_{d+j}G_j(x,y,t),\\ A_2(x,y,t)&:=A(x,y,t)-\sum_{j=1}^{d}x_{d+j}H_j(x,y,t).\end{aligned}$$

Then, for $j=1,2,\ldots,d$,

$$\frac{\partial}{\partial y_{d+j}}A_1(x,y,t)|_{y_{d+1}=\cdots=y_{2d}=0}=0\quad\text{and}\quad\frac{\partial}{\partial x_{d+j}}A_2(x,y,t)|_{x_{d+1}=\cdots=x_{2d}=0}=0,\tag{128}$$

and, for $j=1,2$,

$$A(x,y,t)-A_j(x,y,t)\text{ vanishes to infinite order at diag }\Big((Y\bigcap U)\times(Y\bigcap U)\Big).\tag{129}$$

We also write $u=(u_1,\ldots,u_{2n+1})$ to denote the local coordinates of U. For any smooth function $f\in\mathcal{C}^\infty(U)$, we write $\widetilde{f}\in\mathcal{C}^\infty(U^{\mathbb{C}})$ to denote an almost analytic extension of f, where $U^{\mathbb{C}}$ is an open set in $\mathbb{C}^{2n+1}$ with $U^{\mathbb{C}}\cap\mathbb{R}^{2n+1}=U$. We consider the following two systems

$$\begin{aligned}&\frac{\partial\widetilde{A}_1}{\partial\widetilde{u}_{2d+j}}(\widetilde{x},\widetilde{u''},\widetilde{t})+\frac{\partial\widetilde{A}_2}{\partial\widetilde{x}_{2d+j}}(\widetilde{u''},\widetilde{y},\widetilde{s})=0,\quad j=1,2,\ldots,2n-2d,\\ &\frac{\partial\widetilde{A}_2}{\partial\widetilde{s}}(\widetilde{u''},\widetilde{y},\widetilde{s})=0,\end{aligned}\tag{130}$$

and

$$\begin{aligned}&\frac{\partial\widetilde{A}_1}{\partial\widetilde{u}_{d+j}}(\widetilde{x},\widetilde{u''},\widetilde{t})+\frac{\partial\widetilde{A}_2}{\partial\widetilde{x}_{d+j}}(\widetilde{u''},\widetilde{y},\widetilde{s})=0,\quad j=1,2,\ldots,2n-d,\\ &\frac{\partial\widetilde{A}_2}{\partial\widetilde{s}}(\widetilde{u''},\widetilde{y},\widetilde{s})=0,\end{aligned}\tag{131}$$

where $\widetilde{\widetilde{u}''} = (0, \ldots, 0, \widetilde{u}_{2d+1}, \ldots, \widetilde{u}_{2n+1})$, $\widetilde{u''} = (0, \ldots, 0, \widetilde{u}_{d+1}, \ldots, \widetilde{u}_{2n+1})$. From (128), we can take $\widetilde{A}_1$ and $\widetilde{A}_2$ so that for every $j = 1, 2, \ldots, d$,

$$\frac{\partial \widetilde{A}_1}{\partial \widetilde{u}_{d+j}}(\widetilde{x}, \widetilde{u''}, t) = 0 \quad \text{and} \quad \frac{\partial \widetilde{A}_2}{\partial \widetilde{x}_{d+j}}(\widetilde{u''}, \widetilde{y}, t) = 0, \quad \text{if } \widetilde{u}_{d+1} = \cdots = \widetilde{u}_{2d} = 0, \tag{132}$$

and, for $j = 1, 2$,

$$\widetilde{A}_j(\widetilde{x}, \widetilde{y}, t) = (\widetilde{x}_{2n+1} - \widetilde{y}_{2n+1})\, t + \widetilde{\hat{A}_j}(\widetilde{\mathring{x}''}, \widetilde{\mathring{y}''}, t), \quad \widetilde{\hat{A}_j} \in \mathcal{C}^\infty(U^{\mathbb{C}} \times U^{\mathbb{C}}), \tag{133}$$

where $\widetilde{\mathring{x}''} = (0, \ldots, 0, \widetilde{x}_{d+1}, \ldots, \widetilde{x}_{2n}, 0)$, $\widetilde{\mathring{y}''} = (0, \ldots, 0, \widetilde{y}_{d+1}, \ldots, \widetilde{y}_{2n}, 0)$.

From (101), it is not difficult to see that

$$\begin{aligned}
&\frac{\partial \widetilde{A}_1}{\partial \widetilde{u}_{d+j}}(\widetilde{x}'', \widetilde{x}'', t) + \frac{\partial \widetilde{A}_2}{\partial \widetilde{x}_{d+j}}(\widetilde{x}'', \widetilde{x}'', t) = 0, \quad j = 1, 2, \ldots, 2n-d, \\
&\frac{\partial \widetilde{A}_2}{\partial \widetilde{s}}(\widetilde{x}'', \widetilde{x}'', t) = 0.
\end{aligned}$$

Hence, at $x' = 0, \hat{x}'' = 0$, $(\widetilde{\widetilde{u}''}, \widetilde{s}) = (\widetilde{x}'', t)$ and $(\widetilde{u''}, \widetilde{s}) = (\widetilde{x}'', t)$ are real critical points of (130) and (131) respectively. Let

$$\begin{aligned}
F(\widetilde{x}, \widetilde{y}, \widetilde{\widetilde{u}''}, \widetilde{s}, \widetilde{t}) &:= (\widetilde{A}_1(\widetilde{x}, \widetilde{\widetilde{u}''}, \widetilde{t}) + \widetilde{A}_2(\widetilde{\widetilde{u}''}, \widetilde{y}, \widetilde{s}), \frac{\partial \widetilde{A}_2}{\partial \widetilde{s}}(\widetilde{\widetilde{u}''}, \widetilde{y}, \widetilde{s})), \\
\hat{F}(\widetilde{x}, \widetilde{y}, \widetilde{u}'', \widetilde{s}, \widetilde{t}) &:= (\widetilde{A}_1(\widetilde{x}, \widetilde{u''}, \widetilde{t}) + \widetilde{A}_2(\widetilde{u''}, \widetilde{y}, \widetilde{s}), \frac{\partial \widetilde{A}_2}{\partial \widetilde{s}}(\widetilde{u''}, \widetilde{y}, \widetilde{s})).
\end{aligned}$$

Let $\mathrm{Hess}_{(\widetilde{s}, \widetilde{\widetilde{u}''})} F(\widetilde{x}, \widetilde{y}, \widetilde{\widetilde{u}''}, \widetilde{s}, \widetilde{t})$ denote the complex Hessian of F with respect to $(\widetilde{s}, \widetilde{\widetilde{u}''})$ at $(\widetilde{x}, \widetilde{y}, \widetilde{\widetilde{u}''}, \widetilde{s}, \widetilde{t})$ and let $\mathrm{Hess}_{(\widetilde{s}, \widetilde{u}'')} \hat{F}(\widetilde{x}, \widetilde{y}, \widetilde{u}'', \widetilde{s}, \widetilde{t})$ denote the complex Hessian of $\hat{F}$ with respect to $(\widetilde{s}, \widetilde{u}'')$ at $(\widetilde{x}, \widetilde{y}, \widetilde{u}'', \widetilde{s}, \widetilde{t})$. We can check that the matrices

$$\mathrm{Hess}_{(\widetilde{s}, \widetilde{\widetilde{u}''})} F(\widetilde{x}, \widetilde{y}, \widetilde{\widetilde{u}''}, \widetilde{s}, \widetilde{t})|_{\widetilde{x} = \widetilde{y} = \widetilde{x}'', \widetilde{\widetilde{u}''} = \widetilde{x}'', \widetilde{s} = t}, \quad \mathrm{Hess}_{(\widetilde{s}, \widetilde{u}'')} \hat{F}(\widetilde{x}, \widetilde{y}, \widetilde{u}'', \widetilde{s}, \widetilde{t})|_{\widetilde{x} = \widetilde{y} = \widetilde{x}'', \widetilde{u}'' = \widetilde{x}'', \widetilde{s} = t}$$

are non-singular, for every $t \in I$. Moreover, fix $t_0 \in I$. From Theorem 3.4, it is straightforward to see that

$$\begin{aligned}
&\det \mathrm{Hess}_{(\widetilde{s}, \widetilde{\widetilde{u}''})} F(\widetilde{x}, \widetilde{y}, \widetilde{\widetilde{u}''}, \widetilde{s}, \widetilde{t})|_{\widetilde{x} = \widetilde{y} = p, \widetilde{\widetilde{u}''} = p, \widetilde{s} = t_0} = (-1)(2i\,|\lambda_{d+1}(t_0)| \cdots 2i\,|\lambda_n(t_0)|)^2, \\
&\det \mathrm{Hess}_{(\widetilde{s}, \widetilde{u}'')} \hat{F}(\widetilde{x}, \widetilde{y}, \widetilde{u}'', \widetilde{s}, \widetilde{t})|_{\widetilde{x} = \widetilde{y} = p, \widetilde{u}'' = p, \widetilde{s} = t_0} \\
&= (-1)(2i\,|\lambda_1(t_0)| \cdots 2i\,|\lambda_d(t_0)|)(2i\,|\lambda_{d+1}(t_0)| \cdots 2i\,|\lambda_n(t_0)|)^2.
\end{aligned} \tag{134}$$

Hence, near (p, p), we can solve (130) and (131) and the solutions are unique. Let

$$(\widetilde{\tilde{u}''}, \widetilde{s}) = (\alpha(\widetilde{x}, \widetilde{y}, \widetilde{t}), \gamma(\widetilde{x}, \widetilde{y}, \widetilde{t})),$$
$$\alpha(\widetilde{x}, \widetilde{y}, \widetilde{t}) = (\alpha_{2d+1}(\widetilde{x}, \widetilde{y}, \widetilde{s}), \ldots, \alpha_{2n}(\widetilde{x}, \widetilde{y}, \widetilde{t}) \in \mathcal{C}^\infty(U^{\mathbb{C}} \times U^{\mathbb{C}} \times I^{\mathbb{C}}, \mathbb{C}^{2n-2d}),$$
$$\gamma(\widetilde{x}, \widetilde{y}, \widetilde{t}) \in \mathcal{C}^\infty(U^{\mathbb{C}} \times U^{\mathbb{C}} \times I^{\mathbb{C}}, \mathbb{C}),$$

and

$$(\widetilde{u''}, \widetilde{s}) = (\beta(\widetilde{x}, \widetilde{y}, \widetilde{t}), \delta(\widetilde{x}, \widetilde{y}, \widetilde{t})),$$
$$\beta(\widetilde{x}, \widetilde{y}, \widetilde{t}) = (\beta_{d+1}(\widetilde{x}, \widetilde{y}, \widetilde{s}), \ldots, \beta_{2n}(\widetilde{x}, \widetilde{y}, \widetilde{t}) \in \mathcal{C}^\infty(U^{\mathbb{C}} \times U^{\mathbb{C}} \times I^{\mathbb{C}}, \mathbb{C}^{2n-d}),$$
$$\delta(\widetilde{x}, \widetilde{y}, \widetilde{t}) \in \mathcal{C}^\infty(U^{\mathbb{C}} \times U^{\mathbb{C}} \times I^{\mathbb{C}}, \mathbb{C})$$

be the solutions of (130) and (131), respectively. From (132), it is easy to see that

$$\begin{aligned} &\beta(x, y, t) = (\beta_{d+1}(x, y, t), \ldots, \beta_{2n}(x, y, t)) = (0, \ldots, 0, \alpha_{2d+1}(x, y, t), \ldots, \alpha_{2n}(x, y, t)), \\ &\delta(x, y, t) = \gamma(x, y, t). \end{aligned} \tag{135}$$

From (135), we see that the value of $\widetilde{A}_1(x, \widetilde{\tilde{u}''}, \widetilde{t}) + \widetilde{A}_2(\widetilde{\tilde{u}''}, y, \widetilde{s})$ at critical points $\widetilde{\tilde{u}''} = \alpha(x, y, t)$, $\widetilde{s} = \gamma(x, y, t)$ is equal to the value of $\widetilde{A}_1(x, \widetilde{u''}, t) + \widetilde{A}_2(\widetilde{u''}, y, t)$ at critical points $\widetilde{u''} = \beta(x, y, t)$, $\widetilde{s} = \delta(x, y, t)$. Put

$$\begin{aligned} A_3(x, y, t) :&= \widetilde{A}_1(x, \alpha(x, y, t), t) + \widetilde{A}_2(\alpha(x, y, t), y, \gamma(x, y, t)) \\ &= \widetilde{A}_1(x, \beta(x, y, t), t) + \widetilde{A}_2(\beta(x, y, t), y, \delta(x, y, t)). \end{aligned} \tag{136}$$

$A_3(x, y, t)$ is a complex phase function. From (133), we have

$$A_3(x, y, t) = (x_{2n+1} - y_{2n+1})t + \hat{A}_3(\mathring{x}'', \mathring{y}'', t), \quad \hat{A}_3(\mathring{x}'', \mathring{y}'') \in \mathcal{C}^\infty(U \times U).$$

Definition 4.1. Let $\Phi_1, \Phi_2 \in \mathcal{C}^\infty(U \times U \times I)$. Assume that Φ_1 and Φ_2 satisfy (101) and (106). We say that Φ_1 and Φ_2 are equivalent on U if for any $b_1(x, y, t, k) \in S^{n-\frac{d}{2}}_{\mathrm{loc,cl}}(U \times U \times I)$, $\mathrm{supp}_t b_1(x, y, t, k) \subset I$, we can find $b_2(x, y, t, k) \in S^{n-\frac{d}{2}}_{\mathrm{loc,cl}}(U \times U \times I)$, $\mathrm{supp}_t b_2(x, y, t, k) \subset I$, such that

$$\int e^{ik\Phi_1(x,y,t)} b_1(x, y, t, k) dt = \int e^{ik\Phi_2(x,y,t)} b_2(x, y, t, k) dt + O(k^{-\infty}) \ \text{ on } U \times U$$

and vice versa.

Theorem 4.1. *A_1 and A_3 are equivalent on U in the sense of Definition 4.1.*

Proof. Let s be a local rigid CR trivializing section of L defined on U. We consider the localized kernel of $P_{k,\tau^2} \circ P_{k,\tau^2}$ on U. Let $V \Subset U$ be an open set of p. Let $\chi(x'') \in \mathcal{C}^\infty_c(\Omega_2 \times \Omega_3 \times \Omega_4)$. From (126), we can extend $\chi(x'')$ to $W := \{g \cdot x;\ g \in G, x \in \Omega_2 \times \Omega_3 \times \Omega_4\}$ by

$\chi(g\cdot x'') := \chi(x'')$, for every $g \in G$. Assume that $\chi = 1$ on some neighborhood of V. Let $\chi_1 \in \mathcal{C}^\infty_c(U)$ with $\chi_1 = 1$ on some neighborhood of V and $\operatorname{Supp}\chi_1 \subset \{x \in X;\, \chi(x) = 1\}$. We have

$$\chi_1 P^G_{k,\tau^2} \circ P^G_{k,\tau^2} = \chi_1 P^G_{k,\tau^2}\chi \circ P^G_{k,\tau^2} + \chi_1 P^G_{k,\tau^2}(1-\chi)\circ P^G_{k,\tau^2}. \tag{137}$$

Let's first consider $\chi_1 P^G_{k,\tau^2}(1-\chi)\circ P^G_{k,\tau^2}$. We have

$$(\chi_1 P^G_{k,\tau^2}(1-\chi))(x,u) = \chi_1(x)\int_G P_{k,\tau^2}(x, g\cdot u)(1-\chi(u))d\mu(g). \tag{138}$$

If $u \notin \{x \in X;\, \chi(x) = 1\}$. Since $\operatorname{Supp}\chi_1 \subset \{x \in X;\, \chi(x) = 1\}$ and $\chi(x) = \chi(g\cdot x)$, for every $g \in G$, for every $x \in X$, we conclude that $g \cdot u \notin \operatorname{Supp}\chi_1$, for every $g \in G$. From this observation and notice that P^G_{k,τ^2} is $O(k^{-\infty})$ away from diagonal, we deduce that $\chi_1 P^G_{k,\tau^2}(1-\chi) = O(k^{-\infty})$ and hence

$$\chi_1 P^G_{k,\tau^2}(1-\chi)\circ P^G_{k,\tau^2} = O(k^{-\infty}). \tag{139}$$

From (137) and (139), we get

$$\chi_1 P^G_{k,\tau^2}\circ P^G_{k,\tau^2} = \chi_1 P^G_{k,\tau^2}\chi\circ P^G_{k,\tau^2} + O(k^{-\infty}). \tag{140}$$

We can check that on U,

$$\begin{aligned}
&(\chi_1 P^G_{k,\tau^2}\chi \circ P^G_{k,\tau^2})_s(x,y)\\
&= \int e^{ikA_1(x,u'',t)+ikA_2(u'',y,r)}\chi_1(x)g(x,\mathring{u}'',t,k)\chi(u'')g(u'',\mathring{y}'',r,k)dv(u'')dr\,dt\\
&\quad + O(k^{-\infty}),
\end{aligned} \tag{141}$$

where $(\chi_1 P^G_{k,\tau^2}\chi \circ P^G_{k,\tau^2})_s(x,y)$ denotes the distribution kernel of the localization of $\chi_1 P^G_{k,\tau^2}\chi\circ P^G_{k,\tau^2}$ with respect to s (see the discussion after (49)) and where $d\mu(g)dv(u'') = dV_X(x)$ on U. We use complex stationary phase formula of Melin-Sjöstrand to carry out the integral (141) and get

$$\begin{aligned}
&(\chi_1 P^G_{k,\tau^2}\chi \circ P^G_{k,\tau^2})(x,y) = \int e^{ikA_3(x,y,t)}a(x,y,t,k)dt + O(k^{-\infty}) \ \text{ on } U,\\
&a(x,y,t,k) \in S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U\times I),\\
&a(x,y,t,k) \sim \sum_{j=0}^{\infty} k^{n+1-\frac{d}{2}-j}a_j(x,y,t) \text{ in } S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U\times U\times I),\\
&a_j(x,y,t) \in \mathcal{C}^\infty(U\times U\times I), \quad j = 0,1,2,\ldots,\\
&\operatorname{supp}_t a(x,y,t,k) \subset I, \quad \operatorname{supp}_t a_j(x,y,t) \subset I, \quad j = 0,1,\ldots,\\
&a_0(x,x,t) \neq 0, \quad \text{for every } x \in Y\cap U,\, t \in \{t \in I;\, \tau(t) \neq 0\}.
\end{aligned} \tag{142}$$

From (140), (142) and notice that $(\chi_1 P^G_{k,\tau^2} \circ P^G_{k,\tau^2})(x,y) = (\chi_1 P^G_{k,\tau^4})(x,y)$, we deduce that

$$\int e^{ikA_3(x,y,t)}a(x,y,t,k)dt = \int e^{ikA(x,y,t)}\chi_1(x)\hat{b}(x,y,t,k)dt + O(k^{-\infty}) \quad \text{on } U, \tag{143}$$

where $\hat{b}(x,y,t,k) \in S^{n+1-\frac{d}{2}}_{\rm loc}(1; U\times U\times I)$, $\hat{b}(x,y,t,k) \sim \sum_{j=0}^{\infty} k^{n+1-\frac{d}{2}-j}\hat{b}_j(x,y,t)$ in $S^{n+1-\frac{d}{2}}_{\rm loc}(1; U\times U\times I)$, $\hat{b}_j(x,y,t) \in \mathcal{C}^\infty(U\times U\times I)$, $j=0,1,2,\ldots$, ${\rm supp}_t\,\hat{b}(x,y,t,k) \subset I$, ${\rm supp}_t\,\hat{b}_j(x,y,t) \subset I$, $j=0,1,\ldots$, $\hat{b}_0(x,x,t)\neq 0$, for every $x\in Y\cap U$, $t\in\{t\in I;\ \tau(t)\neq 0\}$. From (143), it is not difficult to check that $A_3(x,y,t)-A(x,y,t)$ vanishes to infinite order at $x=y$, the theorem follows. □

The following two theorems follow from (129), (136), Theorem 4.1, complex stationary phase formula of Melin-Sjöstrand [27] and some straightforward computation. We omit the details.

Theorem 4.2. *With the notations used above, let*

$$\begin{aligned} &F_k(x,y) = \int e^{ikA(x,y,t)}a(x,y,t,k)dt, \quad G_k(x,y) = \int e^{ikA(x,y,t)}b(x,y,t,k)dt,\\ &a(x,y,t,k)\in S^m_{{\rm loc},{\rm cl}}(1;U\times U\times I), \qquad b(x,y,t,k)\in S^\ell_{{\rm loc},{\rm cl}}(1;U\times U\times I),\\ &{\rm supp}_t\,a(x,y,t,k)\subset I, \quad {\rm supp}_t\,b(x,y,t,k)\subset I. \end{aligned}$$

Let $\chi(x'')\in\mathcal{C}^\infty_c(\Omega_2\times\Omega_3\times\Omega_4)$. Then, we have

$$\begin{aligned} &\int F_k(x,u)\chi(u'')G_k(u,y)dV_X(u'') = \int e^{ikA(x,y,t)}c(x,y,t,k)dt + O(k^{-\infty}),\\ &c(x,y,t,k)\in S^{m+\ell-(n+1-\frac{d}{2})}_{{\rm loc},{\rm cl}}(1;U\times U\times I),\\ &c_0(x,x,t) = (2\pi)^{n-\frac{d}{2}+1}\left|\det\left(R^L_x - 2t\mathcal{L}_x\right)\right|^{-1}\left|\det R_x(t)\right|^{\frac{1}{2}}a_0(x,x,t)b_0(x,x,t)\chi(x''),\\ &\quad \forall x\in Y\bigcap U, \end{aligned}$$

where $|\det R_x(t)|$ is in the discussion after (10) and c_0, a_0, b_0 denote the leading terms of c, a, b respectively.

Moreover, if there are $N_1, N_2\in\mathbb{N}$, such that $|a_0(x,y,t)|\leq C\left|(x,y)-(x_0,x_0)\right|^{N_1}$, $|b_0(x,y,t)|\leq C\left|(x,y)-(x_0,x_0)\right|^{N_2}$, for all $x_0\in Y\bigcap U$, $t\in I$, where $C>0$ is a constant, then,

$$|c_0(x,y,t)|\leq\hat{C}\left|(x,y)-(x_0,x_0)\right|^{N_1+N_2},$$

for all $x_0\in Y\bigcap U$, $t\in I$, where $\hat{C}>0$ is a constant.

Theorem 4.3. *With the notations used above, let*

$$\begin{aligned}&\mathcal{F}_k(x,\widetilde{y}'')=\int e^{ikA(x,\widetilde{y}'',t)}\alpha(x,\widetilde{y}'',t,k),\quad \mathcal{G}_k(\widetilde{x}'',y)=\int e^{ikA(\widetilde{x}'',y,t)}\beta(\widetilde{x}'',y,t,k)dt,\\&\alpha(x,\widetilde{y}'',t,k)\in S^m_{\mathrm{loc},\mathrm{cl}}(1;U\times(\Omega_3\times\Omega_4)\times I),\\&\beta(\widetilde{x}'',y,t,k)\in S^\ell_{\mathrm{loc},\mathrm{cl}}(1;(\Omega_3\times\Omega_4)\times U\times I).\end{aligned}$$

Let $\chi_1(\widetilde{x}'')\in\mathcal{C}^\infty_c(\Omega_3\times\Omega_4)$*. Then, we have*

$$\begin{aligned}&\int\mathcal{F}_k(x,\widetilde{u}'')\chi_1(\widetilde{u}'')\mathcal{G}_k(\widetilde{u}'',y)dV_X(\widetilde{u})=e^{ikA(x,y,t)}\gamma(x,y,t,k)+O(k^{-\infty}),\\&\gamma(x,y,t,k)\in S^{m+\ell-(n-d+1)}_{\mathrm{loc},\mathrm{cl}}(1;U\times U\times I),\\&\gamma_0(x,x,t)=(2\pi)^{n-d+1}\left|\det\left(R^L_x-2t\mathcal{L}_x\right)\right|^{-1}\left|\det R_x(t)\right|\alpha_0(x,\widetilde{x}'',t)\beta_0(\widetilde{x}'',x,t)\chi_1(\widetilde{x}''),\\&\quad\forall x\in Y\bigcap U,\end{aligned}$$

where $|\det R_x(t)|$ *is in the discussion after* (10) *and* γ_0*,* α_0*,* β_0 *denote the leading term of* γ*,* α*,* β *respectively.*

Moreover, if there are $N_1,N_2\in\mathbb{N}$*, such that* $|\alpha_0(x,\widetilde{y}'',t)|\leq C\left|(x,\widetilde{y}'')-(x_0,x_0)\right|^{N_1}$*,* $|\beta_0(x,\widetilde{y}'',t)|\leq C\left|(x,\widetilde{y}'')-(x_0,x_0)\right|^{N_2}$*, for all* $x_0\in Y\bigcap U$*,* $t\in I$*, where* $C>0$ *is a constant, then,*

$$|\gamma_0(x,y,t)|\leq\hat{C}\left|(x,y)-(x_0,x_0)\right|^{N_1+N_2},$$

for all $x_0\in Y\bigcap U$*,* $t\in I$*, where* $\hat{C}>0$ *is a constant.*

4.2. Definition of the isomorphism and proof of Theorem 1.4

Let P_{k,X_G,τ^2} be the weighted Fourier-Szegő operator on X_G as in (48). Fix $p\in Y$. Let s be a local CR rigid G-invariant trivializing section of L defined on a G-invariant open set D of p. Let $x=(x_1,\ldots,x_{2n+1})$ be the local coordinates as in Proposition 3.1 defined in an open set U of p, $U\subset D$. We will use the same notations as in Section 4.1. We will identify $\Omega_3\times\Omega_4$ with an open set in X_G and we will identify p as a point in X_G. Let $\varphi_{X_G}(\widetilde{x}'',\widetilde{y}'',t)$ be the phase as in Theorem 2.2. We write $A(\widetilde{x}'',\widetilde{y}'',t):=A(x,y,t)|_{(\Omega_3\times\Omega_4)\times(\Omega_3\times\Omega_4)}$. It is not difficult to see that $\overline{\partial}_{b,X_G}A(\widetilde{x}'',\widetilde{y}'',t)$ vanishes to infinite order at $\widetilde{x}''=\widetilde{y}''$ and $A(\widetilde{x}'',\widetilde{y}'',t)$ satisfies (56). From this observation, we can repeat the process in [15, Section 3.7] with minor change and deduce that

$$\varphi_{X_G}(\widetilde{x}'',\widetilde{y}'',t)\text{ and }A(\widetilde{x}'',\widetilde{y}'',t)\text{ are equivariant on }\Omega_3\times\Omega_4.\tag{144}$$

Let s be a local trivializing G-invariant CR rigid sections of $L|_{X_G}$ defined on $W:=\Omega_3\times\Omega_4$. Thus,

$$\begin{aligned}
&P_{X_G,k,s}(\widetilde{x}'',\widetilde{y}'') = e^{ikA(\widetilde{x}'',\widetilde{y}'',t)}b(\widetilde{x}'',\widetilde{y}'',t,k)dt + O(k^{-\infty}) \text{ on } W,\\
&\beta(\widetilde{x}'',\widetilde{y}'',t,k) \in S^{n+1-d}_{\mathrm{loc}}(1; W\times W\times I),\\
&\beta(\widetilde{x}'',\widetilde{y}'',t,k) \sim \sum_{j=0}^{\infty} k^{n+1-d-j} b_j(\widetilde{x}'',\widetilde{y}'',t) \text{ in } S^{n+1-d}_{\mathrm{loc}}(1; W\times W\times I),\\
&\beta_j(\widetilde{x}'',\widetilde{y}'',t) \in \mathcal{C}^\infty(W\times W\times I), \quad j=0,1,2,\ldots,\\
&\mathrm{supp}\,_t\beta \subset I, \quad \mathrm{supp}\,_t\beta_j \subset I, \quad j=0,1,2,\ldots,\\
&\beta_0(\widetilde{x}'',\widetilde{x}'') = (2\pi)^{-(n-d)-1}\left|\det(R^{L_{X_G}}_{\widetilde{x}''} - 2t\mathcal{L}_{X_G,\widetilde{x}''})\right|\tau^2(t), \quad \forall \widetilde{x}'' \in W.
\end{aligned} \tag{145}$$

Let

$$f(x) = \pi^{\frac{d}{4}}\sqrt{V_{\mathrm{eff}}\,(x)}\,|\det\, R_x(t)|^{-\frac{1}{4}} \in \mathcal{C}^\infty(Y)^G.$$

Recall that R_x is given by (10). We will identify f with a smooth function on X_G, then $f \in \mathcal{C}^\infty(X_G)$. Let

$$\begin{aligned}
\sigma_k : \mathcal{C}^\infty(X, L^k) &\to H^0_b(X_G, L^k_G),\\
u &\to k^{-\frac{d}{4}} P_{k,X_G,\tau^2}\circ f\circ \gamma_G \circ P^G_{k,\tau^2}u,
\end{aligned} \tag{146}$$

where $\gamma_G : \mathcal{C}^\infty(X, L^k)^G \to \mathcal{C}^\infty(X_G, L^k_G)$ is the natural restriction. Let $R^{L_{X_G}}$ be the curvature of $L_{X_G} := L|_{X_G}$ induced by h^L and let $\mathcal{L}_{X_G}$ be the Levi form on X_G induced by $\omega_{0,X_G} := \omega_0|_{X_G}$. We can now prove

Theorem 4.4. *With the notations used above, if $y \notin Y$, then for any open set D of y with $\overline{D}\bigcap Y = \emptyset$, we have*

$$\sigma_k = O(k^{-\infty}) \quad \text{on } X_G \times D. \tag{147}$$

Let $p \in Y$. Let s be a local trivializing G-invariant CR rigid section of L defined on an open set D of p in X. Let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Proposition 3.1 defined in an open set U of p, $U \subset D$. Let $\sigma_{k,s}$ be the localization of σ_k with respect to s. Then,

$$\begin{aligned}
&\sigma_{k,s}(\widetilde{x}'',y) = \int e^{ikA(\widetilde{x}'',y'',t)}\alpha(\widetilde{x}'',y'',k,t) + O(k^{-\infty}) \quad \text{on } W\times U,\\
&\alpha(\widetilde{x}'',y'',t,k) \in S^{n-\frac{3}{4}d+1}_{\mathrm{loc}}(1; W\times U\times I),\\
&\alpha(\widetilde{x}'',y'',t,k) \sim \sum_{j=0}^{\infty} k^{n+1-\frac{3}{4}d-j}\alpha_j(\widetilde{x}'',y'',t) \text{ in } S^{n+1-\frac{3}{4}d}_{\mathrm{loc}}(1; W\times U\times I),\\
&\alpha_j(\widetilde{x}'',y'',t) \in \mathcal{C}^\infty(W\times U\times I), \quad j=0,1,2,\ldots,\\
&\mathrm{supp}\,_t\alpha(\widetilde{x}'',y'',t,k) \subset I, \quad \mathrm{supp}\,_t\alpha_j(\widetilde{x}'',y'',t,k) \subset I, \quad j=0,1,\ldots,
\end{aligned} \tag{148}$$

$$\alpha_0(\widetilde{x}'',\widetilde{x}'',t)=2^{-n-1+d}\pi^{\frac{3d}{4}-n-1}\frac{1}{\sqrt{V_{\text{eff}}(\widetilde{x}'')}}\left|\det\left(R^{L_{X_G}}_{\widetilde{x}''}-2t\mathcal{L}_{\widetilde{x}''}\right)\right|\left|\det R_{\widetilde{x}''}\right|^{\frac{1}{4}}\tau^4(t),\quad \forall \widetilde{x}''\in W, \tag{149}$$

where $W=\Omega_3\times\Omega_4$, Ω_3 and Ω_4 are open sets as in the beginning of Section 4.1.

Proof. Note that $P^G_{k,\tau^2}=O(k^{-\infty})$ away Y. From this observation, we get (147).

Fix $u=(u_1,\ldots,u_{2n+1})\in Y\bigcap U$. From (147), we only need to show that (148) and (149) hold near u and we may assume that $u=(0,\ldots,0,u_{2d+1},\ldots,u_{2n},u_{2n+1})=\widetilde{u}''$. Let V be a small neighborhood of u. Let $\chi(\widetilde{u}'')\in\mathcal{C}^\infty_c(\Omega_3\times\Omega_4)$. From (126), we can extend $\chi(\widetilde{x}'')$ to

$$Q=\{g\cdot x;\ g\in G, x\in\Omega_3\times\Omega_4\}$$

by $\chi(g\cdot\widetilde{x}''):=\chi(\widetilde{x}'')$, for every $g\in G$. Assume that $\chi=1$ on some neighborhood of V. Let $V_G:=V/G$ and let $\pi:V\to V_G$ be the natural projection. Let $\chi_1\in\mathcal{C}^\infty(X_G)$ with $\chi_1=1$ on some neighborhood of V_G and $\operatorname{Supp}\chi_1\subset\{\pi(x)\in Y_G;\ x\in Y,\chi(x)=1\}$. We have

$$\begin{aligned}\chi_1\sigma_k&=k^{-\frac{d}{4}}\chi_1P_{k,X_G,\tau^2}\circ f\circ\gamma_G\circ P^G_{k,\tau^2}\\&=k^{-\frac{d}{4}}\chi_1P_{k,X_G,\tau^2}\circ f\circ\gamma_G\circ\chi P^G_{k,\tau^2}\\&\quad+k^{-\frac{d}{4}}\chi_1P_{k,X_G,\tau^2}\circ f\circ\gamma_G\circ(1-\chi)P^G_{k,\tau^2}.\end{aligned}\tag{150}$$

If $u\in Y$ but $u\notin\{x\in X;\ \chi(x)=1\}$. Since $\operatorname{Supp}\chi_1\subset\{\pi(x)\in X;\ x\in Y,\chi(x)=1\}$ and $\chi(x)=\chi(g\cdot x)$, for every $g\in G$, for every $x\in X$, we conclude that $\pi(u)\notin\operatorname{Supp}\chi_1$. From this observation, we get

$$k^{-\frac{d}{4}}\chi_1P_{k,X_G,\tau^2}\circ f\circ\gamma_G\circ(1-\chi)P^G_{k,\tau^2}=O(k^{-\infty})\ \text{ on } X_G\times X. \tag{151}$$

From (150) and (151), we get

$$\chi_1\sigma_k=k^{-\frac{d}{4}}\chi_1P_{k,X_G,\tau}\circ f\circ\gamma_G\circ\chi P^G_{k,\tau}+O(k^{-\infty})\ \text{ on } X_G\times X.$$

From Theorem 1.1 and (145), we can check that on U,

$$\chi_1\sigma_{k,s}(\widetilde{x}'',y)=\int e^{ikA(\widetilde{x}'',\widetilde{v}'',t)+ikA(\widetilde{v}'',y,u)}\chi_1(\widetilde{x})\beta(\widetilde{x}'',\widetilde{v}'',k,t)\hat{b}(\widetilde{v}'',y,u,k)dV_{X_G}(\widetilde{v}'')dtdu+O(k^{-\infty}), \tag{152}$$

where $\hat{b}(\widetilde{v}'',y,u,k)=\Big(f\circ\gamma_G\circ\chi(\widetilde{v}'')\circ g\Big)(\widetilde{v}'',y,u,k)$, g is the symbol as in Theorem 1.1. We remind the reader that $\sigma_{k,s}$ denotes the localization of σ_k with respect to the given local trivializing G-invariant CR rigid section s (see the discussion after (49)).

From (152) and Theorem 4.3, we see that (148) and (149) hold near u. The theorem follows. □

Let

$$F_k := \sigma_k^*\sigma_k : \mathcal{C}^\infty(X, L^k) \to \mathcal{H}_b^0(X, L^k)^G, \quad \hat{F}_k := \sigma_k\sigma_k^* : \mathcal{C}^\infty(X_G, L^k_{X_G}) \to \mathcal{H}_b^0(X_G, L^k_{X_G}).$$

We explain now that why F_k lands in $\mathcal{H}_b^0(X, L^k)^G$. Fix $u \in \mathcal{C}^\infty(X, L^k)$. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\chi \equiv 1$ near $\operatorname{supp}\tau$. Let $v \in \mathcal{C}^\infty(X, L^k)$. We have

$$\begin{aligned}
&(\,(I - P^G_{k,\chi})F_k u \,|\, v\,)_k \\
&= (\,F_k \,|\, (I - P^G_{k,\chi})v\,)_k \\
&= (\,\sigma_k^*\sigma_k u \,|\, (I - P^G_{k,\chi} v\,)_k \\
&= (\,\sigma_k u \,|\, \sigma_k(I - P^G_{k,\chi})v\,)_k.
\end{aligned} \tag{153}$$

From the definition of σ_k (see (146)), we see that $\sigma_k(I - P^G_{k,\chi})v = 0$. From this observation and (153), we get $(I - P^G_{k,\chi})F_k u = 0$, for all $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$, $\chi \equiv 1$ near $\operatorname{supp}\tau$. Hence, $F_k u \in \mathcal{H}_b^0(X, L^k)^G$.

From Theorem 4.2, Theorem 4.3, we can repeat the proof of Theorem 4.4 with minor change and deduce the following two theorems

Theorem 4.5. *With the notations used above, if $y \notin Y$, then for any open set D of y with $\overline{D}\bigcap Y = \emptyset$, we have $F_k = O(k^{-\infty})$ on $X \times D$. Let $p \in Y$. Let s be a local trivializing G-invariant CR rigid sections of L defined on an open sets D of p in X. Let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Proposition 3.1 defined in an open set U of p, $U \subset D$. Let $F_{k,s}$ be the localization of σ_k with respect to s. Then,*

$$\begin{aligned}
&F_k(x, y) = e^{ikA(x'', y'', t)}a(x'', y'', t, k) + O(k^{-\infty}) \quad \text{on } U \times U, \\
&a(x'', y'', t, k) \in S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\
&a(x'', y'', t, k) \sim \sum_{j=0}^{\infty} k^{n+1-\frac{d}{2}-j} a_j(\widetilde{x}'', y'', t) \text{ in } S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\
&a_j(x'', y'', t) \in \mathcal{C}^\infty(U \times U \times I), \quad j = 0, 1, 2, \ldots, \\
&\operatorname{supp}_t a \subset I, \quad \operatorname{supp} a_j \subset I, \quad j = 0, 1, 2, \ldots,
\end{aligned}$$

and

$$a_0(\widetilde{x}'', \widetilde{x}'', t) = 2^{-n-1+d}\frac{1}{V_{\mathrm{eff}}(\widetilde{x}'')}\left|\det R_{\widetilde{x}''}(t)\right|^{-1/2}\pi^{-n-1+d/2}\left|\det(R^L_x - 2t\mathcal{L}_{\widetilde{x}''})\right|\tau^8(t), \tag{154}$$

for all $\widetilde{x}'' \in U$.

Theorem 4.6. *Let $p \in Y$. Let s be a local trivializing G-invariant CR rigid section of L defined on an open set D of p in X and let $x = (x_1, \dots, x_{2n+1})$ be the local coordinates as in Proposition 3.1 defined in an open set U of p, $U \subset D$. Then,*

$$\begin{aligned}
&\hat{F}_k(\widetilde{x}'', \widetilde{y}'') = \int e^{ikA(\widetilde{x}'', \widetilde{y}'', t)} \hat{a}(\widetilde{x}'', \widetilde{y}'', t, k) dt + O(k^{-\infty}) \quad \textit{on } W \times W, \\
&\hat{a}(\widetilde{x}'', \widetilde{y}'', t, k) \in S^{n+1-d}_{\mathrm{loc}}(1; W \times W \times I), \\
&\hat{a}(\widetilde{x}'', \widetilde{y}'', t, k) \sim \sum_{j=0}^{\infty} k^{n+1-d-j} \hat{a}_j(\widetilde{x}'', \widetilde{y}'', t) \ \textit{in } S^{n-d}_{\mathrm{loc}}(1; W \times W \times I), \\
&\hat{a}_j(\widetilde{x}'', \widetilde{y}'', t) \in \mathcal{C}^\infty(W \times W \times I), \quad j = 0, 1, 2, \dots, \\
&\mathrm{supp}_t \hat{a} \subset I, \quad \mathrm{supp}_t \hat{a}_j \subset I, \quad j = 0, 1, 2, \dots, \\
&\hat{a}_0(\widetilde{x}'', \widetilde{x}'', t) = 2^{-n+\frac{3}{2}d-1} \pi^{d-n-1} |\det(R^{L_{X_G}}_{\widetilde{x}''} - 2t\mathcal{L}_{X_G, \widetilde{x}''})| \tau^8(t), \quad \forall \widetilde{x}'' \in W,
\end{aligned}$$

where $W = \Omega_3 \times \Omega_4$, Ω_3 and Ω_4 are open sets as in the beginning of Section 4.1.

Let

$$R_k := F_k - P^G_{k,\tau^8} : \mathcal{C}^\infty(X, L^k) \to \mathcal{H}^0_b(X, L^k)^G. \tag{155}$$

Our next goal is to show that for k large, $I + R_k : \mathcal{C}^\infty(X, L^k) \to \mathcal{C}^\infty(X, L^k)$ is injective. Let $\|\cdot\|_k$ be the L^2 norm induced by $(\cdot \mid \cdot)_k$.

From Theorem 4.5, we see that if $y \notin Y$, then for any open set D of y with $\overline{D} \bigcap Y = \emptyset$, we have

$$R_k = O(k^{-\infty}) \quad \text{on } X \times D. \tag{156}$$

Let $p \in Y$. Let s be a local trivializing G-invariant CR rigid section of L defined on an open set D of p in X and let $x = (x_1, \dots, x_{2n+1})$ be the local coordinates as in Proposition 3.1 defined in an open set U of p, $U \subset D$. Then,

$$\begin{aligned}
&R_k(x, y) = e^{ikA(x'', y'', t)} r(x'', y'', t, k) + O(k^{-\infty}) \quad \text{on } U \times U, \\
&r(x'', y'', t, k) \in S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\
&r(x'', y'', t, k) \sim \sum_{j=0}^{\infty} k^{n+1-\frac{d}{2}-j} r_j(x'', y'') \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\
&r_j(x'', y'', t) \in \mathcal{C}^\infty(U \times U \times I), \quad j = 0, 1, 2, \dots, \\
&\mathrm{supp}_t r \subset I, \quad \mathrm{supp}_t r_j \subset I, \quad j = 0, 1, \dots.
\end{aligned} \tag{157}$$

Moreover, from (11) and (154), it is easy to see

$$|r_0(x,y)| \le C\,|(x,y)-(x_0,x_0)|\,, \tag{158}$$

for all $x_0 \in Y \bigcap U$, where $C > 0$ is a constant. We use $\|\cdot\|$ to denote the standard L^2 norm on X induced by the given volume form dV_X. We need

Lemma 4.1. *Let $p \in Y$. Let $x = (x_1, \dots, x_{2n+1})$ be the local coordinates as in Proposition 3.1 defined in an open set U of p, $U \subset D$. Let*

$$\begin{aligned}
&H_k(x,y) = \int e^{ikA(x'',y'',t)} h(x,y,t,k)dt \quad \text{on } U \times U,\\
&h(x,y,t,k) \in S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I),\\
&h(x,y,t,k) \sim \sum_{j=0}^{\infty} k^{n-\frac{d}{2}-j} h_j(x,y,t) \text{ in } S^{n-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I),\\
&h(x,y,t,k) \in \mathcal{C}^\infty_c(U \times U \times I),\\
&h_j(x,y,t) \in \mathcal{C}^\infty_c(U \times U \times I), \quad j = 0,1,2,\dots.
\end{aligned}$$

Then,

$$\|H_k u\| \le \delta_k \|u\|\,, \quad \forall u \in \mathcal{C}^\infty(X), \quad \forall k \in \mathbb{N}, \tag{159}$$

where δ_k is a sequence with $\lim_{k\to\infty} \delta_k = 0$.

Proof. Fix $N \in \mathbb{N}$. It is not difficult to see that

$$\|H_k u\| \le \left\| (H_k^* H_k)^{2^N} u \right\|^{\frac{1}{2^{N+1}}} \|u\|^{1-\frac{1}{2^{N+1}}}\,, \quad \forall u \in \mathcal{C}^\infty(X), \tag{160}$$

where H_k^* denotes the adjoint of H_k with respect to the given volume form dV_X. From Theorem 4.2, we can repeat the proof of Theorem 4.4 with minor change and deduce that

$$\begin{aligned}
&(H_k^* H_k)^{2^N}(x,y) = e^{ikA(x'',y'',t)} p(x,y,t,k) + O(k^{-\infty}) \ \text{ on } U \times U,\\
&p(x,y,t,k) \in S^{n+1-2^{N+1}-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I),\\
&p(x,y,t,k) \in \mathcal{C}^\infty_0(U \times U \times I).
\end{aligned}$$

Hence,

$$\left| (H_k^* H_k)^{2^N}(x,y) \right| \le \hat{C} k^{n+1-2^{N+1}-\frac{d}{2}}, \quad \forall (x,y) \in U \times U, \tag{161}$$

where $\hat{C} > 0$ is a constant independent of k. Take N large enough so that $n+1-2^{N+1}-\frac{d}{2} < 0$. From (160) and (161), we get (159). □

We also need

Lemma 4.2. *Let $p \in Y$. Let $x = (x_1, \ldots, x_{2n+1})$ be the local coordinates as in Proposition 3.1 defined in an open set U of p, $U \subset D$. Let*

$$\begin{aligned} &B_k(x,y) = \int e^{ikA(x'',y'',t)} g(x,y,t,k)dt \quad \text{on } U \times U, \\ &g(x,y,t,k) \in S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\ &g(x,y,t,k) \sim \sum_{j=0}^{\infty} k^{n+1-\frac{d}{2}-j} g_j(x,y,t) \ \text{in } S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\ &g_j(x,y,t) \in \mathcal{C}^\infty_c(U \times U \times I), \quad j = 0,1,2,\ldots, \\ &g(x,y,t,k) \in \mathcal{C}^\infty_c(U \times U \times I). \end{aligned}$$

Suppose that

$$|g_0(x,y)| \leq C\,|(x,y) - (x_0,x_0)|\,,$$

for all $x_0 \in Y \bigcap U$, where $C > 0$ is a constant. Then,

$$\|B_k u\| \leq \varepsilon_k \|u\|\,, \quad \forall u \in \mathcal{C}^\infty(X), \quad \forall k \in \mathbb{N}, \tag{162}$$

where ε_k is a sequence with $\lim_{k\to\infty} \varepsilon_k = 0$.

Proof. Fix $N \in \mathbb{N}$. It is not difficult to see that

$$\|B_k u\| \leq \left\| (B_k^* B_k)^{2^N} u \right\|^{\frac{1}{2^{N+1}}} \|u\|^{1-\frac{1}{2^{N+1}}}\,, \quad \forall u \in \mathcal{C}^\infty(X), \tag{163}$$

where B_k^* denotes the adjoint of B_k with respect to the volume form dV_X. From Theorem 4.2, we can repeat the proof of Theorem 4.4 with minor change and deduce that

$$\begin{aligned} &(B_k^* B_k)^{2^N}(x,y) = \int e^{ikA(x'',y'',t)} \hat{g}(x,y,t,k)dt + O(k^{-\infty}) \ \text{ on } U \times U, \\ &\hat{g}(x,y,t,k) \in S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\ &\hat{g}(x,y,t,k) \sim \sum_{j=0}^{\infty} k^{n+1-\frac{d}{2}-j} \hat{g}_j(x,y,t) \ \text{in } S^{n+1-\frac{d}{2}}_{\mathrm{loc}}(1; U \times U \times I), \\ &\hat{g}_j(x,y,t) \in \mathcal{C}^\infty_c(U \times U \times I), \quad j = 0,1,2,\ldots, \\ &\hat{g}(x,y,t,k) \in \mathcal{C}^\infty_c(U \times U \times I), \end{aligned}$$

and

$$|\hat{g}_0(x,y,t)| \leq C\,|(x,y)-(x_0,x_0)|^{2^{N+1}}, \tag{164}$$

for all $x_0 \in Y \bigcap U$, where $C>0$ is a constant.

Let

$$(B_k^* B_k)_0^{2^N}(x,y) = \int e^{ik\Psi(x'',y'',t)}\hat{g}_0(x,y,t,k)dt,$$
$$(B_k^* B_k)_1^{2^N}(x,y) = \int e^{ik\Psi(x'',y'',t)}h(x,y,t,k)dt,$$

where $h(x,y,t,k)=\hat{g}(x,y,t,k)-\hat{g}_0(x,y,t,k)$. It is clear that $h(x,y,t,k)\in S^{n-\frac{d}{2}}_{\rm loc}(1;U\times U\times I)$. From Lemma 4.1, we see that

$$\left\|(B_k^* B_k)_1^{2^N}u\right\| \leq \delta_k\,\|u\|\,,\quad \forall u\in \mathcal{C}^\infty(X),\quad \forall k\in\mathbb{N}, \tag{165}$$

where δ_k is a sequence with $\lim_{k\to\infty}\delta_k=0$.

Let's pause and recall Malgrange preparation theorem (see [13, Theorem 7.57]). Let $F(x)$ be a smooth function defined on an open set U of $\mathbb{R}^N$, $0\in U$. If $F(0)=0$, $\frac{\partial F}{\partial x_N}(0)\neq 0$. Then, in some small neighborhood $\hat{U}\subset U$ of 0 in $\mathbb{R}^N$, we have

$$F(x)=G(x)(x_N-H(x_1,\ldots,x_{N-1})),$$

where G,H are smooth functions on $\hat{U}$ and H is independent of x_N. Let $F(t,x,y):=\partial_t A$. Since $\partial_{x_{2n+1}}F(t,x,y)|_{(x,y)\in Y\times U}=\partial_t\partial_{x_{2n+1}}A|_{(x,y)\in Y\times U}\neq 0$, for all $t\in I$, by Malgrange preparation theorem, we have

$$\partial_t A(x,y,t)=\alpha(x,y,t)(x_{2n+1}-\beta(x',y,t))$$

in $V\times V\times I$, where V is a small open set of p, $\alpha,\beta\in\mathcal{C}^\infty(V\times V\times I)$, $x'=(x_1,\ldots,x_{2n})$. We can consider Taylor expansion of $\widetilde{g}_0(x,y,t)$ at $x_{2n+1}=\beta(x',y,t)$ and by using integration by parts with respect to t, we may take g_0 so that

$$g_0 \text{ is independent of } x_{2n+1}. \tag{166}$$

From (164) and (166), we see that

$$|\hat{g}_0(x,y,t)| \leq C_1\Big(\,|\hat{x}''|+|\hat{y}''|+\left|\widetilde{\mathring{x}}''-\widetilde{\mathring{y}}''\right|\Big)^{2^{N+1}}, \tag{167}$$

where $C_1>0$ is a constant. From (106), we see that

$$|\mathrm{Im}\,A(x,y,t)| \geq c\Big(\,|\hat{x}''|^2+|\hat{y}''|^2+\left|\widetilde{\mathring{x}}''-\widetilde{\mathring{y}}''\right|^2\Big), \tag{168}$$

where $c>0$ is a constant. From (167) and (168), we conclude that

$$\left|(B_k^* B_k)_0^{2^N}(x,y)\right| \leq \hat{C} k^{-2^N+n-\frac{d}{2}+1}, \quad \forall (x,y) \in U \times U, \tag{169}$$

where $\hat{C} > 0$ is a constant independent of k. From (169), we see that if N large enough, then

$$\left\|(B_k^* B_k)_0^{2^N} u\right\| \leq \hat{\delta}_k \left\|u\right\|, \quad \forall u \in \mathcal{C}^\infty(X), \quad \forall k \in \mathbb{N}, \tag{170}$$

where $\hat{\delta}_k$ is a sequence with $\lim_{k\to\infty} \hat{\delta}_k = 0$.

From (163), (165) and (170), we get (162). $\square$

From (156) and Lemma 4.2, we get

Theorem 4.7. *With the notations used above, we have*

$$\left\|R_k u\right\|_k \leq \delta_k \left\|u\right\|_k, \quad \forall u \in \mathcal{C}^\infty(X, L^k), \quad \forall k \in \mathbb{N}, \tag{171}$$

where R_k is as in (155) and δ_k is a sequence with $\lim_{k\to\infty} \delta_k = 0$.

In particular, for $k \gg 1$,

$$I + R_k : \mathcal{C}^\infty(X, L^k) \to \mathcal{C}^\infty(X, L^k) \text{ is injective.} \tag{172}$$

Proof of Theorem 1.4. Fix $\lambda \in \mathrm{Spec}\,(-iT)$, $\tau \equiv 1$ near λ. Let $u \in H^0_{b,\lambda}(X, L^k)^G$. If $\sigma_k u = 0$. Then, $\sigma_k^* \sigma_k u = (P^G_{k,\tau^8} + R_k)u = (I + R_k)u = 0$. From (172), we get $u = 0$ if $k \gg 1$. Note that σ_k maps the space $\mathcal{H}^0_{b,\lambda}(X, L^k)^G$ into $\mathcal{H}^0_{b,\lambda}(X_G, L^k_G)$. Thus, $\sigma_k : \mathcal{H}^0_{b,\lambda}(X, L^k)^G \to \mathcal{H}^0_{b,\lambda}(X_G, L^k_G)$ is injective if $k \gg 1$.

From Theorem 4.6, we have $\sigma_k \sigma_k^* = C_0(P_{k,X_G,\tau^8} + Q_k)$, where Q_k is a semi-classical complex Fourier integral operator of the same type and order of Q_k vanishes at the diagonal, where $C_0 > 0$ is a constant. We can repeat the proof of Theorem 4.7 with minor change and deduce that $I + Q_k : \mathcal{C}^\infty(X_G, L^k_G) \to \mathcal{C}^\infty(X_G, L^k_G)$ is injective, if $k \gg 1$. Note that $(\mathrm{Im}\,\sigma_k)^\perp \cap \mathcal{H}^0_{b,\lambda}(X_G, L^k)^G \subset \mathrm{Ker}\,\sigma_k^* \cap \mathcal{H}^0_{b,\lambda}(X_G, L^k)^G$. We conclude that $\sigma_k : \mathcal{H}^0_{b,\lambda}(X, L^k)^G \to \mathcal{H}^0_{b,\lambda}(X_G, L^k_G)$ is surjective if $k \gg 1$. The theorem follows. $\square$

Acknowledgments

We express our gratitude to the referees for various interesting comments and for suggesting several improvements. Andrea Galasso would like to express his thanks to the Mathematics Institute of Universität zu Köln for their hospitality throughout his scholarship: INdAM (Istituto Nazionale di Alta Matematica) foreign scholarship. Chin-Yu Hsiao was supported by Taiwan Ministry of Science and Technology projects 109-2923-M-001-010-MY4, 113-2115-M-002-011-MY3.

Data availability

No data was used for the research described in the article.

References

[1] M.S. Baouendi, L.P. Rothschild, F. Treves, CR structures with group action and extendability of CR functions, Invent. Math. 82 (2) (1985) 359–396.

[2] X. Dai, K. Liu, X. Ma, A remark on weighted Bergman kernels on orbifolds, Math. Res. Lett. 19 (01) (2012) 143–148.

[3] S. Finski, Semiclassical Ohsawa-Takegoshi extension theorem and asymptotics of the orthogonal Bergman kernel, J. Differ. Geom. 128 (2) (2024) 639–721.

[4] S. Finski, On semiclassical Ohsawa-Takegoshi extension theorem, in: Kengo Hirachi, et al. (Eds.), The Bergman Kernel and Related Topics. Proceedings of the 23rd Hayama Symposium on Complex Analysis of Several Variables, SCV XXIII, Kanagawa, Japan, July 27–28, 2022, in: Springer Proc. Math. Stat., vol. 447, Springer, Singapore, 2024, pp. 141–168.

[5] S. Finski, The asymptotics of the optimal holomorphic extensions of holomorphic jets along submanifolds (English. French summary), J. Math. Pures Appl. (9) 189 (2024) 103586.

[6] S. Finski, Complex embeddings, Toeplitz operators and transitivity of optimal holomorphic extensions, Comment. Math. Helv. (2025), 52 p., https://doi.org/10.4171/CMH/604.

[7] S. Fitzpatrick, On the geometric quantization of contact manifolds, J. Geom. Phys. 61 (2011) 2384–2399.

[8] V. Guillemin, S. Sternberg, Geometric quantization and multiplicities of group representations, Invent. Math. 67 (1982) 515–538.

[9] A. Galasso, Strict quantization for compact pseudo-Kähler manifolds and group actions, SIGMA 21 (2025) 048.

[10] A. Galasso, C.-Y. Hsiao, On the singularities of the Szegő kernels on CR orbifolds, J. Geom. Phys. 209 (2025) 105411, https://doi.org/10.1016/j.geomphys.2024.105411.

[11] Brian C. Hall, William D. Kirwin, Unitarity in "Quantization commutes with reduction", Commun. Math. Phys. 275 (2) (2007) 401–442.

[12] C.D. Hill, M. Nacinovich, A weak pseudoconcavity condition for abstract almost CR manifolds, Invent. Math. 142 (2000) 251–283.

[13] L. Hörmander, The Analysis of Linear Partial Differential Operators I, Classics in Mathematics, Springer-Verlag, Berlin, 2003.

[14] H. Herrmann, C.-Y. Hsiao, X. Li, Szegő kernels and equivariant embedding theorems for CR manifolds, Math. Res. Lett. 29 (1) (2022) 193–246.

[15] C.-Y. Hsiao, Projections in several complex variables, Mém. Soc. Math. Fr., Nouv. Sér. 123 (2010) 131, MR2780123, Zbl 1229.32002.

[16] C.-Y. Hsiao, Szegő kernel asymptotics for high power of CR line bundles and Kodaira embedding theorems on CR manifolds, Mem. Am. Math. Soc. 254 (2018) 1217, v+142.

[17] C.-Y. Hsiao, R.-T. Huang, G-invariant Szegő kernel asymptotics and CR reduction, Calc. Var. Partial Differ. Equ. 60 (1) (2021) 47.

[18] H. Herrmann, C.-Y. Hsiao, X. Li, Szegő kernels and equivariant embedding theorems for CR manifolds, Math. Res. Lett. 29 (1) (2022) 193–245.

[19] C.-Y. Hsiao, G. Marinescu, Szegő kernel asymptotics and Morse inequalities on CR manifolds, Math. Z. 271 (2012) 509–553.

[20] C.-Y. Hsiao, G. Marinescu, Semi-classical spectral asymptotics of Toeplitz operators on strictly pseudoconvex domains, in: Kengo Hirachi, et al. (Eds.), The Bergman Kernel and Related Topics. Proceedings of the 23rd Hayama Symposium on Complex Analysis of Several Variables, SCV XXIII, Kanagawa, Japan, July 27–28, 2022, in: Springer Proc. Math. Stat., vol. 447, Springer, Singapore, 2024, pp. 239–259.

[21] C.-Y. Hsiao, G. Marinescu, X. Li, Equivariant Kodaira embedding for CR manifolds with circle action, Mich. Math. J. 70 (2021) 55–113.

[22] C.-Y. Hsiao, X. Ma, G. Marinescu, Geometric quantization on CR manifolds, Commun. Contemp. Math. 25 (10) (2023) 2250074.

[23] C.-Y. Hsiao, G. Marinescu, X. Li, Semi-classical asymptotics of partial Bergman kernels on $\mathbb{R}$-symmetric complex manifolds with boundary, arXiv:2208.12412.

[24] X. Ma, Geometric quantization on Kähler and symplectic manifolds, in: International Congress of Mathematicians, vol. II, Hyderabad, India, August 19-27, 2010, pp. 785–810.

[25] X. Ma, W. Zhang, Bergman kernels and symplectic reduction, Astérisque 318 (2008), viii+154 pp.

[26] X. Ma, W. Zhang, Geometric quantization for proper moment maps: the Vergne conjecture, Acta Math. 212 (1) (2014) 11–57.

[27] A. Melin, J. Sjöstrand, Fourier Integral Operators with Complex Valued Phase Functions, Springer Lecture Notes in Math., vol. 459 (1074), Springer-Verlag, Berlin, 1975, pp. 120–223.

[28] E. Meinrenken, On Riemann-Roch formulas for multiplicities, J. Am. Math. Soc. 9 (1996) 373–389.

[29] E. Meinrenken, Symplectic surgery and the Spinc-Dirac operator, Adv. Math. 134 (1998) 240–277.

[30] R. Paoletti, The Szegö kernel of a symplectic quotient (English), Adv. Math. 197 (2) (2005) 523–553.

[31] L. Ornea, M. Verbitsky, Sasakian structures on CR-manifolds, Geom. Dedic. 125 (2007) 159–173, https://doi.org/10.1007/s10711-007-9149-5.

[32] J. Ross, R. Thomas, Weighted Bergman kernels on orbifolds, J. Differ. Geom. 88 (1) (May 2011) 87–107, https://doi.org/10.4310/jdg/1317758870.

[33] Y. Tian, W. Zhang, An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg, Invent. Math. 132 (1998) 229–259.

[34] M. Vergne, Multiplicities formula for geometric quantization I, II, Duke Math. J. 82 (1996) 143–179, 181–194.

[35] M. Vergne, Applications of equivariant cohomology, in: International Congress of Mathematicians, vol. I, Eur. Math. Soc., Zürich, 2007, pp. 635–664.