Check for updates

Studies in Applied Mathematics

WILEY

STUDIES IN APPLIED MATHEMATICS

SPECIAL ISSUE ARTICLE OPEN ACCESS

Wave Dynamics, Integrability and Beyond

Gibbons–Tsarev Type Systems and Eventual Identities

Alessandro Arsie[1] | Paolo Lorenzoni[2,3] | Sara Perletti[4,5] | Karoline van Gemst[2,3]

[1]Department of Mathematics and Statistics, The University of Toledo, Toledo, Ohio, USA | [2]Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, Milano, Italy | [3]INFN sezione di Milano-Bicocca, Milano, Italy | [4]Scuola Internazionale Superiore di Studi Avanzati (SISSA), Trieste, Italy | [5]INFN sezione di Trieste, Trieste, Italy

Correspondence: Alessandro Arsie (Alessandro.Arsie@utoledo.edu) | Paolo Lorenzoni (paolo.lorenzoni@unimib.it) | Sara Perletti (sperlett@sissa.it) | Karoline van Gemst (kvgemst@impan.pl)

Received: 30 March 2026 | **Revised:** 17 September 2026 | **Accepted:** 2 October 2026

Keywords: differential geometry | integrable system | mathematical physics

ABSTRACT

We show that non-diagonalizable reductions of the dKP equation associated with regular non-semisimple F-manifolds cannot exist. The proof is based on the derivation and study of a generalized Gibbons–Tsarev system (gGT system) in the non-semisimple/non-diagonalizable setting. Remarkably, a class of solutions of the gGT system is defined by eventual identities of the underlying regular F-manifold structure. Furthermore, we use these vector fields to construct integrable reductions of Pavlov's hydrodynamic chain. In this case, the corresponding solutions are defined for any choice of Jordan block structure of the operator of multiplication by an eventual identity.

1 | Introduction

While the theory of diagonaliaable hydrodynamic type systems, traditionally studied via Riemann invariants,

$$u^i_t = v^i(u)u^i_x, \qquad i \in \{1, \dots, n\},$$

is well established, recent attention has shifted toward non-diagonalizable systems. The theory of F-manifolds provides the geometric tools to extend the theory to the non-diagonalizable case (see, for instance, [1–6] and references therein). In this setting, a set of coordinates introduced by David and Hertling [7] plays the role of Riemann invariants. These coordinates reduce the system to a special block-diagonal form which, remarkably, coincides with the normal form of regular operator fields with vanishing Haantjes tensor (see [8]). Non-diagonalizable systems of hydrodynamic type naturally originate:

- in Nijenhuis geometry (see, for instance, the general construction in [9] and, more recently, [10]),
- in the theory of F-manifolds and Dubrovin–Frobenius manifolds (see references above),
- in the study of confluence of hypergeometric functions [11],
- in the study of parabolic regularization of the gradient catastrophes for the Burgers–Hopf equation [12],
- as hydrodynamic reductions of integrable models (see, for instance, [13–15] and references therein).

This is an open access article under the terms of the Creative Commons Attribution License, which permits use, distribution and reproduction in any medium, provided the original work is properly cited.
© 2026 The Author(s). *Studies in Applied Mathematics* published by Wiley Periodicals LLC.

This paper aims to contribute to the last line of research applying the methods coming from the theory of F-manifolds. In this setting systems of hydrodynamic type have the form

$$u^i_t = c^i_{jk}\mu^k u^j_x, \qquad i \in \{1, \dots, n\}, \tag{1.1}$$

where c^i_{jk} are the structure functions of a commutative associative product $\circ$, with unit e, satisfying Hertling–Manin condition [16] and μ^j are the components of a vector field. There is a unique torsionless connection ∇ defined by the conditions

$$d_\nabla(\mu\circ) = 0, \qquad \nabla e = 0.$$

Via this connection, integrability of a system of the form (1.1) can be written as a tensorial condition involving the Riemann tensor R of ∇ and the product $\circ$:

$$Z \circ R(W, Y)(X) + W \circ R(Y, Z)(X) + Y \circ R(Z, W)(X) = 0, \tag{1.2}$$

for all local vector fields X, Y, Z, W. It was observed in [2] that (1.2) reduces to Tsarev's integrability condition in the semisimple case. It was also shown in [4] that it is a necessary condition for integrability in the non-semisimple regular case. Moreover, it is sufficient for a special class of regular systems of the form (1.1) (called *Darboux–Tsarev systems*) in the general smooth setting [5] and for general systems of the form (1.1) in the analytic setting [1]. Note that the latter result does not require regularity, but only the cyclicity of the vector field μ defining the system. We refer to the above-mentioned references for details.

At first sight, the study of reductions of the form (1.1) might seem very special. In contrast, according to the results of [17] this is a natural choice, since, under mild assumptions, integrability implies the vanishing of the Haantjes tensor.

We will study hydrodynamic reductions of the dispersionless KP hierarchy and of Pavlov's hydrodynamic chain. The dispersionless KP equation is the dispersionless limit of the Kadomtsev–Petviashvili equation [18], and it also appears in nonlinear acoustics as the Khokhlov–Zabolotskaya equation [19, 20]. Pavlov's hydrodynamic chain was introduced in [21] as a new example of an integrable infinite hydrodynamic chain. In the same paper, the author studied and completely classified its diagonal reductions. These include the so-called *ϵ-system* which reduces for specific choices of ϵ to important examples of systems of hydrodynamic type (we refer to [21] for details).

In their seminal paper [22], Gibbons and Tsarev discovered large families of reductions of the dispersionless KP hierarchy, or equivalently of the Benney momentum chain [23]—governed by integrable diagonal systems of hydrodynamic type (semihamiltonian systems [24]). These *diagonal reductions* play a central role in the modern theory of integrable dispersionless equations. In particular, the Gibbons–Tsarev mechanism became one of the fundamental tools in the programme, developed for instance in [25], of characterizing $(2+1)$-dimensional dispersionless integrable PDEs and integrable hydrodynamic chains by means of their hydrodynamic reductions.

In the Gibbons–Tsarev setting, an n-component diagonal reduction is described by functions

$$\mu^1(r^1, \dots, r^n), \dots, \mu^n(r^1, \dots, r^n), \qquad V(r^1, \dots, r^n),$$

satisfying the system

$$\partial_i \mu^j = \frac{\partial_i V}{\mu^i - \mu^j}, \tag{1.3a}$$

$$\partial_i \partial_j V = \frac{2\partial_i V \partial_j V}{(\mu^i - \mu^j)^2}, \tag{1.3b}$$

for $i \neq j$. This is the classical *Gibbons–Tsarev system*. Its solutions determine the characteristic velocities of the reduction together with a density of conservation law V. The same function V is also the Egorov potential of the diagonal metric defining a Hamiltonian structure of the reduction; the existence of such Hamiltonian structures for generic reductions was established in [26]. Since the system (1.3) is Darboux compatible, its solutions depend on n arbitrary functions of one variable.

A second remarkable discovery of Gibbons and Tsarev is that diagonal reductions admit an equivalent description in terms of Landau–Ginzburg superpotentials. This description arises from a system of n chordal Loewner equations governing

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

families of univalent conformal maps from the upper half-plane to slit domains in the upper half-plane [27]. The compatibility conditions for these equations are precisely the Gibbons–Tsarev system. This picture links the hydrodynamic reduction method with complex-analytic and Frobenius-type structures.

The purpose of this paper is to investigate the non-diagonalizable case. In several special examples it is easy to see that genuinely non-diagonalizable reductions of dKP do not occur, but, to the best of our knowledge, a general argument in the regular non-semisimple setting has not been written down explicitly. Our strategy is to translate the problem into the geometry of F-manifolds.

Indeed, reductions of dKP are naturally related to special semisimple F-manifolds with compatible connection [2, 4]. From this point of view, the semisimple case is precisely the diagonal one. It is therefore natural to ask whether regular non-semisimple F-manifolds may produce non-diagonalizable reductions. Starting from this geometric ansatz, one is led to a system generalizing (1.3), namely the *generalized Gibbons–Tsarev system*:

$$\left(\mathcal{L}_\mu c^i_{hj} - c^i_{js}c^s_{ht}[e,\mu]^t\right)\mu^h = e(V)\delta^i_j - e^i\partial_j V, \tag{1.4a}$$

$$\left(c^s_{ih}\partial_s\partial_j V - c^s_{jh}\partial_s\partial_i V\right)\mu^h = (c^h_{sj}\partial_i\mu^s - c^h_{si}\partial_j\mu^s)\partial_h V, \tag{1.4b}$$

where c^i_{jk} are the structure functions of the product, and e^i are the components of the unit field of the F-manifold. In the semisimple case (1.4) reduces to the standard Gibbons–Tsarev system.

It turns out that, in the regular non-semisimple case, (1.4) is too rigid to admit nontrivial solutions of the type relevant for dKP. More precisely, let the local regular structure be described by Jordan blocks of sizes $m_1,\dots,m_r$. We prove that any solution necessarily satisfies

$$\partial_{p(\alpha)}V = 0, \qquad \text{for every } p \geq 2,$$

where the double index $p(\alpha) = m_1 + \dots + m_{\alpha-1} + p$ labels the pth variable associated to the αth block, and

$$\partial_{1(\alpha)}V = 0, \qquad \text{if} \quad m_\alpha \geq 2.$$

Hence V depends only on the coordinates belonging to the blocks of size 1 and the surviving equations are exactly the classical Gibbons–Tsarev system on the first variables. In this precise sense there are no genuinely regular non-diagonalizable reductions of dKP.

Remarkably, the generalized Gibbons–Tsarev system admits a class of solutions related to the geometry of the underlying F-manifold. Indeed, when V is constant, we obtain solutions $(\mu^1,\dots,\mu^n)$ satisfying the condition

$$\mathcal{L}_\mu c^i_{hj} - c^i_{js}c^s_{ht}[e,\mu]^t = 0. \tag{1.5}$$

In the theory of F-manifolds, vector fields satisfying (1.5) are called *eventual identities*; they play an important role in the duality theory of F-manifolds. Thus, although the non-semisimple regular setting does not produce new dKP reductions, it still singles out a geometrically meaningful class of solutions of the generalized Gibbons–Tsarev equations.

In the final part of the paper we study a variant of the system (1.4) where Equations (1.4a) are replaced by (1.5). We show that any solution $(\mu^1,\dots,\mu^n,V)$ of the new system defines a reduction of the form

$$C^\alpha = C^\alpha(r^1,\dots,r^n), \qquad r^i_t = c^i_{jk}(\mu^j - Ve^j)r^k_x$$

of Pavlov's hydrodynamic chain

$$C^{\alpha-1}_t = C^\alpha_x - C^0C^{\alpha-1}_x, \qquad \alpha \in \mathbb{Z},$$

for any choice of product structure. Integrable systems of hydrodynamic type of this form were introduced and studied in [9] from the viewpoint of Nijenhuis geometry and in [4] from the viewpoint of the theory of F-manifolds. The proof relies on the fact that the operator of multiplication by an eventual identity has vanishing Nijenhuis torsion [28]. Using this fact, we can recursively define all functions

$$C^k(r^1,\dots,r^n), \qquad k = \pm1,\pm2,\dots,$$

starting from $C^0 = V(r^1,\dots,r^n)$.

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

The paper is organized as follows. In Section 2.1, we recall basic definitions about regular F-manifolds and their canonical coordinates. In Section 2.2 we recall the definition of an eventual identity $\mathcal{E}$ and prove two alternative characterizations in the regular case: one in terms of a connection ∇ uniquely defined by the eventual identity (Proposition 2.9) and one in terms of the vanishing of Nijenhuis torsion of $\mathcal{E}\circ$ (Proposition 2.14). In Section 3 we derive a system of PDEs (Proposition 3.1), generalizing the usual Gibbons–Tsarev system, whose solutions describe regular reductions of dKP. Section 4 is devoted to the analysis of such a system. In particular, we prove that nontrivial non-diagonalizable reductions of dKP cannot exist, first in the case of a single Jordan block of arbitrary size (Section 4.1, Proposition 4.1) and then in the general case (Section 4.2, Proposition 4.2). In Section 5.1, following [9], we briefly introduce a class of integrable PDEs associated with a tensor field with vanishing Nijenhuis torsion or, in the context of regular F-manifolds, to eventual identities (see [4]). In Section 5.2 we show that reductions of Pavlov's hydrodynamic chain coincide with the systems of PDEs introduced in Section 5.1. These systems are associated with a tensor field L with vanishing Nijenhuis torsion (Theorem 5.3). In particular, due to Proposition 2.14, the special case where $L = \mathcal{E}\circ$ corresponds to eventual identities. Unlike the case of dKP, in the case of reductions of Pavlov's hydrodynamic chain eventual identities define non-trivial reductions for any choice of the Jordan block structure of L. In Section 5.3, we give some examples coming from [4] and in a brief final section we draw some conclusions.

2 | F-Manifolds and Eventual Identities

2.1 | Regular F-Manifolds

F-manifolds were introduced by Hertling and Manin in [16] and are defined as follows.

Definition 2.1. An *F-manifold* is a manifold M equipped with

(i) a commutative associative bilinear product $\circ$ on the module of (local) vector fields, satisfying the *Hertling–Manin condition*:

$$\mathcal{L}_{X\circ Y}\circ = X\circ(\mathcal{L}_Y\circ) + Y\circ(\mathcal{L}_X\circ),$$

for all local vector fields X, Y;

(ii) a distinguished vector field e on M such that

$$e\circ X = X$$

for all local vector fields X.

F-manifolds are often equipped with additional structures. A particularly useful class of F-manifolds, for studying systems of hydrodynamic type, is that of F-manifolds with compatible connection and flat unit.

Definition 2.2 [2]. An *F-manifold with compatible connection and flat unit* is a manifold M equipped with a commutative, associative, and unital product

$$\circ : TM\times TM \to TM,$$

with a torsionless connection ∇ such that:

(i) For R denoting the Riemann tensor of ∇,

$$Z\circ R(W,Y)(X) + W\circ R(Y,Z)(X) + Y\circ R(Z,W)(X) = 0, \tag{2.1}$$

for all local vector fields X, Y, Z, W.

(ii) Let c^k_{ij} denote the structure coefficients of the product $\circ$. Then

$$\nabla_j c^i_{hk} = \nabla_h c^i_{jk}. \tag{2.2}$$

(iii) The identity of the product e is flat: $\nabla e = 0$.

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

F-manifolds with compatible connection naturally encode systems of hydrodynamic type:

$$u^i_t = W^i_j(u)u^j_x, \qquad i \in \{1, \dots, n\}.$$

In the diagonalizable case, any such system can be written in the form

$$u^i_t = (\mu \circ u_x)^i = c^i_{jk}\mu^j u^k_x, \qquad i \in \{1, \dots, n\}, \tag{2.3}$$

where μ is a local vector field and $\circ$ is the product of a semisimple F-manifold. Thus, Riemann invariants coincide with Dubrovin's canonical coordinates in which the structure coefficients are given by

$$c^i_{jk} = \delta^i_j \delta^i_k.$$

Moreover, there is a unique compatible torsionless connection satisfying $d_\nabla(\mu\circ) = 0$ (and $\nabla e = 0$), giving the structure of an F-manifold with compatible connection, where condition (2.1) encodes the integrability of the hydrodynamic-type system.

To pass from the semisimple to the regular non-semisimple setting, one needs the notion of an F-manifold with Euler vector field.

Definition 2.3. An *F-manifold with Euler vector field* is an F-manifold M equipped with a vector field E satisfying

$$\mathcal{L}_E \circ = \circ.$$

Definition 2.4 [7]. An F-manifold with Euler vector field $(M, \circ, e, E)$ is called *regular* if for each $p \in M$ the matrix representing the endomorphism

$$L_p := E_p \circ : T_pM \to T_pM$$

has exactly one Jordan block for each distinct eigenvalue.

If the Jordan normal form of $L = E\circ$ consists of r blocks of sizes $m_1, \dots, m_r$, we label coordinates by $j(\alpha)$, where

$$\alpha \in \{1, \dots, r\}, \qquad j \in \{1, \dots, m_\alpha\}.$$

Thus $u^{j(\alpha)}$ denotes the jth coordinate in the αth Jordan block. In other words, we may identify $u^{j(\alpha)}$ with a single index as

$$j(\alpha) = m_1 + \cdots + m_{\alpha-1} + j.$$

The regular setting reduces to the semisimple one when $r = n$, in which case one retrieves systems of hydrodynamic type admitting a diagonal representation.

Theorem 2.5 [7]. *Let $(M, \circ, e, E)$ be a regular F-manifold of dimension $n \geq 2$ with an Euler vector field E. Furthermore, assume that locally around a point $p \in M$, the Jordan canonical form of the operator L has r Jordan blocks of sizes $m_1, \dots, m_r$ with distinct eigenvalues. Then there exists, locally around p, a distinguished system of coordinates $\{u^1, \dots, u^{m_1+\dots+m_r}\}$ such that*

$$e^{i(\alpha)} = \delta^i_1, \qquad E^{i(\alpha)} = u^{i(\alpha)}, \qquad c^{i(\alpha)}_{j(\beta)k(\gamma)} = \delta^\alpha_\beta \delta^\alpha_\gamma \delta^i_{j+k-1},$$

for all $\alpha, \beta, \gamma \in \{1, \dots, r\}$ and $i \in \{1, \dots, m_\alpha\}$, $j \in \{1, \dots, m_\beta\}$, $k \in \{1, \dots, m_\gamma\}$.

In the regular non-diagonalizable case, we consider systems of hydrodynamic type of the form (2.3) where c^i_{jk} now denote the structure coefficients of a regular F-manifold in the David–Hertling coordinates of Theorem 2.5. In David–Hertling coordinates the defining matrix $W = \mu\circ$ takes a block-diagonal form, $W = \mathrm{diag}(W_{(1)}, \dots, W_{(r)})$, $r \leq n$, whose generic αth

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

block, of size m_α, is of the lower-triangular Toeplitz form

$$W_{(\alpha)} = \begin{bmatrix} \mu^{1(\alpha)} & 0 & \dots & 0 \\ \mu^{2(\alpha)} & \mu^{1(\alpha)} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \mu^{m_\alpha(\alpha)} & \dots & \mu^{2(\alpha)} & \mu^{1(\alpha)} \end{bmatrix}. \tag{2.4}$$

In [4] it was shown that there exists a unique torsionless connection ∇ satisfying

$$\nabla e = 0, \qquad d_\nabla(\mu\circ) = 0, \tag{2.5}$$

provided that $\mu^{1(\alpha)} \neq \mu^{1(\beta)}$ for distinct blocks α, β, and $\mu^{2(\alpha)} \neq 0$ whenever $m_\alpha \geq 2$. This connection endows the manifold with the structure of an F-manifold with compatible connection and flat unit. More precisely, we have the following theorem.

Theorem 2.6 [4]. *Let $\{\mu_{(0)}, \dots, \mu_{(n-1)}\}$ be a set of linearly independent local vector fields on an n-dimensional regular F-manifold $(M, \circ, e)$. Let us assume that the corresponding flows*

$$\mathbf{u}_{t_i} = \mu_{(i)} \circ \mathbf{u}_x, \qquad i \in \{0, \dots, n-1\},$$

pairwise commute, and that there exists a local vector field $\mu \in \{\mu_{(0)}, \dots, \mu_{(n-1)}\}$ such that $\mu^{1(\alpha)} \neq \mu^{1(\beta)}$ for $\alpha \neq \beta$, $\alpha, \beta \in \{1, \dots, r\}$, and $\mu^{2(\alpha)} \neq 0$ for each $\alpha \in \{1, \dots, r\}$ with $m_\alpha \geq 2$. Then an F-manifold $(M, \circ, e, \nabla)$ with compatible connection and flat unit is defined by the unique torsionless connection ∇ realizing $\nabla e = 0$ and $d_\nabla(\mu\circ) = 0$.

In particular, fixing the connection by the condition $d_\nabla(\mu\circ) = 0$ implies that $d_\nabla(\lambda\circ) = 0$ for any other local vector field λ whose flow commutes with that defined by μ. This fact will be important in the subsequent sections. Furthermore, in [6] it was shown that, in the setting of Theorem 2.6, the system for densities of conservation laws is compatible and V is a density of conservation law if and only if V satisfies

$$(\mu\circ)^s_i \nabla_s (dV)_j = (\mu\circ)^s_j \nabla_s (dV)_i. \tag{2.6}$$

2.2 | Eventual Identities

The Euler vector field is a special case of a more general object—an *eventual identity*.

Definition 2.7 [29]. An eventual identity $\mathcal{E}$ for an F-manifold $(M, \circ, e)$ is an invertible vector field $\mathcal{E}$ such that the product

$$X \star Y = \mathcal{E}^{-1} \circ X \circ Y$$

defines an F-manifold $(M, \star, \mathcal{E})$.

Remark 2.8. It was proved in [30] that an invertible vector field $\mathcal{E}$ is an eventual identity if and only if

$$\mathcal{L}_\mathcal{E}(\circ)(X, Y) = [e, \mathcal{E}] \circ X \circ Y. \tag{2.7}$$

Note that (2.7) is precisely the coordinate-free formulation of the condition (1.5), corresponding to constant densities V.

Eventual identities were studied in the setting of regular F-manifolds in [31].

In the regular case, the condition (2.7) can also be written in an equivalent form in terms of the torsionless connection ∇ defined by the conditions

$$\nabla e = 0, \qquad d_\nabla(\mathcal{E}\circ) = 0.$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

We will now state two characterizations which are new in the regular setting.

Proposition 2.9. *Let $(M, \circ, e, E)$ be a regular F-manifold. A vector field $\mathcal{E}$ satisfying the conditions*

$$\mathcal{E}^{1(\alpha)} \neq \mathcal{E}^{1(\beta)}, \qquad \alpha \neq \beta, \alpha, \beta \in \{1, \dots, r\},$$

and

$$\mathcal{E}^{2(\alpha)} \neq 0, \qquad \alpha \in \{1, \dots, r\},\ m_\alpha \geq 2,$$

is an eventual identity if and only if

$$\nabla_{\mathcal{E}} \circ = 0.$$

Proof. This result follows from the identity

$$\nabla_{\mathcal{E}} c^i_{jh} = \mathcal{L}_{\mathcal{E}} c^i_{hj} - c^i_{js} c^s_{ht} [e, \mathcal{E}]^t. \tag{2.8}$$

Indeed, taking into account that, by definition of ∇, we have

$$\partial_j \mathcal{E}^i = -\Gamma^i_{js} \mathcal{E}^s + c^i_{js} \nabla_e \mathcal{E}^s = -\Gamma^i_{js} \mathcal{E}^s + c^i_{js} [e, \mathcal{E}]^s,$$

we obtain

$$\begin{aligned}
\mathcal{L}_{\mathcal{E}} c^i_{hj} &= \mathcal{E}^s \partial_s c^i_{hj} - c^s_{hj} \partial_s \mathcal{E}^i + c^i_{sj} \partial_h \mathcal{E}^s + c^i_{hs} \partial_j \mathcal{E}^s \\
&= \mathcal{E}^s \partial_s c^i_{hj} - c^s_{hj} \left(-\Gamma^i_{st} \mathcal{E}^t + c^i_{st} [e, \mathcal{E}]^t \right) + c^i_{sj} \left(-\Gamma^s_{ht} \mathcal{E}^t + c^s_{ht} [e, \mathcal{E}]^t \right) + c^i_{hs} \left(-\Gamma^s_{jt} \mathcal{E}^t + c^s_{jt} [e, \mathcal{E}]^t \right) \\
&= \nabla_{\mathcal{E}} c^i_{jh} + c^i_{js} c^s_{ht} [e, \mathcal{E}]^t.
\end{aligned}$$

□

The Nijenhuis torsion of a (1,1) tensor field L is given by

$$N_L(X, Y) = [LX, LY] - L[LX, Y] - L[X, LY] + L^2[X, Y], \tag{2.9}$$

where X, Y are vector fields.

Theorem 2.10. *Let $\mathcal{E}$ be a vector field on a regular F-manifold $(M, \circ, e, E)$ and let the Jordan canonical form of $\mathcal{E}\circ$ consist of r Jordan blocks. Assume that $\mathcal{E}^{1(\alpha)} \neq \mathcal{E}^{1(\beta)}$, for all $\alpha \neq \beta$, $\alpha, \beta \in \{1, \dots, r\}$, and that $\mathcal{E}^{2(\alpha)} \neq 0$, for all $\alpha \in \{1, \dots, r\}$ such that $m_\alpha \geq 2$. Then, $\mathcal{E}$ is an eventual identity if $N_{\mathcal{E}\circ} = 0$.*

To prove Theorem 2.10, it will be helpful to first obtain some additional lemmas. Note that, since we assume regularity, we are free to work in David–Hertling canonical coordinates, in which we know that $\mathcal{E}\circ$ is block-diagonal with each block having a lower-triangular Toeplitz form as in (2.4). We will be using the convention that out-of-range indices give zero contribution throughout the remainder of this section.

Lemma 2.11. *Let $\circ$ be the product of a regular F-manifold, and let $\mathcal{E}$ be a vector field for which the Jordan canonical form of $\mathcal{E}\circ$ consists of r blocks. Let $\mathcal{E}^{1(\alpha)} \neq \mathcal{E}^{1(\beta)}$ for any distinct pair $\alpha, \beta \in \{1, \dots, r\}$. If $N_{\mathcal{E}\circ} = 0$, then in David–Hertling canonical coordinates, $\mathcal{E}$ has the form*

$$\mathcal{E} = \sum_{\alpha=1}^{r} \mathcal{E}_{(\alpha)}, \quad \text{where } \mathcal{E}_{(\alpha)} = \sum_{i=1}^{m_\alpha} \mathcal{E}^{i(\alpha)}(u^{1(\alpha)}, \dots, u^{m_\alpha(\alpha)}) \partial_{i(\alpha)},$$

with m_α being the size of block α.

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Proof. Consider (2.9), with $X = \partial_{i(\alpha)}$, $Y = \partial_{j(\beta)}$, and $L = \mathcal{E}\circ$, where $\mathcal{E} = \sum_{\alpha=1}^{r}\sum_{i=1}^{m_\alpha}\mathcal{E}^{i(\alpha)}\partial_{i(\alpha)}$, for $\alpha, \beta \in \{1, \dots, r\}$ with $\alpha \neq \beta$. Note that, by the form of the structure coefficients in Theorem 2.5,

$$LX = \mathcal{E}\circ X = \sum_{\gamma=1}^{r}\sum_{k=1}^{m_\gamma}\mathcal{E}^{k(\gamma)}\partial_{k(\gamma)}\circ\partial_{i(\alpha)} = \sum_{k=1}^{m_\alpha - i+1}\mathcal{E}^{k(\alpha)}\partial_{(k+i-1)(\alpha)}.$$

Thus,

$$N_L(X,Y) = \left[\sum_k \mathcal{E}^{k(\alpha)}\partial_{(k+i-1)(\alpha)}, \sum_l \mathcal{E}^{l(\beta)}\partial_{(l+j-1)(\beta)}\right] - L\left[\sum_k \mathcal{E}^{k(\alpha)}\partial_{(k+i-1)(\alpha)}, \partial_{j(\beta)}\right] -$$
$$- L\left[\partial_{i(\alpha)}, \sum_l \mathcal{E}^{l(\beta)}\partial_{(l+j-1)(\beta)}\right] = T_\alpha + T_\beta,$$

where

$$T_\alpha = -\sum_{k,l}\mathcal{E}^{l(\beta)}\partial_{(l+j-1)(\beta)}(\mathcal{E}^{k(\alpha)})\partial_{(k+i-1)(\alpha)} + \sum_{k,m}\mathcal{E}^{m(\alpha)}\partial_{j(\beta)}(\mathcal{E}^{k(\alpha)})\partial_{(m+k+i-2)(\alpha)},$$

and $T_\beta = -T_\alpha$ after letting $\alpha \leftrightarrow \beta$, $i \leftrightarrow j$. Note that $N_L(X,Y) = 0$ if and only if T_α and T_β vanish independently. Thus, it suffices to consider T_α. The coefficient of $\partial_{s(\alpha)}$ in T_α is given by

$$[\partial_{s(\alpha)}]T_\alpha = -\sum_{l=1}^{m_\beta - j+1}\mathcal{E}^{l(\beta)}\partial_{(l+j-1)(\beta)}(\mathcal{E}^{(s-i+1)(\alpha)}) + \sum_{m=1}^{s-i+1}\mathcal{E}^{m(\alpha)}\partial_{j(\beta)}(\mathcal{E}^{(s-i+2-m)(\alpha)}),$$

and letting $p := s - i + 1$ gives

$$T_\alpha = 0 \Rightarrow (\mathcal{E}^{1(\alpha)} - \mathcal{E}^{1(\beta)})\partial_{j(\beta)}\mathcal{E}^{p(\alpha)} = \sum_{l=2}^{m_\beta - j+1}\mathcal{E}^{l(\beta)}\partial_{(l+j-1)(\beta)}\mathcal{E}^{p(\alpha)} - \sum_{m=2}^{p}\mathcal{E}^{m(\alpha)}\partial_{j(\beta)}\mathcal{E}^{(p+1-m)(\alpha)}. \tag{2.10}$$

We will now perform double induction on p (ascending) and j (descending).

- Let $p = 1$. Then (2.10) becomes

$$(\mathcal{E}^{1(\alpha)} - \mathcal{E}^{1(\beta)})\partial_{j(\beta)}\mathcal{E}^{1(\alpha)} = \sum_{l=2}^{m_\beta - j+1}\mathcal{E}^{l(\beta)}\partial_{(l+j-1)(\beta)}\mathcal{E}^{1(\alpha)}. \tag{2.11}$$

  Let $j = m_\beta$. Then (2.11) gives

$$(\mathcal{E}^{1(\alpha)} - \mathcal{E}^{1(\beta)})\partial_{m_\beta(\beta)}\mathcal{E}^{1(\alpha)} = 0,$$

  since $(l + j - 1)(\beta) = (l - 1 + m_\beta)(\beta) \geq (m_\beta + 1)(\beta)$ (as $l \geq 2$), which implies

$$\partial_{m_\beta(\beta)}\mathcal{E}^{1(\alpha)} = 0,$$

  by the non-coincidence assumption on the first components.
  Now let $\partial_{q(\beta)}\mathcal{E}^{1(\alpha)} = 0, \forall\, q > j$. Then since $l + j - 1 > j$, $\partial_{(l+j-1)(\beta)}\mathcal{E}^{1(\alpha)} = 0, \forall\, l \geq 2$ and

$$(\mathcal{E}^1(\alpha) - \mathcal{E}^{1(\beta)})\partial_{j(\beta)}\mathcal{E}^{1(\alpha)} = 0 \Rightarrow \partial_{j(\beta)}\mathcal{E}^{1(\alpha)} = 0, \quad \forall\, j,\ \alpha \neq \beta.$$

- Let us now assume that for $q < p$,

$$\partial_{j(\beta)}\mathcal{E}^{q(\alpha)} = 0, \quad \forall\, j \in \{1, \dots, m_\beta\},\ \alpha \neq \beta.$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Then (2.10) gives

$$(\mathcal{E}^{1(\alpha)} - \mathcal{E}^{1(\beta)})\partial_{j(\beta)}\mathcal{E}^{p(\alpha)} = \sum_{l=2}^{m_\beta - j + 1} \mathcal{E}^{l(\beta)}\partial_{(l+j-1)(\beta)}\mathcal{E}^{p(\alpha)} \tag{2.12}$$

by the induction hypothesis as $p + 1 - m \leq p - 1$. Let us again perform descending induction on j.

Let $j = m_\beta$. Then the right-hand side of (2.12) vanishes, since $l + j - 1 = l + m_\beta - 1 > m_\beta$, giving the result by the non-coincidence assumption on the first components.

Let us now assume that the statement holds for all $q > j$. Again, the right-hand side of (2.12) vanishes, since $l + j - 1 > j$ for $l \geq 2$, giving

$$(\mathcal{E}^{1(\alpha)} - \mathcal{E}^{1(\beta)})\partial_{j(\beta)}\mathcal{E}^{p(\alpha)} = 0 \Rightarrow \partial_{j(\beta)}\mathcal{E}^{p(\alpha)} = 0,$$

as required.

Therefore, each coefficient $\mathcal{E}^{p(\alpha)}$ depends only on the coordinates of the block α, $\{u^{1(\alpha)}, \ldots, u^{m_\alpha(\alpha)}\}$, concluding the proof of the lemma. □

Lemma 2.12. *For $\mathcal{E}$ as in Lemma 2.11,*

$$N_{\mathcal{E}\circ} = 0 \Leftrightarrow N_{\mathcal{E}_{(\alpha)}\circ} = 0, \quad \forall\, \alpha \in \{1, \ldots, r\}.$$

Proof. Suppose

$$X = \sum_{i=1}^{m_\alpha} f^i(u^{1(\alpha)}, \ldots, u^{m_\alpha(\alpha)})\partial_{i(\alpha)}, \quad Y = \sum_{j=1}^{m_\beta} g^j(u^{1(\beta)}, \ldots, u^{m_\beta(\beta)})\partial_{j(\beta)},$$

then

$$[X, Y] = \sum_{i,j} \left(f^i\partial_{i(\alpha)}(g^j)\partial_{j(\beta)} - g^j\partial_{j(\beta)}(f^i)\partial_{i(\alpha)}\right) = \sum_{i,j} \left(f^i\partial_{i(\alpha)}(g^j)\partial_{j(\alpha)} - g^j\partial_{j(\alpha)}(f^i)\partial_{i(\alpha)}\right),$$

if $\alpha = \beta$, and zero otherwise. Moreover,

$$L(\partial_{i(\alpha)}) = \mathcal{E} \circ \partial_{i(\alpha)} = \sum_{\beta=1}^{r} \mathcal{E}_{(\beta)} \circ \partial_{i(\alpha)} = \sum_{k=1}^{m_\alpha} \mathcal{E}^{k(\alpha)}\partial_{(k+i-1)(\alpha)}.$$

Hence, if $\alpha \neq \beta$, all four terms in $N_L(X, Y)$ vanish, and if $\alpha = \beta$, then $N_L(X, Y) = N_{\mathcal{E}_{(\alpha)}\circ}(X, Y)$. By bilinearity, the result follows. □

In the following lemma we will be dealing with a single fixed block, $\alpha \in \{1, \ldots, r\}$. For brevity we will write i for $i(\alpha)$, let $n := m_\alpha$, and $\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}^i\partial_i$.

Lemma 2.13. *Let $\mathcal{E}^{\circ i}$ denote the i-fold product of $\mathcal{E}$ with itself under $\circ$. If $\mathcal{E}^2 \neq 0$, then*

$$\{\mathcal{E}^{\circ i}\}_{i=0,\ldots n-1}$$

spans the associated block as a vector space.

Proof. The lemma will follow from showing that for $F := \mathcal{E} - \mathcal{E}^1 e = \sum_{a=2}^{n} \mathcal{E}^a\partial_a$

$$\{F^{\circ k}\}_{k=0,\ldots n-1}$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

form a basis of the block as a vector space, together with the fact that each $F^{\circ k}$ is a polynomial in $\mathcal{E}$. Let us first prove the intermediate result. For each $k \in \{0, \dots, n-1\}$, there exist smooth functions $c_{k,s}$ such that

$$F^{\circ k} = (\mathcal{E}^2)^k \partial_{k+1} + \sum_{s=k+2}^{n} c_{k,s} \partial_s.$$

This follows by induction on k. The statement is immediate for $k = 0, 1$ using $e = \partial_1$ and the definition of F. Thus, assume that the statement holds for some $k \in \{1, \dots, n-2\}$. Multiplying both sides by F gives

$$F^{\circ(k+1)} = \left((\mathcal{E}^2)^k \partial_{k+1} + \sum_{s=k+2}^{n} c_{k,s} \partial_s \right) \circ \left(\sum_{a=2}^{n} \mathcal{E}^a \partial_a \right).$$

Now, since

$$\partial_i \circ \partial_j = \partial_{i+j-1}$$

we have that

$$(\mathcal{E}^2)^k \partial_{k+1} \circ \mathcal{E}^2 \partial_2 = (\mathcal{E}^2)^{k+1} \partial_{k+2},$$

and that every other term lies in the span of $\{\partial_{k+3}, \dots \partial_n\}$, giving

$$F^{\circ(k+1)} = (\mathcal{E}^2)^{k+1} \partial_{k+2} + \sum_{s=k+3}^{n} c_{k+1,s} \partial_s,$$

as required. This shows that for each k,

$$F^{\circ k} \in \mathrm{Span}\{\partial_{k+1}, \dots, \partial_n\}$$

with leading term $(\mathcal{E}^2)^k \partial_{k+1}$. It follows that the expansion of $\{F^{\circ i}\}_{i=0,\dots,n-1}$ in the basis $\{\partial_i\}$ is upper triangular, with diagonal entries $1, \mathcal{E}^2, \dots, (\mathcal{E}^2)^{n-1}$. Thus, under the assumption $\mathcal{E}^2 \neq 0$, all diagonal entries are nonzero, and the change-of-basis matrix is invertible. Finally, since $F = \mathcal{E} - \mathcal{E}^1 e$ and e is the unit, each $F^{\circ k}$ is a polynomial in $\mathcal{E}$, which implies

$$\mathrm{Span}\{F^{\circ i}\}_{i=0,\dots,n-1} \subseteq \mathrm{Span}\{\mathcal{E}^{\circ i}\}_{i=0,\dots,n-1}.$$

Since $\{F^{\circ i}\}_{i=0}^{n-1}$ forms a basis of the block, $\mathrm{Span}\{\mathcal{E}^{\circ i}\}_{i=0}^{n-1}$ contains the block, and therefore spans the block. □

We now proceed to prove Theorem 2.10.

Proof. By Lemma 2.12, it is sufficient to consider a single Jordan block $\alpha \in \{1, \dots, r\}$. As above, we will in this proof let $i \equiv i(\alpha)$, $n := m_\alpha$, and $\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}^i \partial_i$.

Define

$$K(X, Y) := \mathcal{L}_{\mathcal{E}}(\circ)(X, Y) - [e, \mathcal{E}] \circ X \circ Y.$$

Then, by Remark 2.8, $\mathcal{E}$ is an eventual identity if and only if

$$K(X, Y) = 0, \quad \forall X, Y.$$

Note that

$$\begin{aligned}
\mathcal{L}_{\mathcal{E}}(\circ)(\partial_i, \partial_j) &= [\mathcal{E}, \partial_i \circ \partial_j] - [\mathcal{E}, \partial_i] \circ \partial_j - \partial_i \circ [\mathcal{E}, \partial_j] \\
&= -\sum_t \partial_{i+j-1}(\mathcal{E}^t) \partial_t + \sum_a \partial_i(\mathcal{E}^a) \partial_{a+j-1} + \sum_a \partial_j(\mathcal{E}^a) \partial_{a+i-1} \\
&= \sum_{t=1}^{n} \left(-\partial_{i+j-1} \mathcal{E}^t + \partial_i \mathcal{E}^{t-j+1} + \partial_j \mathcal{E}^{t-i+1} \right) \partial_t.
\end{aligned}$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Moreover,

$$[e,\mathcal{E}]\circ\partial_i\circ\partial_j=\sum_{a=1}^{n}\partial_1(\mathcal{E}^a)\partial_{a+i+j-2}=\sum_{t=1}^{n}\partial_1(\mathcal{E}^{t-i-j+2})\partial_t,$$

where the second equality follows after re-indexing $t=a+i+j-2$. Thus,

$$K(\partial_i,\partial_j)=\sum_{t=1}^{n}\left(-\partial_{i+j-1}\mathcal{E}^t+\partial_i\mathcal{E}^{t-j+1}+\partial_j\mathcal{E}^{t-i+1}-\partial_1\mathcal{E}^{t-i-j+2}\right)\partial_t. \tag{2.13}$$

Let us show that

$$N_L(X,Y)=K(LX,Y)-K(X,LY).$$

First,

$$K(L\partial_i,\partial_j)-K(\partial_i,L\partial_j)=\sum_{a=1}^{n}\mathcal{E}^a(K(\partial_{a+i-1},\partial_j)-K(\partial_i,\partial_{a+j-1})),$$

which means that, using (2.13), $[\partial_t](K(L\partial_i,\partial_j)-K(\partial_i,L\partial_j))$ is given by

$$\sum_{a=1}^{n}\mathcal{E}^a\left(\partial_{a+i-1}\mathcal{E}^{t-j+1}-\partial_{a+j-1}\mathcal{E}^{t-i+1}+\partial_j\mathcal{E}^{t-a-i+2}-\partial_i\mathcal{E}^{t-a-j+2}\right). \tag{2.14}$$

On the other hand, since $[\partial_i,\partial_j]=0$,

$$N_L(\partial_i,\partial_j)=[L\partial_i,L\partial_j]-L[L\partial_i,\partial_j]-L[\partial_i,L\partial_j],$$

where $[L\partial_i,L\partial_j]$ reads

$$\sum_{a,b}\left(\mathcal{E}^a\partial_{a+i-1}(\mathcal{E}^b)\partial_{b+j-1}-\mathcal{E}^b\partial_{b+j-1}(\mathcal{E}^a)\partial_{a+i-1}\right),$$

which gives (by letting $b+j-1=t$, and $a+i-1=t$ in the first and second sums, respectively, and collecting)

$$[\partial_t][L\partial_i,L\partial_j]=\sum_{s=1}^{n}\mathcal{E}^s\left(\partial_{s+i-1}\mathcal{E}^{t-j+1}-\partial_{s+j-1}\mathcal{E}^{t-i+1}\right).$$

Similarly,

$$[\partial_t](-L[L\partial_i,\partial_j])=\sum_{s=1}^{n}\mathcal{E}^s\partial_j\mathcal{E}^{t-s-i+2},\quad [\partial_t](-L[\partial_i,L\partial_j])=-\sum_{s=1}^{n}\mathcal{E}^s\partial_i\mathcal{E}^{t-s-j+2}.$$

Combining gives precisely (2.14) (after renaming $s\mapsto a$). Hence,

$$N_L(X,Y)=K(LX,Y)-K(X,LY),$$

and, if $N_L(X,Y)=0$, we have

$$K(LX,Y)=K(X,LY),\quad \forall X,Y. \tag{2.15}$$

We will now use this fact together with Lemma 2.13 to complete the proof of the theorem.

Note that $L^pe=\mathcal{E}^{\circ p}$. This follows with a brief induction on p. For $p=0$ we have $L^0e=e$ trivially. If the identity holds for p, then

$$L^{p+1}e=L(L^pe)=\mathcal{E}\circ\mathcal{E}^{\circ p}=\mathcal{E}^{\circ(p+1)}.$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Now, let $p, q \geq 0$. By repeatedly applying (2.15) we obtain a chain of equalities

$$K(L^p e, L^q e) = K(L^{p-1} e, L^{q+1} e) = \cdots = K(e, L^{p+q} e) = 0.$$

The final equality follows from the fact that

$$\begin{aligned} \mathcal{L}_{\mathcal{E}}(\circ)(e, X) &= [\mathcal{E}, e \circ X] - [\mathcal{E}, e] \circ X - e \circ [\mathcal{E}, X] = [\mathcal{E}, X] + [e, \mathcal{E}] \circ X - [\mathcal{E}, X] = [e, \mathcal{E}] \circ X \\ &= [e, \mathcal{E}] \circ e \circ X \Rightarrow K(e, X) = 0. \end{aligned}$$

Therefore,

$$K(L^p e, L^q e) = 0 \quad \forall\, p, q \geq 0 \Rightarrow K(\mathcal{E}^{\circ p}, \mathcal{E}^{\circ q}) = 0, \quad \forall\, p, q \geq 0.$$

Since $\{\mathcal{E}^{\circ i}\}_{i=0,\ldots,n-1}$ spans the block by Lemma 2.13, the theorem holds by the bilinearity of K and its vanishing on the spanning set in both arguments. □

Combining Theorem 2.10 with the converse implication proved in [28], we obtain the following equivalence.

Proposition 2.14. *Let $L \equiv \mathcal{E}\circ$, where $\circ$ is the product of a regular F-manifold $(M, \circ, e, E)$, $\mathcal{E}$ is a vector field, and $\mathcal{E}\circ$ has Jordan canonical form consisting of r blocks in David–Hertling coordinates. Assume that*

$$\mathcal{E}^{1(\alpha)} \neq \mathcal{E}^{1(\beta)}, \quad \text{for all distinct } \alpha, \beta \in \{1, \ldots, r\},$$

$$\mathcal{E}^{2(\alpha)} \neq 0, \quad \text{for every } \alpha \in \{1, \ldots, r\} \text{ with } m_\alpha \geq 2.$$

Then $N_L = 0$ if and only if $\mathcal{E}$ is an eventual identity.

3 | Generalised Gibbons–Tsarev System for Regular Non-Diagonalizable Reductions of dKP

We look for reductions of the dKP equation in potential form

$$U_{yy} - U_{xt} + U_x U_{xx} = 0 \tag{3.1}$$

which can equivalently be written as (the first equation locally gives $V = U_x, W = U_y$):

$$V_y = W_x, \tag{3.2a}$$

$$W_y = V_t - V V_x. \tag{3.2b}$$

We look for n-component reductions $V(r^1, \ldots, r^n), W(r^1, \ldots, r^n)$ governed by n-component compatible systems of hydrodynamic type

$$r^i_y = c^i_{jk}(r) \mu^j(r) r^k_x, \tag{3.3a}$$

$$r^i_t = c^i_{jk}(r) \lambda^j(r) r^k_x, \tag{3.3b}$$

where $c^i_{jk}(r)$ are the structure functions of a regular F-manifold structure with product $\circ$ and unit e. In the semisimple case these are the diagonal reductions of dKP and the present construction drastically simplifies and reduces to the computations given in [25].

From Equation (3.2a) it follows that

$$\partial_i W = \partial_k V \, c^k_{ij} \mu^j \tag{3.4}$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and from Equation (3.2b) it follows that

$$\partial_k W\, c^k_{ij}\mu^j = \partial_k V\, c^k_{ij}\lambda^j - V\partial_i V. \tag{3.5}$$

Using Equation (3.4), Equation (3.5) reduces to

$$c^s_{ij}\partial_s V(\mu\circ\mu - \lambda + Ve)^j = 0,$$

which, by regularity, implies

$$\lambda = \mu\circ\mu + Ve. \tag{3.6}$$

Equation (3.2a) means that V is a density of conservation law for the system (3.3a), and a system (3.3b) compatible with (3.3a) is usually called a symmetry. According to the general theory developed in [4–6], and recalled in Section 2, symmetries and densities of conservation laws can be obtained starting from a torsionless linear connection ∇ uniquely determined by the conditions (2.5). Recall that this connection satisfies the condition (2.2). In this setting, as stated in Section 2.1, the fact that V is a density of conservation law is equivalent to it satisfying the system (2.6), while the fact that the flow (3.3b) is a symmetry of the flow (3.3a) is equivalent to the condition

$$d_\nabla(\lambda\circ) = 0. \tag{3.7}$$

The condition (3.7) in local coordinates reads

$$\partial_j\lambda^i = -\Gamma^i_{js}\lambda^s + c^i_{js}\nabla_e\lambda^s$$

or

$$\partial_j\lambda^i = -\Gamma^i_{js}\lambda^s + c^i_{js}[e,\lambda]^s. \tag{3.8}$$

From (3.6) it follows that

$$\partial_j\lambda^i = \left(\partial_j c^i_{hk}\right)\mu^h\mu^k + 2c^i_{hk}\left(\partial_j\mu^h\right)\mu^k + \left(\partial_j e^i\right)V + e^i\partial_j V. \tag{3.9}$$

Combining (3.8) and (3.9), and taking into account (3.6), we obtain

$$\left(\partial_j c^i_{hk}\right)\mu^h\mu^k + 2c^i_{hk}\left(\partial_j\mu^h\right)\mu^k + \left(\partial_j e^i\right)V + e^i\partial_j V = -\Gamma^i_{js}\left(\mu\circ\mu + Ve\right)^s + c^i_{js}[e,\mu\circ\mu + Ve]^s,$$

or, taking into account $\nabla e = 0$,

$$\left(\partial_j c^i_{hk} + \Gamma^i_{js}c^s_{hk}\right)\mu^h\mu^k + 2c^i_{hk}\left(\partial_j\mu^h\right)\mu^k + e^i\partial_j V = c^i_{js}[e,\mu\circ\mu]^s + e(V)\delta^i_j. \tag{3.10}$$

Thanks to the hypothesis of regularity, we can work in coordinates where the product and the unit are simultaneously constant. In such coordinates, taking into account that

$$\partial_j\mu^h = -\Gamma^h_{js}\mu^s + c^h_{js}[e,\mu]^s,$$

and

$$c^i_{js}[e,\mu\circ\mu]^s = 2c^i_{js}c^s_{lk}(\partial_e\mu^l)\mu^k,$$

we can write (3.10) as

$$\Gamma^i_{js}c^s_{hk}\mu^h\mu^k - 2c^i_{hk}\Gamma^h_{js}\mu^s\mu^k + e^i\partial_j V = e(V)\delta^i_j,$$

which, by using (2.2), can be written as

$$\left(\nabla_\mu c^i_{jh}\right)\mu^h = e(V)\delta^i_j - e^i\partial_j V,$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

which is valid in any system of coordinates. Finally, expanding (2.6), one easily obtains

$$c^s_{it}\mu^t\partial_s\partial_j V - c^s_{jt}\mu^t\partial_s\partial_i V = \left(c^t_{sj}\partial_i\mu^s - c^t_{si}\partial_j\mu^s\right)\partial_t V.$$

Taking into account the identity (2.8), we summarize the result in the following proposition.

Proposition 3.1. *The functions $V(r^1,\dots,r^n), \mu^1(r^1,\dots,r^n),\dots,\mu^n(r^1,\dots,r^n)$, together with*

$$r^i_y = c^i_{jk}(r)\mu^j(r)r^k_x,$$

$$r^i_t = c^i_{jk}(r)\lambda^j(r)r^k_x,$$

define an n-component reduction of the dKP equation in potential form (3.1) if and only if V has a nontrivial dependence on each variable, $\lambda = \mu\circ\mu + Ve$, and the system (1.4) is satisfied.

4 | Nonexistence of Non-Diagonalizable Regular Reductions of dKP

We consider the generalized Gibbons–Tsarev system in David–Hertling canonical coordinates, where both the product and the unit vector field are constant:

$$\mu^t\left(c^s_{ti}\,\partial_s\partial_j V - c^s_{tj}\,\partial_s\partial_i V\right) - \left(c^m_{js}\,\partial_i\mu^s - c^m_{is}\,\partial_j\mu^s\right)\partial_m V = 0, \tag{4.1a}$$

$$\mu^h\left(c^i_{hk}\,\partial_j\mu^k + c^i_{jk}\,\partial_h\mu^k - c^k_{jh}\,\partial_k\mu^i\right) - \mu^h c^i_{js}c^s_{hk}e^m\partial_m\mu^k + e^i\partial_j V - \delta^i_j e^m\partial_m V = 0. \tag{4.1b}$$

We recall that we are working in the regular setting, where μ is assumed to be such that $\mu^{1(\alpha)} \neq \mu^{1(\beta)}$ for $\beta \neq \alpha$ and $\mu^{2(\alpha)} \neq 0$ for $m_\alpha \geq 2$.

4.1 | The One-Block Case

In the extreme case of a single Jordan block, of size $n \geq 2$, we have

$$c^i_{jk} = \delta^{i+1}_{j+k}, \qquad e^i = \delta^i_1.$$

The following proposition shows that a solution V of the system (4.1) must be constant.

Proposition 4.1. *Let V and $\{\mu^i\}_{i\in\{1,\dots,n\}}$ solve the system (4.1). Then*

(i) $\partial_j V = 0, \quad j \in \{1,\dots,n\}$,

(ii) $\partial_j\mu^1 = 0, \quad j \in \{2,\dots,n\}$.

Proof. The proof naturally splits into multiple parts.

STEP 1: $\partial_j V = 0$, $j \geq 2$. Let us fix an index $j \in \{2,\dots,n\}$. By considering the sum over $s \in \{1,\dots,n-j+1\}$ of Equations (4.1b) for $i = s$, replacing j by $j+s-1$, we have

$$0 = \partial_j V + \Sigma_1 + \Sigma_2,$$

where $\partial_j V$ is the single term coming from the only contribution, for $s = 1$, from the last two terms of (4.1b), and where Σ_1 and Σ_2 collect the terms containing derivatives of the functions μ^k. Explicitly,

$$\Sigma_1 = \sum_{s=1}^{n-j+1}\mu^h\left(c^s_{hk}\partial_{j+s-1}\mu^k\right), \qquad \Sigma_2 = -\sum_{s=1}^{n-j+1}\mu^h\left(c^k_{j+s-1,h}\partial_k\mu^s\right).$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

After separating the special cases $s = 1$ and $s = 2$ in the sums Σ_1 and Σ_2, one obtains

$$\Sigma_1 + \Sigma_2 = -\sum_{k=j+2}^{n} \mu^{k-j+1}\partial_k\mu^1 - \sum_{k=j+2}^{n} \mu^{k-j}\partial_k\mu^2 + \sum_{s=3}^{n-j+1} \mu^h c^s_{hk}\partial_{j+s-1}\mu^k - \sum_{s=3}^{n-j+1} \mu^h c^k_{j+s-1,h}\partial_k\mu^s,$$

where the terms in the third sum that correspond to $k \geq 3$ cancel with the fourth sum, by a relabeling of indices. Then

$$\Sigma_1 + \Sigma_2 = -\sum_{k=j+2}^{n} \mu^{k-j+1}\partial_k\mu^1 - \sum_{k=j+2}^{n} \mu^{k-j}\partial_k\mu^2 + \sum_{s=3}^{n-j+1}\sum_{k=1}^{2} \mu^h c^s_{hk}\partial_{j+s-1}\mu^k,$$

where the first (second) sum cancels with the part of the third sum that corresponds to $k = 1$ ($k = 2$), yielding $\Sigma_1 + \Sigma_2 = 0$ and thus $\partial_j V = 0$. Since $j \in \{2, \dots, n\}$ was arbitrary, we have proved

$$\partial_j V = 0, \qquad j \in \{2, \dots, n\}. \tag{4.2}$$

STEP 2: $\partial_j\mu^1 = 0$, $j \geq 3$. We argue by descending induction on j, starting from $j = n$. By inspecting Equation (4.1b) for $i = 1$ and $j = n - 1$, every term vanishes except the one containing $\mu^2\partial_n\mu^1$, and two other terms that cancel with each other. Hence

$$0 = \mu^2\,\partial_n\mu^1.$$

The assumption that $\mu^2 \neq 0$ implies $\partial_n\mu^1 = 0$. Let us now fix $j \in \{3, \dots, n-1\}$ and inductively assume that $\partial_\ell\mu^1 = 0$ for all $\ell \geq j + 1$. We consider (4.1b) for $i = 1$, replacing j by $j - 1$. Up to a simplification, the surviving terms are those involving $\partial_j\mu^1$, as the terms with $\partial_\ell\mu^1$ for $\ell \geq j + 1$ vanish by the induction hypothesis. Therefore

$$0 = \mu^2\,\partial_j\mu^1.$$

As above, this implies $\partial_j\mu^1 = 0$.

STEP 3: $\partial_1 V = 0$ AND $\partial_2\mu^1 = 0$. Let us sum the diagonal equations obtained by considering (4.1b) for $j = i$, where the sum runs over $i = 2, \dots, n$. Because of (4.2) and *Step 2*, the only surviving pieces are those involving $\partial_2\mu^1$ and $\partial_1 V$, and terms that cancel with each other. A straightforward calculation then yields

$$n\,\mu^2\,\partial_2\mu^1 = (n-1)\,\partial_1 V. \tag{4.3}$$

We now use Equation (4.1a) for $(i, j) = (1, 2)$. Again, using (4.2), one sees that all second derivatives of V disappear except those involving $\partial_1 V$, and the equation reduces to

$$(\partial_2\mu^1)(\partial_1 V) = 0. \tag{4.4}$$

Suppose that $\partial_1 V \neq 0$ at some point. Then (4.4) forces $\partial_2\mu^1 = 0$ and (4.3) gives

$$0 = n\,\mu^2\,\partial_2\mu^1 = (n-1)\,\partial_1 V,$$

which is impossible. Hence $\partial_1 V = 0$ everywhere. Similarly, $\partial_2\mu^1 = 0$. □

4.2 | The General Regular Case

We now return to the general block decomposition. The previous summation argument is repeated independently for each block due to the vanishing of mixed structure constants. For r Jordan blocks of sizes $m_1, \dots, m_r$, the system (4.1) reads

$$\mu^{t(\tau)}\left(c^{s(\sigma)}_{t(\tau)i(\alpha)}\partial_{s(\sigma)}\partial_{j(\beta)}V - c^{s(\sigma)}_{t(\tau)j(\beta)}\partial_{s(\sigma)}\partial_{i(\alpha)}V\right) \tag{4.5a}$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$
\begin{aligned}
&-\Big(c^{t(\tau)}_{j(\beta)s(\sigma)}\partial_{i(\alpha)}\mu^{s(\sigma)}-c^{t(\tau)}_{i(\alpha)s(\sigma)}\partial_{j(\beta)}\mu^{s(\sigma)}\Big)\partial_{t(\tau)}V=0,\\
&\mu^{h(\tau)}\Big(c^{i(\alpha)}_{h(\tau)k(\gamma)}\partial_{j(\beta)}\mu^{k(\gamma)}+c^{i(\alpha)}_{j(\beta)k(\gamma)}\partial_{h(\tau)}\mu^{k(\gamma)}-c^{k(\gamma)}_{j(\beta)h(\tau)}\partial_{k(\gamma)}\mu^{i(\alpha)}\\
&-c^{i(\alpha)}_{j(\beta)s(\sigma)}c^{s(\sigma)}_{h(\tau)k(\gamma)}e^{m(\nu)}\partial_{m(\nu)}\mu^{k(\gamma)}\Big)+e^{i(\alpha)}\partial_{j(\beta)}V-\delta^{i(\alpha)}_{j(\beta)}e(V)=0,
\end{aligned}
\tag{4.5b}
$$

for all $\alpha,\beta\in\{1,\dots,r\}$ and $i\in\{1,\dots,m_\alpha\}$, $j\in\{1,\dots,m_\beta\}$.

Proposition 4.2. *Let V and $\{\{\mu^{i(\alpha)}\}_{i\in\{1,\dots,m_\alpha\}}\}_{\alpha\in\{1,\dots,r\}}$ solve the system (4.5). Then, for $\alpha,\beta\in\{1,\dots,r\}$, one has*

(i) for $m_\alpha\geq 2$,

$$
\partial_{j(\alpha)}V=0,\quad j\in\{2,\dots,m_\alpha\},
\tag{4.6}
$$

(ii) for $m_\beta\geq 2$,

$$
\partial_{j(\beta)}\mu^{1(\alpha)}=0,\quad j\in\{2,\dots,m_\beta\},
$$

(iii) the classical Gibbons–Tsarev system is recovered

$$
\partial_{1(\beta)}V=(\mu^{1(\beta)}-\mu^{1(\alpha)})\partial_{1(\beta)}\mu^{1(\alpha)},
\tag{4.7a}
$$

$$
\partial_{1(\alpha)}\partial_{1(\beta)}V=2\frac{(\partial_{1(\alpha)}V)(\partial_{1(\beta)}V)}{(\mu^{1(\beta)}-\mu^{1(\alpha)})^2},
\tag{4.7b}
$$

for the leading variables $\{u^{1(\alpha)}\}_{\alpha\in\{1,\dots,r\}}$, where $\alpha\neq\beta$,

(iv) for $m_\alpha\geq 2$,

$$
\partial_{1(\alpha)}V=0.
$$

Proof. We will prove the proposition in five steps.

STEP 1: $\partial_{j(\alpha)}V=0$ FOR $j\in\{2,\dots,m_\alpha\}$. Let $\alpha\in\{1,\dots,r\}$ such that $m_\alpha\geq 2$ and let $j\geq 2$. We consider the sum, over $s\in\{1,\dots,m_\alpha-j+1\}$, of Equations (4.5b) with $\beta=\alpha$, $i=s$, and j replaced by $j+s-1$. Let us first treat the four terms inside the parentheses. The first and third terms reduce to contributions involving only the αth block. This follows from the fact that the structure constants in David–Hertling coordinates are nonzero only when all three block indices coincide (see Theorem 2.5). More precisely, after summing over s, the expressions occurring in the first and third terms reduce to

$$
\Sigma_1:=\sum_{s=1}^{m_\alpha-j+1}\mu^{h(\alpha)}c^{s(\alpha)}_{h(\alpha)k(\alpha)}\partial_{(j+s-1)(\alpha)}\mu^{k(\alpha)}\quad\text{and}\quad\Sigma_2:=-\sum_{s=1}^{m_\alpha-j+1}\mu^{h(\alpha)}c^{k(\alpha)}_{(j+s-1)(\alpha)h(\alpha)}\partial_{k(\alpha)}\mu^{s(\alpha)},
$$

respectively. Moreover, since $j\geq 2$ and $k\geq 1$, it follows that

$$
c^{s(\alpha)}_{(j+s-1)(\alpha)k(\gamma)}=\delta^\alpha_\gamma\delta^s_{j+s+k-2}=0
$$

and consequently the second term vanishes. The same argument, with t in place of k, shows that the fourth term vanishes as well, when replacing k with t. We now consider the two terms outside the parentheses. Since $e^{s(\alpha)}=\delta^s_1$, we have

$$
\sum_{s=1}^{m_\alpha-j+1}e^{s(\alpha)}\partial_{(j+s-1)(\alpha)}V=\partial_{j(\alpha)}V,
$$

while the final term is zero since

$$
\delta^{s(\alpha)}_{(j+s-1)(\alpha)}=0,
$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

for $j \geq 2$. Thus, the full sum reduces to

$$\partial_{j(\alpha)} V = -\Sigma_1 - \Sigma_2.$$

Again, using the form of the structure coefficients and considering the corresponding index ranges, we get

$$\Sigma_1 = \sum_{\substack{h,k\geq 1 \\ h+k\leq m_\alpha - j+2}} \mu^{h(\alpha)} \partial_{(j+h+k-2)(\alpha)} \mu^{k(\alpha)}, \qquad \Sigma_2 = - \sum_{\substack{h,s\geq 1 \\ h+s\leq m_\alpha - j+2}} \mu^{h(\alpha)} \partial_{(j+h+s-2)(\alpha)} \mu^{s(\alpha)},$$

which coincide after relabeling the dummy index s as k. Hence, $\partial_{j(\alpha)} V = 0$, and since j, α were arbitrary, this proves (i).

STEP 2: $\partial_{j(\alpha)} \mu^{1(\alpha)} = 0$ FOR $j \in \{3, \ldots, m_\alpha\}$. In order to prove that

$$\partial_{j(\alpha)} \mu^{1(\alpha)} = 0, \qquad j \geq 3, \tag{4.8}$$

let us consider (4.5b) with $\beta = \alpha$, $i = 1$ and $j = m_\alpha - 1$. This gives

$$\begin{aligned} 0 =& \mu^{h(\alpha)} \Big(c^{1(\alpha)}_{h(\alpha)k(\alpha)} \partial_{(m_\alpha - 1)(\alpha)} \mu^{k(\alpha)} - c^{k(\alpha)}_{(m_\alpha - 1)(\alpha)h(\alpha)} \partial_{k(\alpha)} \mu^{1(\alpha)} \Big) + e^{1(\alpha)} \partial_{(m_\alpha - 1)(\alpha)} V \\ \overset{(4.6)}{=}& - \mu^{2(\alpha)} \partial_{m_\alpha(\alpha)} \mu^{1(\alpha)}, \end{aligned}$$

yielding $\partial_{m_\alpha(\alpha)} \mu^{1(\alpha)} = 0$. By induction, let us fix $l \in \{3, \ldots, m_\alpha - 1\}$ and assume that

$$\partial_{m(\alpha)} \mu^{1(\alpha)} = 0, \qquad m \in \{l+1, \ldots, m_\alpha\}, \tag{4.9}$$

and show $\partial_{l(\alpha)} \mu^{1(\alpha)} = 0$. Considering (4.5b) with $\beta = \alpha$, $i = 1$ and $j = l - 1$ gives

$$\begin{aligned} 0 =& \mu^{h(\alpha)} \Big(c^{1(\alpha)}_{h(\alpha)k(\alpha)} \partial_{(l-1)(\alpha)} \mu^{k(\alpha)} - c^{k(\alpha)}_{(l-1)(\alpha)h(\alpha)} \partial_{k(\alpha)} \mu^{1(\alpha)} \Big) + e^{1(\alpha)} \partial_{(l-1)(\alpha)} V \\ \overset{(4.6)}{=}& - \sum_{h=2}^{m_\alpha - l+2} \mu^{h(\alpha)} \partial_{(l+h-2)(\alpha)} \mu^{1(\alpha)} \overset{(4.9)}{=} - \mu^{2(\alpha)} \partial_{l(\alpha)} \mu^{1(\alpha)}, \end{aligned}$$

yielding $\partial_{l(\alpha)} \mu^{1(\alpha)} = 0$.

STEP 3: $\partial_{j(\beta)} \mu^{1(\alpha)} = 0$ FOR $j \in \{2, \ldots, m_\beta\}$, $\beta \neq \alpha$. In order to prove that

$$\partial_{j(\beta)} \mu^{1(\alpha)} = 0, \qquad j \geq 2, \quad \beta \neq \alpha, \tag{4.10}$$

let us consider (4.5b) with $\beta \neq \alpha$, $i = 1$ and $j = m_\beta$. This gives

$$0 = (\mu^{1(\alpha)} - \mu^{1(\beta)}) \partial_{m_\beta(\beta)} \mu^{1(\alpha)} + \partial_{m_\beta(\beta)} V \overset{(4.6)}{=} (\mu^{1(\alpha)} - \mu^{1(\beta)}) \partial_{m_\beta(\beta)} \mu^{1(\alpha)},$$

yielding $\partial_{m_\beta(\beta)} \mu^{1(\alpha)} = 0$. By induction, let us fix $l \in \{2, \ldots, m_\beta - 1\}$ and assume that

$$\partial_{m(\beta)} \mu^{1(\alpha)} = 0, \qquad m \in \{l+1, \ldots, m_\beta\}, \tag{4.11}$$

and show $\partial_{l(\beta)} \mu^{1(\alpha)} = 0$. Considering (4.5b) with $\beta \neq \alpha$, $i = 1$ and $j = l$ gives

$$\begin{aligned} 0 =& (\mu^{1(\alpha)} - \mu^{1(\beta)}) \partial_{l(\beta)} \mu^{1(\alpha)} - \sum_{h=2}^{m_\beta} \mu^{h(\beta)} c^{k(\beta)}_{l(\beta)h(\beta)} \partial_{k(\beta)} \mu^{1(\alpha)} + \partial_{l(\beta)} V \\ \overset{(4.6)}{=}& (\mu^{1(\alpha)} - \mu^{1(\beta)}) \partial_{l(\beta)} \mu^{1(\alpha)} - \sum_{h=2}^{m_\beta} \mu^{h(\beta)} c^{k(\beta)}_{l(\beta)h(\beta)} \partial_{k(\beta)} \mu^{1(\alpha)}, \end{aligned}$$

where the sum vanishes by (4.11), yielding $\partial_{l(\beta)} \mu^{1(\alpha)} = 0$.

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

STEP 4: (4.7) HOLDS. Let $\beta \neq \alpha$. Considering $i = j = 1$ in (4.5b) gives

$$\begin{aligned} 0 =& \mu^{h(\tau)}\Big(c^{1(\alpha)}_{h(\tau)k(\gamma)}\partial_{1(\beta)}\mu^{k(\gamma)} - c^{k(\gamma)}_{1(\beta)h(\tau)}\partial_{k(\gamma)}\mu^{1(\alpha)}\Big) + \partial_{1(\beta)}V \\ =& \mu^{h(\alpha)}c^{1(\alpha)}_{h(\alpha)k(\alpha)}\partial_{1(\beta)}\mu^{k(\alpha)} - \mu^{h(\beta)}c^{k(\beta)}_{1(\beta)h(\beta)}\partial_{k(\beta)}\mu^{1(\alpha)} + \partial_{1(\beta)}V \end{aligned}$$

which, by virtue of (4.8)–(4.10), gives (4.7a). Moreover, considering $i = j = 1$ in (4.5a) gives

$$\begin{aligned} 0 =& \mu^{t(\alpha)}c^{s(\alpha)}_{t(\alpha)1(\alpha)}\partial_{s(\alpha)}\partial_{1(\beta)}V - \mu^{t(\beta)}c^{s(\beta)}_{t(\beta)1(\beta)}\partial_{s(\beta)}\partial_{1(\alpha)}V \\ & - c^{t(\beta)}_{1(\beta)s(\beta)}\partial_{1(\alpha)}\mu^{s(\beta)}\partial_{t(\beta)}V + c^{t(\alpha)}_{1(\alpha)s(\alpha)}\partial_{1(\beta)}\mu^{s(\alpha)}\partial_{t(\alpha)}V \\ \overset{(4.6)}{=}& (\mu^{1(\alpha)} - \mu^{1(\beta)})\partial_{1(\alpha)}\partial_{1(\beta)}V - \partial_{1(\alpha)}\mu^{1(\beta)}\partial_{1(\beta)}V + \partial_{1(\beta)}\mu^{1(\alpha)}\partial_{1(\alpha)}V \end{aligned}$$

which, by means of (4.7a), gives (4.7b). This proves (iii).

STEP 5: $\partial_{1(\alpha)}V = 0$, $\partial_{2(\alpha)}\mu^{1(\alpha)} = 0$ FOR $m_\alpha \geq 2$. Let $m_\alpha \geq 2$. We consider the sum, over $i \in \{2, \dots, m_\alpha\}$, of (4.5b) with $\beta = \alpha$ and $j = i$. Let us first treat the four terms inside the parentheses. As in Step 1, the first and third terms reduce to contributions only involving the αth block, by the vanishing of structure coefficients involving indices belonging to different blocks. Moreover, since

$$c^{i(\alpha)}_{i(\alpha)k(\gamma)} = \delta^\alpha_\gamma \delta^k_1, \qquad c^{1(\alpha)}_{h(\tau)k(\gamma)} \neq 0 \text{ iff } \tau = \gamma = \alpha \text{ and } h = k = 1,$$

the second and fourth terms reduce to

$$\mu^{h(\tau)}\partial_{h(\tau)}\mu^{1(\alpha)}, \qquad -\mu^{1(\alpha)}\sum_{\rho=1}^{r}\partial_{1(\rho)}\mu^{1(\alpha)}, \tag{4.12}$$

respectively. Let us now consider the two terms outside the parentheses. Since $i \geq 2$, $e^{i(\alpha)} = 0$, making the first term zero. Moreover, using $\delta^{i(\alpha)}_{i(\alpha)} = 1$, the final term gives

$$-e(V) = -\sum_{\rho=1}^{r}\partial_{1(\rho)}V. \tag{4.13}$$

Notice that (4.12) and (4.13) are independent of i. Thus collecting, the full equation reduces to

$$\begin{aligned} 0 =& \sum_{i=2}^{m_\alpha}\mu^{h(\alpha)}\Big(c^{i(\alpha)}_{h(\alpha)k(\alpha)}\partial_{i(\alpha)}\mu^{k(\alpha)} - c^{k(\alpha)}_{i(\alpha)h(\alpha)}\partial_{k(\alpha)}\mu^{i(\alpha)}\Big) \\ & + (m_\alpha - 1)\sum_{\rho=1}^{r}\Big(\mu^{h(\rho)}\partial_{h(\rho)}\mu^{1(\alpha)} - \mu^{1(\alpha)}\partial_{1(\rho)}\mu^{1(\alpha)} - \partial_{1(\rho)}V\Big). \end{aligned} \tag{4.14}$$

For $\rho \neq \alpha$, using (4.10), the expression inside the sum over ρ is given by

$$\Big(\mu^{1(\rho)} - \mu^{1(\alpha)}\Big)\partial_{1(\rho)}\mu^{1(\alpha)} - \partial_{1(\rho)}V,$$

which vanishes by (4.7a), as proved in Step 4. For $\rho = \alpha$, it is

$$\mu^{h(\alpha)}\partial_{h(\alpha)}\mu^{1(\alpha)} - \mu^{1(\alpha)}\partial_{1(\alpha)}\mu^{1(\alpha)} - \partial_{1(\alpha)}V \overset{(4.8)}{=} \mu^{2(\alpha)}\partial_{2(\alpha)}\mu^{1(\alpha)} - \partial_{1(\alpha)}V.$$

For the i-dependent terms in (4.14), using the form of the structure coefficients, and taking into account the corresponding index ranges, we can write the $k \geq 2$ contribution associated with the first term in the parentheses as

$$\sum_{\substack{h\geq 1, k\geq 2 \\ h+k\leq m_\alpha+1}}\mu^{h(\alpha)}\partial_{(h+k-1)(\alpha)}\mu^{k(\alpha)},$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

which, after a relabeling, cancels with the contribution from the second term. The remaining contribution from the i-sum, corresponding to $k = 1$, is therefore

$$\sum_{h=2}^{m_\alpha} \mu^{h(\alpha)} \partial_{h(\alpha)} \mu^{1(\alpha)} \overset{(4.8)}{=} \mu^{2(\alpha)} \partial_{2(\alpha)} \mu^{1(\alpha)}.$$

All this reduces (4.14) to

$$m_\alpha \mu^{2(\alpha)} \partial_{2(\alpha)} \mu^{1(\alpha)} = (m_\alpha - 1) \partial_{1(\alpha)} V. \tag{4.15}$$

Considering now (4.5a) with $\beta = \alpha$, $i = 1$, and $j = 2$, and using (4.6), we obtain

$$\partial_{2(\alpha)} \mu^{1(\alpha)} \partial_{1(\alpha)} V = 0. \tag{4.16}$$

Finally, since $m_\alpha \geq 2$ and $\mu^{2(\alpha)} \neq 0$, Equations (4.15) and (4.16) imply that

$$\partial_{1(\alpha)} V = 0, \qquad \partial_{2(\alpha)} \mu^{1(\alpha)} = 0,$$

which proves (iv) and, together with Steps 2 and 3, also proves (ii). □

5 | Reductions of Pavlov's Hydrodynamic Chain

In this section, we study reductions

$$C^\alpha = C^\alpha(r^1, \dots, r^n), \qquad r^i_t = c^i_{jk} (\mu^j - V e^j) r^k_x$$

of *Pavlov's hydrodynamic chain*

$$C^{\alpha-1}_t = C^\alpha_x - C^0 C^{\alpha-1}_x, \qquad \alpha \in \mathbb{Z}. \tag{5.1}$$

Throughout this section, the n-component reductions under consideration are assumed to be non-degenerate, that is, locally there exist $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$ such that $\det (\partial_i C^{\alpha_j})^n_{i,j=1} \neq 0$.

Diagonal reductions (corresponding to a semisimple product in our setting) have been studied by Pavlov in [21]. He proved that the characteristic velocities μ^i of the diagonal reductions are given by

$$f^i(r^i) - \sum_{k=1}^n \psi^k(r^k),$$

where $f^i(r^i), \psi_i(r^i)$ for $i = 1, \dots, n$ are arbitrary functions. The starting observation is as follows. Identify the coordinates $(r^1, \dots, r^n)$ with the canonical coordinates of a semisimple product. Then the functions

$$f^1(r^1), \dots, f^n(r^n),$$

define an eventual identity μ, while

$$V = \sum_{k=1}^n \psi^k(r^k)$$

is the general solution of $d \cdot d_L V = 0$ for $L = \mu\circ$. In other words, in the semisimple case, reductions have the form

$$r_t = (\mu - Ve)\circ r_x = (L - VI) r_x,$$

where the vector field μ and the function V are solutions of the system

$$\mathcal{L}_\mu c^i_{hj} - c^i_{js} c^s_{ht} [e, \mu]^t = 0, \tag{5.2a}$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$\left(c^s_{ih}\partial_s\partial_j V - c^s_{jh}\partial_s\partial_i V\right)\mu^h = \left(c^h_{sj}\partial_i\mu^s - c^h_{si}\partial_j\mu^s\right)\partial_h V. \tag{5.2b}$$

Thanks to this interpretation, we can extend this result to the general case, that is, without assuming semisimplicity. Before doing so, however, we briefly recall the Frölicher–Nijenhuis construction associated with a Nijenhuis operator, which provides the natural framework for the main result of this section, Theorem 5.3.

5.1 | A Class of F-Manifolds Related to the Frölicher–Nijenhuis Bicomplex

Let L be a tensor field of type (1,1) with vanishing Nijenhuis torsion. Following [9], we now recall a construction of integrable hierarchies starting from the Frölicher–Nijenhuis bicomplex $(d, d_L, \Omega(M))$ (see [32]). The differential d is the usual de Rham differential, while the differential d_L is defined as

$$\begin{aligned}(d_L\omega)(X_0,\dots,X_k) :=& \sum_{i=0}^{k}(-1)^i(LX_i)(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)) \\ &+ \sum_{0\le i<j\le k}(-1)^{i+j}\,\omega([X_i,X_j]_L,X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots X_k),\end{aligned} \tag{5.3}$$

where $\omega \in \Omega^k(M)$ and

$$[X_i,X_j]_L := [LX_i,X_j] + [X_i,LX_j] - L[X_i,X_j].$$

In [9], an integrable hierarchy of hydrodynamic type was constructed as follows. Starting from a solution a_0 of

$$d\cdot d_L a_0 = 0,$$

one recursively defines the sequence of functions $a_1, a_2, a_3, \dots$ recursively by

$$da_{k+1} = d_L a_k - a_k da_0, \qquad k \ge 0.$$

The corresponding sequence of (1,1) tensor fields, starting from $V_0 = I$, are defined by

$$V_{k+1} := V_k L - a_k I, \qquad k \ge 0.$$

This construction can be applied in particular in the case where $L = \mu\circ$ is the operator of multiplication by an eventual identity. As recalled in Section 2, one can define an F-manifold with compatible connection. This was done in [3] in a special case leading to the definition of Lauricella bi-flat F-manifolds and in the general case (choosing as eventual identity the Euler vector field) in [4].

5.2 | Beyond the Semisimple Case

In order to generalize Pavlov's result, we will need two propositions as auxiliary tools, and to state these we require some further definitions.

Let M be a smooth manifold, let

$$\omega \in \Omega^k(M;TM) = \Gamma(\Lambda^k T^*M \otimes TM),$$

and let $\Omega^\bullet(M)$ denote the de Rham algebra of differential forms on M. Define the insertion operator

$$\iota_\omega : \Omega^p(M) \longrightarrow \Omega^{p+k-1}(M)$$

as the unique derivation of degree $k-1$ characterized on decomposable tensors by

$$\iota_{\alpha\otimes X}\eta = \alpha\wedge\iota_X\eta, \qquad \alpha\in\Omega^k(M),\ X\in\mathfrak{X}(M),\ \eta\in\Omega^\bullet(M).$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Equivalently, if $p \geq 1$, then

$$(\iota_\omega \eta)(X_1, \ldots, X_{p+k-1}) = \sum_{\sigma \in \mathrm{Sh}(k,p-1)} \mathrm{sgn}(\sigma)\, \eta\big(\omega(X_{\sigma(1)}, \ldots, X_{\sigma(k)}), X_{\sigma(k+1)}, \ldots, X_{\sigma(k+p-1)}\big),$$

and $\iota_\omega f = 0$ for every $f \in C^\infty(M)$.
Now define

$$d_\omega := [\iota_\omega, d]_{\mathrm{gr}} = \iota_\omega \circ d - (-1)^{k-1} d \circ \iota_\omega. \tag{5.4}$$

With these definitions at hand, we have the following proposition.

Proposition 5.1. *Let d_ω be as in (5.4), then the following statements hold.*

(1) d_ω is a derivation of degree k, that is,

$$d_\omega : \Omega^p(M) \longrightarrow \Omega^{p+k}(M);$$

(2) if $\omega = \alpha \otimes X$ with $\alpha \in \Omega^k(M)$ and $X \in \mathfrak{X}(M)$, then for every $\eta \in \Omega^\bullet(M)$,

$$d_\omega \eta = \alpha \wedge \mathcal{L}_X \eta + (-1)^k\, d\alpha \wedge \iota_X \eta;$$

(3) the operators d and d_ω always graded-commute:

$$[d, d_\omega]_{\mathrm{gr}} = d \circ d_\omega - (-1)^k d_\omega \circ d = 0.$$

Equivalently,

$$d \circ d_\omega = (-1)^k\, d_\omega \circ d.$$

Hence:
- *if k is even, then d and d_ω commute,*
- *if k is odd, then d and d_ω anticommute.*

(4) Assume now that $k = 1$. Under the canonical identification

$$\Omega^1(M; TM) \cong \Gamma(T^*M \otimes TM) \cong \Gamma(\mathrm{End}(TM)),$$

let

$$L := \omega^\flat \in \Gamma(\mathrm{End}(TM)) \qquad \text{be defined by} \qquad LX = \omega(X).$$

Define the usual insertion operator

$$\iota_L : \Omega^p(M) \longrightarrow \Omega^p(M)$$

by

$$(\iota_L \eta)(X_1, \ldots, X_p) = \sum_{j=1}^{p} \eta(X_1, \ldots, LX_j, \ldots, X_p),$$

and set

$$d_L := \iota_L \circ d - d \circ \iota_L. \tag{5.5}$$

Then

$$\iota_\omega = \iota_L \qquad \text{and therefore} \qquad d_\omega = d_L.$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

(5) *For every $q \geq 0$, every $\eta \in \Omega^q(M)$, and every $X_0, \ldots, X_q \in \mathfrak{X}(M)$,*

$$(d_L\eta)(X_0,\ldots,X_q) = \sum_{i=0}^{q}(-1)^i (LX_i)\big(\eta(X_0,\ldots,\widehat{X_i},\ldots,X_q)\big) + \sum_{0\leq i<j\leq q}(-1)^{i+j}\,\eta([X_i,X_j]_L, X_0,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_q),$$

where

$$[X_i,X_j]_L := [LX_i,X_j] + [X_i,LX_j] - L[X_i,X_j].$$

Thus, the operator d_L defined in (5.5) coincides with the operator d_L defined in (5.3).

Proof. Since ι_ω is a derivation of degree $k-1$ and d is a derivation of degree 1, their graded commutator

$$d_\omega = [\iota_\omega, d]_{\mathrm{gr}}$$

is a derivation of degree k. This proves (1).

For (2), if $\omega = \alpha \otimes X$, then $\iota_\omega = \alpha \wedge \iota_X$, so for every $\eta \in \Omega^\bullet(M)$,

$$\begin{aligned} d_\omega\eta &= \iota_\omega(d\eta) - (-1)^{k-1} d(\iota_\omega\eta) \\ &= \alpha \wedge \iota_X(d\eta) - (-1)^{k-1} d(\alpha \wedge \iota_X\eta) \\ &= \alpha \wedge \big(\iota_X d + d\iota_X\big)\eta + (-1)^k\, d\alpha \wedge \iota_X\eta \\ &= \alpha \wedge \mathcal{L}_X\eta + (-1)^k\, d\alpha \wedge \iota_X\eta, \end{aligned}$$

where we used the graded Leibniz rule and Cartan's formula $\mathcal{L}_X = \iota_X d + d\iota_X$.

For (3), using $d^2 = 0$, we have

$$d\,d_\omega = d\iota_\omega d - (-1)^{k-1} d^2\iota_\omega = d\iota_\omega d,$$

$$d_\omega d = \iota_\omega d^2 - (-1)^{k-1} d\iota_\omega d = (-1)^k\, d\iota_\omega d.$$

This implies

$$d\,d_\omega = (-1)^k\, d_\omega d,$$

or equivalently

$$[d, d_\omega]_{\mathrm{gr}} = 0.$$

The parity statement follows immediately.

For (4), let $\eta \in \Omega^p(M)$. As the statement is immediate for $p = 0$, we may assume in the following that $p \geq 1$. Letting $X_1, \ldots, X_p \in \mathfrak{X}(M)$ and $k = 1$, the shuffle formula gives

$$(\iota_\omega\eta)(X_1,\ldots,X_p) = \sum_{j=1}^{p}(-1)^{j-1}\eta\big(\omega(X_j), X_1,\ldots,\widehat{X_j},\ldots,X_p\big).$$

Because η is alternating, moving $\omega(X_j)$ from the first slot to the jth slot produces another factor $(-1)^{j-1}$. Thus, using the definition of $L = \omega^\flat$ which gives $LX_j = \omega(X_j)$, we have

$$(\iota_\omega\eta)(X_1,\ldots,X_p) = \sum_{j=1}^{p}\eta\big(X_1,\ldots,LX_j,\ldots,X_p\big) = (\iota_L\eta)(X_1,\ldots,X_p),$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

which means that $\iota_\omega = \iota_L$. It follows from the definition of d_ω, since $k = 1$, that $d_\omega = d_L$.
It remains to prove (5). Fix $q \geq 0$, $\eta \in \Omega^q(M)$, and $X_0, \ldots, X_q \in \mathfrak{X}(M)$. By definition,

$$d_L\eta = \iota_L(d\eta) - d(\iota_L\eta).$$

For $r \neq i$, write

$$\eta_i^{(r)} := \eta(X_0, \ldots, \widehat{X_i}, \ldots, LX_r, \ldots, X_q),$$

that is, $\eta_i^{(r)}$ is obtained from $\eta(X_0, \ldots, \widehat{X_i}, \ldots, X_q)$ by replacing the argument X_r by LX_r. Also, for $0 \leq i < j \leq q$, define

$$\eta_{ij}(Y) := \eta(Y, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_q), \tag{5.6}$$

and for $r \neq i, j$ define

$$\eta_{ij}^{(r)}(Y) := \eta(Y, X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, LX_r, \ldots, X_q).$$

We first expand $\iota_L d\eta$, using the usual de Rham formula:

$$\begin{aligned}(\iota_L d\eta)(X_0, \ldots, X_q) = &\sum_{r=0}^{q} (-1)^r\, LX_r\big(\eta(X_0, \ldots, \widehat{X_r}, \ldots, X_q)\big) + \sum_{i=0}^{q} (-1)^i \sum_{r \neq i} X_i(\eta_i^{(r)}) \\ &+ \sum_{0 \leq i < j \leq q} (-1)^{i+j}\, \eta_{ij}([LX_i, X_j] + [X_i, LX_j]) + \sum_{0 \leq i < j \leq q} (-1)^{i+j} \sum_{r \neq i,j} \eta_{ij}^{(r)}\big([X_i, X_j]\big).\end{aligned}$$

Second, we expand $d(\iota_L\eta)$, again applying the de Rham formula, and the fact that

$$\begin{aligned}(d\iota_L\eta)(X_0, \ldots, X_q) &= \sum_{i=0}^{q} (-1)^i X_i\left(\sum_{r \neq i} \eta_i^{(r)}\right) \\ &\quad + \sum_{0 \leq i < j \leq q} (-1)^{i+j} (\iota_L\eta)([X_i, X_j], X_0, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_q) \\ &= \sum_{i=0}^{q} (-1)^i \sum_{r \neq i} X_i\left(\eta_i^{(r)}\right) + \sum_{0 \leq i < j \leq q} (-1)^{i+j} \sum_{r \neq i,j} \eta_{ij}^{(r)}([X_i, X_j]) \\ &\quad + \sum_{0 \leq i < j \leq q} (-1)^{i+j} \eta_{ij}(L[X_i, X_j]).\end{aligned}$$

Subtracting the two expansions gives

$$\begin{aligned}(d_L\eta)(X_0, \ldots, X_q) = &\sum_{r=0}^{q} (-1)^r\, LX_r\big(\eta(X_0, \ldots, \widehat{X_r}, \ldots, X_q)\big) \\ &+ \sum_{0 \leq i < j \leq q} (-1)^{i+j}\, \eta_{ij}\big([LX_i, X_j] + [X_i, LX_j] - L[X_i, X_j]\big),\end{aligned}$$

which, by the definition of $\eta_{ij}(Y)$ as in (5.6), coincides with (5.3). □

Proposition 5.2. *Let L be an invertible tensor field of type (1,1) with vanishing Nijenhuis torsion. Let*

$$\tau_L : \Omega^k(M) \to \Omega^k(M)$$

be the natural action of L on k-forms, defined by

$$(\tau_L\omega)(X_1, \ldots, X_k) := \omega(LX_1, \ldots, LX_k).$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Then the following statements hold.

(1) L^{-1} *also has vanishing Nijenhuis torsion.*

(2) For every $\alpha \in \Omega(M)$*,*

$$d\, d_L\alpha = -\tau_L\big(d\, d_{L^{-1}}(\tau_{L^{-1}}\alpha)\big),$$

or equivalently,

$$d\, d_L = -\tau_L \circ d\, d_{L^{-1}} \circ \tau_{L^{-1}}.$$

Hence, for every $\alpha \in \Omega(M)$*,*

$$d\, d_L\alpha = 0 \quad \Longleftrightarrow \quad d\, d_{L^{-1}}(\tau_{L^{-1}}\alpha) = 0.$$

(3) If $V \in C^\infty(M) = \Omega^0(M)$*, then*

$$d\, d_L V = 0 \quad \Longleftrightarrow \quad d\, d_{L^{-1}} V = 0.$$

Proof. (1) For N_T being the Nijenhuis torsion of a (1,1)-tensor field T as defined in (2.9), we want to prove that $N_{L^{-1}} = 0$, assuming $N_L = 0$. Since L is invertible, it is sufficient to show that

$$N_{L^{-1}}(LX, LY) = 0 \quad \forall X, Y \in \mathfrak{X}(M).$$

We have

$$N_{L^{-1}}(LX, LY) = [X, Y] - L^{-1}\big([X, LY] + [LX, Y]\big) + (L^{-1})^2[LX, LY].$$

Applying L^2 to both sides and using $N_L = 0$, we obtain

$$\begin{aligned} L^2 N_{L^{-1}}(LX, LY) &= L^2[X, Y] - L\big([X, LY] + [LX, Y]\big) + [LX, LY] \\ &= [LX, LY] - L\big([LX, Y] + [X, LY]\big) + L^2[X, Y] \\ &= N_L(X, Y) = 0. \end{aligned}$$

The claim follows by the invertibility of L^2.

(2) First we show that

$$d_L \circ \tau_L = \tau_L \circ d.$$

Indeed, for $\omega \in \Omega^k(M)$ and $X_0, \ldots, X_k \in \mathfrak{X}(M)$, we have

$$\begin{aligned} (d_L(\tau_L\omega))(X_0, \ldots, X_k) = &\sum_{i=0}^{k} (-1)^i (LX_i)\big(\omega(LX_0, \ldots, \widehat{LX_i}, \ldots, LX_k)\big) \\ &+ \sum_{0 \le i < j \le k} (-1)^{i+j}\, \omega\big(L[X_i, X_j]_L, LX_0, \ldots, \widehat{LX_i}, \ldots, \widehat{LX_j}, \ldots, LX_k\big). \end{aligned}$$

Since

$$N_L(X_i, X_j) = [LX_i, LX_j] - L[X_i, X_j]_L,$$

the assumption $N_L = 0$ gives

$$L[X_i, X_j]_L = [LX_i, LX_j].$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Therefore, by substitution, we obtain

$$(d_L(\tau_L\omega))(X_0,\dots,X_k) = (d\omega)(LX_0,\dots,LX_k) = (\tau_L(d\omega))(X_0,\dots,X_k),$$

which indeed implies that

$$d_L\circ\tau_L = \tau_L\circ d.$$

Since L is invertible, τ_L is invertible with inverse $\tau_{L^{-1}}$, and hence

$$d_L = \tau_L\circ d\circ\tau_{L^{-1}}. \tag{5.7}$$

Analogously, for L^{-1}, whose Nijenhuis torsion also vanishes, we obtain

$$d_{L^{-1}} = \tau_{L^{-1}}\circ d\circ\tau_L \qquad \text{and} \qquad d\circ\tau_L = \tau_L\circ d_{L^{-1}}. \tag{5.8}$$

Now let $\alpha\in\Omega(M)$. Then

$$d\,d_L\alpha = d\big(\tau_L(d(\tau_{L^{-1}}\alpha))\big) = \tau_L\big(d_{L^{-1}}d(\tau_{L^{-1}}\alpha)\big) = -\tau_L\big(d\,d_{L^{-1}}(\tau_{L^{-1}}\alpha)\big).$$

Here, the first two equalities follow from (5.7) and the second identity of (5.8), while the final step follows from the anticommutation relation of Proposition 5.1(3), for L^{-1}. This proves the first identity of part (2), and since τ_L is invertible it follows that

$$d\,d_L\alpha = 0 \quad\Longleftrightarrow\quad d\,d_{L^{-1}}(\tau_{L^{-1}}\alpha) = 0,$$

which establishes the second.

(3) Let $V\in C^\infty(M)$. Since V is a 0-form,

$$\tau_{L^{-1}}V = V.$$

Thus, part (2) gives

$$d\,d_LV = -\tau_L(d\,d_{L^{-1}}V).$$

Finally, since $d\,d_{L^{-1}}V$ is a 2-form, and τ_L is invertible on $\Omega^2(M)$, part (3) holds. □

Theorem 5.3. *All non-degenerate reductions*

$$C^\alpha = C^\alpha(r^1,\dots,r^n), \qquad r^i_t = W^i_j(r)r^j_x,$$

of Pavlov's hydrodynamic chain

$$C^{\alpha-1}_t = C^\alpha_x - C^0C^{\alpha-1}_x, \qquad \alpha\in\mathbb{Z},$$

are obtained from an invertible (1,1) tensor field L with vanishing Nijenhuis torsion together with a solution V of the equation $d\cdot d_LV = 0$ by the following procedure.

- *The (1,1) tensor field W is defined as $W = L - VI$;*
- *$C^0 = V$ and the remaining functions C^k are determined by the recursive relations:*

$$dC^{\alpha+1} = d_LC^\alpha, \qquad \alpha = 0,1,2,\dots \tag{5.9a}$$

$$dC^{-\alpha-1} = d_{L^{-1}}C^{-\alpha}, \qquad \alpha = 0,1,2,\dots. \tag{5.9b}$$

Proof. First, we observe that the functions C^α are (locally) well-defined. This follows from the condition $d\cdot d_LV = 0$, Proposition 5.2, and the properties of the bicomplexes of Subsection 5.1.

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

Substituting $C^\alpha = C^\alpha(r^1, \dots, r^n)$ into the Pavlov hydrodynamic chain (5.1) and assuming

$$r_t = (L - VI)r_x,$$

we get the system:

$$\left(L^h_k \partial_h C^{\alpha-1} - \partial_k C^\alpha\right) r^k_x = 0,$$

which is satisfied due to (5.9).

Conversely, we observe that it is not restrictive to write W in the form $W = L - C^0 I$, which gives the system

$$\partial_k C^\alpha = L^h_k \partial_h C^{\alpha-1}.$$

By the non-degeneracy of the reduction, this implies that L is invertible. Moreover, it is immediate to check that Darboux compatibility of this system coincides with the vanishing of the Nijenhuis torsion of L. Indeed

$$\partial_j \partial_k C^\alpha = \left(\partial_j L^h_k\right) \partial_h C^{\alpha-1} + L^h_k \partial_h \partial_j C^{\alpha-1} = \left(\partial_j L^h_k\right) L^i_h \partial_i C^{\alpha-2} + L^h_k \partial_h \left(L^i_j\right) \partial_i C^{\alpha-2} + L^h_k L^i_j \partial_h \partial_i C^{\alpha-2},$$

which gives

$$\partial_j \partial_k C^\alpha - \partial_k \partial_j C^\alpha = \left[\left(\partial_j L^h_k - \partial_k L^h_j\right) L^i_h + L^h_k \partial_h L^i_j - L^h_j \partial_h L^i_k\right] \partial_i C^{\alpha-2} = N^i_{kj} \partial_i C^{\alpha-2}.$$

By the non-degeneracy of the reduction, this implies that $N_L = 0$. Finally, by applying the differential d to both sides, we obtain $d \cdot d_L C^{\alpha-1} = 0$, and letting $\alpha = 1$ with $C^0 = V$ yields $d \cdot d_L V = 0$. □

From Proposition 2.14 we have the following corollary.

Corollary 5.4. *In the special case where $L = \mu\circ$ comes from a regular F-manifold structure $(\circ, e)$, the reductions are defined by solutions*

$$\mu^1(r^1, \dots, r^n), \dots, \mu^n(r^1, \dots, r^n), \qquad V(r^1, \dots, r^n)$$

of the system (5.2).

5.3 | Some Examples

In all the examples, except Example 5.3.14, $\mu = E = \sum_i r^i \frac{\partial}{\partial r^i}$.

5.3.1 | 2 × 2 Jordan Block

$$L = \begin{bmatrix} r_1 & 0 \\ r_2 & r_1 \end{bmatrix}.$$

$$V = F_1(r_1) r_2 + F_2(r_1).$$

5.3.2 | 3 × 3 Jordan Block

$$L = \begin{bmatrix} r_1 & 0 & 0 \\ r_2 & r_1 & 0 \\ r_3 & r_2 & r_1 \end{bmatrix}.$$

$$V = F_1(r_1) r_3 + \frac{1}{2} F_1'(r_1) r_2^2 + F_2(r_1) r_2 + F_3(r_1).$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

5.3.3 | $2\times 2+1\times 1$ Jordan Blocks

$$L=\begin{bmatrix} r_1 & 0 & 0\\ r_2 & r_1 & 0\\ 0 & 0 & r_3\end{bmatrix}.$$

$$V=F_1(r_1)r_2+F_2(r_1)+F_3(r_3).$$

5.3.4 | 4×4 Jordan Block

$$L=\begin{bmatrix} r_1 & 0 & 0 & 0\\ r_2 & r_1 & 0 & 0\\ r_3 & r_2 & r_1 & 0\\ r_4 & r_3 & r_2 & r_1\end{bmatrix}.$$

$$V=\big(F_1'(r_1)r_2+F_2(r_1)\big)r_3+F_1(r_1)r_4+\frac{1}{6}F_1''(r_1)r_2^3+\frac{1}{2}F_2'(r_1)r_2^2+F_3(r_1)r_2+F_4(r_1).$$

5.3.5 | $3\times 3+1\times 1$ Jordan Blocks

$$L=\begin{bmatrix} r_1 & 0 & 0 & 0\\ r_2 & r_1 & 0 & 0\\ r_3 & r_2 & r_1 & 0\\ 0 & 0 & 0 & r_4\end{bmatrix}.$$

$$V=F_1(r_1)r_3+\frac{1}{2}F_1'(r_1)r_2^2+F_2(r_1)r_2+F_3(r_1)+F_4(r_4).$$

5.3.6 | $2\times 2+2\times 2$ Jordan Blocks

$$L=\begin{bmatrix} r_1 & 0 & 0 & 0\\ r_2 & r_1 & 0 & 0\\ 0 & 0 & r_3 & 0\\ 0 & 0 & r_4 & r_3\end{bmatrix}.$$

$$V=F_1(r_1)r_2+F_2(r_1)+F_3(r_3)r_4+F_4(r_3).$$

5.3.7 | $2\times 2+1\times 1+1\times 1$ Jordan Blocks

$$L=\begin{bmatrix} r_1 & 0 & 0 & 0\\ r_2 & r_1 & 0 & 0\\ 0 & 0 & r_3 & 0\\ 0 & 0 & 0 & r_4\end{bmatrix}.$$

$$V=F_1(r_1)r_2+F_2(r_1)+F_3(r_3)+F_4(r_4).$$

5.3.8 | 5×5 Jordan Block

$$L=\begin{bmatrix} r_1 & 0 & 0 & 0 & 0\\ r_2 & r_1 & 0 & 0 & 0\\ r_3 & r_2 & r_1 & 0 & 0\\ r_4 & r_3 & r_2 & r_1 & 0\\ r_5 & r_4 & r_3 & r_2 & r_1\end{bmatrix}.$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

$$V = \left(F_1'(r_1)r_2 + F_2(r_1)\right) r_4 + F_1(r_1)r_5 + \frac{1}{2}F_1'(r_1)r_3^2 + \left(\frac{1}{2}F_1''(r_1)r_2^2 + F_2'(r_1)r_2 + F_3(r_1)\right)r_3$$
$$+ \frac{1}{24}F_1'''(r_1)r_2^4 + \frac{1}{6}F_2''(r_1)r_2^3 + \frac{1}{2}F_3'(r_1)r_2^2 + F_4(r_1)r_2 + F_5(r_1).$$

5.3.9 | $4 \times 4 + 1 \times 1$ Jordan Blocks

$$L = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ r_2 & r_1 & 0 & 0 & 0 \\ r_3 & r_2 & r_1 & 0 & 0 \\ r_4 & r_3 & r_2 & r_1 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix}.$$

$$V = \left(F_1'(r_1)r_2 + F_2(r_1)\right)r_3 + F_1(r_1)r_4 + \frac{1}{6}F_1''(r_1)r_2^3 + \frac{1}{2}F_2'(r_1)r_2^2 + F_3(r_1)r_2 + F_4(r_1) + F_5(r_5).$$

5.3.10 | $3 \times 3 + 2 \times 2$ Jordan Blocks

$$L = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ r_2 & r_1 & 0 & 0 & 0 \\ r_3 & r_2 & r_1 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & r_5 & r_4 \end{bmatrix}.$$

$$V = F_1(r_1)r_3 + \frac{1}{2}F_1'(r_1)r_2^2 + F_2(r_1)r_2 + F_3(r_1) + F_4(r_4)r_5 + F_5(r_4).$$

5.3.11 | $3 \times 3 + 1 \times 1 + 1 \times 1$ Jordan Blocks

$$L = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ r_2 & r_1 & 0 & 0 & 0 \\ r_3 & r_2 & r_1 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix}.$$

$$V = F_1(r_1)r_3 + \frac{1}{2}F_1'(r_1)r_2^2 + F_2(r_1)r_2 + F_3(r_1) + F_4(r_4) + F_5(r_5).$$

5.3.12 | $2 \times 2 + 2 \times 2 + 1 \times 1$ Jordan Blocks

$$L = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 \\ r_2 & r_1 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & r_4 & r_3 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix}.$$

$$V = F_1(r_1)r_2 + F_2(r_1) + F_3(r_3)r_4 + F_4(r_3) + F_5(r_5).$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

5.3.13 | $2\times 2+1\times 1+1\times 1+1\times 1$ Jordan Blocks

$$L=\begin{bmatrix} r_1 & 0 & 0 & 0 & 0\\ r_2 & r_1 & 0 & 0 & 0\\ 0 & 0 & r_3 & 0 & 0\\ 0 & 0 & 0 & r_4 & 0\\ 0 & 0 & 0 & 0 & r_5 \end{bmatrix}.$$

$$V=F_1(r_1)r_2+F_2(r_1)+F_3(r_3)+F_4(r_4)+F_5(r_5).$$

In the regular case, with the Euler vector field chosen as eventual identity, the general solution of Equation (5.2b) was obtained in [4] by combining two results:

- The general solution of the equation $d\cdot d_L V=0$, for L having a single Jordan block of size n, is given by

$$V=F_n(r^1)+\sum_{s>0}\frac{1}{s!}\sum_{k_1=2}^{n}\cdots\sum_{k_s=2}^{n}r^{k_1}\cdots r^{k_s}\left(\frac{d}{dr^1}\right)^{(s-1)}F_{n+s-\sum_{j=1}^{s}k_j}(r^1),$$

  where $F_1,\dots,F_n$ are arbitrary functions of r^1, and $F_{n+s-\sum_{j=1}^{s}k_j}=0$ whenever $n+s-\sum_{j=1}^{s}k_j\le 0$.
- A splitting lemma reducing the solution in the case of multiple Jordan blocks to the single Jordan block case (this lemma has also been proved in [10] by different methods).

5.3.14 | n-Component Jordan Chain

The system of hydrodynamic type

$$\begin{bmatrix} u^1_{t_1}\\ u^2_{t_1}\\ \vdots\\ u^{n-1}_{t_1}\\ u^n_{t_1}\end{bmatrix}=\begin{bmatrix} u^1 & 1 & 0 & \dots & 0\\ 0 & u^1 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & u^1 & 1\\ 0 & \dots & 0 & 0 & u^1\end{bmatrix}\begin{bmatrix} u^1_x\\ u^2_x\\ \vdots\\ u^{n-1}_x\\ u^n_x\end{bmatrix} \tag{5.10}$$

appeared in [11] and also in [12] in the study of parabolic regularization of the gradient catastrophes for the Burgers–Hopf equation. The system (5.10) can be written as $\mathbf{u}_{t_1}=(L-V\,I)\mathbf{u}_x$ with $V=-u^1$ and

$$L=\begin{bmatrix} 0 & 1 & 0 & \dots & 0\\ 0 & 0 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ 0 & \dots & 0 & 0 & 1\\ 0 & \dots & 0 & 0 & 0\end{bmatrix}.$$

While L is not invertible in this case, (5.10) provides an instance of a reduction of Pavlov's hydrodynamic chain with $\alpha\in\mathbb{Z}_{\ge 0}$, as the Nijenhuis torsion clearly vanishes and $d\cdot d_L\,V=0$.

6 | Conclusions

In this paper we study finite-component regular reductions of dKP and Pavlov's hydrodynamic chain defined by systems of hydrodynamic type associated with a regular F-manifold structure $(\circ,e,E)$. They are described by the system

$$\left(\mathcal{L}_\mu c^i_{hj}-c^i_{js}c^s_{ht}[e,\mu]^t\right)\mu^h=e(V)\delta^i_j-e^i\partial_j V, \tag{6.1a}$$

$$\left(c^s_{ih}\partial_s\partial_j V-c^s_{jh}\partial_s\partial_i V\right)\mu^h=\left(c^h_{sj}\partial_i\mu^s-c^h_{si}\partial_j\mu^s\right)\partial_h V, \tag{6.1b}$$

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

and by the system

$$\mathcal{L}_\mu c^i_{hj} - c^i_{js} c^s_{ht} [e, \mu]^t = 0, \tag{6.2a}$$

$$\left(c^s_{ih} \partial_s \partial_j V - c^s_{jh} \partial_s \partial_i V \right) \mu^h = \left(c^h_{sj} \partial_i \mu^s - c^h_{si} \partial_j \mu^s \right) \partial_h V, \tag{6.2b}$$

respectively. For the system (6.1), the tensor field defining the system has the form $W = \mu\circ$, while for the system (6.2) $W = (\mu - Ve)\circ$.

Although the two resulting Gibbons–Tsarev systems closely resemble each other in several respects, their solution spaces differ dramatically. The latter case, (6.2), admits solutions for any given number of Jordan blocks and for Jordan blocks of arbitrary size, whereas in the former, (6.1), no nontrivial non-diagonalizable solutions exist. This finding shows that the existence of non-diagonalizable integrable reductions is not solely tied to the integrability of the underlying model, but depends on some additional property. In [15] the authors suggest that linear degeneracy could play an important role in the existence of such reductions. Remarkably, the same condition is necessary [14], but not sufficient [6], for the existence of a Hamiltonian structure. Identifying which properties beyond integrability ensure the existence of non-diagonalizable integrable reductions remains an open, likely intricate problem that merits deeper study.

Acknowledgments

The authors thank Antonio Moro for useful discussions. Paolo Lorenzoni, Sara Perletti, and Karoline van Gemst are supported by funds of INFN (Istituto Nazionale di Fisica Nucleare) by IS-CSN4 Mathematical Methods of Nonlinear Physics. The authors are thankful to GNFM (Gruppo Nazionale di Fisica Matematica) for supporting activities that contributed to the research reported in this paper.

Open access publishing facilitated by Universita degli Studi di Milano-Bicocca, as part of the Wiley - CRUI-CARE agreement.

Conflicts of Interest

The authors declare no conflicts of interest.

Data Availability Statement

No data sets were generated or analyzed in this work.

References

1. A. Arsie and P. Lorenzoni, "Cyclic F-Manifolds, Distinguished Connections and Integrability," arXiv:2605.25277.

2. P. Lorenzoni, M. Pedroni, and A. Raimondo, "F-Manifolds and Integrable Systems of Hydrodynamic Type," *Archivum Mathematicum* 47 (2011): 163–180.

3. P. Lorenzoni and S. Perletti, "Integrable Systems, Frölicher–Nijenhuis Bicomplexes and Lauricella Bi-Flat Structures," *Nonlinearity* 36, no. 12 (2023): 6925–6990.

4. P. Lorenzoni, S. Perletti, and K. van Gemst, "Integrable Hierarchies and F-Manifolds With Compatible Connection," *Communications in Mathematical Physics* 406 (2025): 1–64.

5. P. Lorenzoni, S. Perletti, and K. van Gemst, "The Generalised Hodograph Method for Non-Diagonalisable Integrable Systems of Hydrodynamic Type," *Nonlinearity* 39 (2026): 1–26.

6. P. Lorenzoni, S. Perletti, and K. van Gemst, "Semi-Hamiltonian Properties of a Class of Non-Diagonalisable Systems of Hydrodynamic Type," *Communications in Mathematical Physics* 407 (2026): 46.

7. L. David and C. Hertling, "Regular F-Manifolds: Initial Conditions and Frobenius Metrics," *Annali della Scuola Normale Superiore di Pisa Classe di Scienze Serie V* XVII (2017): 1121–1152.

8. A. V. Bolsinov, A. Y. Konyaev, and V. S. Matveev, "Local Description of gl-Regular Haantjes Operators," arXiv:2606.04985.

9. P. Lorenzoni and F. Magri, "A Cohomological Construction of Integrable Hierarchies of Hydrodynamic Type," *International Mathematics Research Notices* 2005, no. 34 (2005): 2087–2100.

10. A. Bolsinov, A. Y. Konyaev, and V. S. Matveev, "Nijenhuis Geometry IV: Conservation Laws, Symmetries and Integration of Certain Non-Diagonalisable Systems of Hydrodynamic Type in Quadratures," *Nonlinearity* 37, no. 10 (2024): 105003.

11. Y. Kodama and B. Konopelchenko, "Confluence of Hypergeometric Functions and Integrable Hydrodynamic Type Systems," *Theoretical and Mathematical Physics* 188, no. 3 (2016): 429–455.

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License

12. B. G. Konopelchenko and G. Ortenzi, "Parabolic Regularization of the Gradient Catastrophes for the Burgers–Hopf Equation and Jordan Chain," *Journal of Physics A: Mathematical and Theoretical* 51 (2018): 1–26.

13. E. Ferapontov and M. V. Pavlov, "Kinetic Equation for Soliton Gas: Integrable Reductions," *Journal of Nonlinear Science* 32, no. 2 (2022): 26.

14. P. Vergallo and E. Ferapontov, "Hamiltonian Systems of Jordan Block Type: Delta-Functional Reductions of the Kinetic Equation for Soliton gas," *Journal of Mathematical Physics* 64, no. 10 (2023): 103505.

15. L. Xue and E. Ferapontov, "Quasilinear Systems of Jordan Block Type and the mKP Hierarchy," *Journal of Physics A: Mathematical and Theoretical* 53, no. 20 (2020): 1–14.

16. C. Hertling and Y. I. Manin, "Weak Frobenius Manifolds," *International Mathematics Research Notices* 1999, no. 6 (1999): 277–286.

17. A. Arsie and P. Lorenzoni, "Haantjes Torsion and Integrability: A Proof of Bolsinov-Konyaev-Matveev's Conjecture," arXiv:2607.29373.

18. B. B. Kadomtsev and V. I. Petviashvili, "On the Stability of Solitary Waves in Weakly Dispersing Media," *Soviet Physics - Doklady* 15 (1970): 539–541.

19. O. V. Rudenko and S. I. Soluyan, *Theoretical Foundations of Nonlinear Acoustics* (Consultants Bureau, 1977). Translated from the Russian by Robert T. Beyer, Studies in Soviet Science.

20. E. A. Zabolotskaya and R. V. Khokhlov, "Quasi-Plane Waves in the Nonlinear Acoustics of Confined Beams," *Soviet Physics Acoustics* 15 (1969): 35–40.

21. M. V. Pavlov, "Integrable Hydrodynamic Chains," *Journal of Mathematical Physics* 44 (2003): 4134–4156.

22. J. Gibbons and S. P. Tsarev, "Reductions of the Benney Equations," *Physics Letters A* 211, no. 1 (1996): 19–24.

23. D. J. Benney, "Some Properties of Long Nonlinear Waves," *Studies in Applied Mathematics* 52, no. 1 (1973): 45–50.

24. S. P. Tsarev, "The Geometry of Hamiltonian Systems of Hydrodynamic Type. The Generalised Hodograph Transform," *USSR Izvestiya* 37 (1991): 397–419.

25. E. V. Ferapontov and K. R. Khusnutdinova, "On the Integrability of (2+1)-Dimensional Quasilinear Systems," *Communications in Mathematical Physics* 248 (2004): 187–206.

26. J. Gibbons, P. Lorenzoni, and A. Raimondo, "Hamiltonian Structures of Reductions of the Benney System," *Communications in Mathematical Physics* 287 (2009): 291–322.

27. J. Gibbons and S. P. Tsarev, "Conformal Maps and Reductions of the Benney Equations," *Physics Letters A* 258, no. 4–6 (1999): 263–271.

28. A. Arsie and P. Lorenzoni, "F-Manifolds With Eventual Identities, Bidifferential Calculus and Twisted Lenard-Magri Chains," *International Mathematics Research Notices* 2013, no. 17: 3931–3976.

29. Y. I. Manin, "F-Manifolds With Flat Structure and Dubrovin's Duality," *Advanced Mathematics* 198, no. 1 (2005): 5–26.

30. L. David and I. A. B. Strachan, "Dubrovin's Duality for F-Manifolds With Eventual Identities," *Advanced Mathematics* 226, no. 5 (20 March 2011): 4031–4060.

31. S. Perletti and I. A. B. Strachan, "Regular F-Manifolds With Eventual Identities," *Journal of Physics A: Mathematical and Theoretical* 57 (2024): 475201.

32. A. Frölicher and A. Nijenhuis, "Theory of Vector-Valued Differential Forms," *Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, Series A* 59 (1956): 338–359.

14679590, 2026, 4, Downloaded from https://onlinelibrary.wiley.com/doi/10.1111/sapm.70321 by CochraneItalia, Wiley Online Library on [26/10/2026]. See the Terms and Conditions (https://onlinelibrary.wiley.com/terms-and-conditions) on Wiley Online Library for rules of use; OA articles are governed by the applicable Creative Commons License