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Regular non-semisimple Frobenius manifolds and other flat structures

Sara Perletti

Supervised by Prof. Paolo Lorenzoni

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Summary

Originated in the context of two-dimensional topological field theories, Frobenius manifolds lie at the intersection of several areas of mathematics, embracing integrable systems, algebraic geometry, singularity theory and many more. While most of the theory has been developed under some semisimplicity assumption, we pay our attention to the broader case governed by a milder assumption of regularity. We study regular non-semisimple Frobenius manifolds and other geometric structures progressively generalizing them: F-manifolds, flat F-manifolds and bi-flat F-manifolds.

This thesis presents the results of my research. It unfolds in four chapters, the first of which sets the context and paves the way for the discussion of my own work. The second chapter illustrates a project in its final stage that I tackled together with Prof. Ian Strachan during my visit to the University of Glasgow. The third and the fourth chapters describe the outcome of two problems that I faced in collaboration with Prof. Paolo Lorenzoni, published in [65, 66]. In particular, the main result is the construction of a class of regular non-semisimple bi-flat F-manifolds related to integrable systems of hydrodynamic type, Nijenhuis geometry and the theory of Lauricella functions.

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Introduction

Ever since the scientific revolution, mathematics and physics have been deeply connected and have mutually influenced each other along the centuries in proposing new problems and offering perspectives to handle them.

A notable example of this found place in the twentieth century when Riemannian geometry laid the foundation for the development of general relativity and quantum mechanics stimulated progressions in functional analysis. A second significant example is provided by another physical theory conceived in the past century: quantum field theory. Especially in the last thirty years, quantum field theory has been remarkably building on the interplay between mathematics and physics, leading to the uncovering of very interesting geometrical structures.

Two-dimensional field theories, WDVV equations and Frobenius manifolds

Among quantum field theories, sits the notable class of topological field theories, or topological quantum field theories, characterized by the property of admitting topological invariance. Within this context, in the early 1990's two works by E. Witten [84], in the setting of topological sigma models, and R. Dijkgraaf, H. Verlinde, E. Verlinde [25], in the more general frame of topological field theories obtained by a so-called twisting from supersymmetric quantum field theories, proved themselves as exceptionally relevant in the outlook of detecting new mathematical structures. Here, a remarkable system of nonlinear partial differential equations made its first appearance. B. Dubrovin [28] called such equations *WDVV equations*, after the authors of the two mentioned papers.

Properties of two-dimensional topological field theories turned out to be encoded in specific algebraic structures called *Frobenius algebras*. Such an algebra structure is defined by a commutative, associative and unital product with respect to which a symmetric, non-degenerate bilinear form is invariant. WDVV equations are also known as *associativity equations*, as they express the associativity of the product on the associated Frobenius algebra.

In the physical setting the solutions of WDVV equations describe the moduli space of topological conformal field theories. This is what lead Dubrovin to formally introduce the structure of a Frobenius manifold [27, 28], lately appearing in literature also by the name of *Dubrovin-Frobenius manifold.*

The geometric structure of a Frobenius manifold consists of a commutative and associative product $\circ$ on the tangent bundle, a flat metric η which must be invariant with respect to the product and two distinguished vector fields, denoted by e and E. The first one, e, is known as the unit vector field as it is required to be unit of the product, as well as covariantly constant with respect to the Levi-Civita connection associated with the metric. Additional conditions, as well as axioms expressing compatibility between the metric and the product, are required.

The structure defined on the manifold induces a specific structure on each tangent space. In particular, $\circ$ induces a commutative, associative and unital product with respect to which the symmetric, non-degenerate bilinear form induced by η is invariant. The tangent space to a Frobenius manifold carries then at each point the structure of a Frobenius algebra. Thanks to Dubrovin's axioms, the structure constants of the product, in flat coordinates of the metric, are given in terms of the third order partial derivatives of a function satisfying WDVV equations. In this sense, Frobenius manifolds give a coordinate-free reformulation of WDVV equations. Such a function is known as Frobenius potential. The vector field E takes the name of Euler vector field, since it encodes the homogeneity property of the Frobenius potential. A second product can be defined in terms of the first one and of the Euler vector field, as well as a second metric. This lead Dubrovin to introduce the notion of almost duality for Frobenius manifolds [32].

An additional requirement that can be taken into account is the demand for the Frobenius manifolds to be semisimple, meaning that there exists a holonomic frame of idempotent vector fields. Semisimple Frobenius manifolds are also known as massive, as from a physical point of view they correspond to massive perturbations of two-dimensional topological field theories. Given a semisimple Frobenius manifold, Dubrovin introduced a privileged set of local coordinates [27], called *(Dubrovin's) canonical coordinates*, which reduce the product induced on the manifold by the Frobenius algebras to a constant canonical form.

Under such a semisimplicity assumption, Dubrovin gave a complete classification of three-dimensional Frobenius manifolds in terms of transcendental functions of the Painlevé-VI type [30]. This was possible by means of studying a system of partial differential equations known as Darboux-Egorov system [17, 36].

Extent of Frobenius manifold theory

The theory of Frobenius manifolds appears in several areas of mathematics, including singularity theory, invariant theory of Coxeter groups, integrable systems and algebraic geometry.

The Frobenius manifold structure on the orbit space of a Coxeter group was found by Dubrovin in [31] and relies on previous work of K. Saito and his collaborators [78, 79]. In particular, driven by the study of the structure of parameter spaces of isolated hypersurface singularities, Saito constructed flat coordinates on the orbit spaces of finite real reflection groups. Dubrovin interpreted these results in terms of flat pencils of metrics, which allowed him to define a Frobenius manifold structure on the orbit space of the group. This class of Frobenius manifolds is particularly important, since it provides polynomial solutions of WDVV equations.

In algebraic geometry, Frobenius manifolds naturally appear in the study of intersection theory on moduli spaces of curves and they find applications in enumerative problems concerning, for instance, quantum cohomology and Gromov-Witten theory. In this setting, one of the most celebrated results is the Witten conjecture [85], proved by M. Kontsevich [54]. It establishes a relationship between a tau-function for the Korteweg-de Vries (KdV) integrable hierarchy and the generating function for the intersection numbers of Mumford–Morita–Miller stable classes on the Deligne–Mumford compactification of moduli space of pointed curves [86]. In this context, the Witten conjecture turned out to be just the first and the simplest of several results connecting mathematical and theoretical physics with algebraic geometry. In order to formalize properties on this latter side, in 1994 Kontsevich and Manin introduced cohomological field theories [55], the trivial instance of which underlies the generating function starring in the Witten-Kontsevich result. The geometric structure of a Frobenius manifold turns out to encode the genus zero information of a cohomological field theory. In the semisimple case, a powerful result known as Givental-Teleman classification [82] allows to reconstruct the information at all genera.

Frobenius manifolds also sprout in the ground of integrable hierarchies, as related to relevant classes of integrable partial differential equations, such as integrable PDEs of hydrodynamic type, and they come to light in the theory of partial diffential equations and special functions as closely linked to Painlevé equations, as mentioned above. Moreover, an integrable hierarchy known as the principal hierarchy can be associated with each Frobenius manifold. In the semisimple case, it can be regarded as the dispersionless limit of a more involved bi-Hamiltonian hierarchy which was constructed by Dubrovin and Zhang. After them, the class of integrable hierarchies to which it belongs takes the name of Dubrovin-Zhang hierarchies, or hierarchies of topological type [35]. The KdV hierarchy is the simplest example of this class. Such integrable hierarchies can be equivalently constructed from a cohomological field theory, under an analogous assumption of semisimplicity. In this frame and in the semisimple case, the dispersive perturbation of the principal hierarchy into the full hierarchy of topological type reflects the Givental-Teleman reconstruction of the all-genera information of a cohomological field theory from the genus-zero part encoded in the Frobenius manifold.

The emerging picture is that of Frobenius manifolds as building bridges between different branches of mathematics and physics, providing multiple and convenient connections. Some of the a-priori unrelated constructions where they arise, such as for quantum cohomology and singularity theory, turn out to communicate with each other according to a phenomenon known in literature as mirror symmetry. A first example of this was observed by P. Candelas, X. de la Ossa, P. Green and L. Parkes in studying quantum cohomologies of Calabi-Yau varieties [13].

Generalizing Frobenius manifolds

Given the wide usefulness of Frobenius manifolds throughout mathemathics, it is natural to look for extensions of its rich structure to a more general version which may be applicable to even more situations, hopefully retaining equal convenience.

To this extent, a first step was taken by C. Hertling and Y. Manin in [48]. Here, they introduced the notions of weak Frobenius manifold and of F-manifold. The former is a Frobenius manifold structure without a pre-fixed flat metric, while the latter only consists of a commutative, associative and unital product on the manifold satisfying an additional axiom, known as the Hertling-Manin condition, encoding part of the original potentiality property for Frobenius manifolds, at least in the semisimple case. Further properties of F-manifolds were discussed in [47].

In [77] Sabbah introduced the notion of Saito structures without metric and in [69] Manin defined the closely related structure of a flat F-manifold, or F-manifold with compatible flat structure. As the name suggests, it is an F-manifold enriched with a flat and torsionless connection satisfying some compatibility conditions with respect to the product and its unit. In flat coordinates for the connection, the structure constants of the product can be expressed as the second partial derivatives of a vector potential, which satisfies a system of partial differential equations known as oriented associativity equations [67]. This potentiality feature is not the only one echoing the properties of Frobenius manifolds. For instance, a notion of principal hierarchy and an analog of cohomological field theory can be associated with a flat F-manifold as well, as carried out in [64] and [12] respectively. As for weak Frobenius manifolds, in the semisimple case Dubrovin's almost duality can be also extended.

In the case of flat F-manifolds, the dual structure is an additional flat F-manifold structure satisfying suitable compatibility conditions leading to the notion of *bi-flat F-manifold*, introduced by A. Arsie and P. Lorenzoni in [4]. In particular, the two connections must satisfy a condition known as almost hydrodynamical equivalence, expressed in terms of their exterior covariant derivatives. It follows from this definition that Frobenius manifolds are bi-flat F-manifolds without an invariant metric, where the Euler vector field is unit of the second flat structure.

It turned out that many constructions in the theory of Frobenius manifolds can be extended to flat and bi-flat F-manifolds, including the relation with reflection groups [49, 5] and with Painlevé transcendents that emerge from a Darboux-Egoroff system, once suitably augmented by dropping a symmetry requirement for some quantities, known as Ricci's rotation coefficients [17, 75]. In particular, three-dimensional semisimple bi-flat F-manifolds are parametrized by solutions of Painlevé-VI equation [4, 61, 6].

Regularity

The above overview reflects the fact that most of the theory of Frobenius manifolds and of their generalizations has been developed under the semisimplicity assumption. For instance, it appears as essential for Dubrovin's canonical coordinates and classification of solutions to the WDVV equations, for the Givental-Teleman reconstruction result and in the theory of bi-Hamiltonian deformations of integrable hierarchies.

The easier setting provided by this case is certainly an understandable reason to assume semisimplicity from the beginning when facing a new problem or constructing new objects. However, in some situations it may be pointlessly restrictive.

Results obtained in the non-semisimple case suggest that some of those constructions may actually not rely on semisimplicity. For instance, an integrable hierarchy known by the name of double ramification hierarchy [10] was defined even for non-semisimple cohomological field theories and it is conjectured to be equivalent to the integrable hierarchy of topological type. This DR-DZ conjecture was proved for selected cohomological field theories, including the trivial one realizing Witten's conjecture.

Another empowering result was achieved by L. David and Hertling in [18]. Under a milder assumption of regularity, they provided a generalization of

Dubrovin's canonical coordinates in the setting of F-manifolds with an Euler vector field. An Euler vector field is intended to be a distinguished vector field on the manifold satisfying a suitable compatibility condition with the product. The regularity assumption is the requirement for the operator of multiplication by such an Euler vector field to be regular, in the sense that any two distinct Jordan blocks from its Jordan normal form must have distinct eigenvalues.

With such a precious set of canonical coordinates, working in the regular non-semisimple setting looks more conceivable, despite still being indisputably more involved than the semisimple one. A recent result obtained in the regular non-semisimple setting was the classification of three-dimensional regular non-semisimple bi-flat F-manifolds as parametrized by solutions of the full Painlevé-IV and full Painlevé-V equations [6], as well as the construction of flat structures out of regular generalized Okubo systems [51].

Results

In the wake of David-Hertling's extension of canonical coordinates, I studied some of the above geometric structures in the regular non-semisimple setting. The first two projects which I introduce below were conducted in collaboration with my advisor Prof. Paolo Lorenzoni.

In the first instance, we studied regular non-semisimple Frobenius manifolds and the associated bi-Hamiltonian structures of hydrodynamic type. This work appeared in [65]. We recovered formulas for generic dimension and then we focused on low dimensions, up to 4. In the case corresponding to a single Jordan block in the Jordan canonical form of the operator of multiplication by the Euler vector field, we gave a complete classification. In the cases associated with multiple Jordan blocks, we reduced the classification problem to systems of partial differential equations: a third-order ODE in the three dimensoinal case and to a system of third-order PDEs in the four-dimensional cases. In all of them, we provided explicit examples of Frobenius potentials. An example from our work appeared in [38] in the context of integrable systems of hydrodynamic type which cannot be reduced to a diagonal form.

A second problem we addressed concerned the development of a class of regular bi-flat F-manifolds, called *Lauricella bi-flat F-manifolds* due to their association to the theory of Lauricella functions [56]. The key idea was to combine the construction of integrable hierarchies of hydrodynamic type starting from differential bicomplexes associated with certain Nijenhuis torsionless tensors, known as Frölicher-Nijenhuis bicomplexes, with the construction of flat F-manifolds starting from integrable systems of hydrodynamic type. The torsionless tensor is the operator of multiplication by the Euler vector field and the Jordan blocks of its Jordan canonical form are as many as the parameters associated with the family of regular Lauricella bi-flat F-manifolds which is the result of our construction. These results appeared in [66].

A third piece of work concerning regular F-manifolds is in preparation, in collaboration with Prof. Ian A. B. Strachan. Given a regular F-manifold, we studied a dual multiplication defined by means of an eventual identity, the notion of which generalizes an Euler vector field and guarantees that the dual structure is an F-manifold as well. After solving the equations for an eventual identity, we provided a system of local coordinates preserving the dual multiplication.

Structure of the thesis

The structure of the thesis is outlined below.

After setting the scene for our dissertation in this introductive section, in Chapter 1 we give the definitions of Frobenius manifolds and of their generalizations. We discuss the motivation behind their introduction and we illustrate relevant properties of them, including some of the connections with other areas of mathematics that we mentioned above.

Chapter 2 deals with regular F-manifolds with eventual identities, describing my joint work in preparation with Prof. Ian A. B. Strachan.

Chapter 3 is devoted to regular Frobenius manifolds, presenting the work appearing in [65]. For the sake of readability, some of the formulas for the four-dimensional regular non-semisimple cases, corresponding to a Jordan canonical form of the operator of multiplication by the Euler vector field having at least one Jordan block of size 2, have been moved to Appendix A.

Chapter 4 explains the construction of the regular Lauricella bi-flat F-manifolds relized in [66]. Proofs of some crucial technical lemmas were removed from this chapter in order to provide a neater exposition. They appear in Appendix B.

Finally, we conclude this dissertation by presenting open problems giving way to future work.

Chapter 1

Frobenius manifolds and their generalizations

In the wake of the results by Witten [84], Dijkgraaf and the Verlinde brothers [25] in the context of two-dimensional gravity and by Saito [78, 79] in singularity theory, Dubrovin introduced the notion of Frobenius manifolds [27, 28] in order to provide an intrinsic geometric reformulation of the system of WDVV associativity equations appearing in [84, 25].

In this chapter we unfold this motivation, drawing the relation between two-dimensional topological field theories and Frobenius algebras. We define Frobenius manifolds and describe some of their properties, including the relation with the WDVV equations via the Frobenius potential, the construction of a principal hierarchy which can be recovered as the dispersionless limit of a bi-Hamiltonian hierarchy of topological type [35] and the relations with Painlevé equations [30] and with cohomological field theories [85, 54, 55, 82]. We focus in particular on semisimple Frobenius manifolds, introducing Dubrovin's canonical coordinates [27].

We then define F-manifolds [48, 47] as a generalization of Frobenius manifolds retaining part of their potentiality. We introduce the notions of Euler vector fields and eventual identities on F-manifolds and we state the result from [18] extending Dubrovin's canonical coordinates to F-manifolds with Euler vector field under some regularity assumption, of which semisimplicity is a particular instance. This result is crucial for the present dissertation, as the three main results appearing in the following chapters rely on such generalized canonical coordinates.

Subsequently, we define flat F-manifolds [69] and we show some of the Frobenius properties that are still valid for this more general class. In particular, we treat a potentiality relation involving a system of partial differential equations known as oriented associativity equations [67] and we discuss analogs of the principal hierarchy [64] and of cohomological field theories [12, 2]. Finally, we introduce bi-flat F-manifolds [4], describing Frobenius manifolds without an invariant metric.

1.1 Frobenius manifolds

Let M be a real or complex manifold of finite dimension $n \in \mathbb{N}$. In the first case we will assume that all geometric data are smooth, while in the second case we will assume that all geometric data are holomorphic. TM will denote the smooth or holomorphic tangent bundle, respectively. Let us introduce the notion of a Frobenius structure on M, following [28].

Definition 1.1 *A* Frobenius manifold *structure* $(\eta, \circ, e, E)$ *on* M *is defined by a non-degenerate metric* η[1]*, a commutative and associative product* $\circ$ *on the tangent bundle* TM *and two distinguished vector fields* e *and* E*, satisfying the following conditions:*

1. *the metric is invariant with respect to the product, namely*

$$\eta(X \circ Y, Z) = \eta(X, Y \circ Z)$$

   for X, Y, Z *being vector fields on* M*,*

2. *the metric is flat,*

3. *the tensor* $\nabla \circ$ *is symmetric,*

4. e *is unit of the product,*

5. e *is a flat vector field,*

6. E *is subject to the following homogeneity conditions:*

$$\mathcal{L}_E \circ = \circ, \qquad \mathcal{L}_E e = -e, \qquad \mathcal{L}_E \eta = (2-d)\eta$$

for some constant d*, known as the* charge *of the Frobenius manifold. Here* ∇ *denotes the Levi-Civita connection associated with* η *and* $\mathcal{L}_Z$ *denotes the Lie derivative along a vector field* Z*. The vector field* e *takes the name of* unit vector field*, while* E *takes the name of* Euler vector field.

In local coordinates $t^1, \dots, t^n$, we denote by $\{c^i_{jk}\}_{i,j,k \in \{1,\dots,n\}}$ the structure constants of the product

$$\partial_i \circ \partial_j = c^k_{ij}\, \partial_k, \qquad i, j, k \in \{1, \dots, n\},$$

[1] Such a metric is not required to be positive-definite.

and we set

$$\eta_{ij} = \eta(\partial_i, \partial_j), \qquad i, j \in \{1, \dots, n\},$$

where $\partial_i = \frac{\partial}{\partial t^i}$ for each $i \in \{1, \dots, n\}$. The axioms defining a Frobenius manifold are locally described by the formulas listed below. According to Einstein's convention, a summation symbol is to be intended when repeated indices appear. Commutativity and associativity of the product read

$$c_{ij}^k = c_{ji}^k, \qquad i, j, k \in \{1, \dots, n\}, \tag{1.1}$$

and

$$c_{il}^k c_{jm}^l = c_{jl}^k c_{im}^l, \qquad i, j, k, m \in \{1, \dots, n\}, \tag{1.2}$$

respectively. Invariance of the metric reads

$$\eta_{il}\, c_{jk}^l = \eta_{jl}\, c_{ik}^l, \qquad i, j, k \in \{1, \dots, n\}. \tag{1.3}$$

Flatness of the metric reads

$$R_{ijk}^m = \partial_j \Gamma_{ik}^m - \partial_i \Gamma_{jk}^m + \Gamma_{ik}^s \Gamma_{sj}^m - \Gamma_{jk}^s \Gamma_{is}^m = 0, \qquad i, j, k, m \in \{1, \dots, n\}. \tag{1.4}$$

Symmetry of the tensor $\nabla \circ$ reads

$$\nabla_i c_{jk}^l = \nabla_j c_{ik}^l, \qquad i, j, k, l \in \{1, \dots, n\}. \tag{1.5}$$

The property of e being unit of the product reads

$$c_{jk}^i e^k = \delta_j^i, \qquad i, j \in \{1, \dots, n\}. \tag{1.6}$$

Flatness of e reads

$$\nabla_i e^k = 0, \qquad i, k \in \{1, \dots, n\}. \tag{1.7}$$

The homogeneity conditions read

$$\mathcal{L}_E c_{jk}^i = c_{jk}^i, \quad \mathcal{L}_E e^i = -e^i, \quad \mathcal{L}_E \eta_{ij} = (2-d)\eta_{ij} \tag{1.8}$$

for each $i, j, k \in \{1, \dots, n\}$. In particular, by the homogeneity conditions, the Euler vector field E acts as a conformal Killing vector field on η and its flow preserves the structure constants of the product.

Let $(M, \eta, \circ, e, E)$ be a Frobenius manifold of dimension n.

Remark 1 *As a consequence of the above axioms, the Euler vector field is an affine vector field, namely $\nabla\nabla E = 0$ holds. Moreover*

$$\mathcal{L}_e \eta_{ij} = 0, \qquad i, j \in \{1, \dots, n\}. \tag{1.9}$$

The flatness of the metric endows a Frobenius manifold with a privileged set of local coordinates. More precisely, it implies the existence of local coordinates $t^1, \ldots, t^n$ in which the metric η is constant and the Christoffel symbols of its Levi-Civita connection vanish:

$$\Gamma_{ij}^k = 0, \qquad i, j, k \in \{1, \ldots, n\}.$$

Such coordinates are called *flat* and are defined up to an affine transformation. Using this freedom, one can always reduce the unit vector field to the form

$$e = \frac{\partial}{\partial t^1}$$

which reads $c_{1k}^i = \delta_k^i$ for each $i, k \in \{1, \ldots, n\}$.

Remark 2 *Since $\nabla\nabla E = 0$, the components of the Euler vector field are linear functions of the flat coordinates.*

Remark 3 *By the axiom* (1.3)*, the tensor whose components are defined by*

$$c_{ijk} := \eta_{il} c_{jk}^l, \qquad i, j, k \in \{1, \ldots, n\},$$

is completely symmetric. Moreover, in flat coordinates the tensor with components

$$\partial_l c_{ijk}, \qquad i, j, k, l \in \{1, \ldots, n\},$$

is completely symmetric as well. By successive use of the Poincaré lemma, there must then locally exist a function F of the flat coordinates $t^1, \ldots, t^n$ such that

$$\eta_{il} c_{jk}^l = \partial_i \partial_j \partial_k F \tag{1.10}$$

for each $i, j, k \in \{1, \ldots, n\}$.

Definition 1.2 *The function $F(t^1, \ldots, t^n)$ realizing* (1.10) *is called* Frobenius potential*.*

Let $\{\eta^{ij}\}_{i,j \in \{1,\ldots,n\}}$ denote the components of the contravariant metric being the inverse of η. The Frobenius manifold comes endowed with a second contravariant metric, known as intersection form.

Definition 1.3 *The contravariant metric of components*

$$g^{ij} = \eta^{il} \, c_{lk}^j \, E^k, \qquad i, j \in \{1, \ldots, n\},$$

takes the name of intersection form *of the Frobenius manifold $(M, \eta, \circ, e, E)$.*

Remark 4 *The contravariant metrics of components η^{ij}, g^{ij} satisfy the following relations:*

1. $\mathcal{L}_e \eta^{ij} = 0, \qquad i, j \in \{1, \ldots, n\},$
2. $\mathcal{L}_e g^{ij} = \eta^{ij}, \quad i, j \in \{1, \ldots, n\}.$

Theorem 1.1 *The contravariant pencil of metrics defined by*

$$g_\lambda^{ij} = g^{ij} - \lambda \eta^{ij}, \qquad i, j \in \{1, \ldots, n\}, \tag{1.11}$$

is flat.

Theorem 1.1 suggests a first relation between Frobenius manifolds and flat pencils of metrics, for which we refer to [29]. It is not always possible to produce a Frobenius manifold starting from a flat pencil of metrics. However, such a costruction was carried out by Dubrovin [31], using flat coordinates on the orbit spaces of finite real reflection groups (see [78, 79]). Frobenius manifolds defined this way provide polynomial solutions of WDVV equations, which we introduce below.

1.1.1 WDVV associativity equations

Dubrovin's original motivation for the definition of a Frobenius manifold structure was to give a geometric interpretation to solutions of a remarkable set of equations, introduced in the late '80s in the works of Witten, Dijkgraaf and the Verlinde brothers [84, 25]. After their names, such equations are known as WDVV equations.

Definition 1.4 *The overdetermined system of nonlinear PDEs*

$$\begin{cases} \eta^{ks}(\partial_s\partial_i\partial_j F)\,\eta^{ht}(\partial_t\partial_k\partial_l F) = \eta^{ks}(\partial_s\partial_i\partial_l F)\,\eta^{ht}(\partial_t\partial_k\partial_j F), & i, j, l, h \in \{1, \ldots, n\}, \\ \partial_1\partial_i\partial_j F = \eta_{ij}, & i, j \in \{1, \ldots, n\}, \\ \mathcal{L}_E F = (3-d)F + Q \end{cases} \tag{1.12}$$

for the function $F(t^1, \ldots, t^n)$ takes the name of Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations. *Q denotes a quadratic polynomial in $t^1, \ldots, t^n$:*

$$Q = A_{ij} t^i t^j + B_i t^i + C$$

for some constants $\{A_{ij}\}_{i,j \in \{1,\ldots,n\}}$, $\{B_i\}_{i \in \{1,\ldots,n\}}$, C.

The last condition can be interpreted as the requirement for the Euler vector field to rescale the Frobenius potential in flat coordinates up to quadratic terms.

Proposition 1.5 *The Frobenius potential is a solution to the WDVV equations.*

Proof: The associativity of the product gives

$$\eta^{ks}(\partial_s\partial_i\partial_j F)\,\eta^{ht}(\partial_t\partial_k\partial_l F)\eta^{ks}\eta_{sa}c^a_{ij}\,\eta^{ht}\eta_{sb}c^b_{kl} = c^k_{ij}\,c^h_{kl}$$
$$= c^k_{il}\,c^h_{kj} = \eta^{ks}(\partial_s\partial_i\partial_l F)\,\eta^{ht}(\partial_t\partial_k\partial_j F)$$

for each $i, j, l, h \in \{1, \ldots, n\}$. The property of e being unit of the product implies

$$\partial_1\partial_i\partial_j F = \partial_i\partial_1\partial_j F = \eta_{is}c^s_{1j} = \eta_{is}\delta^s_j = \eta_{ij}$$

for each $i, j \in \{1, \ldots, n\}$. Moreover, for each $i, j \in \{1, \ldots, n\}$ one has

$$\begin{aligned}\partial_1\partial_i\partial_j\big(\mathcal{L}_E F\big) &= \partial_1\partial_i\partial_j\big(E^s\partial_s F\big) = \partial_1\partial_i\big((\partial_j E^s)\partial_s F + E^s\partial_j\partial_s F\big)\\ &= \partial_1\big((\partial_j E^s)\partial_i\partial_s F + (\partial_i E^s)\partial_j\partial_s F + E^s\partial_i\partial_j\partial_s F\big)\\ &= (\partial_j E^s)\partial_1\partial_i\partial_s F + (\partial_i E^s)\partial_1\partial_j\partial_s F + (\partial_1 E^s)\partial_i\partial_j\partial_s F + E^s\partial_1\partial_i\partial_j\partial_s F\end{aligned}$$

as E is a linear functions of the flat coordinates. Since $e = \partial_1$ in flat coordinates and $[e, E] = e$ by the second homogeneity condition (1.8), one gets $\partial_1 E^s = \delta^s_1$ for each $s \in \{1, \ldots, n\}$. It follows that

$$\begin{aligned}\partial_1\partial_i\partial_j\big(\mathcal{L}_E F\big) &= (\partial_j E^s)\partial_1\partial_i\partial_s F + (\partial_i E^s)\partial_1\partial_j\partial_s F + \partial_i\partial_j\partial_1 F + E^s\partial_1\partial_i\partial_j\partial_s F\\ &= (\partial_j E^s)\eta_{is} + (\partial_i E^s)\eta_{sj} + \eta_{ij} + E^s\partial_s\eta_{ij} = \mathcal{L}_E\eta_{ij} + \eta_{ij}\end{aligned}$$

where $\mathcal{L}_E\eta_{ij} = (2-d)\eta_{ij}$ by the third homogeneity condition (1.8). This yields

$$\partial_1\partial_i\partial_j\big(\mathcal{L}_E F\big) = (3-d)\eta_{ij} = \partial_1\partial_i\partial_j\big((3-d)F\big)$$

implying that, up to additive quadratic terms,

$$\mathcal{L}_E F = (3-d)F.$$

■

In the above construction of a Frobenius potential, we explained how it is possible to get a solution to the WDVV equations starting from a Frobenius manifold.

Frobenius manifolds $\to$ WDVV equations

Conversely, a solution to the WDVV equations can be interpreted as a Frobenius potential, in turn defining, at least locally, a Frobenius manifold.

Frobenius manifolds $\rightleftarrows$ WDVV equations

Remark 5 *Let $F(t^1, \dots, t^n)$ be solution to the WDVV equations. By setting*

$$\eta_{ij} = \partial_1 \partial_i \partial_j F, \qquad i, j \in \{1, \dots, n\},$$

$$c^i_{jk} = \eta^{is} \partial_s \partial_j \partial_k F, \qquad i, j, k \in \{1, \dots, n\},$$

and $e = \partial_1$, one gets the components of the metric, the structure constants of the product and the unit and Euler vector fields.

WDVV equations are sometimes referred to as *associativity equations*, as they express the associativity property of the product of the corresponding Frobenius manifold.

1.1.2 Frobenius algebras and two-dimensional topological quantum field theories

Let $\mathbb{K}$ denote a field of characteristic zero.

Definition 1.6 *A* Frobenius algebra *$(A, \eta, \circ, e)$ over $\mathbb{K}$ is a finite-dimensional commutative and associative algebra A over $\mathbb{K}$ with product $\circ$, endowed with a distinguished element $e \in A$ being unit of the product and with a non-degenerate symmetric bilinear form η which is invariant with respect to the product, realizing $\eta(a \circ b, c) = \eta(a, b \circ c)$ for each $a, b, c \in A$. The bilinear form η is called* Frobenius form *or* Frobenius pairing.

A Frobenius manifold can be interpreted as the space of parameters of a family of Frobenius algebras, as at a point m of a Frobenius manifold M the tangent space $T_m M$ inherits a Frobenius algebra structure. More precisely, the product on the tangent spaces is inherited from the product $\circ$ of the Frobenius manifold and the non-degenerate symmetric bilinear form is induced by the metric η. It immediately follows that such bilinear form is invariant with respect to the product. The unit of the algebra is given by the evaluation of the unit vector field e at the point m. We cite [52] for the following examples.

Example 1.7 *The trivial Frobenius algebra is provided by the field $\mathbb{K}$ itself with multiplication $\cdot$ and multiplicative unit $\mathbb{1}$, with the bilinear form defined by*

$$\eta(a, b) = a \cdot b, \qquad a, b \in A.$$

Example 1.8 *Let $G = \{g_0, \dots, g_n\}$ be a finite abelian group with $g_0 = 1$. On the group algebra $V = \mathbb{R}G$ (with elements $\sum_{i=0}^n \alpha_i \, g_i$, $\alpha_i \in \mathbb{R}$) the product is defined by*

$$\left(\sum_{i=0}^n \alpha_i \, g_i \right) \cdot \left(\sum_{j=0}^n \beta_j \, g_j \right) = \left(\sum_{l=0}^n \left(\sum_{k=0}^n \alpha_k \, \beta_{k^{-1} \, l} \right) g_l \right).$$

Let $\vartheta : V \to \mathbb{R}$ satisfy $\vartheta(\sum_{i=0}^{n} \alpha_i\, g_i) = \alpha_0$. A pairing η can be defined as

$$\eta(u, v) = \vartheta(u \cdot v)$$

for each $u, v \in \mathbb{R}G$.

Following [28], we now introduce the notion of a two-dimensional topological field theory (2D TFT), appearing in literature also by the name of two-dimensional topological *quantum* field theory (2D TQFT). M. Atiyah gave this axiomatic definition of two-dimensional topological field theory in [7], inspired by the work of G. Segal on conformal field theories [80].

Definition 1.9 *A* two-dimensional topological field theory *is the assignment to a pair $(\Sigma, \partial\Sigma)$, consisting of a compact oriented surface and its boundary, of a vector $v_{(\Sigma,\partial\Sigma)}$ in the finite-dimensional complex vector space $A_{(\Sigma,\partial\Sigma)}$ defined as*

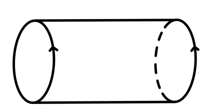
$$A_{(\Sigma,\partial\Sigma)} = \begin{cases} \mathbb{C} & \text{if } \partial\Sigma = \emptyset \\ A_1 \otimes \cdots \otimes A_k & \text{if } \partial\Sigma = \bigsqcup_{i=1}^{k} C_i \end{cases}$$

with $C_1, \ldots, C_k$ being oriented cycles and

$$A_i = \begin{cases} A & \text{if the orientation on } C_i \text{ is coherent with the one induced by } \Sigma \\ A^* & \text{otherwise} \end{cases}$$

for each $i \in \{1, \ldots, k\}$, where A is a fixed finite-dimensional complex vector space and A^* denotes its dual. Such assignment is assumed to only depend on the topology of Σ and of its boundary. Moreover, the three following axioms are required.

(i) *Normalization: the pair*



is associated with id $\in A^* \otimes A$, where in the figure the outward orientation on the surface is assumed.

(ii) *Multiplicativity: the disjoint union of two pairs $(\Sigma_1 \sqcup \Sigma_2, \partial\Sigma_1 \sqcup \partial\Sigma_2)$ is associated with the tensor product of the vectors associated with each pair*

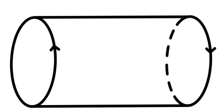
$$v_{(\Sigma_1,\partial\Sigma_1)} \otimes v_{(\Sigma_2,\partial\Sigma_2)} \in A_{(\Sigma_1,\partial\Sigma_1)} \otimes A_{(\Sigma_2,\partial\Sigma_2)}.$$

(iii) *Factorization: if two pairs $(\Sigma, \partial\Sigma)$ and $(\Sigma', \partial\Sigma')$ coincide outside a ball and inside the ball $(\Sigma', \partial\Sigma')$ can be recovered by cutting $(\Sigma, \partial\Sigma)$ along a cycle, splitting in two cycles $\widetilde{C}_1$ and $\widetilde{C}_2$ that are to be accounted as additional connected components of $\partial\Sigma'$, then $v_{(\Sigma,\partial\Sigma)}$ can be obtained by contracting $v_{(\Sigma',\partial\Sigma')}$ with respect to the vector spaces $\widetilde{A}_1$ and $\widetilde{A}_2$ corresponding to $\widetilde{C}_1$ and $\widetilde{C}_2$ respectively, by means of the contraction map that from a tensor product $A_1 \otimes \cdots \otimes A_k$ removes two vector spaces being dual to each other.*

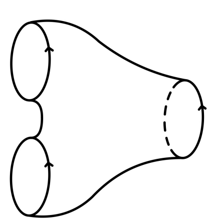
Remark 6 *A two-dimensional topological field theory can be interpreted as a functor from the category of 1-dimensional cobordisms to the one of complex vector spaces. The objects in the category of 1-dimensional cobordisms are closed 1-dimensional manifolds and a morphism between two such objects, a and b, is a closed surface interpolating them, having as a boundary the disjoint union of the boundary of a, with the same orientation, and the boundary of b, with reversed orientation. The composition of morphisms corresponds to gluing surfaces.*

The distinguished space A carries the structure of a Frobenius algebra, as specified in the following result (for instance, see [24]).

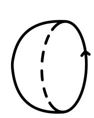
Theorem 1.2 *The space A presents a Frobenius algebra structure with respect to the symmetric bilinear form given by*



and the multiplication given by

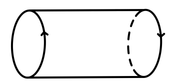


with

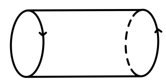


being its unit.

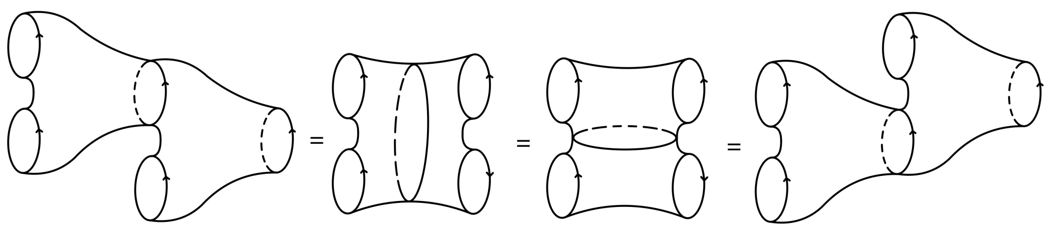
Proof: The bilinear form



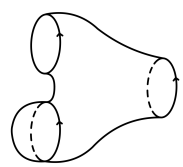
can be inverted to



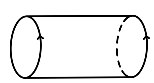
therefore it is non-degenerate. Commutativity of the product is trivial. Associativity is motivated by the relation



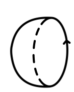
while



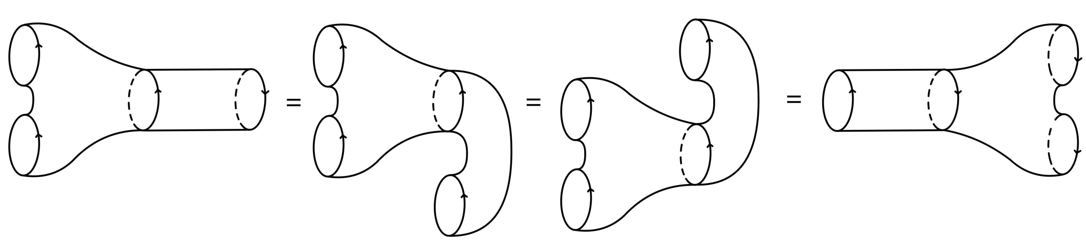
being equivalent to



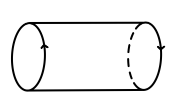
proves that



is unit of the product. The relation



proves the invariance of



with respect to the product. ■

1.1.3 Semisimple Frobenius manifolds

Let $(A, \circ, e)$ denote a commutative and associative algebra of finite dimension n over a field $\mathbb{K}$ of characteristic zero, with $e \in A$ being unit of the product.

Definition 1.10 *An element $x \in A \backslash \{0\}$ is said to be* nilpotent *if there exists some $m \in \mathbb{N}$ realizing $x^m = 0$.*

Definition 1.11 *Elements $\pi_1, \ldots, \pi_n$ of a basis for A are called* idempotents *if*

$$\pi_i \circ \pi_j = \delta_{ij} \pi_i$$

for each $i, j \in \{1, \ldots, n\}$.

Definition 1.12 *An operator of A has a* simple spectrum *if it has n distinct eigenvalues.*

For details about the following result, see for instance [16].

Proposition 1.13 *The following conditions are equivalent for $(A, \circ, e)$:*

a. A does not contain nilpotent elements,

b. A admits a basis of idemponents $\pi_1, \ldots, \pi_n$ realizing

$$\eta(\pi_i, \pi_j) = \eta(\pi_i, \pi_i)\, \delta_{ij}, \qquad i, j \in \{1, \ldots, n\}$$

for some non-degenerate symmetric bilinear form η which is invariant with respect to $\circ$,

c. A is isomorphic to $\mathbb{K}^{\oplus n}$,

d. there exists an element $\mathcal{E} \in A$ such that the operator $\mathcal{E}\circ : A \to A$ has a simple spectrum.

Definition 1.14 *The algebra $(A, \circ, e)$ is* semisimple *if any of the equivalent conditions listed in Proposition 1.13 holds.*

Let $(A, \circ, e, \eta)$ be a semisimple Frobenius algebra.

Remark 7 *The basis of idempotents is orthogonal with respect to the Frobenius form, namely* $\eta(\pi_i, \pi_j) = 0$ *whenever* $i \neq j$*. In fact, for every* $i, j \in \{1, \dots, n\}$*,*

$$\eta(\pi_i, \pi_j) = \eta(\pi_i^2, \pi_j) = \eta(\pi_i, \pi_i \circ \pi_j) = 0$$

as $\pi_1, \dots, \pi_n$ *are idempotents and* η *is invariant with respect to* $\circ$*. Moreover, for any element* $X = X^j \pi_j \in A$ *one has*

$$\sum_{i=1}^{n} \pi_i \circ X = \sum_{i,j=1}^{n} \pi_i \circ X^j \pi_j = \sum_{i,j=1}^{n} X^j \pi_i \circ \pi_j = \sum_{i=1}^{n} X^i \pi_i = X.$$

It follows that the sum of the idempotents is unit of the product: $e = \sum_{i=1}^{n} \pi_i$*.*

Let $(M, \eta, \circ, e, E)$ denote an n-dimensional Frobenius manifold.

Definition 1.15 *A point* m *of the Frobenius manifold* $(M, \eta, \circ, e, E)$ *is* semisimple *if the Frobenius algebra* $T_m M$ *is semisimple. Equivalently,* $(M, \eta, \circ, e, E)$ *is said to be* semisimple *at the point* $m \in M$*. The Frobenius manifold is* semisimple*, or* massive*, if it is semisimple at a generic point.*

Remark 8 *Semisimplicity is an open property. This means that if a Frobenius manifold* $(M, \eta, \circ, e, E)$ *is semisimple at a point* $m \in M$ *then there exists a neighbourhood* $U \subseteq M$ *of* m *such that* $(U, \eta, \circ, e, E)$ *is semisimple at each point.*

Proposition 1.16 *In a neighbourhood of a semisimple point there exists a distinguished set of local coordinates* $u^1, \dots, u^n$ *such that*

$$\pi_i = \frac{\partial}{\partial u^i}, \qquad i = 1, \dots, n. \tag{1.13}$$

Definition 1.17 *Coordinates realizing* (1.13) *are called* canonical coordinates*.*

The proof of Proposition 1.16 relies on the flatness of a connection $\tilde{\nabla}$, known as the *deformed connection*, which can be defined starting from ∇ by considering an additive contribution involving $\circ$. Details about this, as well as about the following result, can be found in [28, 30].

Theorem 1.3 *Let* $(M, \eta, \circ, e, E)$ *be a semisimple* n*-dimensional Frobenius manifold at a point* $m \in M$*. Let* $u^1, \dots, u^n$ *be canonical coordinates in a neighbourhood of* m*. Up to shifts, the product and the vector fields* e*,* E *are described by the following formulas:*

$$\partial_i \circ \partial_j = \delta_{ij}\, \partial_i, \qquad e = \sum_{s=1}^{n} \partial_s, \qquad E = \sum_{s=1}^{n} u^s\, \partial_s \tag{1.14}$$

for each $i, j \in \{1, \dots, n\}$*, where* $\partial_i = \frac{\partial}{\partial u^i}$ *for each* $i \in \{1, \dots, n\}$*.*

Canonical coordinates are defined up to permutations and shifts. A canonical expression for them, which we will adopt here, can be fixed by choosing canonical coordinates to be the eigenvalues of the operator $L := E\circ$ of multiplication by the Euler vector field. In canonical coordinates, such an operator will then be represented by the diagonal matrix

$$L = \text{diag}(u^1, \dots, u^n).$$

The formulas from Theorem 1.3 can be rewritten as

$$c^k_{ij} = \delta^k_i \delta^k_j, \qquad e^i = \delta^i_1, \qquad E^i = u^i$$

for each $i, j, k \in \{1, \dots, n\}$.

Semisimple Frobenius manifolds and Painlevé equations

By spelling out condition (1.3) in canonical coordinates, one sees that the metric η becomes diagonal. We denote its components by

$$\eta_{ij} = H_i^2\, \delta_{ij}, \qquad i, j \in \{1, \dots, n\}.$$

Definition 1.18 *The functions* $\beta_{ij} := \frac{\partial_j H_i}{H_j}$, $i \neq j$, *are called* Ricci rotation coefficients*.*

Given a semisimple Frobenius manifold, the rotation coefficients are symmetric:

$$\beta_{ij} = \beta_{ji}, \qquad i \neq j,$$

and as a consequence the metric is potential in canonical coordinates, meaning that (locally) there exists a function φ such that $H_i^2 = \partial_i \varphi$ for each $i \in \{1, \dots, n\}$. The rotation coefficients satisfy the following overdetermined system of PDEs:

$$\partial_k \beta_{ij} = \beta_{ik}\beta_{kj}, \qquad i \neq j \neq k \neq i, \tag{1.15}$$

$$e(\beta_{ij}) = 0, \qquad i \neq j, \tag{1.16}$$

$$E(\beta_{ij}) = -\beta_{ij}, \qquad i \neq j, \tag{1.17}$$

where

$$e = \sum_{i=1}^n \partial_i, \qquad E = \sum_{i=1}^n u^i \partial_i.$$

In particular, condition (1.16) follows from (1.9). In fact, by spelling out $\mathcal{L}_e \eta_{ij} = 0$ we get $e(\eta_{ij}) = 0$. In particular, for $i = j$ we have $e(H_i^2) = 0$, implying $e(H_i) = 0$.

By differentiating this last condition with respect to u^j for $j \neq i$ we get $e(\partial_j H_i) = 0$. Therefore

$$e(\beta_{ij}) = e\left(\frac{\partial_j H_i}{H_j}\right) = \frac{e\big(\partial_j H_i\big)}{H_j} - \frac{\partial_j H_i}{(H_j)^2}\, e(H_j) = 0.$$

Analogously, condition (1.17) follows from the homogeneity condition of the metric with respect to the Euler vector field (1.8). In fact, by spelling out the condition $\mathcal{L}_E \eta_{ij} = (2-d)\eta_{ij}$ we get $E(\eta_{ij}) = -d\,\eta_{ij}$. In particular, for $i = j$ we have $E(H_i^2) = -d\,H_i^2$, implying $E(H_i) = -\frac{d}{2}H_i$. By differentiating with respect to u^j for $j \neq i$ we get $E(\partial_j H_i) = -\big(\frac{d}{2}+1\big)\partial_j H_i$. Therefore

$$E(\beta_{ij}) = E\left(\frac{\partial_j H_i}{H_j}\right) = \frac{E\big(\partial_j H_i\big)}{H_j} - \frac{\partial_j H_i}{(H_j)^2}\, E(H_j) = -\frac{\partial_j H_i}{H_j} = -\beta_{ij}.$$

Definition 1.19 *The system (1.15, 1.16) is called* Darboux-Egorov system.

The Darboux-Egorov system (see [17, 36]) implies the flatness of the metric η. Given a solution of the above system, the *Lamé coefficients* $H_1, ..., H_n$ are obtained by solving the overdetermined system of PDEs

$$\partial_j H_i = \beta_{ij} H_j, \qquad i \neq j, \tag{1.18}$$
$$e(H_i) = 0, \tag{1.19}$$
$$E(H_i) = D H_i, \tag{1.20}$$

where $D = -\frac{d}{2}$ is an eigenvalue of the skew-symmetric matrix whose entries are defined as $V_{ij} := (u^j - u^i)\beta_{ij}$ for $i \neq j$ [28]. In dimension $n = 3$, on the open set $\{u^1 \neq u^2 \neq u^3 \neq u^1\}$, the general solution of the system (1.16, 1.17) is

$$\begin{aligned}
\beta_{12} &= \frac{1}{u^2 - u^1} F_{12}\left(\frac{u^3 - u^1}{u^2 - u^1}\right),\\
\beta_{23} &= \frac{1}{u^3 - u^2} F_{23}\left(\frac{u^3 - u^1}{u^2 - u^1}\right),\\
\beta_{13} &= \frac{1}{u^3 - u^1} F_{13}\left(\frac{u^3 - u^1}{u^2 - u^1}\right).
\end{aligned} \tag{1.21}$$

The remaining conditions (1.15) are equivalent to the following non-autonomous system of ODEs:

$$\begin{aligned}
\frac{dF_{12}}{dz} &= \frac{1}{z(z-1)} F_{13} F_{23},\\
\frac{dF_{13}}{dz} &= -\frac{1}{z-1} F_{12} F_{23},
\end{aligned} \tag{1.22}$$

$$\frac{dF_{23}}{dz} = \frac{1}{z} F_{12} F_{13},$$

where $z := \frac{u^3 - u^1}{u^2 - u^1}$. It is well-known that three-dimensional Frobenius manifolds are parameterized by solutions of a family of Painlevé VI equation (see [28]). This can be also proved by studying the system (1.22), as illustrated below.

Theorem 1.20 *System* (1.22) *is equivalent to the following sigma form of Painlevé VI equation (see [70]):*

$$z^2(z-1)^2(\sigma'')^2 + 4\left[\sigma'\left(z\sigma' - \sigma\right)^2 - (\sigma')^2(z\sigma' - \sigma)\right] = -2R^2(\sigma')^2 + R^4\sigma', \tag{1.23}$$

where the parameter R^2 is the value of the first integral $I = F_{12}^2 + F_{13}^2 + F_{23}^2$.

Proof: Let us first notice that the quantity $I = F_{12}^2 + F_{13}^2 + F_{23}^2$ is a first integral, as

$$\frac{dI}{dz} = 2\, F_{12} F_{13} F_{23} \left(\frac{1}{z(z-1)} - \frac{1}{z-1} + \frac{1}{z} \right) = 0.$$

We set $I = R^2$. Following [4], let us denote by σ a primitive function of F_{12}^2. We have

$$\begin{aligned} \sigma' &= F_{12}^2 \\ \sigma'' &= 2\, F_{12} \frac{dF_{12}}{dz} = \frac{2\, F_{12} F_{13} F_{23}}{z(z-1)}. \end{aligned} \tag{1.24}$$

By combining (1.24) with the second and third equations in (1.22), we get

$$\begin{aligned} \frac{dF_{13}^2}{dz} &= 2\, F_{13} \frac{dF_{13}}{dz} = -z\, \sigma'' \\ \frac{dF_{23}^2}{dz} &= 2\, F_{23} \frac{dF_{23}}{dz} = (z-1)\, \sigma'' \end{aligned}$$

that is

$$\begin{aligned} F_{12}^2 &= \sigma' \\ F_{13}^2 &= \sigma - z\sigma' + c_1 \\ F_{23}^2 &= -\sigma + (z-1)\sigma' + c_2 \end{aligned}$$

for some constants c_1, c_2. Since we set $I = R^2$, such constants must satisfy the condition

$$c_1 + c_2 = R^2.$$

By choosing $c_1 = c_2 = \frac{R^2}{2}$, we are able to write the squares of the functions F_{12}, F_{13}, F_{23} in terms of the single function $\sigma(z)$ as

$$F_{12}^2 = \sigma', \tag{1.25}$$

$$F_{13}^2 = \sigma - z\sigma' + \frac{R^2}{2}, \tag{1.26}$$

$$F_{23}^2 = -\sigma + (z-1)\sigma' + \frac{R^2}{2}. \tag{1.27}$$

By taking the square of (1.24),

$$z(z-1)\,\sigma'' = 2\,F_{12}F_{13}F_{23}$$

and combining it with (1.25), (1.26), (1.27), we obtain (1.23). ∎

In dimension 4, a special class of Frobenius manifolds that are also related to the Painlevé VI equation was studied in [76].

By dropping the assumption of symmetry of the rotation coefficients and allowing different degrees of homogeneity for the Lamé coefficients, one ends up with the Darboux-Egorov system (1.15, 1.16) with the additional constraint

$$E(\beta_{ij}) = (d_i - d_j - 1)\beta_{ij}, \qquad i \neq j. \tag{1.28}$$

In dimension 3 the system (1.15, 1.16, 1.28) reduces to a system of six ODEs which turns out to be equivalent to the full family of Painlevé VI [61]. The corresponding geometric structure is a generalization of the Frobenius manifold structure and it is called bi-flat structure [4]. A similar result (see [6]) can be obtained by studying the system

$$\partial_k\Gamma^i_{ij} = -\Gamma^i_{ij}\Gamma^i_{ik} + \Gamma^i_{ij}\Gamma^j_{jk} + \Gamma^i_{ik}\Gamma^k_{kj}, \quad i \neq k \neq j \neq i, \tag{1.29}$$

$$e(\Gamma^i_{ij}) = 0, \quad i \neq j, \tag{1.30}$$

$$E(\Gamma^i_{ij}) = -\Gamma^i_{ij}, \quad i \neq j. \tag{1.31}$$

Definition 1.21 *System* (1.29) *is called* Darboux-Tsarev system.

1.1.4 The principal hierarchy of a Frobenius manifold

In [27], Dubrovin showed how to associate to a Frobenius manifold a dispersionless integrable hierarchy that he called the *principal hierarchy*. Following [35], we present the key steps in his construction.

Let $(M, \eta, \circ, e, E)$ be a Frobenius manifold of dimension n and let $v^1, \ldots, v^n$ be flat coordinates of η. Coherently with the previous sections, we denote by ∇ the Levi-Civita connection of η and by $\{c^i_{jk}\}_{i,j,k\in\{1,\ldots,n\}}$ the structure constants of $\circ$. Let us consider the functions

$$\theta_{(\alpha,0)} = v_\alpha, \qquad \alpha \in \{1, \ldots, n\},$$

where we set $v_\alpha = \eta_{\alpha\beta} v^\beta$ for each $\alpha \in \{1,\dots,n\}$. Higher-order functions $\{\theta_{(\alpha,p)}\}_{\alpha\in\{1,\dots,n\},\, p\in\{1,2,3,\dots\}}$ can be constructed by means of the recursive relations

$$\partial_i\partial_j\theta_{(\alpha,p+1)} = c^k_{ij}\,\partial_k\theta_{(\alpha,p)}, \qquad \alpha\in\{1,\dots,n\},\ p\in\{0,1,2,3,\dots\}, \tag{1.32}$$

where additional constraints may be taken into account (we refer to [35] for further details). On the formal loop space $\mathcal{L}(M) = \{S^1 \to M\}$ of M, one can then consider the following infinite family of systems of first-order quasilinear PDEs:

$$v_{t(\alpha,p)} = \nabla\theta_{(\alpha,p)} \circ v_x, \qquad \alpha\in\{1,\dots,n\},\ p\in\{0,1,2,3,\dots\}. \tag{1.33}$$

These equations admit a Hamiltonian structure with respect to the Hamiltonians

$$H_{(\alpha,p)} = \int \theta_{(\alpha,p+1)}\,dx, \qquad \alpha\in\{1,\dots,n\},\ p\in\{0,1,2,3,\dots\}, \tag{1.34}$$

and to the Poisson bracket defined by

$$\big\{v^i(x),v^j(y)\big\}_1 = \eta^{ij}\,\delta'(x-y), \qquad i,j\in\{1,\dots,n\}, \tag{1.35}$$

where $\eta^{ij} = (\eta^{-1})_{ij}$ for each $i,j\in\{1,\dots,n\}$. More precisely, the system (1.33) can be rewritten as

$$v_{t(\alpha,p)} = \big\{v(x), H_{(\alpha,p)}\big\}_1, \qquad \alpha\in\{1,\dots,n\},\ p\in\{0,1,2,3,\dots\}. \tag{1.36}$$

It can be shown (see [27, 35] for details) that the flows associated to Hamiltonians having densities $\{\theta_{(\alpha,p)}\}_{\alpha\in\{1,\dots,n\},\, p\in\{0,1,2,3,\dots\}}$ commute with each other. Moreover, for each $\alpha\in\{1,\dots,n\}$ and $p\in\{0,1,2,3,\dots\}$ there exists a smooth function $\hat\theta_{(\alpha,p)}$ on M making the equation for $v_{t(\alpha,p)}$ Hamiltonian with respect to a second Poisson bracket defined by

$$\big\{v^i(x),v^j(y)\big\}_2 = g^{ij}\big(v(x)\big)\,\delta'(x-y) + \Gamma^{ij}_k\big(v(x)\big)\,u^k_x\,\delta(x-y), \qquad i,j\in\{1,\dots,n\}, \tag{1.37}$$

where $\Gamma^{ij}_k = c^{il}_k\big(\frac{1}{2}-\mu\big)^j_l$ for each $i,j,k\in\{1,\dots,n\}$, with

$$\mu = \frac{2-d}{2} - \nabla E$$

and $c^{ij}_k = \eta^{il}c^j_{lk}$ for each $i,j,k\in\{1,\dots,n\}$. More precisely, for each $\alpha\in\{1,\dots,n\}$ and $p\in\{0,1,2,3,\dots\}$ there exists a smooth function $\hat\theta_{(\alpha,p)}$ on M such that

$$\big\{v(x),H_{(\alpha,p)}\big\}_1 = \big\{v(x),\hat H_{(\alpha,p)}\big\}_2,$$

with the second Hamiltonian being defined as

$$\hat H_{(\alpha,p)} = \int \hat\theta_{(\alpha,p+1)}\,dx.$$

The two Poisson brackets $\{\,,\,\}_1$ and $\{\,,\,\}_2$ are compatible, in the sense that they define a pencil of Poisson brackets. We conclude that the equations (1.33) constitute commuting bi-Hamiltonian flows.

Definition 1.22 *The dispersionless integrable hierarchy of first-order quasilinear PDEs of the form* (1.33) *takes the name of* principal hierarchy *of the Frobenius manifold M.*

Example 1.23 (Dispersionless KdV hierarchy) *In the one-dimensional case with Frobenius potential*

$$F(v) = \frac{v^3}{6},$$

the structure constant of the product is $c_{11}^1 = 1$. Starting from $\theta_{(1,0)} = v$, by means of (1.32)*, we get*

$$\theta_{(1,p)} = \frac{v^{p+1}}{(p+1)!}, \qquad p \in \{0, 1, 2, 3, \dots\}.$$

The equations (1.33) *read*

$$v_{t(1,p)} = \nabla\theta_{(1,p)} \circ v_x = \frac{v^p}{p!}\, v_x, \qquad p \in \{0, 1, 2, 3, \dots\}.$$

Thus, we have obtained the dispersionless KdV hierarchy. In particular, the choice of $p = 1$ recovers the dispersionless KdV equation

$$v_{t(1,1)} = v\, v_x.$$

In [35], Dubrovin and Zhang introduced a way to perturb, by introducing a small parameter ϵ, the principal hierachy of a *semisimple* Frobenius manifold into a dispersive bi-Hamiltonian hierarchy, whose dispersionless limit ($\epsilon \to 0$) retrieves the original principal hiearchy. The dispersive integrable hierarchies constructed this way are known as *hierarchies of topological type* or *Dubrovin-Zhang hierarchies*. The (full) KdV hierarchy can be recovered via this dispersive deformation procedure as well.

1.1.5 Frobenius manifolds and cohomological field theories

In order to define cohomological field theories, we introduce moduli spaces of stable curves. We follow [86, 71, 11] in presenting this.

Moduli spaces of stable curves

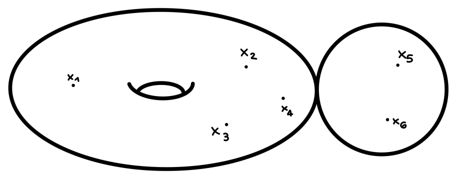
In this section, by a *smooth curve* we will mean a smooth compact complex curve. The objects we are interested in presenting concern curves which are not necessarily smooth but can present nodal singularities.

Definition 1.24 *Given two non-negative integers g, n with $n \geq 1$, a* pointed nodal curve *of type (g, n)*

$$(\mathcal{C}; x_1, \ldots, x_n)$$

is a complex compact curve $\mathcal{C}$ of genus g whose only singularities are simple nodes with n distinct marked points $x_1, \ldots, x_n \in \mathcal{C} \setminus Sing(\mathcal{C})$, where $Sing(\mathcal{C})$ denotes the set of singularities.

An example is the following curve



of type $(1, 6)$.

Definition 1.25 *A pointed nodal curve of type (g, n)*

$$(\mathcal{C}; x_1, \ldots, x_n)$$

is said to be a stable curve *if its automorphism group, namely the group of automorphisims fixing the marked points, is finite.*

Stable curves can be characterized in terms of the genus and the number of marked points, in light of the following result. Given a pointed nodal curve $(\mathcal{C}; x_1, \ldots, x_n)$, let $\widetilde{\mathcal{C}}$ denote its *normalization*, that is the smooth-component curve obtained from $\mathcal{C}$ by ungluing all the nodes. Let $\pi : \widetilde{\mathcal{C}} \to \mathcal{C}$ denote the corresponding projection.

Proposition 1.26 *The following conditions for a pointed nodal curve $(\mathcal{C}; x_1, \ldots, x_n)$ are equivalent:*

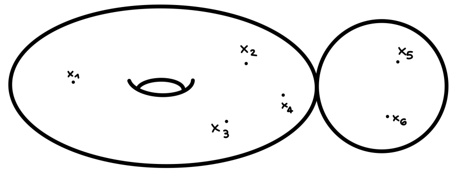
(a) $(\mathcal{C}; x_1, \ldots, x_n)$ *is stable,*

(b) $2g - 2 + n > 0$*,*

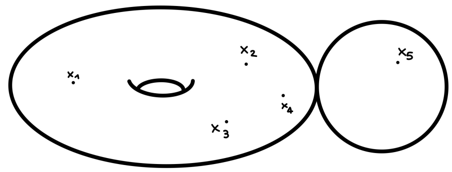
(c) each connected component of $\widetilde{\mathcal{C}}$ with genus 0 has at least 3 special points and each connected component with genus 1 has at least 1 special point.

By special points *we mean either marked points or elements of $\pi^{-1}\big(Sing(\mathcal{C})\big)$.*

Example 1.27 *The curve*

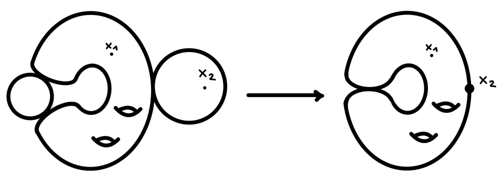


is stable. The curve



is not stable.

Remark 9 *An unstable curve* $(\mathcal{C}; x_1, \ldots, x_n)$ *can be stabilized by contracting its unstable components. We will denote the stabilized curve by* $(\mathcal{C}; x_1, \ldots, x_n)^{st}$*. The figure*



shows an example of such a stabilization process.

Definition 1.28 *We call the* moduli space of stable curves $\overline{\mathcal{M}}_{g,n}$ *the set of isomporphism classes of stable curves of genus* g *with* n *marked points.*

Remark 10 *Some relevant properties of* $\overline{\mathcal{M}}_{g,n}$ *are listed below.*

1. $\overline{\mathcal{M}}_{g,n} \neq \emptyset$ *only if* $2g - 2 + n > 0$*. This follows from the characterization of a stable curve.*

2. $\overline{\mathcal{M}}_{g,n}$ *carries the structure of a smooth complex compact orbifold of dimension*

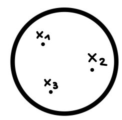
$$\dim_{\mathbb{C}}(\overline{\mathcal{M}}_{g,n}) = 3g - 3 + n.$$

3. $\overline{\mathcal{M}}_{g,n}$ *contains the moduli space of smooth curves* $\mathcal{M}_{g,n}$ *as a smooth open dense suborbifold. Actually, the moduli space of stable curves* $\overline{\mathcal{M}}_{g,n}$ *is the result of the Deligne-Mumford compactification (see [22]) of the moduli space of smooth curves* $\mathcal{M}_{g,n}$ *by adding nodal curves.*

Definition 1.29 *The* boundary *of $\overline{\mathcal{M}}_{g,n}$ is defined as*

$$\partial\overline{\mathcal{M}}_{g,n} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}.$$

Example 1.30 *The moduli space of stable curves of genus* 0 *and* 3 *marked points only consists of a point:* $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3} = \{*\}$.

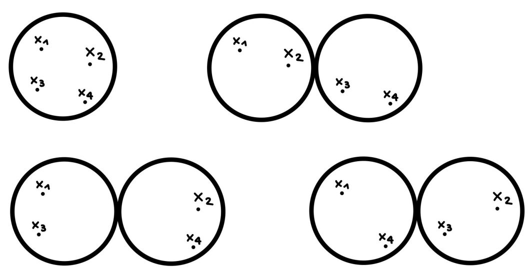


The stability requirement does not allow to arrange the three marked points into more components, so the smooth curve drawn above is the only representative element for $\overline{\mathcal{M}}_{0,3} = \mathcal{M}_{0,3}$. *In particular,* $\overline{\mathcal{M}}_{0,3}$ *has no boundary.*

Example 1.31 *The elements of the moduli space of stable curves of genus* 0 *and* 4 *marked points*

$$\overline{\mathcal{M}}_{0,4} = \mathcal{M}_{0,4} \sqcup \left(\mathcal{M}_{0,3} \times \mathcal{M}_{0,3}\right)^{\sqcup 3}$$

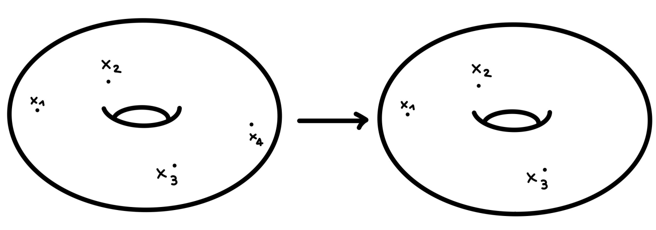
are represented below.



Definition 1.32 *Three natural maps are defined between moduli spaces of stable curves. Together, they take the name of* tautological maps *and are listed in the following.*

- *The* forgetful map *is defined as*

$$p : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$
$$(\mathcal{C}; x_1, \ldots, x_{n+1}) \mapsto (\mathcal{C}; x_1, \ldots, x_n)^{st}.$$

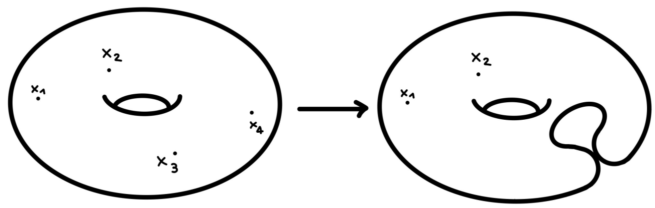


The need to stabilize the resulting curve comes from the fact that dropping one marked point may lead to an unstable curve, as $2g - 2 + n$ *decreases by* 1.

- *The* gluing map of non-separating kind

$$q:\overline{\mathcal{M}}_{g-1,n+2}\to\overline{\mathcal{M}}_{g,n}$$

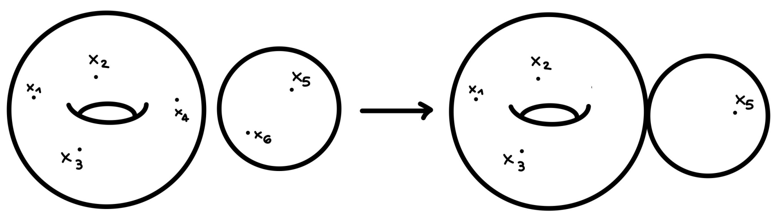
is defined by identifying the last two marked points of a single stable curve.



- *The* gluing map of separating kind

$$r:\overline{\mathcal{M}}_{g_1,n_1+1}\times\overline{\mathcal{M}}_{g_2,n_2+1}\to\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

is defined by identifying the last two marked points of two stable curves.



Definition 1.33 *For each $i\in\{1,\ldots,n\}$, a line bundle $\mathcal{L}_i$ is associated to the i-th marked point by defining its fiber over a point which is represented by a stable curve $(\mathcal{C};x_1,\ldots,x_n)$ as the cotangent space $T^*_{x_i}\mathcal{C}$ of $\mathcal{C}$ at x_i.*

Definition 1.34 *The first Chern classes of these lines bundles*

$$\psi_i:=c_1(\mathcal{L}_i)\in H^2\big(\overline{\mathcal{M}}_{g,n},\mathbb{Q}\big),\qquad\qquad i\in\{1,\ldots,n\},$$

*are called ψ-*classes.

Definition 1.35 *Given some non-negative integers $k_1,\ldots,k_n$, we define* intersection numbers *as the quantities*

$$<\tau_{k_1}\ldots\tau_{k_n}>_g:=\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\ldots\psi_n^{k_n}.$$

Intersection numbers can be arranged in a power series from which they can be generated. Such a generating function is

$$F^{WK}(t_0, t_1, ..., \epsilon) = \sum_{\substack{g\geq 0, n\geq 1,\\ 2g-2+n>0}} \frac{\epsilon^{2g}}{n!} \sum_{k_1,...,k_n\geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \right) t_{k_1} \cdots t_{k_n}.$$

Witten's conjecture [85], later proved by Kontsevich [54], relates the above generating function of intersection numbers with integrable hierachies. It states that such a generating function is the logarithm of a tau-function of the KdV hierarchy.

A way to compute intersection numbers is provided by a machinery known as *topological recursion (TR)*, first developed by B. Eynard and N. Orantin in [37] in the context of random matrix theory. In more general settings, topological recursion allows to compute enumerative invariants by means of recursive formulas, starting from the datum of a so-called spectral curve. Such formulas are based on the structure of moduli spaces of curves and the recursion runs over $2g - 2 + n$.

Cohomological field theories

Let V be a complex vector space of finite dimension. Let $<,>$ be a non-degenerate symmetric bilinear form on V and let $1 \in V$ be a distinguished element. Given a basis $(e_\alpha)_{\alpha\in\{1,...,\dim V\}}$ of V, set $\eta = (\eta_{\alpha\beta})$ where $\eta_{\alpha\beta} =< e_\alpha, e_\beta >$ and $\eta^{-1} = (\eta^{\alpha\beta})$.

Definition 1.36 *A* cohomological field theory *(*CohFT*) on V is a collection of linear maps*

$$\Omega_{g,n} : V^{\otimes n} \to H^{even}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$$

indexed by two non-negative integers g, n such that $2g-2+n > 0$ satisfying the following axioms:

(1) $\mathbb{S}_n$-symmetry: the maps $\Omega_{g,n}$ are equivariant with respect to the action of the symmetric group $\mathbb{S}_n$, which acts on $V^{\otimes n}$ by permuting copies of V and on $H^{even}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ by permuting marked points;

(2) 1 is the unit:

$$< v, w >= \Omega_{0,3}(v \otimes w \otimes 1) \in \mathbb{C}$$

for each $v, w \in V$;

(3) p-compatibility: $p^*\Omega_{g,n} = \iota_1\Omega_{g,n+1}$ namely

$$p^*\Omega_{g,n}(v_1 \otimes \cdots \otimes v_n) = \Omega_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes 1)$$

for each $v_1, \ldots, v_n \in V$;

(4) *gluing-compatibility: (writing $v_{1\dots n} = v_1 \otimes \cdots \otimes v_n$)*

$$q^*\Omega_{g,n}(v_{1\dots n}) = \sum_{1\le\alpha,\beta\le\dim V} \Omega_{g-1,n+2}(v_{1\dots n}\otimes e_\alpha\otimes e_\beta)\,\eta^{\alpha\beta}$$

and

$$r^*\Omega_{g_1+g_2,n_1+n_2}(v_{1\dots n_1+n_2}) = \\ = \sum_{1\le\alpha,\beta\le\dim V} \Omega_{g_1,n_1+1}(v_{1\dots n_1}\otimes e_\alpha)\,\eta^{\alpha\beta}\,\Omega_{g_2,n_2+1}(v_{n_1+1\dots n_1+n_2}\otimes e_\beta)$$

for each $v_1,\dots,v_{n_1+n_2}\in V$.

V is called phase space *and its dimension is known as the* rank *of the cohomological field theory.*

Sometimes in literature CohFTs are defined without requiring conditions (2) and (3) and they are called *CohFTs with unit* when such conditions hold. Here, we will not make such a distinction, as we will only consider CohFTs with unit.

Definition 1.37 *A CohFT defines a* quantum product $\star$ *on its phase space V as follows. The product of $v_1, v_2\in V$ is the unique $v_1\star v_2\in V$ such that*

$$< v_1\star v_2, v_3 >= \Omega_{0,3}(v_1\otimes v_2\otimes v_3)$$

for each $v_3\in V$.

As a consequence of the $\mathbb{S}_n$-symmetry axiom (1) and of the gluing-compatibility axiom (4), the quantum product $\star$ is commutative and associative. Moreover, axiom (2) also implies that 1 is a unit and that the non-degenerate symmetric bilinear form $<,>$ is invariant with respect to the product $\star$. It follows that the phase space $(V,<,>,\star,1)$ carries the structure of a Frobenius algebra. In local coordinates, the quantum product is written as

$$\star^\alpha_{\beta\gamma} = \eta^{\alpha\sigma}\,\Omega_{0,3}(e_\sigma\otimes e_\beta\otimes e_\gamma), \qquad \alpha,\beta,\gamma\in\{1,\dots,\dim V\}.$$

Definition 1.38 *The degree-zero part of a CohFT $\{\Omega_{g,n}\}_{g,n}$*

$$\omega_{g,n} := \deg_0\Omega_{g,n}\in H^0(\overline{\mathcal{M}}_{g,n})\otimes(V^*)^{\otimes n}$$

is called its topological part.

The topological part $\{\omega_{g,n}\}_{g,n}$ of a CohFT $\{\Omega_{g,n}\}_{g,n}$ is uniquely determined by $\omega_{0,3}$ and $<,>$. Equivalently, it is uniquely determined by $\star$ and $<,>$.

Definition 1.39 *A CohFT is* semisimple *if the associated algebra* $(V, \star, 1)$ *is semisimple, namely there exists a basis* $(e_i)_{i\in\{1,\dots,\dim V\}}$ *of idempotents:*

$$e_i \star e_j = \delta_{ij}\, e_i, \qquad\qquad i,j \in \{1,\dots,\dim V\}.$$

Definition 1.40 *The* correlator *of a CohFT* $\{\Omega_{g,n}\}_{g,n}$ *of genus* g *associated to* $v_1,\dots,v_n \in V$ *and the non-negative integers* $k_1,\dots,k_n$ *is*

$$< \tau_{k_1}(v_1)\dots\tau_{k_n}(v_n) >_g := \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(v_1\otimes\cdots\otimes v_n)\,\psi_1^{k_1}\dots\psi_n^{k_n}.$$

It is immediate to notice how correlators generalize intersection numbers. Indeed, intersection numbers can be recovered as the correlators of the trivial CohFT $\{\Omega^{\text{triv}}_{g,n}\}_{g,n}$, which is defined on a phase space of dimension 1, with some basis element e, by setting

$$\Omega^{\text{triv}}_{g,n}\left(\bigotimes_{i=1}^{n} e\right) = 1.$$

Correlators can be organized in a power series, as in the following.

Definition 1.41 *The* potential *of a CohFT* $\{\Omega_{g,n}\}_{g,n}$ *is the power series in the variables* ϵ*,* $\{t^\alpha_d\}^{\alpha\in\{1,\dots,\dim V\}}_{d\in\{0,1,2,\dots\}}$ *defined as*

$$F(t^*_*, \epsilon) := \sum_{g\geq 0} \epsilon^{2g}\, F_g(t^*_*)$$

where

$$F_g(t^*_*) := \sum_{\substack{n\geq 0\\ 2g-2+n>0}} \frac{1}{n!} \sum_{\substack{1\leq\alpha_1,\dots,\alpha_n\leq\dim V\\ k_1,\dots,k_n\geq 0}} < \tau_{k_1}(e_{\alpha_1})\dots\tau_{k_n}(e_{\alpha_n}) >_g\ t^{\alpha_1}_{k_1}\dots t^{\alpha_n}_{k_n}.$$

Proposition 1.42 *Given a CohFT* $\Omega_{g,n}\colon V^{\otimes n} \to H^{even}(\overline{\mathcal{M}}_{g,n})$ *and a basis* $e_1,\dots,e_{\dim V}$ *of its phase space* V*, the function*

$$F(t^1,\dots,t^{\dim V}) := \sum_{n\geq 3}\frac{1}{n!}\sum_{1\leq\alpha_1,\dots,\alpha_n\leq\dim V}\left(\int_{\overline{\mathcal{M}}_{0,n}}\Omega_{0,n}(\otimes_{i=1}^n e_{\alpha_i})\right)\prod_{i=1}^n t^{\alpha_i} \tag{1.38}$$

defines a solution to the WDVV equations.

A Frobenius manifold then encodes the information about the genus-zero part of a CohFT.

$$\boxed{\text{CohFTs} \overset{\substack{\text{genus}\\ g=0}}{\to} \text{Frobenius manifolds}}$$

In the semisimple case, the information about all genera of a CohFT can be fully reconstructed starting from the underlying Frobenius manifold. This was proved by C. Teleman in [82], extending to CohFTs a construction which A. Givental introduced in the setting of Gromov-Witten theory [43, 44, 45]. In particular, the uniqueness feature in such a construction is provided in the case where the CohFT satisfies an additional assumption of homogeneity with respect to the Euler vector field (see, for instance, [72]). Such a homogeneity property is implicitly required by the Frobenius manifold axioms (see, for instance, [82]). We refer to [82, 72] for an accurate proof of the following theorem.

Theorem 1.4 (Givental-Teleman classification) *Any homogeneous semisimple CohFT can be uniquely determined starting from its topological part.*

The key ideas in this reconstruction process are illustrated as follows. The genus-zero information about a CohFT is encoded in a Frobenius manifold structure. In particular, its topological part is uniquely determined by a Frobenius algebra structure. In the semisimple case, the Givental-Teleman result provides a recipe to reconstruct the whole CohFT starting from its topological part, by means of a so-called R-matrix action. R-matrices are elements of a group acting on semisimple CohFTs in a transitive way. More precisely, the Givental-Teleman result states that there exists a unique R-matrix that recovers the whole CohFT when applied to its topological part. As a consequence, higher genus information of a semisimple CohFT is determined by the genus zero information contained in the underlying Frobenius manifold. In particular, the R-matrix is uniquely specified in terms of $\Omega_{0,3}$ and the Euler vector field.

$$\boxed{\text{CohFTs} \underset{\substack{GT\\(ss)}}{\overset{\substack{\text{genus}\\g=0}}{\rightleftarrows}} \text{Frobenius manifolds}}$$

Building on the relation between Frobenius manifolds and CohFTs, the construction of integrables hiearchies of topological type [35] can be rephrased as a construction starting from semisimple CohFTs (for instance, see [11]). As generalizing the Witten-Kontsevich result, the potential of the CohFT is the logarithm of a tau-function of the resulting hierarchy of topological type. In the particular case of the trivial CohFT one recovers the KdV hierarchy as the hierarchy and, as the potential of the CohFT, the Witten-Kontsevich generating function.

Another construction of integrable hierarchies starting from (even non-semisimple) CohFTs, or from solutions of the WDVV equations, was proposed by A. Buryak in [10]. Such hierarchies are known as *double ramification (DR) hierarchies*. As well as for hierarchies of topological type, the dispersionless limit of the DR hierarchy associated with some solution of the WDVV equations is the corresponding principal hierarchy. The DR hierarchy is conjectured to be equivalent to the Dubrovin-Zhang hierarchy, meaning that the two of them are related by a change of coordinates known as a *Miura transformation*. Such a conjecture has so far been proved for selected CohFTs.

Below, referring to [33], we list some Frobenius potentials associated to well-known integrable hierarchies.

KdV	$F(u) = \frac{1}{6}u^3$
Boussinesq	$F(u,v) = \frac{1}{2}uv^2 + u^4$
Toda	$F(u,v) = \frac{1}{2}uv^2 + e^u$
NLS	$F(u,v) = \frac{1}{2}uv^2 + \frac{1}{2}u^2\big(\log u - \frac{3}{2}\big)$

1.2 F-manifolds

Introduced by Hertling and Manin in [48], F-manifolds have proved to be an extremely versatile concept, whose outreach embraces, for instance, the theory of integrable systems and even information theory [14].

Definition 1.43 *An* F-manifold *is a triple* $(M, \circ, e)$ *consisting of a manifold* M*, a commutative and associative multiplication* $\circ$ *on the tangent bundle and a distinguished vector field* e *on* M *being unit of* $\circ$*, namely such that* $e \circ X = X$ *for all vector fields* X*, satisfying the identity*

$$[X\circ Y, W\circ Z] - [X\circ Y, Z]\circ W - [X\circ Y, W]\circ Z - X\circ[Y, Z\circ W] + X\circ[Y,Z]\circ W \tag{1.39}$$
$$+ X\circ[Y,W]\circ Z - Y\circ[X, Z\circ W] + Y\circ[X,Z]\circ W + Y\circ[X,W]\circ Z = 0,$$

for all vector fields X, Y, W, Z*, where* $[X, Y]$ *is the Lie bracket.*

Remark 11 *Condition* (1.39) *is known as the* Hertling-Manin condition *and it is equivalent to*

$$\mathcal{L}_{X\circ Y}(\circ) = X\circ\mathcal{L}_Y(\circ) + Y\circ\mathcal{L}_X(\circ) \tag{1.40}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$*, where* $\mathcal{L}_X$ *denotes the Lie derivative along* X*.*

Frobenius manifolds constitute particular instances of F-manifolds, as satisfying the Hertling-Manin condition (see, for instance, [47]).

Definition 1.44 *An* Euler vector field *on an F-manifold* $(M, \circ, e)$ *is a vector field* E *satisfying the condition*

$$\mathcal{L}_E(\circ)(X,Y) = d\, X \circ Y \tag{1.41}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$*, for some constant* d *which is referred to as the* weight *of the Euler vector field. This means that* E *preserves the multiplication up to a constant.*

Many properties of F-manifolds have been derived (see [47]), however their classification remains an open problem, even in three-dimensions [8]. Motivated by Dubrovin's construction of almost-dual Frobenius manifolds [32], Manin introduced a new commutative and associative multiplication:

$$X * Y = \mathcal{E}^{-1} \circ X \circ Y, \qquad X, Y \in \mathfrak{X}(M), \tag{1.42}$$

for an arbitrary invertible vector field $\mathcal{E}$, where *invertible* means that the vector field $\mathcal{E}^{-1}$ satisfies the condition $\mathcal{E} \circ \mathcal{E}^{-1} = e$. In general, $\mathcal{E}$ will not be defined everywhere on the manifold, but only on the complementary set $M \backslash \Sigma$ of some submanifold Σ where it is not invertible. For simplicity, we will just refer to the manifold as M rather than separately to the manifolds M and $M \backslash \Sigma$. It is immediate to see that $\mathcal{E}$ is a unit for $*$. Manin then defined an eventual identity as a vector field $\mathcal{E}$ that preserves the F-manifold structure [69].

Definition 1.45 *An* eventual identity *for an F-manifold* $(M, \circ, e)$ *is an invertible vector field* $\mathcal{E}$ *such that the dual multiplication* (1.42) *defines an F-manifold structure* $(M, *, \mathcal{E})$*.*

Eventual identities appeared in the definition of multi-flat structures given in [6]. A characterization of eventual identities was given in [20].

Theorem 1.5 *Given an F-manifold* $(M, \circ, e)$*, an invertible vector field* $\mathcal{E}$ *is an eventual identity if and only if*

$$\mathcal{L}_{\mathcal{E}}(\circ)(X,Y) = [e, \mathcal{E}] \circ X \circ Y, \qquad X, Y \in \mathfrak{X}(M). \tag{1.43}$$

1.2.1 Regular F-manifolds

Let $(M, \circ, e, E)$ be a F-manifold of dimension n. The multiplication $\circ$ is said to be semisimple if, at a generic point in the manifold, there exists a set of idempotent vector fields π_i with the property

$$\pi_i \circ \pi_j = \delta_{ij} \pi_i, \qquad i \in \{1, \ldots, n\},$$

namely the multiplication decomposes into one-dimensional blocks. It may be shown that *canonical coordinates* $\{u^1, \ldots, u^n\}$ exist in which the idempotent vector fields are $\pi_i = \partial_i$ for each $i \in \{1, \ldots, n\}$. In the Frobenius manifold case, they coincide with Dubrovin's canonical coordinates.

Without semisimplicity, it is much harder to study F-manifolds, as well as their generalisations. Indeed, there is no classification of F-manifolds beyond two dimensions, with only a partial classification in three dimensions [8]. However, in [18], David and Hertling gave the definition of a so-called regular F-manifold and extended the notion of canonical coordinates to such a case.

Definition 1.46 *An F-manifold $(M, \circ, e, E)$ with Euler field is called* regular *at a point $m \in M$ if the endomorphism $E \circ |_m : T_mM \to T_mM$ is regular, namely each of its Jordan blocks is associated to a different eigenvalue. The F-manifold is* (generically) regular *if it is regular at any (generic) point.*

Theorem 1.6 *Let $(M, \circ, e, E)$ be a regular F-manifold of dimension n and let $m \in M$ be a point around which the operator $E\circ$ has r Jordan blocks, of sizes $m_1, \ldots, m_r$. Then, locally around m, there exists a distinguished system of coordinates*

$$\left\{u^{j(\alpha)} \,\middle|\, \alpha \in \{1, \ldots, r\},\, j \in \{1, \ldots, m_\alpha\}\right\}$$

where the structure constants of the product $\circ$ are given by

$$c^{i(\alpha)}_{j(\beta)k(\gamma)} = \delta^\alpha_\beta \delta^\alpha_\gamma \delta^i_{j+k-1} \tag{1.44}$$

for all suitable indices and where the unit and the Euler vector fields are given respectively by

$$e = \sum_{\alpha=1}^{r} \partial_{u^{1(\alpha)}}, \qquad E = \sum_{\alpha=1}^{r} \sum_{s=1}^{m_\alpha} u^{s(\alpha)}\, \partial_{u^{s(\alpha)}}. \tag{1.45}$$

The above formulas appear slightly simpler than the original ones in [18], which can be recovered from these after a shift in the first two variables of each block. We will refer to the coordinates provided by Theorem 1.6 by calling them *generalized canonical coordinates*, or simply *canonical coordinates*.

Under the assumptions of Theorem 1.6, the canonical coordinates $u^1, \ldots, u^n$ for M can be re-labelled by means of the following notation: for each $\alpha \in \{2, \ldots, r\}$ and for each $j \in \{1, \ldots, m_\alpha\}$ we write

$$j(\alpha) = m_1 + \cdots + m_{\alpha-1} + j$$

(for $\alpha = 1$ we set $j(\alpha) = j$) so that $u^{j(\alpha)}$ denotes the j-th coordinate associated to the α-th Jordan block. When dealing with regular F-manifolds of this form, we will write u^i when seeing the coordinate as running from 1 to the dimension of the manifold and we will write $u^{i(\alpha)}$ when in need to highlight the Jordan block to which the coordinate refers. According to this notation, ∂_i and $\partial_{i(\alpha)}$ will denote the partial derivative with respect to u^i and $u^{i(\alpha)}$ respectively.

1.3 Flat F-manifolds

Related to the notion of Saito structures without metric introduced by Sabbah in [77] and conveying a potentiality relating them to a system of PDEs known as oriented associativity equations [67], flat F-manifolds were introduced by Manin in [69] as F-manifold endowed with a compatible flat structure. A more general definition was given in [64] for an F-manifold with compatible connection, which is not necessarily flat.

Definition 1.47 *A* flat F-manifold $(M, \circ, \nabla, e)$ *is an F-manifold M equipped with a connection ∇ related to the product $\circ$ and to the unit vector field e by the following axioms:*

(i) *the one-parameter family of connections*

$$\nabla^{(\lambda)} := \nabla - \lambda \circ \tag{1.46}$$

is flat and torsionless for any λ,

(ii) *the vector field e is covariantly constant, namely $\nabla e = 0$.*

The requirement for the family of connections (1.46) to be torsionless amounts to the request for ∇ to be torsionless and for the product $\circ$ to be commutative. The flatness of (1.46) amounts to the flatness of ∇, the associativity of the product $\circ$ and the symmetry of ∇c with respect to the lower indices.

In particular, the last condition and the commutativity of the product imply that for every $m \in \{1, \dots, n\}$, where $n = \dim M$ and $t^1, \dots, t^n$ denote flat coordinates for ∇, locally there exists a function $F^m(t^1, \dots, t^n)$ such that

$$c^m_{ij} = \partial_i \partial_j F^m, \qquad i, j, \in \{1, \dots, n\}. \tag{1.47}$$

Definition 1.48 *The n-tuple $(F^1, \dots, F^n)$ of functions defined by* (1.47) *takes the name of* vector potential *of the flat F-manifold.*

In flat coordinates for ∇, the fact that the vector field e is unit of the product $\circ$ implies that

$$\partial_1 \partial_j F^m = \delta^m_j, \qquad m, j \in \{1, \dots, n\}, \tag{1.48}$$

and the associativity of the product $\circ$ gives

$$\partial_i \partial_s F^m \, \partial_j \partial_k F^s = \partial_j \partial_s F^m \, \partial_i \partial_k F^s, \qquad i, j, k, m \in \{1, \dots, n\}. \tag{1.49}$$

Conversely, any n-tuple $(F^1, \dots, F^n)$ of functions satisfying (1.48) and (1.49) defines a flat F-manifold, with the product being given by (1.47) and the connection being given by $\nabla_{\partial_i}\partial_j = 0$ for each $i, j \in \{1, \dots, n\}$.

In the particular case where the flat F-manifold is a Frobenius manifold, the invariant metric implies the existence of a Frobenius potential, as seen above (1.10), from which the vector potential can be recovered. Equations (1.48) and (1.49) play the role of WDVV associativity equations in the flat F-manifold context. Together with a third condition expressing homogeneity with respect to some Euler vector field, they take the name of *extended WDVV equations* or *oriented WDVV equations* (for instance, see [67, 5, 51]).

1.3.1 The principal hierarchy of a flat F-manifold

Given a flat F-manifold, it was shown in [64] that an integrable hierarchy of hydrodynamic type can be constructed. The flatness of the connection gives a basis of flat vector fields and in turn a set of flows, known as primary flows. Higher flows can be obtained recursively. The resulting integrable hierarchy generalizes Dubrovin's principal hierarchy for Frobenius manifolds to the case where the connection is not associated with a metric.

Let $(M, \circ, \nabla, e)$ be a flat F-manifold of dimension n. Let

$$X_{(1,-1)}, \dots, X_{(n,-1)}$$

be a frame of flat vector fields, the first one being $e = X_{(1,-1)}$. They are called *primary vector fields*. According to [64], higher order vector fields can be constructed by imposing the condition

$$\nabla X_{(\alpha,l+1)} = X_{(\alpha,l)}\,\circ, \qquad l \in \{-1, 0, 1, 2, \dots\}, \quad \alpha \in \{1, \dots, n\}. \tag{1.50}$$

Then

$$c^i_{js}\, \nabla_k X^s_{(\alpha,l)} = c^i_{ks}\, \nabla_j X^s_{(\alpha,l)}$$

for all suitable indices. In fact, the primary vector fields trivially yield $\nabla X_{(\alpha,-1)} = 0$ for each $\alpha \in \{1, \dots, n\}$ and for $l \geq 0$ we have

$$\begin{aligned} c^i_{js}\, \nabla_k X^s_{(\alpha,l)} &= c^i_{js}\, c^s_{km} X^m_{(\alpha,l-1)} \\ &= c^i_{ks}\, c^s_{jm} X^m_{(\alpha,l-1)} = c^i_{ks}\, \nabla_j X^s_{(\alpha,l)} \end{aligned}$$

for all suitable indices. As a consequence, the flows

$$u^i_t = c^i_{jk} X^k_{(\alpha,l)} u^j_x, \qquad i \in \{1, \dots, n\},$$

and

$$u^i_\tau = c^i_{jk} X^k_{(\beta,m)} u^j_x, \qquad\qquad i \in \{1,\dots,n\},$$

associated to different solutions of (1.50) commute. The hierarchy of quasilinear PDEs of the form

$$u^i_{t(\alpha,l)} = c^i_{jk}\, X^k_{(\alpha,l)}\, u^j_x, \qquad\qquad i \in \{1,\dots,n\}, \tag{1.51}$$

is defined as the *principal hierarchy* of the F-manifold.

Remark 12 *In the particular case of a Frobenius manifold, the above integrable hierarchy coincides with the principal hierarchy. More precisely, by setting*

$$X_{(\alpha,l)} = \nabla\theta_{(\alpha,l)}, \qquad l \in \{-1,0,1,2\dots\}, \quad \alpha \in \{1,\dots,n\},$$

and letting the second index start at -1, the equations (1.33) *coincide with* (1.51) *and the recursive relations* (1.32) *coincide with* (1.50)*.*

1.3.2 F-cohomological field theories

The relation between Frobenius manifolds and cohomological field theories persists in the flat F-manifold setting. This was observed in [2], where Givental's reconstruction to higher genera was extended to the case of an F-cohomological field theory (F-CohFT), introduced in [12] and generalizing cohomological field theories, under the semisimplicity assumption.

Following [2], we define F-cohomological field theories. Let V be a complex vector space of finite dimension. Let $<,>$ be a non-degenerate symmetric bilinear form on V and let $1 \in V$ be a distinguished element. Given a basis $(e_\alpha)_{\alpha\in\{1,\dots,\dim V\}}$ of V with $(e^\alpha)_{\alpha\in\{1,\dots,\dim V\}}$ dual basis of V^*, set $\eta = (\eta_{\alpha\beta})$ where $\eta_{\alpha\beta} =< e_\alpha, e_\beta >$ and $\eta^{-1} = (\eta^{\alpha\beta})$.

Definition 1.49 *An* F-cohomological field theory *(*F-CohFT*) on V is a collection of linear maps*

$$\Omega_{g,n+1} : V^* \otimes V^{\otimes n} \;\to\; H^{even}\big(\overline{\mathcal{M}}_{g,n+1}, \mathbb{C}\big)$$

indexed by two non-negative integers g, n such that $2g-1+n>0$ verifying the following axioms:

(1) $\mathbb{S}_n$-symmetry: the maps $\Omega_{g,n+1}$ are equivariant with respect to the action of the symmetric group $\mathbb{S}_n$, which acts on $V^* \otimes V^{\otimes n}$ by permuting copies of V and on $H^{even}\big(\overline{\mathcal{M}}_{g,n+1}, \mathbb{C}\big)$ by permuting the last n marked points;

(2) *1 is the unit:*

$$\delta^\alpha_\beta = \Omega_{0,3}(e^\alpha \otimes e_\beta \otimes 1)$$

for each $\alpha, \beta \in \{1, \dots, \dim V\}$*;*

(3) *p-compatibility:*

$$p^*\Omega_{g,n+1}(e^{\alpha_0} \otimes e_{\alpha_1} \cdots \otimes e_{\alpha_n}) = \Omega_{g,n+2}(e^{\alpha_0} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} \otimes 1)$$

for each $\alpha_0, \alpha_1, \dots, \alpha_n \in \{1, \dots, \dim V\}$*, where*

$$p : \overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g,n+1}$$
$$(\mathcal{C}; x_1, \dots, x_{n+2}) \mapsto (\mathcal{C}; x_1, \dots, x_{n+1})^{st}$$

is the forgetful map;

(4) *gluing-compatibility:*

$$\begin{aligned} &r^*\Omega_{g_1+g_2,n_1+n_2+1}(e^{\alpha_0} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{n_1+n_2}}) \\ &= \sum_{1 \le \mu \le \dim V} \Omega_{g_1,n_1+2}(e^{\alpha_0} \otimes \bigotimes_{i\in I} e_{\alpha_i} \otimes e_\mu) \otimes \Omega_{g_2,n_2+1}(e^\mu \otimes \bigotimes_{j \in J} e_{\alpha_j}) \end{aligned}$$

for each $\alpha_0, \alpha_1, \dots, \alpha_{n_1+n_2} \in \{1, \dots, \dim V\}$*, where* $I \sqcup J = \{2, \dots, n_1 + n_2 + 1\}$ *with* $|I| = n_1$*,* $|J| = n_2$ *and where*

$$r : \overline{\mathcal{M}}_{g_1,n_1+2} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2+1}$$

is the gluing map of separating kind.

Let $\{\Omega_{g,n+1}\}_{g,n}$ be an F-CohFT on $(V, <,>, 1)$ with $(e_\alpha)_{\alpha \in \{1,\dots,\dim V\}}$ being a basis of V and $(e^\alpha)_{\alpha\in\{1,\dots,\dim V\}}$ being a dual basis of V^*. Let us set $\dim V = N$. One can prove that for each $\alpha \in \{1, \dots, N\}$ the function

$$F^\alpha(t^1, \dots, t^N) := \sum_{n\ge 2} \frac{1}{n!} \sum_{1\le\alpha_1,\dots,\alpha_n\le N} \left(\int_{\overline{\mathcal{M}}_{0,n+1}} \Omega_{0,n+1}\big(e^\alpha \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}\big) \right) t^{\alpha_1} \dots t^{\alpha_n} \tag{1.52}$$

is a solution of (1.48) and (1.49). It follows that $(F^1, \dots, F^N)$ is the vector potential of a flat F-manifold structure defined in a neighbourhood of 0 in V.

We now give the definition of a partial CohFT, as introduced in [58].

Definition 1.50 *A* partial cohomological field theory *(*partial CohFT*) on V is a collection of linear maps*

$$\Omega_{g,n} : V^{\otimes n} \to H^{even}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$$

indexed by two non-negative integers g, n such that $2g-2+n > 0$ verifying the following axioms:

(1) $\mathbb{S}_n$*-symmetry: the maps* $\Omega_{g,n}$ *are equivariant with respect to the action of the symmetric group* $\mathbb{S}_n$*, which acts on* $V^{\otimes n}$ *by permuting copies of* V *and on* $H^{even}(\overline{\mathcal{M}}_{g,n}, \mathbb{C})$ *by permuting marked points;*

(2) 1 *is the unit:*

$$< e_\alpha, e_\beta >= \Omega_{0,3}(e_\alpha \otimes e_\beta \otimes 1)$$

for each $\alpha, \beta \in \{1, \ldots, \dim V\}$*;*

(3) p*-compatibility:*

$$p^*\Omega_{g,n}(\otimes e_{\alpha_1} \cdots \otimes e_{\alpha_n}) = \Omega_{g,n+1}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} \otimes 1)$$

for each $\alpha_1, \ldots, \alpha_n \in \{1, \ldots, \dim V\}$*, where*

$$\begin{aligned} p : \overline{\mathcal{M}}_{g,n+1} &\to \overline{\mathcal{M}}_{g,n} \\ (\mathcal{C}; x_1, \ldots, x_{n+1}) &\mapsto (\mathcal{C}; x_1, \ldots, x_n)^{st} \end{aligned}$$

is the forgetful map;

(4) gluing-compatibility:

$$\begin{aligned} &r^*\Omega_{g_1+g_2,n_1+n_2}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{n_1+n_2}}) \\ &= \sum_{1 \le \mu,\nu \le \dim V} \eta^{\mu\nu}\, \Omega_{g_1,n_1+1}(\bigotimes_{i\in I} e_{\alpha_i} \otimes e_\mu) \otimes \Omega_{g_2,n_2+1}(\bigotimes_{j \in J} e_{\alpha_j} \otimes e_\nu) \end{aligned}$$

for each $\alpha_1, \ldots, \alpha_{n_1+n_2} \in \{1, \ldots, \dim V\}$*, where* $I \sqcup J = \{1, \ldots, n_1 + n_2\}$ *with* $|I| = n_1$*,* $|J| = n_2$ *and where*

$$r : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}$$

is the gluing map of separating kind.

Any partial CohFT $\{\Omega_{g,n}\}_{g,n}$ defines an F-CohFT $\{\widetilde{\Omega}_{g,n+1}\}_{g,n}$, by setting

$$\widetilde{\Omega}_{g,n+1}(e^{\alpha_0} \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}) = \sum_{1 \le \mu \le \dim V} \eta^{\alpha_0 \mu}\, \Omega_{g,n+1}(e_\mu \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n})$$

for each $\alpha_0, \alpha_1, \ldots, \alpha_n \in \{1, \ldots, \dim V\}$. Moreover, a partial CohFT can be promoted to a CohFT by requiring compatibility with respect to the gluing map of non-separating kind

$$q : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$$

namely by imposing that

$$q^*\Omega_{g,n}(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}) = \sum_{1\leq\mu,\nu\leq\dim V}\Omega_{g-1,n+2}(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n}\otimes e_\mu\otimes e_\nu)\,\eta^{\mu\nu}$$

for each $\alpha_1,\ldots,\alpha_n\in\{1,\ldots,\dim V\}$.

Let $\{\Omega_{g,n}\}_{g,n}$ be a partial CohFT on $(V,<,>,1)$ with $(e_\alpha)_{\alpha\in\{1,\ldots,\dim V\}}$ being a basis of V and $(e^\alpha)_{\alpha\in\{1,\ldots,\dim V\}}$ being a dual basis of V^*. Let us set $\dim V = N$. One can prove that the function

$$F(t^1,\ldots,t^N) := \sum_{n\geq 3}\frac{1}{n!}\sum_{1\leq\alpha_1,\ldots,\alpha_n\leq N}\left(\int_{\overline{\mathcal{M}}_{0,n}}\Omega_{0,n}(e_{\alpha_1}\otimes\cdots\otimes e_{\alpha_n})\right)t^{\alpha_1}\ldots t^{\alpha_n} \quad (1.53)$$

is a solution of the first two sets of equations in (1.12). It follows that F is the potential of a Frobenius-manifold structure defined in a neighbourhood of 0 in V, consistently with the theory presented above as the function (1.53) appeared in (1.38).

Similarly to the Frobenius setting, a flat F-manifold then encodes the information about the genus-zero part of an F-CohFT. In the case of semisimple flat F-manifolds, Givental's reconstruction of higer-genus information was extended in [2]. The double ramification hierarchy construction was extended as well to F-CohFTs in [1] and it was there used to deform the principal hierarchy of a semisimple flat F-manifold into a dispersive integrable hierarchy.

1.4 Bi-flat F-manifolds

The notion of bi-flat F-manifold was introduced in [4], motivated by the study of dual structures on flat F-manifolds and building on the generalizations of Frobenius manifolds. We give it as follows.

Definition 1.51 *A* bi-flat F-manifold $(M,\nabla,\nabla^*,\circ,*,e,E)$ *is a manifold M equipped with a pair of connections ∇ and ∇^*, a pair of products $\circ$ and $*$ on the tangent spaces T_mM and a pair of vector fields e and E satisfying the following conditions:*

(i) $(\nabla,\circ,e)$ *defines a flat F-manifold structure on M,*

(ii) $(\nabla^*,*,E)$ *defines a flat F-manifold structure on M,*

(iii) *the two flat F-manifold structures are related by the conditions*

- $X * Y = (E\circ)^{-1}X\circ Y,$ (1.54)

- $[e,E]=e,$ (1.55)
- $\mathcal{L}_E\circ=\circ,$ (1.56)
- $(d_\nabla - d_{\nabla^*})(X\circ)=0$ (1.57)

where X and Y are arbitrary vector fields and at a generic point the operator $E\circ$ is assumed to be invertible. Here d_∇ denotes the exterior covariant derivative with respect to ∇ and $\mathcal{L}_E$ denotes the Lie derivative along E.

We recall the definition of *exterior covariant derivative* d_∇ with respect to ∇, which extends the notion of differential to vector-valued differential forms (see, for instance, [57]). The exterior covariant derivative d_∇ of a k-differential form ω with values in TM is defined by

$$\begin{aligned}(d_\nabla\omega)(X_0,\dots,X_k) := &\sum_{i=0}^{k}(-1)^i\,\nabla_{X_i}\big(\omega(X_0,\dots,\hat{X}_i,\dots,X_k)\big)\\ &+\sum_{0\le i<j\le k}(-1)^{i+j}\,\omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)\end{aligned}$$

for $X_0,\dots,X_k\in\mathfrak{X}(M)$, where $\hat{X}$ denotes the absence of a vector field X in the arguments of ω.

The axiom (1.57) expresses the condition of *almost hydrodynamical equivalence* between the two connections. One may notice that not all of the axioms are independent. For instance, the compatibility between the dual connection and the dual product follows from the other axioms (see [6]).

Remark 13 *The dual connection is defined only at the points where the operator $E\circ$ is invertible. At these points the condition*

$$(d_\nabla - d_{\nabla^*})(X\circ)=0,\qquad X\in\mathfrak{X}(M), \tag{1.58}$$

is equivalent to the condition

$$(d_\nabla - d_{\nabla^*})(X*)=0,\qquad X\in\mathfrak{X}(M). \tag{1.59}$$

This implies

$$\Gamma^{*k}_{ij}=\Gamma^k_{ij}-c^{*l}_{ji}\,\nabla_l E^k \tag{1.60}$$

for all suitable indices. Moreover, the flatness of the dual connection follows from the linearity of the Euler vector field (see Theorem 4.4 in [5] for the semisimple case and Lemmas 4.2 and 4.3 in [50] for the general case).

Frobenius manifolds are particular instances of bi-flat F-manifolds. In this specific case, the connection of the first flat structure is the Levi-Civita connection associated with the invariant metric and the unit of the second flat structure is provided by the Euler vector field.

Chapter 2

Regular F-manifolds with eventual identities

This chapter concerns regular F-manifolds with eventual identities and it is based on the content of a work in preparation with Prof. Ian A. B. Strachan.

Our main aim is to solve the equations (1.43) for an eventual identity on a regular F-manifold with Euler vector field E. In the semisimple case, in canonical coordinates eventual identities can be easily proved to be of the form

$$\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}^i(u^i)\frac{\partial}{\partial u^i}.$$

Thus, if the multiplication is semisimple, eventual identities are defined by n functions of one variable. As one may expect, the regular non-semisimple case turns out to be more involved.

We first solve the equations for an eventual identity $\mathcal{E}$ in the case corresponding to a single Jordan block of the operator of multiplication by the Euler vector field, providing explicit examples. We then examine the dual structure defined via the multiplication (1.42), constructing a new basis of vector fields with respect to which the dual product preserves the regular structure of the original product on the F-manifold. We finally extend these considerations to the general case of multiple Jordan blocks.

As a consequence, families of Nijenhuis operators are constructed. Other directions which may be pursued starting from the results of the present chapter involve the construction of classes of examples, possibly fitting the classification provided in [9], as well as the specialization to the regular case of the constructions from [21] (see also [19]) about the analogues of the first and second structural connections of a Frobenius manifold in the case of F-manifolds with an eventual identity. Finally, non-semisimple Frobenius manifolds with an underlying regular

F-manifold structure may also be constructed, as Chapter 3 will show.

2.1 Eventual identities and dual coordinates

Let $(M, \circ, e, E)$ be a regular F-manifold of dimension n and let $m \in M$ be a point around which the operator $E\circ$ has r Jordan blocks, of sizes $m_1, \ldots, m_r$. According to Theorem 1.6, we denote by

$$\left\{u^{j(\alpha)} \,\middle|\, \alpha \in \{1, \ldots, r\},\, j \in \{1, \ldots, m_\alpha\}\right\}$$

the canonical coordinates realizing (1.44) and (1.45).

2.1.1 The case of a single block

We first study the case corresponding to a single Jordan block of the operator $E\circ$, namely $r = 1$. In order to ease the notation, we drop the greek indices.

Eventual identities

Proposition 2.1 *A vector field $\mathcal{E} \in \mathfrak{X}(M)$ is an eventual identity if and only if*

$$\partial_l \mathcal{E}^m = \begin{cases} (l-1)\,\partial_2 \mathcal{E}^{m-l+2} - (l-2)\,\partial_1 \mathcal{E}^{m-l+1} & \text{for } l \leq m, \\ 0 & \text{for } l > m, \end{cases} \tag{2.1}$$

for each $m \in \{1, \ldots, n\}$.

Proof: By picking $X = \partial_j$ and $Y = \partial_k$, (1.43) becomes

$$\mathcal{L}_{\mathcal{E}}(\partial_j \circ \partial_k) - [\mathcal{E}, \partial_j] \circ \partial_k - [\mathcal{E}, \partial_k] \circ \partial_j = [e, \mathcal{E}] \circ \partial_j \circ \partial_k$$

that is

$$\mathcal{L}_{\mathcal{E}} \partial_{j+k-1} - [\mathcal{E}, \partial_j] \circ \partial_k - [\mathcal{E}, \partial_k] \circ \partial_j = [\partial_1, \mathcal{E}] \circ \partial_{j+k-1}$$

namely

$$-\,\partial_{j+k-1}\mathcal{E} + \partial_j \mathcal{E} \circ \partial_k + \partial_k \mathcal{E} \circ \partial_j = \partial_1 \mathcal{E} \circ \partial_{j+k-1}$$

or, for each $i \in \{1, \ldots, n\}$,

$$-\,\partial_{j+k-1}\mathcal{E}^i + \partial_j \mathcal{E}^{i-k+1} + \partial_k \mathcal{E}^{i-j+1} = \partial_1 \mathcal{E}^{i-j-k+2}. \tag{2.2}$$

Each of these quantities is intended to vanish for negative or zero indices. Condition (2.2) immediately follows from (2.1). We now show that (2.2) implies (2.1). Let us fix $m \leq n$. First, we show that $\partial_l \mathcal{E}^m = 0$ for each $l > m$. We proceed by induction over m.

- **m=1** By choosing $i = j = 1$, $k = 2$ in (2.2), we get $\partial_2 \mathcal{E}^1 = 0$. Let us suppose $\partial_l \mathcal{E}^1 = 0$ for every $l \in \{2, \ldots, \bar{l} - 1\}$ (for some fixed $\bar{l} \geq 3$). By choosing $i = 1$, $j = \bar{l} - 1$, $k = 2$ in (2.2), we get $\partial_{\bar{l}} \mathcal{E}^1 = 0$. Thus $\partial_l \mathcal{E}^1 = 0$ for each $l \geq 2$.

- **m=h-1** Let us suppose $\partial_l \mathcal{E}^m = 0$ for every $m \in \{1, \ldots, h-1\}$ (for some fixed $h \geq 2$) and $l > m$.

- **m=h** Let us fix $l > h$. By choosing $i = h$, $j = l - 1$, $k = 2$ in (2.2), we get

  $$-\partial_l \mathcal{E}^h + \partial_{l-1} \mathcal{E}^{h-1} + \partial_2 \mathcal{E}^{h-l+2} = \partial_1 \mathcal{E}^{h-l+1}$$

  where $\partial_{l-1} \mathcal{E}^{h-1} = \partial_2 \mathcal{E}^{h-l+2} = \partial_1 \mathcal{E}^{h-l+1}$ by inductive hypothesis, so $\partial_l \mathcal{E}^h = 0$.

We are now left with showing that

$$\partial_l \mathcal{E}^m = (l-1)\, \partial_2 \mathcal{E}^{m-l+2} - (l-2)\, \partial_1 \mathcal{E}^{m-l+1}$$

for each $l \leq m$. We proceed by induction over l, starting from $l = m$.

- **l=m** We need to prove

  $$\partial_m \mathcal{E}^m = (m-1) \partial_2 \mathcal{E}^2 - (m-2) \partial_1 \mathcal{E}^1. \tag{2.3}$$

  We proceed by induction over m.

  - **m=2** Condition (2.3) trivially holds for $m = 2$.
  - **m=h-1** Let us fix some $h \geq 3$ and assume that for each $m \in \{2, \ldots, h-1\}$ we have $\partial_m \mathcal{E}^m = (m-1) \partial_2 \mathcal{E}^2 - (m-2) \partial_1 \mathcal{E}^1$.
  - **m=h** By choosing $i = h$, $j = h - 1$, $k = 2$ in (2.2), we get

    $$-\partial_h \mathcal{E}^h + \partial_{h-1} \mathcal{E}^{h-1} + \partial_2 \mathcal{E}^2 = \partial_1 \mathcal{E}^1$$

    which by inductive hypothesis yields

    $$\partial_h \mathcal{E}^h = \partial_{h-1} \mathcal{E}^{h-1} + \partial_2 \mathcal{E}^2 - \partial_1 \mathcal{E}^1 = (h-1) \partial_2 \mathcal{E}^2 - (h-2) \partial_1 \mathcal{E}^1.$$

  Condition (2.3) is proved.

- **l=t+1** Let us suppose

  $$\partial_l \mathcal{E}^m = (l-1)\, \partial_2 \mathcal{E}^{m-l+2} - (l-2)\, \partial_1 \mathcal{E}^{m-l+1}$$

  for every $l \geq t + 1$, for some fixed $t \geq 2$.

- **l=t** We show

$$\partial_t \mathcal{E}^m = (t-1)\,\partial_2 \mathcal{E}^{m-t+2} - (t-2)\,\partial_1 \mathcal{E}^{m-t+1}.$$

By choosing $i = m+1$, $j = t$, $k = 2$ in (2.2), we get

$$\partial_{t+1} \mathcal{E}^{m+1} - \partial_t \mathcal{E}^m = \partial_2 \mathcal{E}^{m-t+2} - \partial_1 \mathcal{E}^{m-t+1} \tag{2.4}$$

that yields

$$\partial_t \mathcal{E}^m = \partial_{t+1} \mathcal{E}^{m+1} - \partial_2 \mathcal{E}^{m-t+2} + \partial_1 \mathcal{E}^{m-t+1} = (t-1)\,\partial_2 \mathcal{E}^{m-t+2} - (t-2)\,\partial_1 \mathcal{E}^{m-t+1}$$

by means of the inductive assumption.

■

Remark 14 *The m-th component of an eventual identity $\mathcal{E}$ must only depend on the first m coordinates.*

Proposition 2.1 leads to the following result.

Theorem 2.2 *Given a regular F-manifold with (generalized) canonical coordinates $u^1, \ldots, u^n$ where the structure constants of the product and the components of the unit vector field respectively read $c^i_{jk} = \delta^i_{j+k-1}$ and $e^i = \delta^i_1$, an eventual identity must be of the form*

$$\mathcal{E} = \sum_{i=1}^{n} \mathcal{E}^i(u^1, \ldots, u^i) \frac{\partial}{\partial u^i}$$

where the functions $\{\mathcal{E}^i\}_{i \in \{1,\ldots,n\}}$ are solutions to (2.1).

Remark 15 *Condition* (2.1) *gives a compatible system of PDEs, namely*

$$\partial_i \partial_j \mathcal{E}^m = \partial_j \partial_i \mathcal{E}^m, \qquad i, j \in \{1, \ldots, n\}, \tag{2.5}$$

for each $m \in \{1, \ldots, n\}$. In fact, condition (2.1) *becomes trivial when the lower index l equals 1 or 2, which means we only have to prove* (2.5) *for $i, j \geq 3$. Let us fix $m \in \{1, \ldots, n\}$. Without loss of generality, let us assume $i, j \leq m$, as for $i > m$ (or equivalently $j > m$) both the left and the right-hand side of* (2.5) *trivially vanish. The quantity*

$$\begin{aligned}
\partial_i \partial_j \mathcal{E}^m &= \partial_i \big((j-1)\,\partial_2 \mathcal{E}^{m-j+2} - (j-2)\,\partial_1 \mathcal{E}^{m-j+1} \big) \\
&= (j-1)\,\partial_2 \big((i-1)\,\partial_2 \mathcal{E}^{m-i-j+4} - (i-2)\,\partial_1 \mathcal{E}^{m-i-j+3} \big) \\
&\quad - (j-2)\,\partial_1 \big((i-1)\,\partial_2 \mathcal{E}^{m-i-j+3} - (i-2)\,\partial_1 \mathcal{E}^{m-i-j+2} \big) \\
&= (j-1)(i-1)\,\partial_2^2 \mathcal{E}^{m-i-j+4} - (j-1)(i-2)\,\partial_1 \partial_2 \mathcal{E}^{m-i-j+3} \\
&\quad - (j-2)(i-1)\,\partial_1 \partial_2 \mathcal{E}^{m-i-j+3} + (j-2)(i-2)\,\partial_1^2 \mathcal{E}^{m-i-j+2}
\end{aligned}$$

is symmetric with respect to exchanging the indices i, j, proving (2.5).

Remark 16 *The first components of an eventual identity are of the form*

$$
\begin{aligned}
\mathcal{E}^1 &= f_1(u^1)\\
\mathcal{E}^2 &= f_2(u^1,u^2)\\
\mathcal{E}^3 &= (2\partial_2 f_2 - f_1')u^3 + f_3(u^1,u^2)\\
\mathcal{E}^4 &= (3\partial_2 f_2 - 2f_1')u^4 + 2(\partial_2^2 f_2)(u^3)^2 + (2\partial_2 f_3 - \partial_1 f_2)u^3 + f_4(u^1,u^2)\\
\mathcal{E}^5 &= (4\partial_2 f_2 - 3f_1')u^5 + (3\partial_2 f_3 - 2\partial_1 f_2)u^4 + 6(\partial_2^2 f_2)u^3u^4\\
&\quad + \frac{4}{3}(\partial_2^3 f_2)(u^3)^3 + \frac{1}{2}\big(4\partial_2^2 f_3 - 4\partial_1\partial_2 f_2 + f_1''\big)(u^3)^2\\
&\quad + (2\partial_2 f_4 - \partial_1 f_3)u^3 + f_5(u^1,u^2)
\end{aligned}
$$

for some functions $\{f_i\}_{i\in\{1,\dots,5\}}$ of the coordinates u^1,u^2. In particular, f_1 only depends on u^1.

Proposition 2.3 *For each $m\in\{1,\dots,n\}$ the m-th component of an eventual identity $\mathcal{E}$ is a polynomial function in the variables $\{u^i\}_{i\in\{3,\dots,n\}}$ with coefficients being functions of the coordinates u^1,u^2. In particular, $\mathcal{E}^1,\dots,\mathcal{E}^m$ only depend on a function f_1 of the coordinate u^1 and $m-1$ functions $\{f_i\}_{i\in\{2,\dots,m\}}$ of the coordinates u^1,u^2.*

Proof: For each $m\in\{1,\dots,n\}$, let us denote by $\mathcal{P}_{1,2}^{3,\dots,m}$ the set of polynomial functions in the variables $\{u^i\}_{i\in\{3,\dots,m\}}$ with coefficients being functions of the coordinates u^1,u^2. We want to prove that for each $m\in\{1,\dots,n\}$ we have

$$\mathcal{E}^m = P^m(u^1,u^2;u^3,\dots,u^m) + f_m(u^1,u^2) \tag{2.6}$$

for some function f_m of u^1,u^2 and a function $P^m(u^1,u^2;u^3,\dots,u^m)\in\mathcal{P}_{1,2}^{3,\dots,m}$ which is uniquely determined up to $f_1,\dots,f_{m-1}$. The above example proves (2.6) for $m\le 5$. Let us assume (2.6) holds for $m\le M$, for some fixed $M\le n$, and show it holds for $m=M$. For each $l\ge 3$ (without loss of generality $l\le M$) we have

$$\partial_l\mathcal{E}^M \overset{(2.1)}{=} (l-1)\,\partial_2\mathcal{E}^{M-l+2} - (l-2)\,\partial_1\mathcal{E}^{M-l+1}$$

thus, since $M-l+2\le M-1$, by induction we get

$$
\begin{aligned}
\partial_l\mathcal{E}^M &= (l-1)\,\partial_2 P^{M-l+2}(u^1,u^2;u^3,\dots,u^{M-l+2}) + (l-1)\,\partial_2 f_{M-l+2}(u^1,u^2)\\
&\quad - (l-2)\,\partial_1 P^{M-l+1}(u^1,u^2;u^3,\dots,u^{M-l+1}) - (l-2)\,\partial_1 f_{M-l+1}(u^1,u^2).
\end{aligned}
$$

In particular $\partial_l\mathcal{E}^M\in\mathcal{P}_{1,2}^{3,\dots,M-l+2}\subseteq\mathcal{P}_{1,2}^{3,\dots,M}$ for each $l\ge 3$, thus

$$\mathcal{E}^M = P^M(u^1,u^2;u^3,\dots,u^M) + f_M(u^1,u^2)$$

for a function f_M of u^1,u^2 and a function $P^M\in\mathcal{P}_{1,2}^{3,\dots,M}$ whose coefficents only depend on $f_1,\dots,f_{M-1}$. ■

Remark 17 *Proposition 2.3 implies that an eventual identity can be fully determined starting from a function f_1 of the coordinate u^1 and $n-1$ functions $\{f_i\}_{i\in\{2,\dots,n\}}$ of the coordinates u^1, u^2.*

Below, we present some examples of eventual identites.

Example 2.4 *The Euler vector field*

$$E = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i}$$

is an eventual identity. In fact, given $\mathcal{E}^1(u^1) = u^1$ and $\mathcal{E}^2(u^2) = u^2$, condition (2.1) *gives for each $i \in \{3, \dots, n\}$*

$$\mathcal{E}^i(u^1, \dots, u^i) = u^i + f_i(u^1, \dots, u^{i-1})$$

for some function $f_i(u^1, \dots, u^{i-1})$. In particular, for each $j \in \{3, \dots, i-1\}$ we have

$$\partial_j f_i = \partial_j \mathcal{E}^i = (j-1)\partial_2 \mathcal{E}^{i-j+2} - (j-2)\partial_1 \mathcal{E}^{i-j+1}.$$

Since $i-j+2 \leq i-1$ (and a fortiori $i-j+1 \leq i-1$) for each $j \geq 3$, by induction we get

$$\partial_j f_i = (j-1)\partial_2 u^{i-j+2} - (j-2)\partial_1 u^{i-j+1}, \qquad j \in \{3, \dots, i-1\}.$$

Since $i-j+2 \geq 3$ and $i-j+1 \geq 2$ for each $j \leq i-1$, we get $\partial_j f_i = 0$ for each $j \in \{3, \dots, i-1\}$, proving the function f_i only depends on u^1, u^2. As expected, a first example of eventual identity is then provided by the Euler vector field (more generally, up to additive functions of u^1 and u^2).

The following examples live in dimension $n = 4$.

Example 2.5 *Let us consider two functions F, G of u^1 and a function H of u^2. An eventual identity $\mathcal{E}$ with*

$$\mathcal{E}^1 = F(u^1)$$
$$\mathcal{E}^2 = G(u^1) + H(u^2)$$

has higher components of the form

$$\begin{aligned}
\mathcal{E}^3 =& \big(2H'(u^2) - F'(u^1)\big)u^3 + f_3(u^1, u^2) \\
\mathcal{E}^4 =& \big(3H'(u^2) - 2F'(u^1)\big)u^4 + 2(u^3)^2\, H''(u^2) \\
&+ \big(2\partial_2 f_3(u^1, u^2) - G'(u^1)\big)u^3 + f_4(u^1, u^2)
\end{aligned}$$

for some functions f_3, f_4 of u^1, u^2.

Example 2.6 *Let us consider two functions F, G of u^1 and a function H of u^2. An eventual identity $\mathcal{E}$ with*

$$\begin{aligned}\mathcal{E}^1 &= F(u^1)\\ \mathcal{E}^2 &= G(u^1)\, H(u^2)\end{aligned}$$

has higher components of the form

$$\begin{aligned}\mathcal{E}^3 =&\big(2G(u^1)\, H'(u^2) - F'(u^1)\big)u^3 + f_3(u^1, u^2)\\ \mathcal{E}^4 =&\big(3G(u^1)\, H'(u^2) - 2F'(u^1)\big)u^4 + 2(u^3)^2\, G(u^1)\, H''(u^2)\\ &+ \big(2\partial_2 f_3(u^1, u^2) - G'(u^1)\, H(u^2)\big)u^3 + f_4(u^1, u^2)\end{aligned}$$

for some functions f_3, f_4 of u^1, u^2. For instance, for $F(u^1) = G(u^1) = u^1$ and $H(u^2) = u^2$ we get

$$\begin{aligned}\mathcal{E}^3 =&\big(2u^1 - 1\big)u^3 + f_3(u^1, u^2)\\ \mathcal{E}^4 =&\big(3u^1 - 2\big)u^4 + \big(2\partial_2 f_3(u^1, u^2) - u^2\big)u^3 + f_4(u^1, u^2)\end{aligned}$$

for some functions f_3, f_4 of u^1, u^2. For $F(u^1) = u^1$, $G(u^1) = (u^1)^2$ and $H(u^2) = (u^2)^2$ we get

$$\begin{aligned}\mathcal{E}^3 =&\big(4(u^1)^2\, u^2 - 1\big)u^3 + f_3(u^1, u^2)\\ \mathcal{E}^4 =&\big(6(u^1)^2\, u^2 - 2\big)u^4 + 4(u^3)^2\, (u^1)^2 + \big(2\partial_2 f_3(u^1, u^2) - 2u^1\, (u^2)^2\big)u^3 + f_4(u^1, u^2)\end{aligned}$$

for some functions f_3, f_4 of u^1, u^2.

Example 2.7 *An eventual identity $\mathcal{E}$ with*

$$\begin{aligned}\mathcal{E}^1 &= u^1\\ \mathcal{E}^2 &= (u^1)^2\, (u^2)^3 + u^2\end{aligned}$$

has higher components of the form

$$\begin{aligned}\mathcal{E}^3 =&\big(6(u^1)^2(u^2)^2 + 1\big)u^3 + f_3(u^1, u^2)\\ \mathcal{E}^4 =&\big(9(u^1)^2(u^2)^2 + 1\big)u^4 + 12(u^1)^2\, u^2\, (u^3)^2\\ &+ \big(2\partial_2 f_3(u^1, u^2) - 2u^1(u^2)^3\big)u^3 + f_4(u^1, u^2)\end{aligned}$$

for some functions f_3, f_4 of u^1, u^2.

Example 2.8 *An eventual identity $\mathcal{E}$ with*

$$\mathcal{E}^1 = \sin(u^1)$$

$$\mathcal{E}^2 = u^1 \cos(u^2) + u^2$$

has higher components of the form

$$\begin{aligned}\mathcal{E}^3 =& \big(2 - 2u^1 \sin(u^2) - \cos(u^1)\big)u^3 + f_3(u^1, u^2)\\ \mathcal{E}^4 =& \big(3 - 3u^1 \sin(u^2) - 2\cos(u^1)\big)u^4 - 2u^1 \cos(u^2)\,(u^3)^2\\ &+ \big(2\partial_2 f_3(u^1, u^2) - \cos(u^2)\big)u^3 + f_4(u^1, u^2)\end{aligned}$$

for some functions f_3, f_4 of u^1, u^2.

Proposition 2.9 *Let $\mathcal{E}$ be an eventual identity. Its inverse is of the form*

$$\mathcal{E}^{-1} = \sum_{i=1}^{n} (\mathcal{E}^{-1})^i(u^1, \ldots, u^i)\frac{\partial}{\partial u^i}$$

where

$$(\mathcal{E}^{-1})^1 = \frac{1}{\mathcal{E}^1}$$

and

$$(\mathcal{E}^{-1})^{k+1} = -\frac{1}{\mathcal{E}^1}\sum_{s=1}^{k}(\mathcal{E}^{-1})^{k-s+1}\mathcal{E}^{s+1} \tag{2.7}$$

for each $k \in \{1, \ldots, n-1\}$.

Proof: By spelling out the condition $\mathcal{E}^{-1} \circ \mathcal{E} = e$ in canonical coordinates, we get

$$\delta_1^i = \sum_{j=1}^{i}(\mathcal{E}^{-1})^j\mathcal{E}^{i-j+1} \tag{2.8}$$

By choosing $i = 1$ in (2.8) we get

$$(\mathcal{E}^{-1})^1 = \frac{1}{\mathcal{E}^1}.$$

We prove (2.7) by induction over k.

- **k=1** By choosing $i = 2$ in (2.8) we get

$$(\mathcal{E}^{-1})^2 = -\frac{1}{\mathcal{E}^1}(\mathcal{E}^{-1})^1\mathcal{E}^2.$$

- **k=h-1** Let us suppose

$$(\mathcal{E}^{-1})^{k+1} = -\frac{1}{\mathcal{E}^1}\sum_{s=1}^{k}(\mathcal{E}^{-1})^{k-s+1}\mathcal{E}^{s+1}$$

for every $k \in \{1, \ldots, h-1\}$ (for a fixed $h \geq 2$).

- **k=h** By choosing $i = h+1$ in (2.8) we get

$$0 = \sum_{j=1}^{h+1} (\mathcal{E}^{-1})^j \mathcal{E}^{h-j+2} = \sum_{j=1}^{h} (\mathcal{E}^{-1})^j \mathcal{E}^{h-j+2} + (\mathcal{E}^{-1})^{h+1} \mathcal{E}^1$$

thus

$$(\mathcal{E}^{-1})^{h+1} = -\frac{1}{\mathcal{E}^1} \sum_{j=1}^{h} (\mathcal{E}^{-1})^j \mathcal{E}^{h-j+2} = -\frac{1}{\mathcal{E}^1} \sum_{s=1}^{h} (\mathcal{E}^{-1})^{k-s+1} \mathcal{E}^{s+1}.$$

■

Remark 18 *The inverse of an eventual identity must be an eventual identity as well.*

Remark 19 *The structure constants of the dual product*

$$X * Y = \mathcal{E}^{-1} \circ X \circ Y, \qquad X,Y \in \mathfrak{X}(M),$$

are given by

$$\tilde{c}{}^i_{jk} = (\mathcal{E}^{-1})^s c^i_{st} c^t_{jk} = (\mathcal{E}^{-1})^{i-j-k+2}.$$

for all suitable indices. The dual product is expressed on the coordinate vector fields as

$$\partial_i * \partial_j = \tilde{c}{}^k_{ij} \, \partial_k = \sum_{k=i+j-1}^{n} (\mathcal{E}^{-1})^{k-i-j+2} \, \partial_k, \qquad i, j \in \{1, \dots, n\}.$$

In particular, $\partial_i * \partial_j = 0$ for $i + j \geq n + 2$.

Vector fields preserving the regular F-manifold decomposition for the dual structure

We look for vector fields $v_1, \dots, v_n$ such that

$$v_i * v_j = v_{i+j-1}$$

for $i + j \leq n + 1$ and

$$v_i * v_j = 0$$

for $i + j \geq n + 2$.

Proposition 2.10 *By choosing $v_1 = \mathcal{E}$ we get $v_1 * v_j = v_j$ for each $j \in \{1, \dots, n\}$.*

Proof: This directly follows from $\mathcal{E}$ being unit of the product $*$. ■

Proposition 2.11 *By choosing*

$$v_m = \sum_{i=m}^{n} (v_m)^i \, \partial_i$$

for some functions $(v_m)^m, \ldots, (v_m)^n$ *for each* $m \in \{2, \ldots, n\}$ *we have* $v_i * v_j = 0$ *for* $i + j \geq n + 2$.

Proof: We have

$$v_i * v_j = \left(\sum_{a=i}^{n} (v_i)^a \partial_a \right) * \left(\sum_{b=j}^{n} (v_j)^b \partial_b \right) = \sum_{a=i}^{n} \sum_{b=j}^{n-a+1} (v_i)^a (v_j)^b \partial_a * \partial_b$$

where the second sum only survives for $n - a + 1 \geq j$ that is $a \leq n - j + 1$. The first sum then only survives for $n - j + 1 \geq i$ that is $i + j \leq n + 1$. Therefore $v_i * v_j = 0$ for $i + j \geq n + 2$. ∎

Proposition 2.12 *By choosing*

$$v_{i+1} = \sum_{k=i+1}^{n} (v_{i+1})^k \partial_k, \qquad i \in \{1, \ldots, n\},$$

with

$$(v_{i+1})^k = \sum_{a=i}^{k-1} \sum_{b=2}^{k-a+1} (v_i)^a (v_2)^b (\mathcal{E}^{-1})^{k-a-b+2} \tag{2.9}$$

for each $i \geq 2$ *and* $k \geq i$, *we have*

$$v_i * v_j = v_{i+j-1}$$

for $i + j \leq n + 1$.

Proof: We already know that $v_i * v_j = v_{i+j-1}$ when $i = 1$ or $j = 1$. By imposing $v_i * v_2 = v_{i+1}$ for $i \geq 2$ we get

$$\begin{aligned}
\sum_{k=i+1}^{n} (v_{i+1})^k \partial_k &= \sum_{a=i}^{n} \sum_{b=2}^{n-a+1} (v_i)^a (v_2)^b \partial_a * \partial_b \\
&= \sum_{a=i}^{n} \sum_{b=2}^{n-a+1} (v_i)^a (v_2)^b \sum_{k=a+b-1}^{n} (\mathcal{E}^{-1})^{k-a-b+2} \partial_k \\
&= \sum_{k=i+1}^{n} \left(\sum_{a=i}^{k-1} \sum_{b=2}^{k-a+1} (v_i)^a (v_2)^b (\mathcal{E}^{-1})^{k-a-b+2} \right) \partial_k
\end{aligned}$$

thus for each $k \in \{i+1, \ldots, n\}$

$$(v_{i+1})^k = \sum_{a=i}^{k-1} \sum_{b=2}^{k-a+1} (v_i)^a (v_2)^b (\mathcal{E}^{-1})^{k-a-b+2}. \tag{2.10}$$

We now have to show that this ensures $v_i * v_j = v_{i+j-1}$ for $i \geq 2$ and $j \geq 3$ such that $i + j \leq n+1$. The condition $v_i * v_j = v_{i+j-1}$ amounts to

$$\begin{aligned}
\sum_{k=i+j-1}^{n} (v_{i+j-1})^k \partial_k &= \sum_{a=i}^{n} \sum_{b=j}^{n-a+1} (v_i)^a (v_j)^b \partial_a * \partial_b \\
&= \sum_{a=i}^{n} \sum_{b=j}^{n-a+1} (v_i)^a (v_j)^b \sum_{k=a+b-1}^{n} (\mathcal{E}^{-1})^{k-a-b+2} \partial_k \\
&= \sum_{k=i+j-1}^{n} \left(\sum_{a=i}^{k-j+1} \sum_{b=j}^{k-a+1} (v_i)^a (v_j)^b (\mathcal{E}^{-1})^{k-a-b+2} \right) \partial_k
\end{aligned}$$

that is for each $k \in \{i+j-1, \ldots, n\}$

$$(v_{i+j-1})^k = \sum_{a=i}^{k-j+1} \sum_{b=j}^{k-a+1} (v_i)^a (v_j)^b (\mathcal{E}^{-1})^{k-a-b+2}.$$

Let us first observe that $v_i * v_j = v_{i+1} * v_{j-1}$. Let us define

$$\begin{aligned}
A_{ij}^m &:= (v_i * v_j)^m - (v_{i+1} * v_{j-1})^m \\
&= \sum_{a=i}^{m-j+1} \sum_{b=j}^{m-a+1} v_i^a \, v_j^b \, (\mathcal{E}^{-1})^{m-a-b+2} - \sum_{a=i+1}^{m-j+2} \sum_{b=j-1}^{m-a+1} v_{i+1}^a \, v_{j-1}^b \, (\mathcal{E}^{-1})^{m-a-b+2}
\end{aligned}$$

for each $m \geq i+j-1$. We need to prove that $A_{ij}^m = 0$ for each $m \geq i+j-1$. We proceed by induction over m.

- **m=i+j-1** We have

  $$A_{ij}^{i+j-1} = v_i^i \, v_j^j \, (\mathcal{E}^{-1})^1 - v_{i+1}^{i+1} \, v_{j-1}^{j-1} \, (\mathcal{E}^{-1})^1 = 0$$

  as

  $$v_j^j = v_{j-1}^{j-1} \frac{a_2}{\mathcal{E}^1}, \qquad j \in \{2, \ldots, n\}. \tag{2.11}$$

- **m=M** Let us suppose $A_{ij}^m = 0$ for each $m \in \{i+j-1, \ldots, M\}$ for a given $M \geq i+j-1$.

- **m=M+1** We show $A_{ij}^{M+1} = 0$. We have

$$A_{ij}^{M+1} := \sum_{a=i}^{M-j+2} \sum_{b=j}^{M-a+2} v_i^a \, v_j^b \, (\mathcal{E}^{-1})^{M-a-b+3} - \sum_{a=i+1}^{M-j+3} \sum_{b=j-1}^{M-a+2} v_{i+1}^a \, v_{j-1}^b \, (\mathcal{E}^{-1})^{M-a-b+3}$$

$$
\begin{aligned}
&\overset{(2.7)}{=} -\frac{1}{\mathcal{E}^1}\sum_{a=i}^{M-j+2}\sum_{b=j}^{M-a+2} v_i^a\, v_j^b \sum_{s=1}^{M-a-b+2} (\mathcal{E}^{-1})^{M-a-b-s+3}\,\mathcal{E}^{s+1} \\
&+ \frac{1}{\mathcal{E}^1}\sum_{a=i+1}^{M-j+3}\sum_{b=j-1}^{M-a+2} v_{i+1}^a\, v_{j-1}^b \sum_{s=1}^{M-a-b+2} (\mathcal{E}^{-1})^{M-a-b-s+3}\,\mathcal{E}^{s+1} \\
&= -\frac{1}{\mathcal{E}^1}\sum_{s=1}^{M-i-j+2} \mathcal{E}^{s+1}\left[\sum_{a=i}^{M-s-j+2}\sum_{b=j}^{M-a-s+2} v_i^a\, v_j^b\,(\mathcal{E}^{-1})^{M-a-b-s+3}\right. \\
&\left. - \sum_{a=i+1}^{M-s-j+3}\sum_{b=j-1}^{M-a-s+2} v_{i+1}^a\, v_{j-1}^b\,(\mathcal{E}^{-1})^{M-a-b-s+3}\right] \\
&= -\frac{1}{\mathcal{E}^1}\sum_{s=1}^{M-i-j+2} \mathcal{E}^{s+1}\, A_{ij}^{M-s+1} = 0
\end{aligned}
$$

as $A_{ij}^{M-s+1} = 0$ by inductive hypothesis for each $s \geq 1$.

We have then proved that $v_i * v_j = v_{i+1} * v_{j-1}$. This implies $v_i * v_j = v_{\bar{i}} * v_{\bar{j}}$ whenever $\bar{i} + \bar{j} = i + j$. In particular, for the choice of $\bar{j} = 2$ and $\bar{i} = i + j - 2$ we have

$$
\sum_{a=i}^{k-j+1}\sum_{b=j}^{k-a+1} (v_i)^a (v_j)^b (\mathcal{E}^{-1})^{k-a-b+2} = \sum_{a=i+j-2}^{k-1}\sum_{b=2}^{k-a+1} (v_{i+j-2})^a (v_2)^b (\mathcal{E}^{-1})^{k-a-b+2}. \quad (2.12)
$$

We have

$$
\begin{aligned}
v_{i+j-1} &= \sum_{k=i+j-1}^{n} (v_{i+j-1})^k \partial_k \\
&\overset{(2.10)}{=} \sum_{k=i+j-1}^{n}\sum_{a=i+j-2}^{k-1}\sum_{b=2}^{k-a+1} (v_{i+j-2})^a (v_2)^b (\mathcal{E}^{-1})^{k-a-b+2}\partial_k \\
&\overset{(2.12)}{=} \sum_{k=i+j-1}^{n}\sum_{a=i}^{k-j+1}\sum_{b=j}^{k-a+1} (v_i)^a (v_j)^b (\mathcal{E}^{-1})^{k-a-b+2}\partial_k = v_i * v_j.
\end{aligned}
$$

■

This result determines $v_3, \ldots, v_n$. We already chose $v_1 = \mathcal{E}$. There is still arbitrariness in choosing v_2. Propositions 2.10, 2.11, 2.12 yield the following result.

Theorem 2.1 *By setting*

- $v_1 = \mathcal{E}$
- $v_2 = \sum\limits_{i=2}^{n} a_i\, \partial_i$ *for some functions* $a_2 \neq 0, a_3, \ldots, a_n$

- $v_{i+1} = \sum_{k=i+1}^{n} (v_{i+1})^k \, \partial_k$ *for* $i \geq 2$ *with*

$$(v_{i+1})^k = \sum_{a=i}^{k-1} \sum_{b=2}^{k-a+1} (v_i)^a (v_2)^b (\mathcal{E}^{-1})^{k-a-b+2}, \qquad k \geq i+1,$$

we have

$$v_i * v_j = v_{i+j-1} \, \mathbb{1}_{\{i+j \leq n+1\}}, \qquad i, j \in \{1, \dots, n\}.$$

Coordinates preserving the regular F-manifold decomposition for the dual structure

We now show that there exists a set of coordinates $w^1, \dots, w^n$ such that

$$v_i = \frac{\partial}{\partial w^i}, \qquad i \in \{1, \dots, n\}.$$

We assume v_2 to be an eventual identity. This means that we require v_2 to be solution to (2.1), that is

$$\partial_l v_2^m = \begin{cases} (l-1)\, \partial_2 v_2^{m-l+2} - (l-2)\, \partial_1 v_2^{m-l+1} & \text{for } l \leq m, \\ 0 & \text{for } l > m, \end{cases} \tag{2.13}$$

or

$$\partial_l a_m = \begin{cases} (l-1)\, \partial_2 a_{m-l+2} - (l-2)\, \partial_1 a_{m-l+1} & \text{for } l \leq m, \\ 0 & \text{for } l > m, \end{cases} \tag{2.14}$$

for each $m \in \{1, \dots, n\}$. In particular, for each $m \in \{2, \dots, n\}$ the function a_m must only depend on the first m coordinates.

Lemma 2.13 *For each* $j \geq 3$ *and* $J \geq j$ *we have*

$$v_j^J = -\frac{1}{\mathcal{E}^1} \sum_{s=1}^{J-j} v_j^{J-s} \mathcal{E}^{s+1} + \frac{1}{\mathcal{E}^1} \sum_{s=1}^{J-1} v_{j-1}^a v_2^{J-a+1}. \tag{2.15}$$

Proof: For each $j \geq 3$ and $J \geq j$ we have

$$\begin{aligned} v_j^J &\overset{(2.9)}{=} \sum_{a=j-1}^{J-1} \sum_{b=2}^{J-a+1} (v_{j-1})^a (v_2)^b (\mathcal{E}^{-1})^{J-a-b+2} \\ &= \sum_{a=j-1}^{J-1} \sum_{b=2}^{J-a} (v_{j-1})^a (v_2)^b (\mathcal{E}^{-1})^{J-a-b+2} + \sum_{a=j-1}^{J-1} (v_{j-1})^a (v_2)^{J-a+1} (\mathcal{E}^{-1})^1 \\ &\overset{(2.7)}{=} -\frac{1}{\mathcal{E}^1} \sum_{a=j-1}^{J-1} \sum_{b=2}^{J-a} (v_{j-1})^a (v_2)^b \sum_{s=1}^{J-a-b+1} (\mathcal{E}^{-1})^{J-a-b-s+2} \mathcal{E}^{s+1} \end{aligned}$$

$$
+\frac{1}{\mathcal{E}^1}\sum_{a=j-1}^{J-1}(v_{j-1})^a(v_2)^{J-a+1}
$$

$$
\overset{(2.9)}{=} -\frac{1}{\mathcal{E}^1}\sum_{s=1}^{J-j}\mathcal{E}^{s+1}(v_j)^{J-s}+\frac{1}{\mathcal{E}^1}\sum_{a=j-1}^{J-1}(v_{j-1})^a(v_2)^{J-a+1}.
$$

■

Lemma 2.14 *For each $j \geq 3$ and $J \geq j$ we have*

$$
v_j^J = \frac{a_2}{\mathcal{E}^1}v_{j-1}^{J-1} - \frac{1}{\mathcal{E}^1}\sum_{s=1}^{J-j}\left(v_j^{J-s}\mathcal{E}^{s+1} - v_{j-1}^{J-s-1}a_{s+2}\right). \tag{2.16}
$$

Proof: For each $j \geq 3$ and $J \geq j$ we have

$$
\begin{aligned}
v_j^J &\overset{(2.15)}{=} -\frac{1}{\mathcal{E}^1}\sum_{s=1}^{J-j}\mathcal{E}^{s+1}(v_j)^{J-s}+\frac{1}{\mathcal{E}^1}\sum_{a=j-1}^{J-1}(v_{j-1})^a(v_2)^{J-a+1} \\
&= -\frac{1}{\mathcal{E}^1}\sum_{s=1}^{J-j}\mathcal{E}^{s+1}(v_j)^{J-s}+\frac{1}{\mathcal{E}^1}\sum_{s=0}^{J-j}(v_{j-1})^{J-s-1}(v_2)^{s+2} \\
&= \frac{a_2}{\mathcal{E}^1}(v_{j-1})^{J-1} - \frac{1}{\mathcal{E}^1}\sum_{s=1}^{J-j}\mathcal{E}^{s+1}(v_j)^{J-s}+\frac{1}{\mathcal{E}^1}\sum_{s=1}^{J-j}(v_{j-1})^{J-s-1}a_{s+2}.
\end{aligned}
$$

■

Remark 20 *For each $j \geq 3$ we have*

$$
v_j^j = \frac{a_2}{\mathcal{E}^1}v_{j-1}^{j-1} = \frac{(a_2)^2}{(\mathcal{E}^1)^2}v_{j-2}^{j-2} = \cdots = \frac{(a_2)^{j-2}}{(\mathcal{E}^1)^{j-2}}v_2^2 = \frac{(a_2)^{j-1}}{(\mathcal{E}^1)^{j-2}} \neq 0. \tag{2.17}
$$

The vector fields $v_1, \ldots, v_n$ are then linearly independent.

Proposition 2.15 *For each $j \in \{1, \ldots, n\}$ and $J \geq j$, the function v_j^J only depends on the first $J - j + 2$ coordinates.*

Proof: We show $\partial_k v_j^J = 0$ for $k \geq J - j + 3$. For $j = 1$ and $j = 2$, this trivially follows from (2.1) and (2.13) respectively. Let us now fix $j \geq 3$. We proceed by induction over $J \geq j$.

- **J=j** For each $k \geq 3$ we have

$$
\partial_k v_j^j \overset{(2.17)}{=} \partial_k\left(\frac{(a_2)^{j-1}}{(\mathcal{E}^1)^{j-2}}\right) = 0.
$$

- **J-1** Let us suppose $\partial_k v_j^{\overline{J}} = 0$ for each $k \geq \overline{J} - j + 3$, for each $\overline{J} \in \{j, \dots, J-1\}$ (for a fixed $J \geq j$).

- **J** Let us fix $k \geq J - j + 3$. We have

$$\partial_k v_j^J \overset{(2.16)}{=} \frac{a_2}{\mathcal{E}^1}(\partial_k v_{j-1}^{J-1}) - \frac{1}{\mathcal{E}^1}\sum_{s=1}^{J-j}\Big((\partial_k v_j^{J-s})\mathcal{E}^{s+1} + v_j^{J-s}(\partial_k \mathcal{E}^{s+1}) - (\partial_k v_{j-1}^{J-s-1})a_{s+2} - v_{j-1}^{J-s-1}(\partial_k a_{s+2})\Big)$$

where $\partial_k v_{j-1}^{J-1}$ and $\partial_k v_{j-1}^{J-s-1}$ vanish by induction over j, $\partial_k v_j^{J-s}$ vanishes by induction over J. By the steps $j = 1$ and $j = 2$, the quantities $\partial_k \mathcal{E}^{s+1}$ and $\partial_k a_{s+2}$ vanish as well for each $s \leq J - j$. Then $\partial_k v_j^J = 0$.

■

Remark 21 *As observed in [20], the product of eventual identities is an eventual identity:*

$$\begin{aligned}(\mathcal{L}_{\mathcal{E}_1 \circ \mathcal{E}_2}\circ)(X,Y) &= \mathcal{E}_1 \circ (\mathcal{L}_{\mathcal{E}_2}\circ)(X,Y) + \mathcal{E}_2 \circ (\mathcal{L}_{\mathcal{E}_1}\circ)(X,Y)\\ &= \mathcal{E}_1 \circ [e, \mathcal{E}_2] \circ X \circ Y + \mathcal{E}_2 \circ [e, \mathcal{E}_1] \circ X \circ Y\\ &= [e, \mathcal{E}_1 \circ \mathcal{E}_2] \circ X \circ Y\end{aligned}$$

(as $\mathcal{L}_e \circ = 0$) for any $X, Y \in \mathfrak{X}(M)$, for $\mathcal{E}_1$ and $\mathcal{E}_2$ being eventual identities.

Proposition 2.16 *The vector fields $\{v_i\}_{i\geq 3}$ are eventual identities.*

Proof: Let us fix $i \geq 3$. The vector field v_i is defined as

$$v_i = v_2 * v_{i-1} = \mathcal{E}^{-1} \circ v_2 \circ v_{i-1}$$

Since v_2 is assumed to be an eventual identity, v_i is by induction an eventual identity, as product of eventual identities. ■

Let us now assume $[v_1, v_2] = 0$, that is for each $m \in \{2, \dots, n\}$

$$\partial_1 a_m = -\frac{1}{\mathcal{E}^1}\sum_{l=2}^{m}\big(\mathcal{E}^l\, \partial_l a_m - a_l\, \partial_l \mathcal{E}^m\big). \tag{2.18}$$

Remark 22 *We assumed two properites: that v_2 is an eventual identity and that it commutes with $\mathcal{E}$. These two requirements impose conditions on the partial derivatives $\{\partial_l a_m\}_{m\in\{2,\dots,n\}}$ for $l \neq 2$. More precisely, the condition of v_2 being an eventual identity amounts to* (2.14)

$$\partial_l a_m = \begin{cases}(l-1)\,\partial_2 a_{m-l+2} - (l-2)\,\partial_1 a_{m-l+1} & \textit{for } l \leq m\\ 0 & \textit{for } l > m\end{cases}$$

thus for each $m \in \{3, \dots, n\}$ *it fixes the quantities* $\{\partial_l a_m\}_{l\in\{3,\dots,m\}}$ *in terms of the quantities* $\{\partial_2 a_j,\ \partial_1 a_j\}_{j\in\{2,\dots,m-1\}}$. *The condition* $[v_1, v_2] = 0$ *amounts to* (2.18)

$$\begin{aligned}\partial_1 a_m &= -\frac{1}{\mathcal{E}^1}\sum_{l=2}^{m}\big(\mathcal{E}^l\,\partial_l a_m - a_l\,\partial_l\mathcal{E}^m\big)\\ &\overset{(2.1)}{\underset{(2.14)}{=}} -\frac{1}{\mathcal{E}^1}\sum_{l=2}^{m}\Big((l-1)\big(\mathcal{E}^l\,\partial_2 a_{m-l+2} - a_l\,\partial_2\mathcal{E}^{m-l+2}\big)\\ &\quad - (l-2)\big(\mathcal{E}^l\,\partial_1 a_{m-l+1} - a_l\,\partial_1\mathcal{E}^{m-l+1}\big)\Big)\end{aligned}$$

thus for each $m \in \{2, \dots, n\}$ *it fixes the quantity* $\partial_1 a_m$ *in terms of the quantities* $\{\partial_2 a_j,\ \partial_2\mathcal{E}^j\}_{j\in\{2,\dots,m\}}$ *and* $\{\partial_1 a_j,\ \partial_1\mathcal{E}^j\}_{j\in\{1,\dots,m-1\}}$ *where* $a_1 = 0$.

In order to assume v_2 *both to be an eventual identity and to commute with* $v_1 = \mathcal{E}$, *then, one must verify that for each* $m \in \{2, \dots, n\}$ *the conditions* (2.14) *and*

$$\partial_1 a_m = -\frac{1}{\mathcal{E}^1}\sum_{t=2}^{m}\big(\mathcal{E}^t\,\partial_t a_m - a_t\,\partial_t\mathcal{E}^m\big) \tag{2.19}$$

*define a compatible system of PDEs. Since we already checked the compatibility conditions for the system defining an eventual identity ((*2.5*) for* (2.1)*, where* (2.1) *rewrites as* (2.14) *for the eventual identity* v_2*), we only need to prove the condition* $\partial_1\partial_l a_m = \partial_l\partial_1 a_m$ *for each* $l \in \{2, \dots, m\}$ *(for* $l > m$ *the condition becomes trivial) for each* $m \in \{2, \dots, n\}$.

Let us start from considering $m = 2$. *Since* (2.14) *becomes an empty condition, the only equation for* a_2 *is*

$$\partial_1 a_2 = -\frac{1}{\mathcal{E}^1}\big(\mathcal{E}^2\,\partial_2 a_2 - a_2\,\partial_2\mathcal{E}^2\big)$$

that is

$$\mathcal{E}^1\partial_1 a_2 + \mathcal{E}^2\,\partial_2 a_2 - a_2\,\partial_2\mathcal{E}^2 = 0. \tag{2.20}$$

Since this is the only equation for a_2, *no compatibility condition is required. In other words, we automatically write*

$$\partial_1\partial_2 a_2 = \partial_2\partial_1 a_2.$$

We will now show how (2.20) *is the only non-trivial relation appearing among the compatibility conditions for the above system.*

Let us now consider $m = 3$. *We need to show that* $\partial_1\partial_l a_3 = \partial_l\partial_1 a_3$ *for* $l \in \{2, 3\}$. *Let us start from* $l = 3$. *We have*

$$\partial_1\partial_3 a_3 = \partial_1(2\partial_2 a_2) = 2\partial_2\partial_1 a_2 = -\frac{2}{\mathcal{E}^1}\partial_2\big(\mathcal{E}^2\,\partial_2 a_2 - a_2\,\partial_2\mathcal{E}^2\big) = -\frac{2}{\mathcal{E}^1}\big(\mathcal{E}^2\,\partial_2^2 a_2 - a_2\,\partial_2^2\mathcal{E}^2\big)$$

and

$$\partial_3\partial_1 a_3 = -\frac{1}{\mathcal{E}^1}\sum_{t=2}^{3}\partial_3\big(\mathcal{E}^t\,\partial_t a_3 - a_t\,\partial_t\mathcal{E}^3\big) = -\frac{1}{\mathcal{E}^1}\big(\mathcal{E}^2\,\partial_3\partial_2 a_3 - a_2\,\partial_3\partial_2\mathcal{E}^3\big)$$
$$= -\frac{2}{\mathcal{E}^1}\big(\mathcal{E}^2\,\partial_2^2 a_2 - a_2\,\partial_2^2\mathcal{E}^2\big)$$

proving $\partial_1\partial_3 a_3 = \partial_3\partial_1 a_3$. Let us now consider $l = 2$. In order to prove the compatibility condition $\partial_1\partial_2 a_3 = \partial_2\partial_1 a_3$, let us rewrite (2.14) *for $m = 3$, namely $\partial_3 a_3 = 2\,\partial_2 a_2$ for $a_3(u^1, u^2, u^3)$, as $a_3 = 2\,(\partial_2 a_2)\,u^3 + g_3(u^1, u^2)$ for some function g_3 of u^1, u^2. In particular, let g_3 vanish. We have*

$$\partial_1\partial_2 a_3 = \partial_1(2\,(\partial_2^2 a_2)\,u^3) = 2\,(\partial_1\partial_2^2 a_2)\,u^3$$

and

$$\partial_2\partial_1 a_3 = -\frac{1}{\mathcal{E}^1}\sum_{t=2}^{3}\partial_2\big(\mathcal{E}^t\,\partial_t a_3 - a_t\,\partial_t\mathcal{E}^3\big)$$
$$= -\frac{1}{\mathcal{E}^1}\big(\partial_2\mathcal{E}^2\,\partial_2 a_3 + \mathcal{E}^2\,\partial_2^2 a_3 + \partial_2\mathcal{E}^3\,\partial_3 a_3 + \mathcal{E}^3\,\partial_2\partial_3 a_3$$
$$- \partial_2 a^2\,\partial_2\mathcal{E}_3 - a^2\,\partial_2^2\mathcal{E}_3 - \partial_2 a^3\,\partial_3\mathcal{E}_3 - a^3\,\partial_2\partial_3\mathcal{E}_3\big)$$

where $a_3 = 2\,(\partial_2 a_2)\,u^3$ and $\mathcal{E}^3 = (2\,\partial_2\mathcal{E}^2 - \partial_1\mathcal{E}^1)\,u^3 + f_3(u^1, u^2)$ for some function f_3 of u^1, u^2. In particular, let f_3 vanish. We get

$$\partial_2\partial_1 a_3 = -\frac{2u^3}{\mathcal{E}^1}\big(\partial_2\mathcal{E}^2\,\partial_2^2 a_2 + \mathcal{E}^2\,\partial_2^3 a_2 - \partial_2^2\mathcal{E}^2\,\partial_2 a_2 - a_2\,\partial_2^3\mathcal{E}^2\big)$$

which, by taking into account that applying ∂_2 twice to (2.20) *gives*

$$\mathcal{E}^1\partial_1\partial_2^2 a_2 + \partial_2\mathcal{E}^2\,\partial_2^2 a_2 + \mathcal{E}^2\,\partial_2^3 a_2 - \partial_2 a_2\,\partial_2^2\mathcal{E}^2 - a_2\,\partial_2^3\mathcal{E}^2 = 0, \tag{2.21}$$

becomes

$$\partial_2\partial_1 a_3 = 2u^3\left(\partial_1\partial_2^2 a_2\right) = \partial_1\partial_2 a_3.$$

We want to prove by induction over m that $\partial_1\partial_l a_m = \partial_l\partial_1 a_m$ for $l \in \{2, \ldots, m\}$, starting from the above case of $m = 3$. Let us then fix $m \in \{3, \ldots, n\}$ and assume

$$\partial_1\partial_l a_k = \partial_l\partial_1 a_k, \qquad l \in \{2, \ldots, k\}, \tag{2.22}$$

for each $k \in \{3, \ldots, m-1\}$. We are now going to show that $\partial_1\partial_l a_m = \partial_l\partial_1 a_m$ for each $l \in \{2, \ldots, m\}$. In order to prove this, we use an inner induction over the difference $m - l$, starting from $m - l = 0$. Let us then consider $l = m$. We have

$$\partial_1\partial_m a_m = \partial_1\big((m-1)\partial_2 a_2\big) = (m-1)\partial_1\partial_2 a_2 = -\frac{m-1}{\mathcal{E}^1}\,\partial_2\big(\mathcal{E}^2\,\partial_2 a_2 - a_2\,\partial_2\mathcal{E}^2\big)$$

$$= -\frac{m-1}{\mathcal{E}^1}\left(\mathcal{E}^2\,\partial_2^2 a_2 - a_2\,\partial_2^2\mathcal{E}^2\right)$$

and

$$\begin{aligned}
\partial_m\partial_1 a_m &= -\frac{1}{\mathcal{E}^1}\sum_{t=2}^{m}\partial_m\big(\mathcal{E}^t\,\partial_t a_m - a_t\,\partial_t\mathcal{E}^m\big) = -\frac{1}{\mathcal{E}^1}\sum_{t=2}^{m}\big(\mathcal{E}^t\,\partial_m\partial_t a_m - a_t\,\partial_m\partial_t\mathcal{E}^m\big)\\
&= -\frac{1}{\mathcal{E}^1}\sum_{t=2}^{m}\big(\mathcal{E}^t\,\partial_t((m-1)\partial_2 a_2) - a_t\,\partial_t((m-1)\partial_2\mathcal{E}^2 - (m-2)\partial_1\mathcal{E}^1)\big)\\
&= -\frac{m-1}{\mathcal{E}^1}\big(\mathcal{E}^2\,\partial_2^2 a_2 - a_2\,\partial_2^2\mathcal{E}^2\big) = \partial_1\partial_m a_m.
\end{aligned}$$

Let us fix $s \in \{1,\ldots,m-3\}$ and let us inductively assume $\partial_1\partial_l a_m = \partial_l\partial_1 a_m$ for each $l \in \{m-s+1,\ldots,m\}$. We need to show that $\partial_1\partial_l a_m = \partial_l\partial_1 a_m$ for $l = m-s$, which in turn will imply $\partial_1\partial_l a_m = \partial_l\partial_1 a_m$ for each $l \in \{3,\ldots,m\}$. We have

$$\partial_1\partial_l a_m = \partial_1\big((l-1)\,\partial_2 a_{s+2} - (l-2)\,\partial_1 a_{s+1}\big)$$

where $\partial_1\partial_2 a_{s+2} = \partial_2\partial_1 a_{s+2}$ since $s+2 \le m-1$, thus

$$\begin{aligned}
\partial_1\partial_l a_m &= -\frac{l-1}{\mathcal{E}^1}\sum_{t=2}^{s+2}\partial_2\big(\mathcal{E}^t\,\partial_t a_{s+2} - a_t\,\partial_t\mathcal{E}^{s+2}\big) - (l-2)\,\partial_1^2 a_{s+1}\\
&= -\frac{l-1}{\mathcal{E}^1}\sum_{t=2}^{s+2}\big(\partial_2\mathcal{E}^t\,\partial_t a_{s+2} - \partial_2 a_t\,\partial_t\mathcal{E}^{s+2}\big)\\
&\quad - \frac{l-1}{\mathcal{E}^1}\sum_{t=2}^{s+2}\big(\mathcal{E}^t\,\partial_2\partial_t a_{s+2} - a_t\,\partial_2\partial_t\mathcal{E}^{s+2}\big) - (l-2)\,\partial_1^2 a_{s+1}.
\end{aligned}$$

On the other hand

$$\begin{aligned}
\partial_l\partial_1 a_m &= -\frac{1}{\mathcal{E}^1}\sum_{t=2}^{m}\partial_l\big(\mathcal{E}^t\,\partial_t a_m - a_t\,\partial_t\mathcal{E}^m\big)\\
&= -\frac{1}{\mathcal{E}^1}\sum_{t=l}^{m}\big(\partial_l\mathcal{E}^t\,\partial_t a_m - \partial_l a_t\,\partial_t\mathcal{E}^m\big) - \frac{l-1}{\mathcal{E}^1}\sum_{t=2}^{s+2}\big(\mathcal{E}^t\,\partial_t\partial_2 a_{s+2} - a_t\,\partial_t\partial_2\mathcal{E}^{s+2}\big)\\
&\quad + \frac{l-2}{\mathcal{E}^1}\sum_{t=2}^{s+1}\big(\mathcal{E}^t\,\partial_t\partial_1 a_{s+1} - a_t\,\partial_t\partial_1\mathcal{E}^{s+1}\big).
\end{aligned}$$

Then

$$\begin{aligned}
-\partial_1\partial_l a_m + \partial_l\partial_1 a_m &= \frac{l-1}{\mathcal{E}^1}\sum_{t=2}^{s+2}\big(\partial_2\mathcal{E}^t\,\partial_t a_{s+2} - \partial_2 a_t\,\partial_t\mathcal{E}^{s+2}\big) + (l-2)\,\partial_1^2 a_{s+1}\\
&\quad - \frac{1}{\mathcal{E}^1}\sum_{t=l}^{m}\big(\partial_l\mathcal{E}^t\,\partial_t a_m - \partial_l a_t\,\partial_t\mathcal{E}^m\big)
\end{aligned}$$

$$+\frac{l-2}{\mathcal{E}^1}\sum_{t=2}^{s+1}\big(\mathcal{E}^t\,\partial_t\partial_1 a_{s+1}-a_t\,\partial_t\partial_1\mathcal{E}^{s+1}\big)$$

where

$$\begin{aligned}\partial_1^2 a_{s+1}=&\partial_1\Big(-\frac{1}{\mathcal{E}^1}\sum_{t=2}^{s+1}\big(\mathcal{E}^t\,\partial_t a_{s+1}-a_t\,\partial_t\mathcal{E}^{s+1}\big)\Big)\\ =&\frac{\partial_1\mathcal{E}^1}{(\mathcal{E}^1)^2}\sum_{t=2}^{s+1}\big(\mathcal{E}^t\,\partial_t a_{s+1}-a_t\,\partial_t\mathcal{E}^{s+1}\big)-\frac{1}{\mathcal{E}^1}\sum_{t=2}^{s+1}\big(\partial_1\mathcal{E}^t\,\partial_t a_{s+1}\\ &+\mathcal{E}^t\,\partial_1\partial_t a_{s+1}-\partial_1 a_t\,\partial_t\mathcal{E}^{s+1}-a_t\,\partial_1\partial_t\mathcal{E}^{s+1}\big)\end{aligned}$$

with $\partial_1\partial_t a_{s+1}=\partial_t\partial_1 a_{s+1}$ *for each* $t\geq 2$*. We get*

$$\begin{aligned}&-\partial_1\partial_l a_m+\partial_l\partial_1 a_m=\frac{l-1}{\mathcal{E}^1}\sum_{t=2}^{s+2}\big(\partial_2\mathcal{E}^t\,\partial_t a_{s+2}-\partial_2 a_t\,\partial_t\mathcal{E}^{s+2}\big)\\ &+(l-2)\frac{\partial_1\mathcal{E}^1}{(\mathcal{E}^1)^2}\sum_{t=2}^{s+1}\big(\mathcal{E}^t\,\partial_t a_{s+1}-a_t\,\partial_t\mathcal{E}^{s+1}\big)\\ &-\frac{l-2}{\mathcal{E}^1}\sum_{t=2}^{s+1}\big(\partial_1\mathcal{E}^t\,\partial_t a_{s+1}-\partial_1 a_t\,\partial_t\mathcal{E}^{s+1}\big)-\frac{1}{\mathcal{E}^1}\sum_{t=l}^{m}\big(\partial_l\mathcal{E}^t\,\partial_t a_m-\partial_l a_t\,\partial_t\mathcal{E}^m\big)\\ =&\frac{l-1}{\mathcal{E}^1}\sum_{t=2}^{s+2}\Bigg(\partial_2\mathcal{E}^t\,\big((t-1)\partial_2 a_{s-t+4}-(t-2)\partial_1 a_{s-t+3}\big)\\ &-\partial_2 a_t\,\big((t-1)\partial_2\mathcal{E}^{s-t+4}-(t-2)\partial_1\mathcal{E}^{s-t+3}\big)\Bigg)\\ &+(l-2)\frac{\partial_1\mathcal{E}^1}{(\mathcal{E}^1)^2}\sum_{t=2}^{s+1}\big(\mathcal{E}^t\,\partial_t a_{s+1}-a_t\,\partial_t\mathcal{E}^{s+1}\big)\\ &-\frac{l-2}{\mathcal{E}^1}\sum_{t=2}^{s+1}\Bigg(\partial_1\mathcal{E}^t\,\big((t-1)\partial_2 a_{s-t+3}-(t-2)\partial_1 a_{s-t+2}\big)\\ &-\partial_1 a_t\,\big((t-1)\partial_2\mathcal{E}^{s-t+3}-(t-2)\partial_1\mathcal{E}^{s-t+2}\big)\Bigg)\\ &-\frac{1}{\mathcal{E}^1}\sum_{t=l}^{m}\big((l-1)\partial_2\mathcal{E}^{t-l+2}-(l-2)\partial_1\mathcal{E}^{t-l+1}\big)\big((t-1)\partial_2 a_{m-t+2}-(t-2)\partial_1 a_{m-t+1}\big)\\ &+\frac{1}{\mathcal{E}^1}\sum_{t=l}^{m}\big((l-1)\partial_2 a_{t-l+2}-(l-2)\partial_1 a_{t-l+1}\big)\big((t-1)\partial_2\mathcal{E}^{m-t+2}-(t-2)\partial_1\mathcal{E}^{m-t+1}\big)\end{aligned}$$

which, by means of suitable changes of variables (we want to collect terms of the form $\partial_2\mathcal{E}^h$ *and* $\partial_1\mathcal{E}^{h-1}$*), becomes*

$$-\partial_1\partial_l a_m+\partial_l\partial_1 a_m=(l-2)\frac{\partial_1\mathcal{E}^1}{(\mathcal{E}^1)^2}\sum_{t=2}^{s+1}\big(\mathcal{E}^t\,\partial_t a_{s+1}-a_t\,\partial_t\mathcal{E}^{s+1}\big)$$

$$
\begin{aligned}
&+\frac{l-1}{\mathcal{E}^1}\sum_{h=2}^{s+2}\bigg(\partial_2\mathcal{E}^h\left((h-1)\partial_2 a_{s-h+4}-(h-2)\partial_1 a_{s-h+3}\right)\bigg)\\
&-\frac{l-1}{\mathcal{E}^1}\sum_{h=2}^{s+2}\bigg(\partial_2 a_{s-h+4}\left((s-h+3)\partial_2\mathcal{E}^h-(s-h+2)\partial_1\mathcal{E}^{h-1}\right)\bigg)\\
&-\frac{l-2}{\mathcal{E}^1}\sum_{h=3}^{s+2}\bigg(\partial_1\mathcal{E}^{h-1}\left((h-2)\partial_2 a_{s-h+4}-(h-3)\partial_1 a_{s-h+3}\right)\bigg)\\
&+\frac{l-2}{\mathcal{E}^1}\sum_{h=2}^{s+1}\bigg(\partial_1 a_{s-h+3}\left((s-h+2)\partial_2\mathcal{E}^h-(s-h+1)\partial_1\mathcal{E}^{h-1}\right)\bigg)\\
&-\frac{1}{\mathcal{E}^1}\sum_{h=2}^{s+2}\big((l-1)\partial_2\mathcal{E}^h-(l-2)\partial_1\mathcal{E}^{h-1}\big)\big((h+l-3)\partial_2 a_{s-h+4}-(h+l-4)\partial_1 a_{s-h+3}\big)\\
&+\frac{1}{\mathcal{E}^1}\sum_{h=2}^{s+2}\big((l-1)\partial_2 a_{s-h+4}-(l-2)\partial_1 a_{s-h+3}\big)\big((m-h+1)\partial_2\mathcal{E}^h-(m-h)\partial_1\mathcal{E}^{h-1}\big)
\end{aligned}
$$

where the fifth sum (originally over $h \le s+1$) can be extended to $h \le s+2$ (as the contribute is null for $h = s+2$) and where the fourth sum (originally starting from $h = 3$) can be made into starting from $h = 2$, up to properly add the corresponding term

$$
\frac{l-2}{\mathcal{E}^1}\,\partial_1\mathcal{E}^1\left(\partial_1 a_{s+1}\right)=-\frac{l-2}{(\mathcal{E}^1)^2}\,\partial_1\mathcal{E}^1\sum_{t=2}^{s+1}\left(\mathcal{E}^t\,\partial_t a_{s+1}-a_t\,\partial_t\mathcal{E}^{s+1}\right)
$$

which cancels out with the first sum. This yields

$$
\begin{aligned}
&-\partial_1\partial_l a_m+\partial_l\partial_1 a_m\\
&=\frac{l-1}{\mathcal{E}^1}\sum_{h=2}^{s+2}\bigg(\partial_2\mathcal{E}^h\left((h-1)\partial_2 a_{s-h+4}-(h-2)\partial_1 a_{s-h+3}\right)\bigg)\\
&-\frac{l-1}{\mathcal{E}^1}\sum_{h=2}^{s+2}\bigg(\partial_2 a_{s-h+4}\left((s-h+3)\partial_2\mathcal{E}^h-(s-h+2)\partial_1\mathcal{E}^{h-1}\right)\bigg)\\
&-\frac{l-2}{\mathcal{E}^1}\sum_{h=2}^{s+2}\bigg(\partial_1\mathcal{E}^{h-1}\left((h-2)\partial_2 a_{s-h+4}-(h-3)\partial_1 a_{s-h+3}\right)\bigg)\\
&+\frac{l-2}{\mathcal{E}^1}\sum_{h=2}^{s+2}\bigg(\partial_1 a_{s-h+3}\left((s-h+2)\partial_2\mathcal{E}^h-(s-h+1)\partial_1\mathcal{E}^{h-1}\right)\bigg)\\
&-\frac{1}{\mathcal{E}^1}\sum_{h=2}^{s+2}\big((l-1)\partial_2\mathcal{E}^h-(l-2)\partial_1\mathcal{E}^{h-1}\big)\big((h+l-3)\partial_2 a_{s-h+4}-(h+l-4)\partial_1 a_{s-h+3}\big)\\
&+\frac{1}{\mathcal{E}^1}\sum_{h=2}^{s+2}\big((l-1)\partial_2 a_{s-h+4}-(l-2)\partial_1 a_{s-h+3}\big)\big((m-h+1)\partial_2\mathcal{E}^h-(m-h)\partial_1\mathcal{E}^{h-1}\big).
\end{aligned}
$$

By collecting $\partial_2\mathcal{E}^h$ and $\partial_1\mathcal{E}^{h-1}$ for $h \in \{2,\dots,s+2\}$ we get $-\partial_1\partial_l a_m + \partial_l\partial_1 a_m = 0$. Therefore we proved $\partial_1\partial_l a_m = \partial_l\partial_1 a_m$ for each $l \in \{3,\dots,m\}$.

In order to prove that the same relation holds for $l = 2$ as well, we consider the following result.

For each $k \in \{3,\dots,n\}$, let us denote by $\hat{\mathcal{P}}_k$ the set of polynomials in the variables $u^3,\dots,u^k$ with coefficients being polynomials in the derivatives (up to some positive integer order) of a_2 with respect to u^1, u^2.

Lemma 2.17 *For each $m \in \{3,\dots,n\}$ and $k \in \{3,\dots,m\}$ the function a_m can be written as*

$$a_m(u^1,\dots,u^m) = P_m^{(k)}(u^1,\dots,u^m) + C_m^{(m-k+2)}(u^1,\dots,u^{m-k+2}) \tag{2.23}$$

for some $P_m^{(k)} \in \hat{\mathcal{P}}_m$ and some function $C_m^{(m-k+2)}$ of $u^1,\dots,u^{m-k+2}$.

Proof: *For $m = 3$, by* (2.14) *we have $a_3 = 2\,\partial_2 a_2\, u^3$, immediately proving* (2.23) *for*

$$P_3^{(3)}(u^1,u^2,u^3) = 2\,\partial_2 a_2\, u^3$$

and some function $C_3^{(2)}$ of u^1,u^2. Let us fix $m \in \{4,\dots,n\}$ and assume that for each $h \in \{3,\dots,m-1\}$ and $k \in \{3,\dots,h\}$ there exist some $P_h^{(k)} \in \hat{\mathcal{P}}_h$ and some function $C_h^{(h-k+2)}$ of $u^1,\dots,u^{h-k+2}$ such that

$$a_h(u^1,\dots,u^h) = P_h^{(k)}(u^1,\dots,u^h) + C_h^{(h-k+2)}(u^1,\dots,u^{h-k+2}). \tag{2.24}$$

We have to show (2.23) *for each $k \in \{3,\dots,m\}$. We proceed by induction over k, starting from $k = 3$. By* (2.14) *we have*

$$\partial_m a_m = (m-1)\,\partial_2 a_2$$

thus

$$a_m = (m-1)\,\partial_2 a_2\, u^m + A_m(u^1,\dots,u^{m-1})$$

for some function A_m of $u^1,\dots,u^{m-1}$. This proves (2.23) *for $k = 3$, with*

$$P_m^{(3)}(u^1,\dots,u^m) = (m-1)\,\partial_2 a_2\, u^m$$

and

$$C_m^{(m-1)}(u^1,\dots,u^{m-1}) = A_m(u^1,\dots,u^{m-1}).$$

Let us fix $k \in \{4,\dots,m\}$ and assume that for each $h \in \{3,\dots,k-1\}$ there exist some $P_m^{(h)} \in \hat{\mathcal{P}}_m$ and some function $C_m^{(m-h+2)}$ of $u^1,\dots,u^{m-h+2}$ such that

$$a_m(u^1,\dots,u^m) = P_m^{(h)}(u^1,\dots,u^m) + C_m^{(m-h+2)}(u^1,\dots,u^{m-h+2}). \tag{2.25}$$

By choosing $h = k - 1$ *in* (2.25) *we get*

$$a_m(u^1, \dots, u^m) = P_m^{(k-1)}(u^1, \dots, u^m) + C_m^{(m-k+3)}(u^1, \dots, u^{m-k+3}) \tag{2.26}$$

for some $P_m^{(k-1)} \in \hat{\mathcal{P}}_m$ *and some function* $C_m^{(m-k+3)}$ *of* $u^1, \dots, u^{m-k+3}$. *By combining* (2.26) *with* (2.14), *we have*

$$\begin{aligned}\partial_{m-k+3} C_m^{(m-k+3)} &= \partial_{m-k+3} a_m - \partial_{m-k+3} P_m^{(k-1)} \\ &= (m-k+2)\partial_2 a_{k-1} - (m-k+1)\partial_1 a_{k-2} - \partial_{m-k+3} P_m^{(k-1)} \in \hat{\mathcal{P}}_m\end{aligned}$$

as $a_{k-1}, a_{k-2}, P_m^{(k-1)} \in \hat{\mathcal{P}}_m$ *by inductive assumption. Then*

$$C_m^{(m-k+3)}(u^1, \dots, u^{m-k+3}) = Q_m^{(m-k+3)}(u^1, \dots, u^m) + B_m^{(m-k+2)}(u^1, \dots, u^{m-k+2})$$

for some $Q_m^{(m-k+3)} \in \hat{\mathcal{P}}_m$ *and some function* $B_m^{(m-k+2)}$ *of* $u^1, \dots, u^{m-k+2}$. *Therefore we get*

$$a_m(u^1, \dots, u^m) = P_m^{(k)}(u^1, \dots, u^m) + C_m^{(m-k+2)}(u^1, \dots, u^{m-k+2})$$

for

$$P_m^{(k)}(u^1, \dots, u^m) = P_m^{(k-1)}(u^1, \dots, u^m) + Q_m^{(m-k+3)}(u^1, \dots, u^m) \in \hat{\mathcal{P}}_m$$

and

$$C_m^{(m-k+2)}(u^1, \dots, u^{m-k+2}) = B_m^{(m-k+2)}(u^1, \dots, u^{m-k+2})$$

proving (2.23). ■

By choosing $k = m$ *in* (2.23), *for each* $m \in \{3, \dots, n\}$ *we get*

$$a_m(u^1, \dots, u^m) = P_m^{(m)}(u^1, \dots, u^m) + C_m^{(2)}(u^1, u^2). \tag{2.27}$$

Up to an additive function of u^1, u^2, *then* $a_m \in \hat{\mathcal{P}}_m$ *and, as a consequence,*

$$\partial_1 \partial_2 a_m = \partial_2 \partial_1 a_m$$

for each $m \in \{3, \dots, n\}$.

Below we present some examples of how, given an eventual identity $\mathcal{E}$ whose components do not include additive functions of u^1, u^2, a second eventual identity v_2 commuting with $\mathcal{E}$ can be found. In canonical coordinates v_2 will be written as

$$v_2 = \sum_{i=2}^{n} a_i(u^1, \dots, u^i)\, \partial_i$$

for some functions $a_2, \dots, a_n$. The function a_2 is determined as a solution to (2.20) and higher components $a_3, \dots, a_n$ are uniquely constructed by means of (2.14). In particular, being v_2 an eventual identity, up to additive functions of u^1, u^2, its first components read

$$\begin{aligned}
a_2 &= a_2(u^1, u^2)\\
a_3 &= 2(\partial_2 a_2)u^3\\
a_4 &= 3(\partial_2 a_2)u^4 + 2(\partial_2^2 a_2)(u^3)^2 - (\partial_1 a_2)u^3\\
a_5 &= 4(\partial_2 a_2)u^5 - 2(\partial_1 a_2)u^4 + 6(\partial_2^2 a_2)u^3u^4 + \frac{4}{3}(\partial_2^3 a_2)(u^3)^3 - 2\big(\partial_1\partial_2 a_2\big)(u^3)^2.
\end{aligned}$$

Example 2.18 *When $\mathcal{E}$ is the Euler vector field, namely its components are $\mathcal{E}^i(u^i) = u^i$ for each $i \in \{1, \dots, n\}$, a solution to* (2.20) *is*

$$a_2 = u^1\,\varphi\left(\frac{u^2}{u^1}\right)$$

for some function φ of the ratio $\xi = \frac{u^2}{u^1}$. Up to additive functions of u^1, u^2, the first higher components of v_2 are given by

$$\begin{aligned}
a_3 &= 2\,u^3\,\varphi'(\xi)\\
a_4 &= 3\,u^4\,\varphi'(\xi) + 2\,\frac{(u^3)^2}{u^1}\,\varphi''(\xi) + \frac{u^2u^3}{u^1}\,\varphi'(\xi) - u^3\,\varphi(\xi).
\end{aligned}$$

The following examples live in dimension $n = 4$.

Example 2.19 *Let us set*

$$\begin{aligned}
\mathcal{E}^1 &= u^1\\
\mathcal{E}^2 &= u^1\,u^2
\end{aligned}$$

so that higher components of the eventual identity $\mathcal{E}$ read

$$\begin{aligned}
\mathcal{E}^3 &= \big(2\,u^1 - 1\big)u^3\\
\mathcal{E}^4 &= \big(3\,u^1 - 2\big)u^4 - u^2u^3.
\end{aligned}$$

A solution to (2.20) *is*

$$a_2 = e^{u^1}\,\varphi\big(u^2\,e^{-u^1}\big)$$

for some function φ of $\xi = u^2\,e^{-u^1}$. Up to additive functions of u^1, u^2, the first higher components of v_2 are given by

$$\begin{aligned}
a_3 &= 2\,u^3\,\varphi'(\xi)\\
a_4 &= (3\,u^4 + u^2u^3)\varphi'(\xi) + 2\,(u^3)^2e^{-u^1}\,\varphi''(\xi) - u^3\,\varphi(\xi)\,e^{u^1}.
\end{aligned}$$

Example 2.20 *Let us set*

$$\begin{aligned}\mathcal{E}^1 &= \sin(u^1)\\ \mathcal{E}^2 &= u^2\end{aligned}$$

so that the higher components of the eventual identity $\mathcal{E}$ read

$$\begin{aligned}\mathcal{E}^3 =&\big(2-\cos(u^1)\big)u^3\\ \mathcal{E}^4 =&\big(3-2\,\cos(u^1)\big)u^4.\end{aligned}$$

A solution to (2.20) *is*

$$a_2 =\big(\csc(u^1)-\cot(u^1)\big)\,\varphi\bigg(\frac{u^2}{\csc(u^1)-\cot(u^1)}\bigg)$$

for some function φ of

$$\xi = \frac{u^2}{\csc(u^1)-\cot(u^1)} = \frac{u^2\,\sin(u^1)}{1-\cos(u^1)}.$$

Up to additive functions of u^1, u^2, the higher components of v_2 are given by

$$\begin{aligned}a_3 =&2\,u^3\,\varphi'(\xi)\\ a_4 =&\frac{1}{\sin(u^1)^2}\Bigg(\big(u^2u^3\,\sin(u^1)+3(\sin(u^1)^2)u^4\big)\varphi'(\xi)\\ &+2\big((u^3)^2\sin(u^1)\,(1+\cos(u^1))\big)\varphi''(\xi)+\big(\cos(u^1)-1\big)u^3\,\varphi(\xi)\Bigg).\end{aligned}$$

Proposition 2.21 *For each $j \geq 3$ we have $[v_1, v_j] = 0$.*

Proof: Let us fix $j \geq 3$. We have

$$\begin{aligned}[v_1, v_j] &= \mathcal{L}_{\mathcal{E}} v_j = \mathcal{L}_{\mathcal{E}}\big(\mathcal{E}^{-1}\circ v_2 \circ v_{j-1}\big)\\ &= (\mathcal{L}_{\mathcal{E}}\circ)(\mathcal{E}^{-1}, v_2\circ v_{j-1}) + [\mathcal{E},\mathcal{E}^{-1}]\circ v_2\circ v_{j-1}\\ &+\mathcal{E}^{-1}\circ\Bigg((\mathcal{L}_{\mathcal{E}}\circ)(v_2, v_{j-1}) + [\mathcal{E}, v_2]\circ v_{j-1} + v_2\circ[\mathcal{E}, v_{j-1}]\Bigg)\end{aligned}$$

where $[\mathcal{E}, v_2] = [v_1, v_2] = 0$ by assumption and $[\mathcal{E}, v_{j-1}] = [v_1, v_{j-1}] = 0$ by induction. Since $\mathcal{E}$ is an eventual identity, we also have

$$(\mathcal{L}_{\mathcal{E}}\circ)(\mathcal{E}^{-1}, v_2\circ v_{j-1}) = [e,\mathcal{E}]\circ\mathcal{E}^{-1}\circ v_2\circ v_{j-1}$$

and

$$(\mathcal{L}_{\mathcal{E}}\circ)(v_2, v_{j-1}) = [e,\mathcal{E}]\circ v_2\circ v_{j-1}.$$

We get

$$[v_1, v_j] = 2[e, \mathcal{E}] \circ \mathcal{E}^{-1} \circ v_2 \circ v_{j-1} + [\mathcal{E}, \mathcal{E}^{-1}] \circ v_2 \circ v_{j-1}$$

where

$$[\mathcal{E}, \mathcal{E}^{-1}] = -2\mathcal{E}^{-1} \circ [e, \mathcal{E}] \tag{2.28}$$

as shown in [20]. This proves $[v_1, v_j] = 0$. ■

Proposition 2.22 *For each $i \in \{1, \dots, n\}$ we have*

$$[v_i, \mathcal{E}^{-1}] = -2\mathcal{E}^{-1} \circ [e, v_i]. \tag{2.29}$$

Proof: For $i = 1$ this amounts to (2.28). Let us fix $i \geq 2$. The quantity $\mathcal{L}_{v_i} e$ can either be written as

$$\begin{aligned}\mathcal{L}_{v_i} e = \mathcal{L}_{v_i}(\mathcal{E} \circ \mathcal{E}^{-1}) &= (\mathcal{L}_{v_i} \circ)(\mathcal{E}, \mathcal{E}^{-1}) + [v_i, \mathcal{E}] \circ \mathcal{E}^{-1} + \mathcal{E} \circ [v_i, \mathcal{E}^{-1}] \\ &= [e, v_i] \circ \mathcal{E} \circ \mathcal{E}^{-1} + [v_i, \mathcal{E}] \circ \mathcal{E}^{-1} + \mathcal{E} \circ [v_i, \mathcal{E}^{-1}]\end{aligned}$$

or as

$$\mathcal{L}_{v_i} e = [v_i, e] = -[e, v_i].$$

We get

$$\mathcal{E} \circ [v_i, \mathcal{E}^{-1}] = -2[e, v_i] - [v_i, \mathcal{E}] \circ \mathcal{E}^{-1}$$

where $[v_i, \mathcal{E}] = [v_i, v_1] = 0$, which proves (2.29). ■

Proposition 2.23 *For each $j \in \{2, \dots, n\}$ we have*

$$[v_2, v_j] = 0. \tag{2.30}$$

Proof: We proceed by induction over j, starting from the trivial case $j = 2$. Let us fix $j \geq 3$. We have

$$\begin{aligned}[v_2, v_j] = \mathcal{L}_{v_2} v_j &= \mathcal{L}_{v_2}(\mathcal{E}^{-1} \circ v_2 \circ v_{j-1}) \\ &= (\mathcal{L}_{v_2} \circ)(\mathcal{E}^{-1}, v_2 \circ v_{j-1}) + [v_2, \mathcal{E}^{-1}] \circ v_2 \circ v_{j-1} \\ &\quad + \mathcal{E}^{-1} \circ \big((\mathcal{L}_{v_2} \circ)(v_2, v_{j-1}) + [v_2, v_2] \circ v_{j-1} + v_2 \circ [v_2, v_{j-1}]\big)\end{aligned}$$

where $[v_2, v_2] = 0$ trivially and $[v_2, v_{j-1}] = 0$ by induction. Since v_2 is assumed to be an eventual identity, we get

$$\begin{aligned}[v_2, v_j] &= [e, v_2] \circ \mathcal{E}^{-1} \circ v_2 \circ v_{j-1} + [v_2, \mathcal{E}^{-1}] \circ v_2 \circ v_{j-1} + \mathcal{E}^{-1} \circ [e, v_2] \circ v_2 \circ v_{j-1} \\ &= 2[e, v_2] \circ \mathcal{E}^{-1} \circ v_2 \circ v_{j-1} + [v_2, \mathcal{E}^{-1}] \circ v_2 \circ v_{j-1} \overset{(2.29)}{=} 0.\end{aligned}$$

■

Proposition 2.24 *For each $i, j \in \{2, \dots, n\}$ we have*

$$[v_i, v_j] = 0. \tag{2.31}$$

Proof: Let us fix $i \geq 2$. We proceed by induction over j, starting from the case $j = 2$ which holds true by (2.30). Let us then consider $j \geq 3$. We have

$$\begin{aligned}[v_i, v_j] &= \mathcal{L}_{v_i} v_j = \mathcal{L}_{v_i}(\mathcal{E}^{-1} \circ v_2 \circ v_{j-1}) \\ &= (\mathcal{L}_{v_i}\circ)(\mathcal{E}^{-1}, v_2 \circ v_{j-1}) + [v_i, \mathcal{E}^{-1}] \circ v_2 \circ v_{j-1} \\ &+ \mathcal{E}^{-1} \circ \big((\mathcal{L}_{v_i}\circ)(v_2, v_{j-1}) + [v_i, v_2] \circ v_{j-1} + v_2 \circ [v_i, v_{j-1}]\big)\end{aligned}$$

where $[v_i, v_2] = 0$ by (2.30), $[v_i, v_{j-1}] = 0$ by induction and $[v_i, \mathcal{E}^{-1}] = -2\mathcal{E}^{-1} \circ [e, v_i]$ by (2.29). Since v_i is an eventual identity, we get

$$\begin{aligned}[v_i, v_j] = [e, v_i] \circ \mathcal{E}^{-1} \circ v_2 \circ v_{j-1} - 2\mathcal{E}^{-1} \circ [e, v_i] \circ v_2 \circ v_{j-1} \\ + \mathcal{E}^{-1} \circ [e, v_i] \circ v_2 \circ v_{j-1} = 0.\end{aligned}$$

■

We have then proved the following.

Theorem 2.2 *The vector fields $\{v_i\}_{i \in \{1,\dots,n\}}$ pairwise commute.*

Corollary 2.25 *There exist coordinates $w^1, \dots, w^n$ such that*

$$v_i = \frac{\partial}{\partial w^i}, \qquad i \in \{1, \dots, n\}.$$

2.1.2 The case of multiple Jordan blocks

Let us now consider the more general case where $r \geq 1$. Condition (1.43) reads

$$\begin{aligned}&- \delta_{\beta\gamma}\partial_{(j+k-1)(\beta)}\mathcal{E}^{i(\alpha)} + \delta^{\alpha}_{\gamma}\partial_{j(\beta)}\mathcal{E}^{(i-k+1)(\alpha)} + \delta^{\alpha}_{\beta}\partial_{k(\gamma)}\mathcal{E}^{(i-j+1)(\alpha)} \\ &= \delta^{\alpha}_{\beta}\delta^{\alpha}_{\gamma}\sum_{\sigma=1}^{r}\partial_{1(\sigma)}\mathcal{E}^{(i-j-k+2)(\alpha)}\end{aligned} \tag{2.32}$$

for each choice of $\alpha, \beta, \gamma \in \{1, \dots, r\}$ and for each $i \in \{1, \dots, m_\alpha\}$, $j \in \{1, \dots, m_\beta\}$, $k \in \{1, \dots, m_\gamma\}$. If $\beta = \gamma$ in (2.32) we get

$$\begin{aligned}&- \partial_{(j+k-1)(\beta)}\mathcal{E}^{i(\alpha)} + \delta^{\alpha}_{\beta}\partial_{j(\beta)}\mathcal{E}^{(i-k+1)(\alpha)} + \delta^{\alpha}_{\beta}\partial_{k(\beta)}\mathcal{E}^{(i-j+1)(\alpha)} \\ &= \delta^{\alpha}_{\beta}\sum_{\sigma=1}^{r}\partial_{1(\sigma)}\mathcal{E}^{(i-j-k+2)(\alpha)}\end{aligned}$$

which yields

$$\partial_{(j+k-1)(\beta)}\mathcal{E}^{i(\alpha)} = 0$$

for $\alpha \neq \beta$ (implying that $\mathcal{E}^{i(\alpha)}$ only depends on $u^{1(\alpha)}, \dots, u^{m_\alpha(\alpha)}$) and

$$-\partial_{(j+k-1)(\alpha)}\mathcal{E}^{i(\alpha)} + \partial_{j(\alpha)}\mathcal{E}^{(i-k+1)(\alpha)} + \partial_{k(\alpha)}\mathcal{E}^{(i-j+1)(\alpha)} = \partial_{1(\alpha)}\mathcal{E}^{(i-j-k+2)(\alpha)} \tag{2.33}$$

for $\alpha = \beta$. If $\beta \neq \gamma$ in (2.32) we get

$$\delta^\alpha_\gamma \partial_{j(\beta)}\mathcal{E}^{(i-k+1)(\alpha)} + \delta^\alpha_\beta \partial_{k(\gamma)}\mathcal{E}^{(i-j+1)(\alpha)} = 0$$

which trivially holds. An eventual identity must then be of the form

$$\mathcal{E} = \sum_{\alpha=1}^{r}\sum_{i=1}^{m_\alpha}\mathcal{E}^{i(\alpha)}(u^{1(\alpha)}, \dots, u^{m_\alpha(\alpha)})\frac{\partial}{\partial u^{i(\alpha)}}$$

where the functions $\mathcal{E}^{i(\alpha)}(u^{1(\alpha)}, \dots, u^{m_\alpha(\alpha)})$ are solutions to (2.33). Since for every $\alpha \in \{1, \dots, r\}$ condition (2.33) is analogous to (2.2) and the functions $\{\mathcal{E}^{i(\alpha)}\}_{i\in\{1,\dots,m_\alpha\}}$ only depend on the coordinates $u^1, \dots, u^{m_\alpha}$, the results about eventual identites from the previous section naturally extend to the general case where $r \geq 1$.

Theorem 2.26 *Given a regular F-manifold with (generalized) canonical coordinates* $\{u^{1(\alpha)}, \dots, u^{m_\alpha(\alpha)}\}_{\alpha\in\{1,\dots,r\}}$ *where the structure constants of the product and the components of the unit vector field respectively read* $c^{i(\alpha)}_{j(\beta)k(\gamma)} = \delta^\alpha_\beta\delta^\alpha_\gamma\delta^i_{j+k-1}$ *and* $e^{i(\alpha)} = \delta^i_1$*, an eventual identity must be of the form*

$$\mathcal{E} = \sum_{\alpha=1}^{r}\sum_{i=1}^{m_\alpha}\mathcal{E}^{i(\alpha)}(u^{1(\alpha)}, \dots, u^{m_\alpha(\alpha)})\frac{\partial}{\partial u^{i(\alpha)}}$$

where for each $\alpha \in \{1, \dots, r\}$ *the functions* $\{\mathcal{E}^{i(\alpha)}\}_{i\in\{1,\dots,m_\alpha\}}$ *are solutions to*

$$\partial_{l(\alpha)}\mathcal{E}^{m(\alpha)} = \begin{cases} (l-1)\,\partial_{2(\alpha)}\mathcal{E}^{(m-l+2)(\alpha)} - (l-2)\,\partial_{1(\alpha)}\mathcal{E}^{(m-l+1)(\alpha)} & \text{for } l \leq m, \\ 0 & \text{for } l > m, \end{cases} \tag{2.34}$$

for each $m \in \{1, \dots, m_\alpha\}$.

Remark 23 *Condition* (2.34) *gives a compatible system of PDEs, namely*

$$\partial_{i(\alpha)}\partial_{j(\beta)}\mathcal{E}^{m(\gamma)} = \partial_{j(\beta)}\partial_{i(\alpha)}\mathcal{E}^{m(\gamma)}, \quad i \in \{1, \dots, m_\alpha\}, j \in \{1, \dots, m_\beta\}, \alpha, \beta \in \{1, \dots, r\}, \tag{2.35}$$

for each $\gamma \in \{1, \dots, r\}$ *and* $m \in \{1, \dots, m_\gamma\}$*. In fact, both the sides of such condition trivially vanish as soon as at least two of the indices* α, β, γ *are different (as each of the functions* $\{\mathcal{E}^{i(\alpha)}\}_{i\in\{1,\dots,m_\alpha\},\,\alpha\in\{1,\dots,r\}}$ *only depends on the coordinates associated to the corresponding block). The only non-trivial case is then recovered when all of the greek indices coincide, a case in which the proof of* (2.5) *can be adapted to prove* (2.35).

Proposition 2.27 *Let us fix* $\alpha \in \{1, \dots, r\}$*. For each* $m \in \{1, \dots, m_\alpha\}$ *the* $m(\alpha)$*-th component of an eventual identity* $\mathcal{E}$ *is a polynomial function in the variables* $\{u^{i(\alpha)}\}_{i\in\{3,\dots,m_\alpha\}}$ *with coefficients being functions of the coordinates* $u^{1(\alpha)}, u^{2(\alpha)}$*. In particular,* $\mathcal{E}^{1(\alpha)}, \dots, \mathcal{E}^{m(\alpha)}$ *only depend on a function* $f_{1(\alpha)}$ *of the coordinate* $u^{1(\alpha)}$ *and* $m-1$ *functions* $\{f_{i(\alpha)}\}_{i\in\{2,\dots,m_\alpha\}}$ *of the coordinates* $u^{1(\alpha)}, u^{2(\alpha)}$*.*

Proof: The argument for the case of a single Jordan block clearly extends to the more general case. ∎

Remark 24 *Proposition 2.27 implies that an eventual identity can be fully determined starting from* r *functions* $\left\{f_{1(\alpha)}(u^{1(\alpha)})\right\}_{\in\{1,\dots,r\}}$ *of one variable and* $n-r$ *functions* $\left\{f_{i(\alpha)}(u^{1(\alpha)}, u^{2(\alpha)})\right\}_{i\in\{2,\dots,m_\alpha\},\,\alpha\in\{1,\dots,r\}}$ *of two variables.*

Example 2.28 *The Euler vector field*

$$E = \sum_{i=1}^{n} u^i \frac{\partial}{\partial u^i}$$

is an eventual identity. In fact, fixed $\alpha \in \{1, \dots, r\}$ *(without loss of generality, we consider* α *such that* $m_\alpha \geq 3$*), given* $\mathcal{E}^{1(\alpha)}(u^{1(\alpha)}) = u^{1(\alpha)}$ *and* $\mathcal{E}^{2(\alpha)}(u^{2(\alpha)}) = u^{2(\alpha)}$*, condition* (2.34) *gives for each* $i \in \{3, \dots, m_\alpha\}$

$$\mathcal{E}^{i(\alpha)}(u^{1(\alpha)}, \dots, u^{i(\alpha)}) = u^{i(\alpha)} + f_{i(\alpha)}(u^{1(\alpha)}, \dots, u^{(i-1)(\alpha)})$$

for some function $f_{i(\alpha)}(u^{1(\alpha)}, \dots, u^{(i-1)(\alpha)})$*. Analogoulsy to the case of a single Jordan block, the function* $f_{i(\alpha)}$ *actually depends only on* $u^{1(\alpha)}, u^{2(\alpha)}$*. As expected, a first example of eventual identity is then provided by the Euler vector field.*

A less trivial example is provided as follows, which we suggest to compare with the examples discussed in the case of a single Jordan block.

Example 2.29 *Let us consider the case of dimension* $n = 7$ *with two Jordan blocks of sizes* 4 *and* 3 *respectively. We have* $r = 2$ *and*

$$u^{1(1)} = u^1,\ u^{2(1)} = u^2,\ u^{3(1)} = u^3,\ u^{4(1)} = u^4,$$
$$u^{1(2)} = u^5,\ u^{2(2)} = u^6,\ u^{3(2)} = u^7.$$

Let $\mathcal{E}$ *be an eventual identity with*

$$\mathcal{E}^1 = u^1$$
$$\mathcal{E}^2 = (u^1)^2\,(u^2)^3 + u^2$$

and

$$\mathcal{E}^5 = \sin(u^5)$$
$$\mathcal{E}^6 = u^5\,\cos(u^6) + u^6.$$

The remaining components of $\mathcal{E}$ must be of the form

$$\begin{aligned}\mathcal{E}^3 =& \big(6(u^1)^2(u^2)^2+1\big)u^3 + f_3(u^1,u^2)\\ \mathcal{E}^4 =& \big(9(u^1)^2(u^2)^2+1\big)u^4 + 12(u^1)^2\,u^2\,(u^3)^2\\ &+\big(2\partial_2 f_3(u^1,u^2) - 2u^1(u^2)^3\big)u^3 + f_4(u^1,u^2)\end{aligned}$$

for some functions f_3, f_4 of u^1, u^2 and

$$\mathcal{E}^7 = \big(2 - 2u^5\,\sin(u^6) - \cos(u^5)\big)u^7 + f_7(u^5,u^6)$$

for some function f_7 of u^5, u^6.

Theorem 2.3 *Let $\mathcal{E}$ be an eventual identity. Its inverse is of the form*

$$\mathcal{E}^{-1} = \sum_{\alpha=1}^{r}\sum_{i=1}^{m_\alpha}(\mathcal{E}^{-1})^{i(\alpha)}(u^{1(\alpha)},\ldots,u^{i(\alpha)})\frac{\partial}{\partial u^{i(\alpha)}}$$

where

$$(\mathcal{E}^{-1})^{1(\alpha)} = \frac{1}{\mathcal{E}^{1(\alpha)}}$$

and

$$(\mathcal{E}^{-1})^{(k+1)(\alpha)} = -\frac{1}{\mathcal{E}^{1(\alpha)}}\sum_{s=1}^{k}(\mathcal{E}^{-1})^{(k-s+1)(\alpha)}\mathcal{E}^{(s+1)(\alpha)}$$

for each $\alpha \in \{1,\ldots,r\}$ and $k \in \{1,\ldots,m_\alpha - 1\}$.

Remark 25 *The structure constants of the dual product*

$$X * Y = \mathcal{E}^{-1}\circ X \circ Y, \qquad X,Y \in \mathfrak{X}(M),$$

are given by

$$\tilde{c}{}^{i(\alpha)}_{j(\beta)k(\gamma)} = (\mathcal{E}^{-1})^{s(\sigma)}c^{i(\alpha)}_{s(\sigma)t(\tau)}c^{t(\tau)}_{j(\beta)k(\gamma)} = \delta^\alpha_\beta\delta^\alpha_\gamma(\mathcal{E}^{-1})^{(i-j-k+2)(\alpha)}$$

for all suitable indices. The dual product is expressed on the coordinate vector fields as

$$\partial_{i(\alpha)} * \partial_{j(\beta)} = \tilde{c}{}^{k(\gamma)}_{i(\alpha)j(\beta)}\partial_{k(\gamma)} = \delta_{\alpha\beta}\sum_{k=i+j-1}^{n}(\mathcal{E}^{-1})^{(k-i-j+2)(\alpha)}\partial_{k(\alpha)}$$

for all suitable indices. In particular, $\partial_{i(\alpha)} * \partial_{j(\beta)} = 0$ for $\alpha \neq \beta$ and $\partial_{i(\alpha)} * \partial_{j(\alpha)} = 0$ for $i + j \geq m_\alpha + 2$.

Analogously to the case of the previous section, we introduce vector fields $\{v_{1(\alpha)},\dots,v_{m_\alpha(\alpha)}\}_{\alpha\in\{1,\dots,r\}}$ such that

$$v_{i(\alpha)} * v_{j(\beta)} = \delta_{\alpha\beta}\, v_{(i+j-1)(\alpha)}\, \mathbb{1}_{\{i+j\leq m_\alpha+1\}}$$

for each $\alpha,\beta\in\{1,\dots,r\}$ and $i\in\{1,\dots,m_\alpha\}$, $j\in\{1,\dots,m_\beta\}$.

Theorem 2.4 *By setting*

- $v_{1(\alpha)} = \sum_{i=1}^{m_\alpha}\mathcal{E}^{i(\alpha)}\partial_{i(\alpha)}$
- $v_{2(\alpha)} = \sum_{\alpha=1}^{r}\sum_{i=2}^{m_\alpha} a_{i(\alpha)}\,\partial_{i(\alpha)}$ *for some functions* $a_{2(\alpha)}\neq 0, a_{3(\alpha)},\dots,a_{m_\alpha(\alpha)}$
- $v_{(i+1)(\alpha)} = \sum_{\alpha=1}^{r}\sum_{k=i+1}^{m_\alpha}(v_{(i+1)(\alpha)})^{k(\alpha)}\,\partial_{k(\alpha)}$ *for* $i\geq 2$ *with*

$$(v_{(i+1)(\alpha)})^{k(\alpha)} = \sum_{a=i}^{k-1}\sum_{b=2}^{k-a+1}(v_{i(\alpha)})^{a(\alpha)}(v_{2(\alpha)})^{b(\alpha)}(\mathcal{E}^{-1})^{(k-a-b+2)(\alpha)}, \qquad k\geq i+1,$$

for each $\alpha\in\{1,\dots,r\}$*, we have*

$$v_{i(\alpha)} * v_{j(\beta)} = \delta_{\alpha\beta}\, v_{(i+j-1)(\alpha)}\, \mathbb{1}_{\{i+j\leq m_\alpha+1\}}$$

for each $\alpha,\beta\in\{1,\dots,r\}$ *and* $i\in\{1,\dots,m_\alpha\}$, $j\in\{1,\dots,m_\beta\}$.

Proof: The proof for the case $r=1$ can be adapted to the general case where $r\geq 1$, as for each $\alpha\in\{1,\dots,r\}$ the functions $\{\mathcal{E}^{i(\alpha)}\}_{i\in\{1,\dots,m_\alpha\}}$ and $\{a_{i(\alpha)}\}_{i\in\{2,\dots,m_\alpha\}}$ only depend on the coordinates $u^{1(\alpha)},\dots,u^{m_\alpha(\alpha)}$. ■

In particular, we assume $v_{2(\alpha)}$ to be solution to (2.34) for each $\alpha\in\{1,\dots,r\}$. In other words, we are assuming

$$\sum_{\alpha=1}^{r}\sum_{i=1}^{m_\alpha} a_{i(\alpha)}\frac{\partial}{\partial u^{i(\alpha)}}$$

to be an eventual identity.

Remark 26 *Analogous formulas to* (2.15) *and* (2.16) *apply here as well. Precisely, for each* $\alpha\in\{1,\dots,r\}$ *we have*

$$v_{j(\alpha)}^{J(\alpha)} = -\frac{1}{\mathcal{E}^{1(\alpha)}}\sum_{s=1}^{J-j} v_{j(\alpha)}^{(J-s)(\alpha)}\mathcal{E}^{(s+1)(\alpha)} + \frac{1}{\mathcal{E}^{1(\alpha)}}\sum_{s=1}^{J-1} v_{(j-1)(\alpha)}^{a(\alpha)} v_{2(\alpha)}^{(J-a+1)(\alpha)} \tag{2.36}$$

for each $j \geq 3$ and $J \geq j$ and

$$v_{j(\alpha)}^{J(\alpha)} = \frac{a_{2(\alpha)}}{\mathcal{E}^{1(\alpha)}} v_{(j-1)(\alpha)}^{(J-1)(\alpha)} - \frac{1}{\mathcal{E}^{1(\alpha)}} \sum_{s=1}^{J-j} \left(v_{j(\alpha)}^{(J-s)(\alpha)} \mathcal{E}^{(s+1)(\alpha)} - v_{(j-1)(\alpha)}^{(J-s-1)(\alpha)} a_{(s+2)(\alpha)} \right) \tag{2.37}$$

for each $j \geq 3$ and $J \geq j$. In particular, for each $j \geq 3$ we have

$$v_{j(\alpha)}^{j(\alpha)} = \frac{a_{2(\alpha)}}{\mathcal{E}^{1(\alpha)}} v_{(j-1)(\alpha)}^{(j-1)(\alpha)} = \frac{\left(a_{2(\alpha)}\right)^{(j-1)(\alpha)}}{\left(\mathcal{E}^{1(\alpha)}\right)^{(j-2)(\alpha)}} \neq 0. \tag{2.38}$$

The vector fields $v_1, \ldots, v_n$ are then linearly independent.

The same considerations about the independence on coordinates which correspont to different blocks lead to the following.

Proposition 2.30 *For each $\alpha \in \{1, \ldots, r\}$ the vector fields $\{v_{i(\alpha)}\}_{i \geq 3}$ are solutions to* (2.34)*, namely*

$$\sum_{\alpha=1}^{r} \sum_{I=i}^{m_\alpha} v_{i(\alpha)}^{I(\alpha)} \frac{\partial}{\partial u^{I(\alpha)}}$$

is an eventual identity for each $i \geq 3$. In particular, for each $j \in \{1, \ldots, m_\alpha\}$ and $J \geq j$ the function $v_{j(\alpha)}^{J(\alpha)}$ only depends on the coordinates $u^{1(\alpha)}, \ldots, u^{(J-j+2)(\alpha)}$.

Let us assume $[v_{1(\alpha)}, v_{2(\alpha)}] = 0$ for each $\alpha \in \{1, \ldots, r\}$, that is for $m \in \{2, \ldots, m_\alpha\}$

$$\partial_1 a_{m(\alpha)} = -\frac{1}{\mathcal{E}^{1(\alpha)}} \sum_{l=2}^{m} \left(\mathcal{E}^{l(\alpha)} \, \partial_{l(\alpha)} a_{m(\alpha)} - a_{l(\alpha)} \, \partial_{l(\alpha)} \mathcal{E}^{m(\alpha)} \right). \tag{2.39}$$

Analogously to the case of a single Jordan block, the request for v_2 to both be an eventual identity and to commute with $v_1 = \mathcal{E}$ gives rise to a compatible system of PDEs for the components of v_2.

Theorem 2.5 *The vector fields $\{v_{i(\alpha)}\}_{i \in \{1, \ldots, m_\alpha\}, \, \alpha \in \{1, \ldots, r\}}$ pairwise commute.*

Proof: For each $\alpha \in \{1, \ldots, r\}$, the results about the commutativity of the vector fields $\{v_{i(\alpha)}\}_{i \in \{1, \ldots, m_\alpha\}}$ extend as well, thus the vector fields $\{v_{i(\alpha)}\}_{i \in \{1, \ldots, m_\alpha\}}$ pairwise commute. Since $[v_{i(\alpha)}, v_{j(\beta)}]$ trivially vanishes for $\alpha \neq \beta$, the result is proved. ■

Corollary 2.31 *There exist coordinates $\{w^{1(\alpha)}, \ldots, w^{m_\alpha(\alpha)}\}_{\alpha \in \{1, \ldots, r\}}$ such that*

$$v_{i(\alpha)} = \frac{\partial}{\partial w^{i(\alpha)}}$$

for each $\alpha \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, m_\alpha\}$.

2.1.3 Applications to Nijenhuis geometry

Among the directions toward which the results of this chapter may be extended, one way concerns Nijenhuis operators (we refer to Chapter 4 for further details), characterized by the property of having vanishing Nijenhuis torsion (see (4.17)). Other recent studies in Nijenhuis geometry were conducted in [3, 9].

It was proved in [20] that for any F-manifold with eventual identity $\mathcal{E}$ the endomorphism $\mathcal{E}\circ$ is an Nijenhuis operator. In the semisimple case this is just a diagonal endomorphism, but for the eventual identities constructed in this chapter, the Nijenhuis operator takes the block-diagonal form where each block assumes the form

$$\mathcal{L}_\alpha = \begin{bmatrix} \mathcal{E}^{1(\alpha)} & 0 & 0 & \dots & 0 & 0 \\ \mathcal{E}^{2(\alpha)} & \mathcal{E}^{1(\alpha)} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{E}^{(m_\alpha-1)(\alpha)} & \mathcal{E}^{(m_\alpha-2)(\alpha)} & \mathcal{E}^{(m_\alpha-3)(\alpha)} & \dots & \mathcal{E}^{1(\alpha)} & 0 \\ \mathcal{E}^{m_\alpha(\alpha)} & \mathcal{E}^{(m_\alpha-1)(\alpha)} & \mathcal{E}^{(m_\alpha-2)(\alpha)} & \dots & \mathcal{E}^{2(\alpha)} & \mathcal{E}^{1(\alpha)} \end{bmatrix}.$$

Chapter 3

Regular Frobenius manifolds

This chapter is dedicated to regular non-semisimple Frobenius manifolds and it is based on [65]. We recover formulas for a generic dimension, before focusing on dimensions 2, 3 and 4. We give a complete classification in the case corresponding to a single Jordan block in the Jordan canonical form of the operator of multiplication by the Euler vector field. In the cases associated with multiple Jordan blocks, we reduce the classification problem to systems of partial differential equations: a third-order ODE in the three-dimensional case and to a system of third-order PDEs in the four-dimensional cases. In all of the cases, we provide explicit examples of Frobenius potentials.

Further details about the four-dimensional regular non-semisimple cases corresponding to a Jordan canonical form of the operator of multiplication by the Euler vector field having at least one Jordan block of size 2 are contained in Appendix A.

Since given a Frobenius manifold $(M, \eta, \circ, e, E)$ the contravariant metrics η^{-1} and $L\eta^{-1}$, with $L = E\circ$, define a flat pencil of metrics, a by-product of our results is a list of non-semisisimple flat pencils of metrics that define the bi-Hamiltonian structures of the principal hierarchies of the associated Frobenius manifolds. The study of this class of bi-Hamiltonian structures and of their bi-Hamiltonian deformations in the non-semisimple case is at a preliminary stage. One of the few available results is [23].

3.1 Generic dimension

Let $(M, \eta, \circ, e, E)$ be a regular Frobenius manifold of dimension n. Let r be the number of Jordan blocks of L and let $m_1, \dots, m_r$ be their sizes. According to Theorem 1.6, we denote by

$$\left\{u^{j(\alpha)} \,\middle|\, \alpha \in \{1, \dots, r\},\, j \in \{1, \dots, m_\alpha\}\right\}$$

the canonical coordinates realizing (1.44) and (1.45). Therefore the product has the following form:

$$\partial_{i(\alpha)} \circ \partial_{j(\beta)} = \begin{cases} \delta_{\alpha\beta}\, \partial_{(i+j-1)(\alpha)}, & i+j \leq m_\alpha + 1, \\ 0, & i+j \geq m_\alpha + 2, \end{cases} \tag{3.1}$$

for all $i \in \{1,\dots,m_\alpha\}$, $j \in \{1,\dots,m_\beta\}$ for each $\alpha,\beta \in \{1,\dots,r\}$. The unit vector field takes the form

$$e = \sum_{\alpha=1}^{r} \partial_{1(\alpha)} \tag{3.2}$$

and the Euler vector field becomes

$$E = \sum_{s=1}^{n} u^s\, \partial_s\,. \tag{3.3}$$

The operator $L = E\,\circ$ is given by

$$L = L^{i(\alpha)}_{j(\beta)}\, \partial_{i(\alpha)} \otimes du^{j(\beta)} \tag{3.4}$$

where

$$L^{i(\alpha)}_{j(\beta)} = \begin{cases} \delta_{\alpha\beta}\, u^{(i-j+1)(\alpha)}, & i \geq j, \\ 0, & i < j, \end{cases} \tag{3.5}$$

for $\alpha,\beta \in \{1,\dots,r\}$ and $i \in \{1,\dots,m_\alpha\}$, $j \in \{1,\dots,m_\beta\}$. In fact, for any given $\alpha,\beta \in \{1,\dots,r\}$ and $i \in \{1,\dots,m_\alpha\}$, $j \in \{1,\dots,m_\beta\}$ we have

$$\begin{aligned} L^{i(\alpha)}_{j(\beta)} &= \left(E \circ \partial_{j(\beta)}\right)^{i(\alpha)} = u^{k(\gamma)} \left(\partial_{k(\gamma)} \circ \partial_{j(\beta)}\right)^{i(\alpha)} \\ &= \begin{cases} u^{k(\gamma)}\, \delta_{\beta\gamma} \left(\partial_{(j+k-1)(\beta)}\right)^{i(\alpha)}, & 1 \leq k \leq m_\beta - j + 1, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} u^{k(\beta)}\, \delta_{\alpha\beta}\, \delta^i_{j+k-1}, & 1 \leq i - j + 1, \\ 0, & \text{otherwise}, \end{cases} \\ &= \begin{cases} \delta_{\alpha\beta}\, u^{(i-j+1)(\alpha)}, & i \geq j, \\ 0, & i < j. \end{cases} \end{aligned}$$

Remark 27 *Due to the regularity condition, we are implicitly assuming that* $u^{2(\alpha)} \neq 0$ *for each* $\alpha \in \{1,\dots,r\}$ *and* $u^{1(\alpha)} \neq u^{1(\beta)}$ *if* $\alpha \neq \beta$ *for each* $\alpha,\beta \in \{1,\dots,r\}$.

In order for the data $(\eta, \circ, e, E)$ to define an actual Frobenius manifold, we have to impose all of the axioms entering Definition 1.1. In particular, we want to study conditions (1.3)–(1.9) in canonical coordinates. As stated in [18], condition (1.3) implies that the metric η is represented by a block diagonal matrix, each block of which is an upper triangular Hankel matrix (for instance, in the case of a single Jordan block see (3.27)). Precisely,

$$\eta = \delta_{\alpha\beta}\, \overline{\eta}_{(i+j-1)(\alpha)}\, du^{i(\alpha)} \otimes du^{j(\beta)} \tag{3.6}$$

for some functions $\left\{\overline{\eta}_{(i)(\alpha)} \,|\, 1 \leq \alpha \leq r,\ 1 \leq i \leq m_\alpha\right\}$ and $\overline{\eta}_{(i)(\alpha)} = 0$ for $i \geq m_\alpha + 1$. In fact, condition (1.3) spells out as

$$\eta_{i(\alpha)s(\sigma)}\, \delta^\sigma_\beta \delta^\sigma_\gamma\, \delta^s_{j+k-1} = \eta_{j(\beta)s(\sigma)}\, \delta^\sigma_\alpha \delta^\sigma_\gamma\, \delta^s_{i+k-1} \tag{3.7}$$

for each $\alpha, \beta, \gamma \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, m_\alpha\}$, $j \in \{1, \ldots, m_\beta\}$, $k \in \{1, \ldots, m_\gamma\}$. By picking $\alpha \neq \beta = \gamma$ in (3.7), we see how the components of the metric vanish when corresponding to different Jordan blocks of L. By picking $\alpha = \beta = \gamma$ in (3.7), we get

$$\eta_{i(\alpha)(j+k-1)(\alpha)} = \eta_{j(\alpha)(i+k-1)(\alpha)}$$

for suitable $i, j, k \in \{1, \ldots, m_\alpha\}$, proving (3.6). Moreover, condition (1.7) implies the existence of a *metric potential* H such that

$$\overline{\eta}_{i(\alpha)} = \partial_{i(\alpha)} H \tag{3.8}$$

for all $i \in \{1, \ldots, m_\alpha\}$ for each $\alpha \in \{1, \ldots, r\}$. In fact, condition (1.7) spells out as

$$\sum_{\sigma=1}^{r} \Gamma^{k(\gamma)}_{i(\alpha)1(\sigma)} = 0 \tag{3.9}$$

that amounts to

$$\begin{aligned}
&\sum_{\sigma=1}^{r}\sum_{s=1}^{m_\gamma} \eta^{k(\gamma)s(\gamma)} \left(\partial_{i(\alpha)}\eta_{s(\gamma)1(\sigma)} + \partial_{1(\sigma)}\eta_{i(\alpha)s(\gamma)} - \partial_{s(\gamma)}\eta_{i(\alpha)1(\sigma)}\right) \\
&= \sum_{s=1}^{m_\gamma} \eta^{k(\gamma)s(\gamma)} \left(\partial_{i(\alpha)}\eta_{s(\gamma)1(\sigma)} - \partial_{s(\gamma)}\eta_{i(\alpha)1(\alpha)}\right) + \sum_{s=1}^{m_\gamma} \eta^{k(\gamma)s(\gamma)} \sum_{\sigma=1}^{r} \partial_{1(\sigma)}\eta_{i(\alpha)s(\gamma)} \\
&\overset{(1.9)}{=} \sum_{s=1}^{m_\gamma} \eta^{k(\gamma)s(\gamma)} \left(\partial_{i(\alpha)}\overline{\eta}_{s(\gamma)} - \partial_{s(\gamma)}\overline{\eta}_{i(\alpha)}\right) = 0
\end{aligned} \tag{3.10}$$

for each $\alpha, \gamma \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, m_\alpha\}$, $k \in \{1, \ldots, m_\gamma\}$. By picking $k = m_\gamma$ in (3.10) we get

$$\eta^{m_\gamma(\gamma)1(\gamma)} \left(\partial_{i(\alpha)}\overline{\eta}_{1(\gamma)} - \partial_{1(\gamma)}\overline{\eta}_{i(\alpha)}\right) = 0$$

implying $\partial_{i(\alpha)}\overline{\eta}_{1(\gamma)} = \partial_{1(\gamma)}\overline{\eta}_{i(\alpha)}$. Let us fix $t \in \{2, \dots, m_\gamma\}$ and inductively assume that $\partial_{i(\alpha)}\overline{\eta}_{s(\gamma)} = \partial_{s(\gamma)}\overline{\eta}_{i(\alpha)}$ for each $1 \leq s \leq t-1$. By picking $k = m_\gamma - t + 1$ in (3.10) we get

$$\sum_{s=1}^{t} \eta^{(m_\gamma - t+1)(\gamma)s(\gamma)} \left(\partial_{i(\alpha)}\overline{\eta}_{s(\gamma)} - \partial_{s(\gamma)}\overline{\eta}_{i(\alpha)}\right) = 0.$$

By means of the inductive assumption, only the term for $s = t$ survives, giving $\partial_{i(\alpha)}\overline{\eta}_{t(\gamma)} = \partial_{t(\gamma)}\overline{\eta}_{i(\alpha)}$ and therefore proving

$$\partial_{i(\alpha)}\overline{\eta}_{k(\gamma)} = \partial_{k(\gamma)}\overline{\eta}_{i(\alpha)}$$

for each $\alpha, \gamma \in \{1, \dots, r\}$ and $i \in \{1, \dots, m_\alpha\}$, $k \in \{1, \dots, m_\gamma\}$. There must then exist a potential H realizing condition (3.8).

Since we consider non-semisimple Frobenius manifolds, there must exist at least one Jordan block of size greater or equal than 2. Without loss of generality we then assume that the size of the first Jordan block is greater than 1. By dropping this assumption, analogous results will hold, where different coordinates will play the roles here played by u^1, u^2.

If we take into account that the metric must be homogeneous with respect to the Euler vector field and constant with respect to the unity vector field, a further expression for the terms $\overline{\eta}_{(i)(\alpha)}$ can be recovered.

Theorem 3.1 *The functions $\overline{\eta}_i$ appearing in* (3.6) *can be written as*

$$\overline{\eta}_i = (u^2)^{-d}\, F_i, \qquad i \in \{1, \dots, n\}, \tag{3.11}$$

for some functions $F_1, \dots, F_n$ of the variables

$$z^j = \frac{u^{j+2} - u^1 \sum\limits_{\alpha=2}^{r} \delta^{j+2}_{1(\alpha)}}{u^2}, \qquad j \in \{1, \dots, n-2\}, \tag{3.12}$$

such that

$$F_1 = -\sum_{\alpha=2}^{r} \partial_{z^{1(\alpha)-2}} f + C_1, \tag{3.13}$$

$$F_2 = -z^j\, \partial_{z^j} f - (d-1)\, f + C_2, \tag{3.14}$$

$$F_j = \partial_{z^{j-2}} f, \qquad j \in \{3, \dots, n\}, \tag{3.15}$$

for some function f of $z^1, \dots, z^{n-2}$ and constants C_1, C_2. In particular, the quantity

$$\sum_{\alpha=1}^{r} F_{1(\alpha)} = C_1 \tag{3.16}$$

is a constant that vanishes whenever $d \neq 0$.

Proof: By imposing (1.9) we get

$$\sum_{\alpha=1}^{r} \partial_{1(\alpha)}\overline{\eta}_i = \mathcal{L}_e\overline{\eta}_i = 0$$

for $i \in \{1,\dots,n\}$. It follows that each $\overline{\eta}_i$ can be written as

$$\overline{\eta}_i = \varphi_i\left(u^2, u^3 - u^1\sum_{\alpha=2}^{r}\delta^3_{1(\alpha)}, \dots, u^n - u^1\sum_{\alpha=2}^{r}\delta^n_{1(\alpha)}\right) \tag{3.17}$$

for some function φ_i of $n-1$ variables. By the homogeneity condition (1.8), it can be rewritten as in (3.11) for some function F_i of the variables defined in (3.12).

The flatness of e with respect to ∇ implies that $d\big(\eta(e,\cdot)\big) = 0$ (see [18]), that is

$$\partial_{j(\beta)}\overline{\eta}_{i(\alpha)}\, du^{j(\beta)} \wedge du^{i(\alpha)} = 0$$

thus

$$\partial_{j(\beta)}\overline{\eta}_{i(\alpha)} - \partial_{i(\alpha)}\overline{\eta}_{j(\beta)} = 0 \tag{3.18}$$

for all $i \in \{1,\dots,m_\alpha\}$, $j \in \{1,\dots,m_\beta\}$ and $\alpha,\beta \in \{1,\dots,r\}$. In particular, for $i(\alpha), j(\beta) \in \{3,\dots,n\}$ we get

$$\partial_{z^{j(\beta)-2}}F_{i(\alpha)} = \partial_{z^{i(\alpha)-2}}F_{j(\beta)}.$$

There must then exist a function f of the variables $z^1,\dots,z^{n-2}$ realizing (3.15). By fixing $j(\beta) = 2$ and $i(\alpha) \in \{3,\dots,n\}$ in (3.18) we obtain the following relation:

$$\partial_{i(\alpha)}\big((u^2)^{-d}\,F_2\big) = \partial_2\big((u^2)^{-d}\,F_{i(\alpha)}\big)$$

which amounts to

$$(u^2)^{-1}\,\partial_{z^{i(\alpha)-2}}F_2 = -d\,(u^2)^{-1}\,F_{i(\alpha)} + \partial_2 F_{i(\alpha)}$$

and, by the chain rule and (3.12),

$$\partial_{z^{i(\alpha)-2}}F_2 = -d\,F_{i(\alpha)} - \sum_{j=1}^{n-2} z^j\,\partial_{z^j}F_{i(\alpha)}.$$

By taking into account (3.15) we get

$$\partial_{z^{i(\alpha)-2}}F_2 = -d\,\partial_{z^{i(\alpha)-2}}f - \sum_{j=1}^{n-2} z^j\,\partial_{z^j}\partial_{z^{i(\alpha)-2}}f.$$

Then for each $i \in \{1,\dots,n-2\}$

$$\partial_{z^i}F_2 = -d\,\partial_{z^i}f - z^j\,\partial_{z^j}\partial_{z^i}f$$

that is

$$\partial_{z^i}\left[F_2 - (1-d)\, f + z^j\, \partial_{z^j} f\right] = 0.$$

Therefore the quantity $F_2 - (1-d)\, f + z^j\, \partial_{z^j} f$ equals some constant C_2, proving (3.14).

By taking $i(\alpha) = 1(\alpha)$, $j(\beta) \in \{3, \ldots, n\}$ in (3.18) and summing over all the indices $\alpha \in \{1, \ldots, r\}$, we get

$$\sum_{\alpha=1}^{r} \partial_{j(\beta)} \overline{\eta}_{1(\alpha)} = \sum_{\alpha=1}^{r} \partial_{1(\alpha)} \overline{\eta}_{j(\beta)}$$

that is

$$(u^2)^{-d}\, \partial_{j(\beta)} \left(\sum_{\alpha=1}^{r} F_{1(\alpha)} \right) = \mathcal{L}_e \overline{\eta}_{j(\beta)}$$

thus

$$\partial_{z^{j(\beta)-2}} \left(\sum_{\alpha=1}^{r} F_{1(\alpha)} \right) = 0.$$

This means that

$$\partial_{z^j} \left(\sum_{\alpha=1}^{r} F_{1(\alpha)} \right) = 0$$

for all $j \in \{1, \ldots, n-2\}$, proving that $\sum\limits_{\alpha=1}^{r} F_{1(\alpha)}$ must be equal to some constant C_1. Condition (3.13) follows.

On the other hand, by taking $i(\alpha) = 1(\alpha)$, $j(\beta) = 2$ in (3.18) and summing over all $\alpha \in \{1, \ldots, r\}$ we get

$$\partial_2 \left(\sum_{\alpha=1}^{r} (u^2)^{-d}\, F_{1(\alpha)} \right) = 0$$

which, since $\sum\limits_{\alpha=1}^{r} F_{1(\alpha)} = C_1$, amounts to

$$\partial_2 \big((u^2)^{-d}\, C_1 \big) = 0.$$

This implies $d\, C_1 = 0$, meaning that the constant C_1 must vanish whenever $d \neq 0$.

■

Proposition 3.1 *Up to constants, the function f appearing in* (3.13), (3.14), (3.15) *is related to the metric potential H by the following formula:*

$$H = (u^2)^{1-d}\, f + C_2\, \varphi(u^2) + C_1\, u^1 \tag{3.19}$$

where

$$\varphi(u^2) = \begin{cases} \frac{(u^2)^{1-d}}{1-d}, & \textit{if } d \neq 1, \\ \ln u^2, & \textit{if } d = 1. \end{cases} \tag{3.20}$$

Proof: By (3.8) and (3.11) we have

$$\partial_i H = (u^2)^{-d} \, F_i\big(z^1, \ldots, z^{n-2}\big) \tag{3.21}$$

for each $i \in \{1, \ldots, n\}$. For $i \geq 3$ we get

$$\partial_i H = (u^2)^{-d} \, \partial_{z^{i-2}} f$$

that is

$$\partial_{z^{i-2}} H = (u^2)^{1-d} \, \partial_{z^{i-2}} f$$

or

$$\partial_{z^{i-2}}\big(H - (u^2)^{1-d} \, f\big) = 0.$$

It follows that

$$H = (u^2)^{1-d} \, f + K(u^1, u^2) \tag{3.22}$$

for some function $K(u^1, u^2)$. For $i = 2$ in (3.21) we get

$$\partial_2 H = (u^2)^{-d} \, \big(- z^j \, \partial_{z^j} f - (d-1) \, f + C_2\big)$$

that is, by the chain rule and (3.12),

$$(u^2)^{-d}\big((1-d) \, f - z^j \, \partial_{z^j} f\big) + \partial_2 K = (u^2)^{-d} \, \big(- z^j \, \partial_{z^j} f - (d-1) \, f + C_2\big)$$

yielding

$$\partial_2 K(u^1, u^2) = C_2 \, (u^2)^{-d}.$$

Then

$$K(u^1, u^2) = \begin{cases} C_2 \, \frac{(u^2)^{1-d}}{1-d} + k(u^1), & \text{if } d \neq 1, \\ C_2 \, \ln u^2 + k(u^1), & \text{if } d = 1. \end{cases} \tag{3.23}$$

for some function $k(u^1)$. By putting together (3.22) and (3.23) one gets

$$H = (u^2)^{1-d} \, f + C_2 \, \varphi(u^2) + k(u^1)$$

for

$$\varphi(u^2) = \begin{cases} \frac{(u^2)^{1-d}}{1-d}, & \text{if } d \neq 1, \\ \ln u^2, & \text{if } d = 1. \end{cases} \tag{3.24}$$

For $i = 1$ in (3.21) we finally get

$$\partial_1 H = (u^2)^{-d} \, F_1$$

that is, by the chain rule and (3.12),

$$-(u^2)^{-d} \sum_{\alpha=2}^{r} \partial_{z^{1(\alpha)-2}} f + \partial_1 k(u^1) = -(u^2)^{-d} \sum_{\alpha=2}^{r} \partial_{z^{1(\alpha)-2}} f + (u^2)^{-d} \, C_1.$$

Thus

$$\partial_1 k(u^1) = (u^2)^{-d}\, C_1 = \begin{Bmatrix} 0 & \text{if } d \neq 0 \\ C_1 & \text{if } d = 0 \end{Bmatrix} = C_1$$

implying

$$k(u^1) = C_1\, u^1 + C_3$$

for some constant C_3. We conclude that

$$H = (u^2)^{1-d}\, f + C_2\, \varphi(u^2) + C_1\, u^1 + C_3$$

for $\varphi(u^2)$ as in (3.24). ■

3.2 The case of a single Jordan block: explicit results up to dimension 4

In this section we classify regular non-semisimple Frobenius manifold structures up to dimension 4 in the case where the operator L has a single Jordan block. Due to the results of the previous section in the specific case where L has a single Jordan block of size n the unit vector field becomes $e = \partial_1$ and in canonical coordinates we have

$$\partial_i \circ \partial_j = \begin{cases} \partial_{i+j-1}, & i+j \leq n+1, \\ 0, & i+j \geq n+2, \end{cases} \tag{3.25}$$

for all $i, j \in \{1, \ldots, n\}$ and $u^i = u^{i(1)}$ for each $i \in \{1, \ldots, n\}$. The operator L is described by the following lower triangular Toeplitz matrix:

$$L = \begin{bmatrix} u^1 & 0 & 0 & \ldots & 0 & 0 \\ u^2 & u^1 & 0 & \ldots & 0 & 0 \\ u^3 & u^2 & u^1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ u^{n-1} & u^{n-2} & u^{n-3} & \ldots & u^1 & 0 \\ u^n & u^{n-1} & u^{n-2} & \ldots & u^2 & u^1 \end{bmatrix}. \tag{3.26}$$

The metric is represented by an upper triangular Hankel matrix that only depends on the coordinate u^2 and on n functions $F_1, \ldots, F_n$ of the variables

$$z^i = \frac{u^{i+2}}{u^2}, \qquad i \in \{1, \ldots, n-2\}.$$

It takes the following form:

$$\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & F_3 & \dots & F_{n-1} & F_n \\ F_2 & F_3 & F_4 & \dots & F_n & 0 \\ F_3 & F_4 & F_5 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ F_{n-1} & F_n & 0 & \dots & 0 & 0 \\ F_n & 0 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{3.27}$$

In particular, F_1 is equal to a constant C_1 that vanishes whenever $d \neq 0$ and the other F_is are expressed in terms of a function $f(z^1, \dots, z^{n-2})$ by

$$F_2 = -z^i \, \partial_{z^i} f - (d-1)\, f + C_2, \tag{3.28}$$

$$F_j = \partial_{z^{j-2}} f, \qquad j \in \{3, \dots, n\}, \tag{3.29}$$

for some constant C_2.

3.2.1 Dimension $n = 2$

Let M be a two-dimensional Frobenius manifold with product $\circ$, metric η, unit vector field e and Euler vector field E. Let us require M to be regular and the operator $L = E\circ$ to have a single Jordan block near a point $m \in M$. The unit and the Euler vector fields read respectively $e = \partial_1$ and $E = u^1\partial_1 + u^2\partial_2$. It follows directly from (3.27) that the metric has the form

$$\eta = (u^2)^{-d} \begin{bmatrix} C_1 & C_2 \\ C_2 & 0 \end{bmatrix} \tag{3.30}$$

for some constant C_1 which vanishes whenever $d \neq 0$ and for some non-zero constant C_2.

We are able to recover flat coordinates and an explicit expression for the Frobenius prepotential, as pointed out in the following result.

Theorem 3.2 *Flat coordinates coincide with the canonical ones when $d = 0$. Otherwise, they are given by*

$$x^1(u^1, u^2) = u^1$$
$$x^2(u^1, u^2) = \frac{(u^2)^{1-d}}{1-d}$$

when $d \neq 1$ and by

$$x^1(u^1, u^2) = u^1$$

$$x^2(u^1, u^2) = \ln u^2$$

when $d = 1$. In all the cases, the prepotential is given by

$$F(x^1, x^2) = \frac{C_1}{6}\,(x^1)^3 + \frac{C_2}{2}\,(x^1)^2\,x^2 \tag{3.31}$$

up to second-order polynomial terms. In flat coordinates the unit and the Euler vector fields are respectively given by $e = \tilde{\partial}_1$ and

$$E = \begin{cases} x^1\tilde{\partial}_1 + \tilde{\partial}_2, & \textit{if } d = 1, \\ x^1\tilde{\partial}_1 + x^2(1-d)\tilde{\partial}_2, & \textit{if } d \neq 1. \end{cases}$$

Proof: If $d = 0$ then the metric in (3.30) is constant, thus flat coordinates coincide with the canonical ones. Let us now fix $d \neq 0$. In this case the flat coordinates are

$$x^1(u^1, u^2) = u^1$$
$$x^2(u^1, u^2) = \frac{(u^2)^{1-d}}{1-d}$$

when $d \neq 1$ and

$$x^1(u^1, u^2) = u^1$$
$$x^2(u^1, u^2) = \ln u^2$$

when $d = 1$. In both cases, in flat coordinates the metric becomes

$$\tilde{\eta} = \begin{bmatrix} C_1 & C_2 \\ C_2 & 0 \end{bmatrix}$$

and the structure constants equal the ones in canonical coordinates:

$$\tilde{c}_{ij}^k = c_{ij}^k, \qquad i, j, k \in \{1, 2\}.$$

It follows that up to second-order polynomial terms the Frobenius prepotential F is of the form

$$F(x^1, x^2) = \frac{C_1}{6}(x^1)^3 + \frac{C_2}{2}(x^1)^2\,x^2$$

and that in flat coordinates the unit and the Euler vector fields become of the form stated above. ■

3.2.2 Dimension $n = 3$

Let M be a three-dimensional Frobenius manifold with product $\circ$, metric η, unit vector field e and Euler vector field E. Let us require M to be regular and the operator $L = E\circ$ to have a single Jordan block near a point $m \in M$. The unit and the Euler vector fields read respectively $e = \partial_1$ and $E = u^1\partial_1 + u^2\partial_2 + u^3\partial_3$. We already know from (3.27) that the metric is of the form

$$\eta = (u^2)^{-d} \begin{bmatrix} F_1(\frac{u^3}{u^2}) & F_2(\frac{u^3}{u^2}) & F_3(\frac{u^3}{u^2}) \\ F_2(\frac{u^3}{u^2}) & F_3(\frac{u^3}{u^2}) & 0 \\ F_3(\frac{u^3}{u^2}) & 0 & 0 \end{bmatrix} \tag{3.32}$$

for some functions F_1, F_2, F_3 and that F_1 is equal to a constant C_1 that vanishes whenever $d \neq 0$. It turns out from the zero-curvature conditions that the functions F_2, F_3 must be solutions to the following system of ODEs

$$\begin{cases} F_2' + z\,F_3' + d\,F_3 = 0 \\ 2\,F_3\,F_3'' - 3\,(F_3')^2 = 0. \end{cases} \tag{3.33}$$

In fact, let us introduce the variable $z = \frac{u^3}{u^2}$. We have already seen that there exists a function $f(z)$ such that

$$\begin{aligned} F_2(z) &= -z\,f'(z) - (d-1)\,f(z) + C_2, \\ F_3(z) &= f'(z), \end{aligned}$$

for some constant C_2. It follows that

$$F_2' + z\,F_3' + d\,F_3 = 0.$$

Moreover, by requiring that $R^1_{232} = 0$ one obtains the Liouville-type differential equation

$$2\,F_3\,F_3'' - 3\,(F_3')^2 = 0.$$

This suffices to make all of the conditions in (1.3), (1.5), (1.7), (1.8), (1.9), (1.4) hold without imposing more. So far, what we know about the functions F_1, F_2, F_3 is that F_1 equals some constant C_1 and that F_2, F_3 are solutions to the system (3.33). Two expressions for the function f appering in (3.28), (3.29) are then possible, as shown below.

Theorem 3.3 *The function f realizing* (3.28), (3.29) *is either provided by*

$$f(z) = C_3\,z + C_4 \tag{3.34}$$

for some constants C_3, C_4 or by

$$f(z) = -\frac{C_4}{z + C_3} + C_5 \tag{3.35}$$

for some constants C_3, C_4, C_5.

Proof: The first condition in (3.33) amounts to (3.28) and (3.29), while the second one can be rewritten as

$$2\, f'(z)\, f'''(z) - 3\,(f''(z))^2 = 0. \tag{3.36}$$

If we assume $f''(z) \neq 0$ then the solutions to equation (3.36) can be written in the form of (3.35), while (3.34) is recovered by considering solutions corresponding to $f''(z) = 0$. ∎

Summarizing, two cases may occur: either

$$\begin{cases} F_1(z) = C_1 \\ F_2(z) = -C_3\, d\, z + C_2 \\ F_3(z) = C_3 \end{cases} \tag{3.37}$$

for some constant C_1 that vanishes for $d \neq 0$ and some constants C_2, C_3 or

$$\begin{cases} F_1(z) = C_1 \\ F_2(z) = \frac{C_3\, C_4}{(z+C_3)^2} - \frac{(2-d)\, C_4}{z+C_3} + C_2 \\ F_3(z) = \frac{C_4}{(z+C_3)^2} \end{cases} \tag{3.38}$$

for some constant C_1 that vanishes for $d \neq 0$ and some constants C_2, C_3, C_4.

Proposition 3.2 *In the case of* (3.37) *flat coordinates are given by*

$$\begin{aligned} x^1(u^1, u^2, u^3) &= u^1 \\ x^2(u^1, u^2, u^3) &= (u^2)^{-d}\, u^3 + \frac{C_2 (u^2)^{1-d}}{C_3 (1-d)} \\ x^3(u^1, u^2, u^3) &= \frac{2}{2-d} (u^2)^{\frac{2-d}{2}} \end{aligned}$$

when $d \notin \{0, 1, 2\}$, by

$$\begin{aligned} x^1(u^1, u^2, u^3) &= u^1 \\ x^2(u^1, u^2, u^3) &= \frac{u^3}{(u^2)^2} - \frac{C_2}{C_3\, u^2} \\ x^3(u^1, u^2, u^3) &= \ln u^2 \end{aligned}$$

when $d = 2$, *by*

$$\begin{aligned} x^1(u^1,u^2,u^3) &= u^1 \\ x^2(u^1,u^2,u^3) &= \frac{u^3}{u^2} + \frac{C_2}{C_3}\,\ln u^2 \\ x^3(u^1,u^2,u^3) &= 2\sqrt{u^2} \end{aligned}$$

when $d = 1$ *and, trivially, by*

$$\begin{aligned} x^1(u^1,u^2,u^3) &= u^1 \\ x^2(u^1,u^2,u^3) &= u^2 \\ x^3(u^1,u^2,u^3) &= u^3 \end{aligned}$$

when $d = 0$.

The proof is a straightforward computation.

Proposition 3.3 *Let* x^1, x^2, x^3 *denote flat coordinates. Up to second-order polynomial terms, in the case of* (3.37) *the prepotential is given by*

$$F(x^1,x^2,x^3) = \frac{C_3}{2}\,(x^1)^2\,x^2 + \frac{C_3}{2}\,x^1\,(x^3)^2 \tag{3.39}$$

when $d \neq 0$ *and by*

$$F(x^1,x^2,x^3) = \frac{C_1}{6}\,(x^1)^3 + \frac{C_2}{2}\,(x^1)^2\,x^2 + \frac{C_3}{2}\,(x^1)^2\,x^3 + \frac{C_3}{2}\,x^1\,(x^2)^2 \tag{3.40}$$

when $d = 0$ *(in this latter case flat coordinates coincide with the canonical ones). If* $d \neq 0$ *then in flat coordinates the multiplication is written as*

$$\begin{aligned} \tilde\partial_1 \circ \tilde\partial_1 &= \tilde\partial_1 \\ \tilde\partial_1 \circ \tilde\partial_2 &= \tilde\partial_2 \\ \tilde\partial_1 \circ \tilde\partial_3 &= \tilde\partial_3 \\ \tilde\partial_2 \circ \tilde\partial_2 &= 0 \\ \tilde\partial_2 \circ \tilde\partial_3 &= 0 \\ \tilde\partial_3 \circ \tilde\partial_3 &= \tilde\partial_2 \end{aligned}$$

and the Euler vector field reads

$$E = x^1\,\tilde\partial_1 + (1-d)\,x^2\,\tilde\partial_2 + \frac{2-d}{2}\,x^3\,\tilde\partial_3$$

if $d \notin \{0,1,2\}$,

$$E = x^1\,\tilde\partial_1 + \frac{C_2}{C_3}\,\tilde\partial_2 + \frac{1}{2}\,x^3\,\tilde\partial_3$$

if $d = 1$ *and*

$$E = x^1\,\tilde{\partial}_1 - x^2\,\tilde{\partial}_2 + \tilde{\partial}_3$$

if $d = 2$*. In flat coordinates the unit vector field is* $e = \tilde{\partial}_1$ *for each value of* d*.*

The proof is a straightforward computation.

Analogous results can be achieved for the case of (3.38), as presented below.

Proposition 3.4 *In the case of* (3.38) *flat coordinates and the Euler vector field are respectively given by*

$$\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{C_2C_3(u^2)^2 + C_2\,u^2u^3 - C_4(u^2)^2}{C_1\,(C_3\,u^2+u^3)}\\
x^2(u^1,u^2,u^3) &= \big[\big(-C_2C_3\sqrt{C_1C_4} - C_4(C_1 - C_4\\
&\quad + C_2C_3)\big)(u^2)^{\frac{2C_4-\sqrt{C_1C_4}}{C_4}}\\
&\quad - C_2\,u^3(u^2)^{\frac{C_4-\sqrt{C_1C_4}}{C_4}}(C_4\\
&\quad + \sqrt{C_1C_4})\big]\frac{1}{C_4(C_1-C_4)\,(C_3\,u^2+u^3)}\\
x^3(u^1,u^2,u^3) &= \big[\big(C_2C_3\sqrt{C_1C_4} - C_4(C_1 - C_4\\
&\quad + C_2C_3)\big)(u^2)^{\frac{2C_4+\sqrt{C_1C_4}}{C_4}}\\
&\quad - C_2\,u^3(u^2)^{\frac{C_4+\sqrt{C_1C_4}}{C_4}}(C_4\\
&\quad - \sqrt{C_1C_4})\big]\frac{1}{C_4(C_1-C_4)\,(C_3\,u^2+u^3)}\\
E &= x^1\,\tilde{\partial}_1 + b\,x^2\,\tilde{\partial}_2 + c\,x^3\,\tilde{\partial}_3
\end{aligned}$$

where

$$\begin{aligned}
b &= \frac{(C_2C_3+C_1)(C_4)^{\frac{3}{2}} - C_1C_2C_3\sqrt{C_4} + (C_4)^2\sqrt{C_1}}{C_4\,(C_2C_3\sqrt{C_1} + C_2C_3\sqrt{C_4} + C_1\sqrt{C_4} - C_4\sqrt{C_4})}\\
&\quad + \frac{-(C_1)^{\frac{3}{2}}C_4 - (C_4)^{\frac{5}{2}}}{C_4\,(C_2C_3\sqrt{C_1} + C_2C_3\sqrt{C_4} + C_1\sqrt{C_4} - C_4\sqrt{C_4})}\\
c &= \frac{\sqrt{C_4}\,(C_1 - C_4)}{\sqrt{C_1}C_4 - (C_4)^{\frac{3}{2}}}
\end{aligned}$$

when $d = 0$*,* $C_1 \neq 0$ *and* $C_1 \neq C_4$*, by*

$$x^1(u^1,u^2,u^3) = u^1 + \frac{(u^2)^2(\ln u^2)^2}{2\,(C_3\,u^2+u^3)}$$

$$
\begin{aligned}
&+ \frac{C_2\, u^2\big(2\,\ln u^2 - (\ln u^2)^2 - 2\big)}{2\,C_4}\\
x^2(u^1,u^2,u^3) &= -\frac{(u^2)^2\,\ln u^2}{C_3\,u^2+u^3} + \frac{C_2\,u^2\big(\ln u^2 - 1\big)}{C_4}\\
x^3(u^1,u^2,u^3) &= -\frac{(u^2)^2}{C_3\,u^2+u^3} + \frac{C_2\,u^2}{C_4}\\
E &= (x^1-x^2)\,\tilde\partial_1 + (x^2+x^3)\,\tilde\partial_2 + x^3\,\tilde\partial_3
\end{aligned}
$$

when $d=0$ and $C_1=0$, by

$$
\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 - \frac{(u^2)^2}{C_3\,u^2+u^3} + \frac{C_2\,u^2}{C_4}\\
x^2(u^1,u^2,u^3) &= -\frac{(u^2)^3}{2\,\big(C_3\,u^2+u^3\big)} + \frac{C_2\,(u^2)^2}{4\,C_4}\\
x^3(u^1,u^2,u^3) &= -\frac{u^2}{C_3\,u^2+u^3} + \frac{C_2\,\ln u^2}{C_4}\\
E &= x^1\,\tilde\partial_1 + 2\,x^2\,\tilde\partial_2 + \frac{C_2}{C_4}\,\tilde\partial_3
\end{aligned}
$$

when $d=0$ and $C_1=C_4$, by

$$
\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{2(u^2)^2\big(C_4 - C_2\,C_3\big)}{C_4(d)^2(C_3\,u^2+u^3)}\\
x^2(u^1,u^2,u^3) &= \frac{\big(C_2C_3 - C_4\,(1-d)\big)(u^2)^{2-d}}{C_4\,(1-d)\big(C_3\,u^2+u^3\big)}\\
&+ \frac{C_2\,u^3(u^2)^{1-d}}{C_4\,(1-d)\big(C_3\,u^2+u^3\big)}\\
x^3(u^1,u^2,u^3) &= \frac{2C_2\,u^3(u^2)^{\frac{2-d}{2}}}{C_4\,(2-d)\big(C_3\,u^2+u^3\big)}\\
&+ \frac{-\big(C_4(2-d) - 2C_2C_3\big)(u^2)^{\frac{4-d}{2}}}{C_4\,(2-d)\big(C_3\,u^2+u^3\big)}\\
E \;&=\; x^1\,\tilde\partial_1 - \frac{d-2}{2}\,x^2\,\tilde\partial_2 - (d-1)\,x^3\,\tilde\partial_3
\end{aligned}
$$

when $d \notin \{0,1,2\}$ and $C_3 \neq 0$, by

$$
\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{2\big(-C_2\,u^2\,u^3 + C_4(u^2)^2\big)}{C_4\,(d)^2\,u^3}\\
x^2(u^1,u^2,u^3) &= \frac{2C_2\,u^3(u^2)^{\frac{2-d}{2}} - C_4\,(2-d)(u^2)^{\frac{4-d}{2}}}{C_4\,(2-d)\,u^3}\\
x^3(u^1,u^2,u^3) &= \frac{C_2\,u^3(u^2)^{1-d} - C_4\,(1-d)(u^2)^{2-d}}{C_4\,(1-d)\,u^3}
\end{aligned}
$$

$$E \;=\; x^1\,\tilde\partial_1 - \frac{d-2}{2}\,x^2\,\tilde\partial_2 - (d-1)\,x^3\,\tilde\partial_3$$

when $d \notin \{0,1,2\}$ and $C_3 = 0$, by

$$\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{2(u^2)^2}{C_3\,u^2+u^3} - \frac{2C_2\,u^2}{C_4}\\
x^2(u^1,u^2,u^3) &= -\frac{u^2}{C_3\,u^2+u^3} + \frac{C_2\,\ln u^2}{C_4}\\
x^3(u^1,u^2,u^3) &= -\frac{(u^2)^{\frac{3}{2}}}{C_3\,u^2+u^3} + \frac{2C_2\sqrt{u^2}}{C_4}\\
E &= x^1\,\tilde\partial_1 - \frac{d-2}{2}\,x^2\,\tilde\partial_2 + \left(\frac{C_2}{C_4} - (d-1)\,x^3\right)\tilde\partial_3
\end{aligned}$$

when $d = 1$, by

$$\begin{aligned}
x^1(u^1,u^2,u^3) &= u^1 + \frac{(u^2)^2}{2\big(C_3\,u^2+u^3\big)} - \frac{C_2\,u^2}{2C_4}\\
x^2(u^1,u^2,u^3) &= -\frac{u^2}{C_3\,u^2+u^3} + \frac{C_2\,\ln u^2}{C_4}\\
x^3(u^1,u^2,u^3) &= -\frac{1}{C_3\,u^2+u^3} - \frac{C_2}{C_4\,u^2}\\
E &= x^1\,\tilde\partial_1 + \left(\frac{C_2}{C_4} - \frac{d-2}{2}\,x^2\right)\tilde\partial_2 - (d-1)\,x^3\,\tilde\partial_3
\end{aligned}$$

when $d = 2$. In each of these cases the unit vector field reads $e = \tilde\partial_1$.

Here are explicit expressions for the Frobenius potential in some selected cases.

Example 3.5 *Let us fix $d = 0$, $C_1 = C_4$ and $C_2 = 0$. In flat coordinates the metric becomes*

$$\tilde\eta = \begin{bmatrix} C_4 & 0 & 0\\ 0 & 0 & -C_4\\ 0 & -C_4 & 0 \end{bmatrix},$$

the multiplication is given by

$$\begin{aligned}
\tilde\partial_1 \circ \tilde\partial_1 &= \tilde\partial_1\\
\tilde\partial_1 \circ \tilde\partial_2 &= \tilde\partial_2\\
\tilde\partial_1 \circ \tilde\partial_3 &= \tilde\partial_3\\
\tilde\partial_2 \circ \tilde\partial_2 &= -\frac{3\sqrt{2}}{4}\sqrt{\frac{x^3}{x^2}}\tilde\partial_2 + \frac{\sqrt{2}}{4}\left(\frac{x^3}{x^2}\right)^{\frac{3}{2}}\tilde\partial_3\\
\tilde\partial_2 \circ \tilde\partial_3 &= -\tilde\partial_1 - \frac{3\sqrt{2}}{4}\sqrt{\frac{x^2}{x^3}}\tilde\partial_2 - \frac{3\sqrt{2}}{4}\sqrt{\frac{x^3}{x^2}}\tilde\partial_3
\end{aligned}$$

$$\tilde{\partial}_3 \circ \tilde{\partial}_3 = \frac{\sqrt{2}}{4}\left(\frac{x^2}{x^3}\right)^{\frac{3}{2}}\tilde{\partial}_2 - \frac{3\sqrt{2}}{4}\sqrt{\frac{x^2}{x^3}}\tilde{\partial}_3$$

and the prepotential reads

$$F(x^1, x^2, x^3) = \frac{2\sqrt{2}}{3}C_4\,(x^2)^{\frac{3}{2}}(x^3)^{\frac{3}{2}} + \frac{C_4}{6}(x^1)^3 - C_4\,x^1\,x^2\,x^3$$

up to second-order polynomial terms. In flat coordinates the unit and the Euler vector fields are respectively written as

$$e = \tilde{\partial}_1$$

and

$$E = x^1\,\tilde{\partial}_1 + 2\,x^2\,\tilde{\partial}_2.$$

Example 3.6 *Let us fix $d = 2$ and $C_2 = 0$. In flat coordinates the metric becomes*

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & C_4 \\ 0 & C_4 & 0 \\ C_4 & 0 & 0 \end{bmatrix},$$

the multiplication is given by

$$\begin{aligned}
\tilde{\partial}_1 \circ \tilde{\partial}_1 &= \tilde{\partial}_1 \\
\tilde{\partial}_1 \circ \tilde{\partial}_2 &= \tilde{\partial}_2 \\
\tilde{\partial}_1 \circ \tilde{\partial}_3 &= \tilde{\partial}_3 \\
\tilde{\partial}_2 \circ \tilde{\partial}_2 &= -\frac{3}{2}\left(\frac{x^2}{x^3}\right)^2\tilde{\partial}_1 + 3\frac{x^2}{x^3}\tilde{\partial}_2 + \tilde{\partial}_3 \\
\tilde{\partial}_2 \circ \tilde{\partial}_3 &= \left(\frac{x^2}{x^3}\right)^3\tilde{\partial}_1 - \frac{3}{2}\left(\frac{x^2}{x^3}\right)^2\tilde{\partial}_2 \\
\tilde{\partial}_3 \circ \tilde{\partial}_3 &= -\frac{3}{4}\left(\frac{x^2}{x^3}\right)^4\tilde{\partial}_1 + \left(\frac{x^2}{x^3}\right)^3\tilde{\partial}_2
\end{aligned}$$

and the prepotential reads

$$F(x^1, x^2, x^3) = \frac{C_4}{2}\,(x^1)^2\,x^3 + \frac{C_4}{2}\,x^1\,(x^2)^2 + \frac{C_4}{8}\,\frac{(x^2)^4}{x^3} \tag{3.41}$$

up to second-order polynomial terms. In flat coordinates the unit and the Euler vector fields are respectively written as

$$e = \tilde{\partial}_1$$

and

$$E = x^1\,\tilde{\partial}_1 - x^3\,\tilde{\partial}_3.$$

Example 3.7 *Let us fix $d = 2$ and $C_2 = 1$. In flat coordinates the metric becomes*

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & C_4 \\ 0 & C_4 & 0 \\ C_4 & 0 & 0 \end{bmatrix},$$

the multiplication is given by

$$\begin{aligned}
\tilde{\partial}_1 \circ \tilde{\partial}_1 &= \tilde{\partial}_1 \\
\tilde{\partial}_1 \circ \tilde{\partial}_2 &= \tilde{\partial}_2 \\
\tilde{\partial}_1 \circ \tilde{\partial}_3 &= \tilde{\partial}_3 \\
\tilde{\partial}_2 \circ \tilde{\partial}_2 &= -\frac{3}{2(C_4)^2(x^3)^2} W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right)^2 \tilde{\partial}_1 \\
&\quad + \frac{3}{C_4 x^3} W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right) \tilde{\partial}_2 + \tilde{\partial}_3 \\
\tilde{\partial}_2 \circ \tilde{\partial}_3 &= \frac{1}{(C_4)^3(x^3)^3} W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right)^3 \tilde{\partial}_1 \\
&\quad - \frac{3}{2(C_4)^2(x^3)^2} W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right)^2 \tilde{\partial}_2 \\
\tilde{\partial}_3 \circ \tilde{\partial}_3 &= -\frac{3}{4(C_4)^4(x^3)^4} W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right)^4 \tilde{\partial}_1 \\
&\quad + \frac{1}{(C_4)^3(x^3)^3} W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right)^3 \tilde{\partial}_2
\end{aligned}$$

and the prepotential reads

$$\begin{aligned}
F(x^1, x^2, x^3) = \frac{1}{24(C_4)^3 x^3} &\Big(3\, W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right)^4 \\
&+ 22\, W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right)^3 + 63\, W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right)^2 \\
&+ 72\, W\left(C_4\, x^3\, e^{C_4\, x^2 - 1}\right) \Big) + \frac{C_4}{2}\, (x^1)^2\, x^3 + \frac{C_4}{2}\, x^1\, (x^2)^2
\end{aligned}$$

up to second-order polynomial terms, where W denotes the principal branch of the Lambert W function, defined as the multivalued inverse of the function $w \mapsto w e^w$ (see [15] and references therein). In flat coordinates the unit and the Euler vector fields are respectively written as

$$e = \tilde{\partial}_1$$

and

$$E = x^1\, \tilde{\partial}_1 + \frac{1}{C_4}\, \tilde{\partial}_2 - x^3\, \tilde{\partial}_3.$$

3.2.3 Dimension $n = 4$

Let M be a four-dimensional Frobenius manifold with product $\circ$, metric η, unit vector field e and Euler vector field E. Let us require M to be regular and the operator $L = E\circ$ to have a single Jordan block near a point $m \in M$. The unit and the Euler vector fields read respectively $e = \partial_1$ and $E = u^1\partial_1 + u^2\partial_2 + u^3\partial_3 + u^4\partial_4$. We already know from (3.27) that the metric is of the form

$$\eta = (u^2)^{-d} \begin{bmatrix} F_1(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & F_2(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & F_3(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & F_4(\frac{u^3}{u^2}, \frac{u^4}{u^2}) \\ F_2(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & F_3(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & F_4(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & 0 \\ F_3(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & F_4(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & 0 & 0 \\ F_4(\frac{u^3}{u^2}, \frac{u^4}{u^2}) & 0 & 0 & 0 \end{bmatrix} \tag{3.42}$$

for some functions F_1, F_2, F_3, F_4 of the variables $z = \frac{u^3}{u^2}$, $w = \frac{u^4}{u^2}$. In particular, F_1 is equal to a constant C_1 which vanishes whenever $d \neq 0$ and from (3.28) and (3.29) we know that F_2, F_3, F_4 can be expressed as

$$F_2(z,w) = -z\,\partial_z f(z,w) - w\,\partial_w f(z,w) - (d-1)\,f(z,w) + C_2 \tag{3.43}$$

$$F_3(z,w) = \partial_z f(z,w) \tag{3.44}$$

$$F_4(z,w) = \partial_w f(z,w) \tag{3.45}$$

for some function $f(z,w)$ and some constant C_2. By the flatness conditions, two expressions for f are possible, as shown below. This fully classifies regular four-dimensional Frobenius manifolds whose operator $L = E\circ$ has a single Jordan block.

Theorem 3.4 *The function f realizing* (3.28), (3.29) *is either provided by*

$$f(z,w) = C_3\,w\,e^{C_4\,z} + h(z) \tag{3.46}$$

for some constants C_3, C_4 and some functiton $h(z)$ which is solution to

$$h'''(z) - 2\,C_4\,h''(z) + C_4^2\,h'(z) + 2\,C_3\,C_4\,e^{C_4\,z} = 0 \tag{3.47}$$

or by

$$f(z,w) = C_3 - \frac{A(z)}{2\,B(z) + w} \tag{3.48}$$

for some constant C_3 and solutions $A(z)$, $B(z)$ to the following system of ODEs:

$$A''\,A - (A')^2 + 2\left(C_2 + (1-d)\,C_3\right)A = 0 \tag{3.49}$$

$$A\,B''' - A'\left(B''+1\right) + 2\left(C_2 + (1-d)\,C_3\right)\left(B'+z\right) + C_1 = 0. \tag{3.50}$$

Proof: By requiring that $R^1_{243} = 0$ we get

$$2\,\partial_w f\,\partial_w^3 f - 3(\partial_w^2 f)^2 = 0. \tag{3.51}$$

Let us distinguish two cases: $\partial_w^2 f \neq 0$ and $\partial_w^2 f = 0$. In the first case we obtain

$$f(z,w) = C(z) - \frac{A(z)}{2\,B(z) + w} \tag{3.52}$$

for some functions $A(z)$, $B(z)$, $C(z)$ while in the second one we obtain

$$f(z,w) = w\,h_1(z) + h_2(z) \tag{3.53}$$

for some functions $h_1(z)$, $h_2(z)$.

If f is as in (3.52) then condition $R^3_{343} = 0$ implies that the function $C(z)$ must be equal to a constant C_3. Conditions $R^3_{234} = 0$ and $R^2_{322} = 0$ yields respectively

$$A''\,A - (A')^2 + 2\left(C_2 + (1-d)\,C_3\right)A = 0$$

and

$$A\,B''' - A'\left(B'' + 1\right) + 2\left(C_2 + (1-d)\,C_3\right)\left(B' + z\right) + C_1 = 0.$$

All the other conditions in (1.3), (1.5), (1.7), (1.8), (1.9), (1.4) hold without imposing more.

If, on the other hand, f is as in (3.53), condition $R^3_{234} = 0$ implies that

$$h_1(z)\,h_1''(z) - (h_1'(z))^2 = 0. \tag{3.54}$$

Solutions to (3.54) are given by $h_1(z) = C_3\,e^{C_4\,z}$ for some constants C_3 and C_4, so that

$$f(z,w) = C_3\,w\,e^{C_4\,z} + h_2(z).$$

By imposing condition $R^2_{322} = 0$ we get

$$h_2'''(z) - 2\,C_4\,h_2''(z) + C_4^2\,h_2'(z) + 2\,C_3\,C_4\,e^{C_4\,z} = 0$$

that yields

$$h_2(z) = C_7 - \frac{e^{C_4\,z}}{C_4^2}\left[C_3\,C_4^2\,z^2 - C_4\,(2\,C_3 + C_5)\,z - C_4\,C_6 + 2\,C_3 + C_5\right]$$

when $C_4 \neq 0$ and

$$h_2(z) = C_5\,z^2 + C_6\,z + C_7$$

when $C_4 = 0$ for some constants C_5, C_6, C_7, so that f becomes respectively

$$f(z,w) = C_3\,w\,e^{C_4\,z} + C_7 - \frac{e^{C_4\,z}}{C_4^2}\left[C_3\,C_4^2\,z^2 - C_4\,(2\,C_3 + C_5)\,z - C_4\,C_6 + 2\,C_3 + C_5\right] \tag{3.55}$$

and

$$f(z,w) = C_3\, w\, e^{C_4\, z} + C_5\, z^2 + C_6\, z + C_7. \tag{3.56}$$

In both cases it turns out that all the other conditions in (1.3), (1.5), (1.7), (1.8), (1.9), (1.4) hold without imposing more. ■

Proposition 3.8 *The functions $A(z)$ and $B(z)$ appearing in* (3.49) *and* (3.50) *are expressed via hyperbolic functions and second-order polynomials:*

$$\begin{aligned} A(z) &= \frac{C_2 + (1-d)\,C_3}{C_4^2}\,\sinh^2\big(C_4(z+C_5)\big) \\ B(z) &= C_6\cosh(2C_4(z+C_5)) + C_7\sinh(2C_4(z+C_5)) \\ &\quad -\frac{z}{2}\Big(\frac{C_1}{C_2+(1-d)\,C_3} + 4C_4C_7\Big) - \frac{z^2}{2} + C_8 \end{aligned}$$

for some constants C_4, C_5, C_6, C_7, C_8 if $C_2 + (1-d)\,C_3 \neq 0$ and

$$A(z) = C_5\,\big(\cosh(C_4\, z) + \sinh(C_4\, z)\big) \tag{3.57}$$

$$\begin{aligned} B(z) = &\frac{1}{2\,(C_4)^3\, C_5}\big((2\, C_6\, C_4\, C_5 + C_1)\cosh(C_4\, z) \\ &+ (2\, C_6\, C_4\, C_5 - C_1)\sinh(C_4\, z)\big) - \frac{z^2}{2} + C_7\, z + C_8 \end{aligned} \tag{3.58}$$

for some constants C_4, C_5, C_6, C_7, C_8 if $C_2 + (1-d)\,C_3 = 0$.

Below flat coordinates are computed for selected other cases, together with some Frobenius prepotentials.

Example 3.9 *Let us consider the case* (3.46) *with $C_3 = 1$, $C_4 = 0$ and $d \neq 0$. Equation* (3.47) *becomes $h'''(z) = 0$ yielding $h(z) = a\,z^2 + b\,z + c$ for some constants a, b, c. In particular we choose $a = c = 0$ and $b = 1$, so that $h(z) = z$ and $f(z,w) = z + w$. When $d \neq 1$, in the flat coordinates*

$$\begin{aligned} x^1(u^1,u^2,u^3,u^4) &= u^1 \\ x^2(u^1,u^2,u^3,u^4) &= (u^2)^{-d}\big(u^3+u^4\big) \\ x^3(u^1,u^2,u^3,u^4) &= \frac{1}{2}u^2 + u^3 \\ x^4(u^1,u^2,u^3,u^4) &= \frac{1}{1-d}(u^2)^{1-d} \end{aligned}$$

we have

$$\tilde{\eta} = \begin{bmatrix} 0 & 1 & 0 & C_2 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ C_2 & 0 & 1 & 0 \end{bmatrix},$$

$$
\begin{aligned}
e &= \tilde{\partial}_1, \\
E &= x^1\,\tilde{\partial}_1 + (1-d)\,x^2\,\tilde{\partial}_2 + x^3\,\tilde{\partial}_3 + (1-d)\,x^4\,\tilde{\partial}_4.
\end{aligned}
$$

Up to second-order polynomial terms, the prepotential is given by

$$
F(x^1,x^2,x^3,x^4) = \frac{C_2}{2}(x^1)^2x^4 + x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2 + \frac{(d-1)^{\frac{d-3}{d-1}}\,e^{\frac{2\pi i}{d-1}}\,(x^4)^{\frac{d-3}{d-1}}}{2(d+1)(d-3)}
$$

when $d \notin \{-1,0,3\}$*, by*

$$
F(x^1,x^2,x^3,x^4) = \frac{1}{4}(x^4)^2\ln x^4 + \frac{C_2}{2}(x^1)^2x^4 + x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2
$$

when $d=-1$*, by*

$$
\begin{aligned}
F(x^1,x^2,x^3,x^4) =& \frac{C_2}{2}(x^1)^2x^2 + x^1x^2x^3 + \frac{1}{2}(x^1)^2x^3 \\
&+ \frac{1}{2}(x^1)^2x^4 + \frac{1}{2}x^1(x^2)^2 + \frac{1}{6}(x^2)^3
\end{aligned}
$$

when $d=0$*, by*

$$
F(x^1,x^2,x^3,x^4) = \frac{C_2}{2}(x^1)^2x^4 + x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2 - \frac{1}{16}\ln x^4
$$

when $d=3$*. The case where* $d=1$ *must be treated separately. In the flat coordinates*

$$
\begin{aligned}
x^1(u^1,u^2,u^3,u^4) &= u^1 \\
x^2(u^1,u^2,u^3,u^4) &= \frac{u^3+u^4}{u^2} \\
x^3(u^1,u^2,u^3,u^4) &= \frac{1}{2}u^2 + u^3 \\
x^4(u^1,u^2,u^3,u^4) &= \ln u^2
\end{aligned}
$$

the unit and the Euler vector fields are given by

$$
e = \tilde{\partial}_1, \quad E = x^1\,\tilde{\partial}_1 + x^3\,\tilde{\partial}_3 + \tilde{\partial}_4.
$$

The metric is as the one for $d \neq 1$ *and up to second-order polynomial terms the prepotential is*

$$
F(x^1,x^2,x^3,x^4) = \frac{1}{8}e^{2x^4} + \frac{C_2}{2}(x^1)^2x^4 + x^1x^3x^4 + \frac{1}{2}(x^1)^2x^2.
$$

Example 3.10 *Let us consider the case* (3.46) *with* $C_3 = C_4 = 1$ *and* $d \neq 0$*. Equation* (3.47) *becomes* $h'''(z) - 2h''(z) + h'(z) + 2e^z = 0$ *yielding* $h(z) = a - (z^2 + b\,z + c)e^z$ *for* some constants a, b, c. In particular we choose $a = b = c = 0$, so that $h(z) = -z^2 e^z$ and $f(z, w) = (w - z^2)\, e^z$. When $d \notin \{0, 1, 2\}$ the flat coordinates are

$$\begin{aligned} x^1(u^1, u^2, u^3, u^4) &= u^1 + \frac{u^2}{2(1-d)} \\ x^2(u^1, u^2, u^3, u^4) &= C_2 \ln u^2 - 2(1-d)\, e^{\frac{u^3}{u^2}} - \frac{(u^3)^2 - u^2 u^4}{(u^2)^2}\, e^{\frac{u^3}{u^2}} \\ x^3(u^1, u^2, u^3, u^4) &= (u^2)^{-d-1}\big(u^2 u^4 - (u^3)^2\big)\, e^{\frac{u^3}{u^2}} + \frac{C_2 (u^2)^{1-d}}{1-d} \\ x^4(u^1, u^2, u^3, u^4) &= \frac{(u^2)^{2-d}}{2-d}. \end{aligned}$$

In such coordinates the unit and the Euler vector fields are respectively written as $e = \tilde{\partial}_1$ *and*

$$E = x^1\, \tilde{\partial}_1 + C_2\, \tilde{\partial}_2 - (d-1)\, x^3\, \tilde{\partial}_3 - (d-2)\, x^4\, \tilde{\partial}_4.$$

For $d = -1$ *the metric is*

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{4} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \end{bmatrix}$$

and up to second-order polynomial terms the prepotential is given by

$$\begin{aligned} F(x^1, x^2, x^3, x^4) = &-\frac{1}{32}\sqrt[3]{3} C_2 (x^4)^{\frac{4}{3}} \ln\left(3\, x^4\right) + \frac{15}{128}\sqrt[3]{3} C_2 (x^4)^{\frac{4}{3}} + \frac{1}{32}\sqrt[3]{9}(x^4)^{\frac{2}{3}} x^3 \\ &+ \frac{3}{32}\sqrt[3]{3}(x^4)^{\frac{4}{3}} x^2 + \frac{1}{2}(x^1)^2 x^3 - \frac{1}{4} x^1 x^2 x^4. \end{aligned}$$

For $d = -2$ *the metric is*

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{6} \\ 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{6} & 0 & 0 \end{bmatrix}$$

and up to second-order polynomial terms the prepotential is given by

$$\begin{aligned} F(x^1, x^2, x^3, x^4) = &-\frac{1}{45}\sqrt{2} C_2 (x^4)^{\frac{5}{4}} \ln\left(2\sqrt{x^4}\right) + \frac{4}{75}\sqrt{2} C_2 (x^4)^{\frac{5}{4}} + \frac{2}{45}\sqrt{2}(x^4)^{\frac{5}{4}} x^2 \\ &+ \frac{1}{72}\sqrt{x^4}\, x^3 + \frac{1}{2}(x^1)^2 x^3 - \frac{1}{6} x^1 x^2 x^4. \end{aligned}$$

The case $d = 2$ *must be treated separately. In the flat coordinates*

$$x^1(u^1, u^2, u^3, u^4) = u^1 - \frac{u^2}{2}$$

$$
\begin{aligned}
x^2(u^1,u^2,u^3,u^4) &= \frac{2(u^2)^2 + u^2u^4 - (u^3)^2}{(u^2)^2}\, e^{\frac{u^3}{u^2}} \\
x^3(u^1,u^2,u^3,u^4) &= \frac{u^2u^4 - (u^3)^2}{(u^2)^3}\, e^{\frac{u^3}{u^2}} - \frac{C_2}{u^2} \\
x^4(u^1,u^2,u^3,u^4) &= \ln u^2
\end{aligned}
$$

we have

$$
e = \tilde{\partial}_1,\ \ E = x^1\,\tilde{\partial}_1 - x^3\,\tilde{\partial}_3 + \tilde{\partial}_4,\ \ \tilde{\eta} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & C_2 \end{bmatrix}.
$$

Up to second-order polynomial terms the prepotential is

$$
\begin{aligned}
F(x^1,x^2,x^3,x^4) = {} & -\frac{1}{16}x^3\, e^{2x^4} + \left(C_2 + \frac{x^2}{2}\right) e^{x^4} \\
& + \frac{C_2}{2}x^1(x^4)^2 + \frac{1}{2}x^1x^2x^4 + \frac{1}{2}(x^1)^2x^3.
\end{aligned}
$$

In the case where $d = 1$, which must be handled separately as well, flat coordinates are given by

$$
\begin{aligned}
x^1(u^1,u^2,u^3,u^4) &= u^1 + \frac{u^2}{2} - \frac{u^2}{2}\ln u^2 \\
x^2(u^1,u^2,u^3,u^4) &= \frac{u^2u^4 - (u^3)^2}{(u^2)^2}\, e^{\frac{u^3}{u^2}} + C_2\,\ln u^2 \\
x^3(u^1,u^2,u^3,u^4) &= \left(\frac{u^2u^4 - (u^3)^2}{(u^2)^2}\ln u^2 + 2\right) e^{\frac{u^3}{u^2}} + \frac{C_2}{2}\big(\ln u^2\big)^2 \\
x^4(u^1,u^2,u^3,u^4) &= u^2
\end{aligned}
$$

and

$$
e = \tilde{\partial}_1,\ \ E = \left(x^1 - \frac{x^4}{2}\right)\tilde{\partial}_1 + C_2\,\tilde{\partial}_2 + x^2\,\tilde{\partial}_3 + x^4\,\tilde{\partial}_4,
$$

$$
\tilde{\eta} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{bmatrix}.
$$

Up to second-order polynomial terms the prepotential is

$$
\begin{aligned}
F(x^1,x^2,x^3,x^4) = {} & \frac{C_2}{24}(x^4)^2\big(\ln x^4\big)^3 - \frac{3}{16}\left(C_2 + \frac{2}{3}x^2\right)(x^4)^2\big(\ln x^4\big)^2 \\
& + \frac{7}{16}\left(C_2 + \frac{6}{7}x^2 + \frac{4}{7}x^3\right)(x^4)^2\ln x^4
\end{aligned}
$$

$$+\frac{1}{2}x^1x^3x^4+\frac{1}{2}(x^1)^2x^2-\frac{7\,x^2+6\,x^3}{16}(x^4)^2.$$

The last case to be considered separatley is the one for $d=0$, as C_1 may not vanish. Here the flat coordinates are

$$\begin{aligned}
x^1(u^1,u^2,u^3,u^4) &= u^1+\frac{u^2}{2}\\
x^2(u^1,u^2,u^3,u^4) &= \frac{u^2u^4-(u^3)^2}{u^2}\,e^{\frac{u^3}{u^2}}-\frac{C_1-2\,C_2}{2}\,u^2\\
x^3(u^1,u^2,u^3,u^4) &= \frac{u^2u^4-(u^3)^2-2(u^2)^2}{(u^2)^2}\,e^{\frac{u^3}{u^2}}\\
&\quad-\frac{C_1-4\,C_2}{4}\,\ln u^2\\
x^4(u^1,u^2,u^3,u^4) &= \frac{(u^2)^2}{2}.
\end{aligned}$$

In such coordinates the unit and the Euler vector fields are respectively $e=\tilde{\partial}_1$ and

$$E=x^1\,\tilde{\partial}_1+x^2\,\tilde{\partial}_2-\left(C_2-\frac{C_1}{4}\right)\tilde{\partial}_3-2\,x^4\,\tilde{\partial}_4$$

and up to second-order polynomial terms the prepotential reads

$$\begin{aligned}
F(x^1,x^2,x^3,x^4) &= \frac{3(C_1-4C_2)\ln(2\,x^4)-8C_1}{72}\,\sqrt{2}(x^4)^{\frac{3}{2}}\\
&\quad+\frac{32C_2+24\,x^3}{72}\,\sqrt{2}(x^4)^{\frac{3}{2}}-\frac{1}{8}\,x^2\,x^4\,\ln(x^4)\\
&\quad+\frac{C_1}{6}(x^1)^3+\frac{1}{2}(x^1)^2x^2-\frac{1}{2}\,x^1\,x^3\,x^4.
\end{aligned}$$

3.3 The multiple-block cases

As seen above, an expression for the Frobenius metric in terms of a function f realizing (3.13), (3.14), (3.15) can be achieved in the case where the operator $L=E\,\circ$ has multiple Jordan blocks as well. This section is devoted to show how, in this case, it is possible to reduce the conditions defining a Frobenius manifold to a single ODE in dimension 3 and to a system of PDEs in dimension 4. We will then provide explicit examples of solutions and their respective Frobenius potentials.

3.3.1 The three-dimensional case

In dimension 3, the only regular non-semisimple case with multiple Jordan blocks is the one of two blocks, of sizes 2 and 1 respectively.

We already know that there exists a function f of the variable

$$z = \frac{u^3 - u^1}{u^2}$$

such that the metric can be written as

$$\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & 0 \\ F_2 & 0 & 0 \\ 0 & 0 & F_3 \end{bmatrix}$$

for

$$\begin{aligned} F_1(z) &= -f'(z) + C_1 \\ F_2(z) &= -z\, f'(z) - (d-1)\, f(z) + C_2 \\ F_3(z) &= f'(z) \end{aligned}$$

where C_1, C_2 are constants. In particular, the quantity $F_1 + F_3 = C_1$ must vanish whenever $d \neq 0$. The flatness condition amounts to the following equation:

$$\begin{aligned} 2\, z^2\, (z\, f' + (d-1) f - C_2)\, f'\, f''' = {}& z^2 (f'')^2 \big(3\, z\, f' + (d-1) f - C_2\big) \\ & + 4\big((d-1)\, z\, f' - (d-1) f + C_2\big)\, z\, f'\, f'' \\ & + d\, (f')^2 \big((3\, d-2)\, z\, f' + (d-2)\, ((d-1) f - C_2)\big). \end{aligned} \tag{3.59}$$

By solving (3.59), one can determine explicitly the function f, which turns out to be expressed in terms of hyperbolic functions.

Example 3.11 *When $d = 0$ a solution to* (3.59) *is provided by choosing $f(z) = a\, z + b$ for some constants a, b. The metric is constant in canonical coordinates and reads*

$$\eta = \begin{bmatrix} C_1 - a & C_2 + b & 0 \\ C_2 + b & 0 & 0 \\ 0 & 0 & a \end{bmatrix}.$$

Up to second-order polynomial terms the Frobenius potential is

$$F(u^1, u^2, u^3) = \frac{C_1 - a}{6}\, (u^1)^3 + \frac{C_2 + b}{2}\, (u^1)^2\, u^2 + \frac{a}{6}\, (u^3)^3.$$

3.3.2 The four-dimensional case

In dimension 4, three rearrangements in Jordan blocks are possible:

- two blocks, of sizes 3 and 1 respectively;

- two blocks, both of size 2;
- three blocks, of sizes 2, 1 and 1 respectively.

Two blocks, of sizes 3 and 1 respectively. In this case metric is given by

$$\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & F_3 & 0 \\ F_2 & F_3 & 0 & 0 \\ F_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & F_4 \end{bmatrix}$$

where

$$\begin{aligned}
F_1(z,w) &= -\partial_w f(z,w) + C_1 \\
F_2(z,w) &= -z\,\partial_z f(z,w) - w\,\partial_w f(z,w) - (d-1)\,f(z,w) + C_2 \\
F_3(z,w) &= \partial_z f(z,w) \\
F_4(z,w) &= \partial_w f(z,w)
\end{aligned}$$

for some constants C_1, C_2 and a function f of the variables

$$z = \frac{u^3}{u^2}, \qquad w = \frac{u^4 - u^1}{u^2}.$$

In particular, the quantity $F_1 + F_4$ is a constant that must vanish whenever $d \neq 0$. The flatness conditions amount to the following system of PDEs for the third derivatives of f:

$$\partial_z^3 f = \frac{3(\partial_z^2 f)^2}{2\partial_z f} \tag{3.60}$$

$$\partial_z^2 \partial_w f = \frac{\partial_z\partial_w f\,\big(\partial_z f\,\partial_z\partial_w f + 2\,\partial_z^2 f\,\partial_w f\big)}{2\partial_z f\,\partial_w f} \tag{3.61}$$

$$\begin{aligned}
\partial_z\partial_w^2 f = {} & \frac{1}{2w(\partial_z f)^2\,\partial_w f}\big(2w\,\partial_z f\,\partial_w f(\partial_z\partial_w f)^2 + \big(-\partial_w f\,\partial_z^2 f\,\big(w\,\partial_w f \\
& + (d-1)\,f - C_2\big) + (\partial_z f)^2\big((d-2)\,\partial_w f + w\,\partial_w^2 f\big)\big)\,\partial_z\partial_w f \\
& + \partial_z f\,\partial_z^2 f\,\partial_w f\,\big(w\,\partial_w^2 f + d\,\partial_w f\big)\big)
\end{aligned} \tag{3.62}$$

$$\begin{aligned}
\partial_w^3 f = {} & \frac{1}{2\,w^3\,\partial_w f\,(\partial_z f)^3}(-\partial_z f\,\partial_w f\,w^2\,(w\,\partial_w f + (d-1)\,f \\
& - C_2)\,(\partial_z\partial_w f)^2 - ((w^3\,(\partial_w f)^2 - 2\,(-(d-1)\,f + C_2)\,w^2\,\partial_w f \\
& + (-(d-1)\,f - w\,C_1 + C_2)\,\partial_z f + (-(d-1)\,f \\
& + C_2)^2\,w)\,\partial_z^2 f - 3\,(\partial_z f)^2\,(w^3\,\partial_w^2 f - (-4\,w^2\,d\,\partial_w f)\tfrac{1}{3} \\
& - \tfrac{d-2}{3}\,\partial_z f + (-\tfrac{d}{3}\,(-(d-1)\,f + C_2)\,w)))\,\partial_w f\,\partial_z\partial_w f
\end{aligned} \tag{3.63}$$

$$
\begin{aligned}
&+ \partial_z f\, ((w^2\, (w\, \partial_w f + (d-1)\, f - C_2)\, \partial_w^2 f \\
&+ (w^2\, \partial_w f - \partial_z f + (d-1)\, w\, f \\
&- w\, C_2)\, d\, \partial_w f)\, \partial_w f\, \partial_z^2 f + (\partial_z f)^2\, (w^2\, (\partial_w^2 f)^2 \\
&- w\, \partial_w f\, (d+4)\, \partial_w^2 f - 2\, (\partial_w f)^2\, d)\, w)).
\end{aligned}
$$

Two blocks, both of size 2. In this case the metric is given by

$$
\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & 0 & 0 \\ F_2 & 0 & 0 & 0 \\ 0 & 0 & F_3 & F_4 \\ 0 & 0 & F_4 & 0 \end{bmatrix}
$$

where

$$
\begin{aligned}
F_1(z,w) &= -\partial_z f(z,w) + C_1 \\
F_2(z,w) &= -z\, \partial_z f(z,w) - w\, \partial_w f(z,w) - (d-1)\, f(z,w) + C_2 \\
F_3(z,w) &= \partial_z f(z,w) \\
F_4(z,w) &= \partial_w f(z,w)
\end{aligned}
$$

for some constants C_1, C_2 and a function f of the variables

$$
z = \frac{u^3 - u^1}{u^2}, \qquad w = \frac{u^4}{u^2}.
$$

In particular, the quantity $F_1 + F_3$ is a constant that must vanish whenever $d \neq 0$. The flatness conditions amount to a system of PDEs for the third derivatives of f which is presented in Appendix A.

Three blocks, of sizes 2, 1 and 1 respectively. In this case the metric is given by

$$
\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & 0 & 0 \\ F_2 & 0 & 0 & 0 \\ 0 & 0 & F_3 & 0 \\ 0 & 0 & 0 & F_4 \end{bmatrix}
$$

where

$$
\begin{aligned}
F_1(z,w) &= -\partial_z f(z,w) - \partial_w f(z,w) + C_1 \\
F_2(z,w) &= -z\, \partial_z f(z,w) - w\, \partial_w f(z,w) - (d-1)\, f(z,w) + C_2 \\
F_3(z,w) &= \partial_z f(z,w)
\end{aligned}
$$

$$F_4(z,w) = \partial_w f(z,w)$$

for some constants C_1, C_2 and a function f of the variables

$$z = \frac{u^3 - u^1}{u^2}, \qquad w = \frac{u^4 - u^1}{u^2}.$$

In particular, the quantity $F_1 + F_3 + F_4$ is a constant that must vanish whenever $d \neq 0$. The flatness conditions amount to a system of PDEs for the third derivatives of f which is presented in Appendix A.

In all of the three cases there exist solutions corresponding to a linear expression for the function $f(z,w)$. More precisely, when $d = 0$ the function

$$f(z,w) = a\,z + b\,w + c \tag{3.64}$$

(where a, b and c are constants) is a solution to the respective system of PDEs for the third derivatives of f. With this choice of f, the Frobenius metric turns out to be constant in canonical coordinates. In the following example we provide such a metric and the Frobenius potential in the case when $L = E\circ$ has two Jordan blocks of sizes 3 and 1. We refer to Appendix A for the remaining cases.

Example 3.12 *Let the function $f(z,w)$ be of the form* (3.64), $d = 0$. *When $L = E\circ$ has two Jordan blocks of sizes 3 and 1 the metric is given by*

$$\eta = \begin{bmatrix} C_1 - b & C_2 + c & a & 0 \\ C_2 + c & a & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \end{bmatrix}$$

and up to second-order polynomial terms the Frobenius potential is

$$\begin{aligned} F(u^1,u^2,u^3,u^4) =& \frac{C_1 - b}{6}\,(u^1)^3 + \frac{C_2 + c}{2}\,(u^1)^2\,u^2 + \frac{a}{2}\,(u^1)^2\,u^3 \\ &+ \frac{a}{2}\,u^1\,(u^2)^2 + \frac{b}{6}\,(u^4)^3. \end{aligned}$$

When $L = E\circ$ has two Jordan blocks of sizes 3 and 1, a less trivial example is presented as follows.

Example 3.13 *Let $L = E\circ$ have two Jordan blocks of sizes 3 and 1. When looking for a function f of the form*

$$f(z,w) = a\,z + g(w)$$

for some function $g(w)$, the system (3.60)–(3.63) *comes down to a single ODE for $g(w)$:*

$$2\,w^2\,g'(w)\,g'''(w) - w^2\,\big(g''(w)\big)^2 + (d+4)\,w\,g'(w)\,g''(w) + 2\,d\,\big(g'(w)\big)^2 = 0. \tag{3.65}$$

This yields

$$g(w) = a_1 + \frac{d^2\,(a_2)^2}{16\,a_3\,(d-1)\,w} + a_2\,w^{-\frac{d}{2}} + a_3\,w^{1-d}$$

when $d \neq 1$ *and*

$$g(w) = a_1 - \frac{(a_2)^2\,\ln w}{16\,a_3} + \frac{a_2}{\sqrt{w}} + \frac{a_3}{w}$$

when $d = 1$*, for some constants* a_1*,* a_2*,* a_3*. For instance, when* $d = 2$ *in the flat coordinates*

$$\begin{aligned}
x^1(u^1,u^2,u^3,u^4) &= \frac{u^3}{(u^2)^2} + \frac{a - C_2}{a\,u^2} + \frac{(a_2)^2}{4\,a\,a_3\,(u^4-u^1)} + \frac{a_2+a_3}{a\,(u^4-u^1)}\\
x^2(u^1,u^2,u^3,u^4) &= \ln u^2\\
x^3(u^1,u^2,u^3,u^4) &= \ln\left(u^4-u^1\right)\\
x^4(u^1,u^2,u^3,u^4) &= u^1
\end{aligned}$$

the metric becomes

$$\widetilde{\eta} = \begin{bmatrix} 0 & 0 & 0 & a\\ 0 & a & 0 & 0\\ 0 & 0 & -\frac{(2\,a_3+a_2)^2}{4\,a_3} & 0\\ a & 0 & 0 & 0 \end{bmatrix}$$

and up to second-order polynomial terms the Frobenius potential is

$$F(x^1,x^2,x^3,x^4) = -\frac{(2\,a_3+a_2)^2}{8\,a_3}\left(2\,e^{x^3} + (x^3)^2\,x^4\right) + \frac{a}{2}\left(x^1\,(x^4)^2 + (x^2)^2\,x^4\right).$$

In flat coordinates the unit and the Euler vector fields are respectively written as

$$e = \tilde{\partial}_4$$

and

$$E = -x^1\,\tilde{\partial}_1 + \tilde{\partial}_2 + \tilde{\partial}_3 + x^4\,\tilde{\partial}_4.$$

Chapter 4

Regular Lauricella bi-flat F-manifolds

This chapter is devoted to the construction of a class of regular bi-flat F-manifolds, called *Lauricella bi-flat F-manifolds*, and it is based on [66].

The starting point of such a construction is a $(1,1)$-type tensor field L with vanishing Nijenhuis torsion (see (4.17) below), to which a bidifferential complex (d, d_L) on the Grasmann algebra of differential forms, known by the name of Frölicher-Nijenhuis bicomplex [40], can be associated. Such a complex plays an important role in the theory of integrable systems, in both finite [68] and infinite-dimensional case [62]. More in general, in recent years there has been a growing interest in applications of Nijenhuis geometry to integrable systems of hydrodynamic type (see [9] and references therein).

Another ingredient in our construction is a function a_0 which is solution to equation (4.18). By means of a generalized Lenard-Magri chain, it has been shown in [62] that an integrable hierachy of quasilinear systems of PDEs can be defined, starting from L and a_0, in terms of tensor fields being polynomials in L.

We choose L to be the operator of multiplication by the Euler vector field on a flat F-manifold and we impose that the flows of the resulting integrable hierarchy define symmetries of its principal hierarchy [64], in the sense that the flows of the two hierarchies must pairwise commute. This leads to the additional condition (4.25).

Our main Theorem 4.13 states that for any choice of the Jordan canonical form of the operator of multiplication by the Euler vector field and for any choice of some weights parametrizing the Jordan blocks, there is a unique associated regular bi-flat F-manifold structure. We end up with a multi-parameter family of regular bi-flat F-manifolds, the parameters being as many as the Jordan blocks appearing in the Jordan canonical form of the operator of multiplication by the Euler vector field. In conformity with the semisimple case studied in [6], we call bi-flat structures obtained by this procedure *Lauricella bi-flat F-manifolds*, as related to the theory of Lauricella functions [56] and Lauricella connections [59].

Given $\varepsilon_1, \dots, \varepsilon_n \in (0,1)$, the *Lauricella function* of weight $\varepsilon := (\varepsilon_1, \dots, \varepsilon_n)$ at a point $u := (u^1, \dots, u^n) \in \mathbb{C}^n \setminus \mathcal{H}$, where $\mathcal{H} := \underset{1\leq i<j\leq n}{\cup} \{u \in \mathbb{C}^n | u^i = u^j\}$, is defined by

$$\int_{\gamma_u} \eta_u = \int_{\gamma_u} (u^1 - \zeta)^{-\varepsilon_1} \dots (u^n - \zeta)^{-\varepsilon_n} \, d\zeta.$$

Here γ_u is an oriented piecewise differentiable arc whose end-points lie in $\{u^1, \dots, u^n\}$, but such that γ_u does not meet this set elsewhere, and a determination of the multivalued differential η_u is fixed. Let δ_k denote the oriented piecewise differentiable arc connecting u^{k-1} with u^k and let L_u^ε denote the $(n-1)$-dimensional vector space generated by $\left\{ \int_{\delta_k} \eta_u \right\}_{k\in\{2,\dots,n\}}$. Any $f \in L_u^\epsilon$ satisfies the following conditions (see [59] for details):

1. $e(f) = 0$, where $e = \sum_{i=1}^n \frac{\partial}{\partial u^i}$,
2. f is homogeneous of degree $1 - \sum_{i=1}^n \varepsilon_i$,
3. f satisfies the system of differential equations

$$(u^i - u^j) \frac{\partial^2 f}{\partial u^i \partial u^j} = \varepsilon_j \frac{\partial f}{\partial u^i} - \varepsilon_i \frac{\partial f}{\partial u^j}, \quad 1 \leq i < j \leq n. \tag{4.1}$$

The *Euler-Poisson-Darboux system* (4.1) can be rewritten in the form

$$d d_L f = d a_0 \wedge df. \tag{4.2}$$

By definition, the flat coordinates of the connection ∇ of the associated bi-flat F-manifold are the solutions of the Euler-Poisson-Darboux system (4.1) satisfying the condition (1). The homogeneity condition (2) selects $n-1$ flat coordinates that are Lauricella functions, the remaining flat coordinate being the function a_0.

Our work strongly relies on some technical lemmas, whose proofs were moved to Appendix B in order to provide a cleaner presentation.

4.1 Integrable systems of hydrodynamic type

Integrable diagonal systems of hydrodynamic type

$$r^i_t = v^i(r) r^i_x, \qquad i \in \{1, \dots, n\}, \tag{4.3}$$

have been studied by Tsarev in [83]. Assuming $v^i \neq v^j$ for $i \neq j$, Tsarev proved that the whole information about the integrability of such systems is contained in the $n(n-1)$ functions

$$\Gamma^i_{ij} = \frac{\partial_j v^i}{v^j - v^i}, \qquad i \neq j. \tag{4.4}$$

The integrability of this system amounts to the conditions

$$\partial_j \left(\frac{\partial_k v^i}{v^k - v^i} \right) = \partial_k \left(\frac{\partial_j v^i}{v^j - v^i} \right), \qquad i \neq j \neq k \neq i. \tag{4.5}$$

Systems satisfying the condition (4.5) are called *semi-Hamiltonian systems* or *rich systems*. They possess infinitely many symmetries, depending on n functions of a single variable,

$$r^i_\tau = w^i(r) r^i_x, \qquad i \in \{1, \dots, n\}, \tag{4.6}$$

obtained by solving the linear system

$$\partial_j w^i = \Gamma^i_{ij}(w^j - w^i), \qquad i \neq j, \tag{4.7}$$

and infinitely many densities of conservation laws obtained by solving the linear system

$$\partial_i \partial_j h - \Gamma^i_{ij} \partial_i h - \Gamma^j_{ji} \partial_j h = 0, \qquad i \neq j. \tag{4.8}$$

Tsarev's integrability condition is the compatibility of the systems (4.7) and (4.8).

Let us consider now a general system of hydrodynamic type

$$u^i_t = V^i_j(u) u^j_x, \qquad i \in \{1, \dots, n\}. \tag{4.9}$$

Assuming that at each point the $(1,1)$-tensor field V has pairwise distict eigenvalues, the diagonalizability of the system is equivalent to the vanishing of the *Haantjes tensor* of V [46]

$$H_V(X,Y) := N_V(VX, VY) - V\, N_V(X, VY) - V\, N_V(VX, Y) + V^2\, N_V(X,Y) = 0$$

for any choice of vector fields X, Y, where N_V is defined as in (4.17), and the semi-Hamiltonian condition (4.5) is equivalent to the vanishing of a tensor field, called the *semi-Hamiltonian tensor* [74]. The diagonalizing coordinates $(r^1, ..., r^n)$ are called *Riemann invariants* and the diagonal entries of the $(1,1)$-tensor field V in such coordinates are called *characteristic velocities* of the system. Given a semi-Hamiltonian system and n functional independent solutions $(h^1, ..., h^n)$ of the system (4.8), one has

$$h^i_t = \partial_x K^i, \qquad i \in \{1, \dots, n\}, \tag{4.10}$$

for some functions $K^i(r^1, ..., r^n)$, $i \in \{1, \dots, n\}$. In other words the system can be written as a system of conservation laws. It turns out that also the converse statement it is true: a system of conservation laws admitting Riemann invariants is semi-Hamiltonian. In this way, following Sevennec, one can equivalently define semi-Hamiltonian systems as systems of hydrodynamic type that can be written both in the diagonal and in the conservative forms (4.3) and (4.10). We refer to [81] for further details.

The main equations of Tsarev's theory can be also formulated in terms of a family of torsionless connections.

Definition 4.1 *Given a* $(1,1)$*-tensor field* V*, a torsionless connection* ∇ *satisfying*

$$d_\nabla V = 0 \tag{4.11}$$

will be called a Tsarev's connection *associated with* V*.*

Here $d_\nabla V$ is the the exterior covariant derivative of the $(1,1)$-tensor field V:

$$(d_\nabla V)^i_{jk} = \nabla_j V^i_k - \nabla_k V^i_j = \partial_j V^i_k - \partial_k V^i_j + \Gamma^i_{js} V^s_k - \Gamma^i_{ks} V^s_j, \quad i,j,k \in \{1,\dots,n\}.$$

Tsarev's connections are not uniquely defined: in the Riemann invariants $r^1, ..., r^n$, where $V = \mathrm{diag}(v^1, ..., v^n)$, the above condition is equivalent to $\Gamma^i_{jk} = 0$ for pairwise distinct indices and to (4.4). Moreover, $\Gamma^i_{ji} = \Gamma^i_{ij}$ for $i \neq j$ due to the vanishing of the torsion. All of the remaining Christoffel symbols $\{\Gamma^i_{jj},\, \Gamma^i_{ii}\}_{i,j\in\{1,\dots,n\},\, i\neq j}$ are free. In order to prove this fact, we have to spell out the condition

$$(d_\nabla V)^i_{jk} = \partial_j V^i_k + \Gamma^i_{mj} V^m_k - \partial_k V^i_j - \Gamma^i_{mk} V^m_j = 0, \quad i,j,k \in \{1,\dots,n\},$$

in the Riemann invariants. For pairwise distinct indices $i,j,k \in \{1,\dots,n\}$ we obtain

$$(d_\nabla V)^i_{jk} = \Gamma^i_{kj}(v^k - v^j) = 0$$

and as a consequence, taking into account that the characteristic velocities are pairwise distinct, we get $\Gamma^i_{kj} = 0$ for $i \neq j \neq k \neq i$. If $i = k$ (or equivalently $i = j$) we get

$$(d_\nabla V)^i_{ji} = \partial_j v^i + \Gamma^i_{ij}(v^i - v^j) = 0, \quad i,j \in \{1,\dots,n\},\, i \neq j.$$

Theorem 4.2 *A diagonalizable system of hydrodynamic type with pairwise distinct characteristic velocities is* semi-Hamiltonian *if and only if the Tsarev's connections associated with* V *satisfy the condition* $d^2_\nabla W = 0$ *for any* $(1,1)$*-type tensor field* W *commuting with* V*.*

Proof. By straightforward computation, for any choice of the indices we get

$$[d^2_\nabla W]^l_{jik} = R^l_{ijn} W^n_k + R^l_{jkn} W^n_i + R^l_{kin} W^n_j, \quad i,j,l,k \in \{1,\dots,n\},$$

where R is the Riemann tensor of ∇:

$$R^k_{ijl} = \partial_j \Gamma^k_{il} - \partial_i \Gamma^k_{jl} + \Gamma^k_{js}\Gamma^s_{il} - \Gamma^k_{is}\Gamma^s_{jl}, \quad i,j,l,k \in \{1,\dots,n\}.$$

In the Riemann invariants the set of matrices W are diagonal and the condition $d^2_\nabla W = 0$ reads

$$[d^2_\nabla W]^l_{jik} = R^l_{ijk} w^k + R^l_{jki} w^i + R^l_{kij} w^j = 0, \quad i,j,k,l \in \{1,\dots,n\},$$

for any $(w^1, ..., w^n)$. Due to the arbitrariness of W, this is equivalent to $R^l_{ijk} = 0$ for pairwise distinct indices i,j,k. If at least two out of the three indices i,j,k are equal to each other then the condition is automatically satisfied, since $R^l_{jik} = -R^l_{ijk}$ for all indices. Thus, assuming the indices i,j,k pairwise distinct, we need to consider the case $l = i$ (the cases $l = j$ and $l = k$ are equivalent to this one). Due to the arbitrariness of W, we obtain the conditions

$$R^i_{jki} = 0, \quad R^i_{ijk} = 0$$

for all suitable indices. The second condition, also known as *Darboux-Tsarev system*, reads

$$\partial_i \Gamma^k_{kj} + \Gamma^k_{ki}\Gamma^k_{kj} - \Gamma^k_{kj}\Gamma^j_{ji} - \Gamma^k_{ik}\Gamma^i_{ij} = 0, \qquad i \neq j \neq k \neq i, \tag{4.12}$$

while the first condition reads

$$\partial_j \Gamma^i_{ik} = \partial_k \Gamma^i_{ij}, \qquad i \neq j \neq k \neq i, \tag{4.13}$$

Clearly (4.13) follows from (4.12). Remarkably, if the functions $\{\Gamma^i_{ij}\}_{i,j\in\{1,\dots,n\},\, i\neq j}$ are given by (4.4) then both conditions (4.5) and (4.12) are equivalent to (4.13), due to the identity [83]

$$\partial_i \Gamma^k_{kj} + \Gamma^k_{ki}\Gamma^k_{kj} - \Gamma^k_{kj}\Gamma^j_{ji} - \Gamma^k_{ki}\Gamma^i_{ij} = \frac{v^i - v^k}{v^j - v^i}\left[\partial_j\left(\frac{\partial_i v^k}{v^i - v^k}\right) - \partial_i\left(\frac{\partial_j v^k}{v^j - v^k}\right)\right]. \tag{4.14}$$

for all suitable indices. ■

The Sevennec's result can be formulated as follows.

Theorem 4.3 *Let V be a $(1,1)$-type tensor field with pairwise distinct eigenvalues and with vanishing Haantjies tensor. Then V defines a semi-Hamiltonian system if and only if among the associated Tsarev's connections there is a flat connection.*

Proof. Let us assume that ∇ is a flat Tsarev's connection. In flat coordinates, for suitable indices, the condition $d_\nabla V = 0$ reads $\partial_k V^i_j = \partial_j V^i_k$, which implies that in flat coordinates (locally) we have $V^i_j = \partial_j X^i$ and thus $V^i_j u^j_x = \partial_x X^i$. Since existence of Riemann invariants is assumed by hypothesis, the Sevennec's result implies that the system defined by V is semi-Hamiltonian.

Let us now assume that V defines a semi-Hamiltonian system. Then, due to Sevennec's result, there exist coordinates $u^1, ..., u^n$ where $V^i_j u^j_x = \partial_x X^i$ for suitable indices, implying $\partial_k V^i_j = \partial_j V^i_k$. Let us define ∇ in the coordinates $(u^1, ..., u^n)$ as $\Gamma^i_{jk} = 0$. Then the condition $\partial_k V^i_j = \partial_j V^i_k$ can be written as $d_\nabla V = 0$. In other words, ∇ is a flat Tsarev's connection. ■

The symmetries

$$u^i_\tau = W^i_j(\mathbf{u})u^j_x, \qquad i \in \{1,\dots,n\}, \tag{4.15}$$

of the system (4.9) are defined by $(1,1)$-type tensor fields $W(u)$ commuting with V and satisfying the condition

$$d_\nabla W = 0. \tag{4.16}$$

The full hierarchy is thus defined by the solutions of the system (4.16).

4.2 Frölicher-Nijenhuis bicomplex and integrable systems

Let L be a tensor field of type $(1,1)$ with vanishing Nijenhuis torsion. This means that for any pair of vector fields X and Y we have

$$N_L(X,Y) := [LX, LY] - L\,[X, LY] - L\,[LX, Y] + L^2\,[X,Y] = 0. \tag{4.17}$$

Following [62], we recall a construction of integrable hierarchies starting from the *Frölicher-Nijenhuis bicomplex* $(d, d_L, \Omega(M))$, which we introduce as follows. The differential d is the usual de Rham differential, while the differential d_L is defined as

$$\begin{aligned}(d_L\omega)(X_0,\dots,X_k) := &\sum_{i=0}^{k}(-1)^i(LX_i)(\omega(X_0,\dots,\hat{X}_i,\dots,X_k))\\ &+\sum_{0\le i<j\le k}(-1)^{i+j}\omega([X_i,X_j]_L, X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots X_k)\end{aligned}$$

for $X_0,\dots,X_k$ vector fields and a k-differential form $\omega \in \Omega^k(M)$, where $\hat{X}$ denotes the absence of a vector field X in the arguments of ω and

$$[X_i, X_j]_L := [LX_i, X_j] + [X_i, LX_j] - L[X_i, X_j]$$

extends the usual commutator of vector fields by involving the presence of L. For $L = I$ the vector field $[X_i, X_j]_L$ reduces to the commutator $[X_i, X_j]$ and the differential d_L reduces to d. The vanishing of the Nijenhuis torsion of L translates to the fact that $d_L^2 = 0$. The differentials d and d_L anticommute, thus the pair (d, d_L) defines a bidifferential complex. Starting from a solution of the equation

$$dd_La_0 = 0 \tag{4.18}$$

a sequence of functions $a_1, a_2, a_3, \dots$ can be recursively defined by means of a generalized Lenard-Magri chain:

$$da_{k+1} = d_La_k - a_kda_0, \qquad k \ge 0.$$

In turn, a sequence of $(1,1)$-type tensor fields $V_1, V_2, V_3, \dots$ can be constructed starting from the identity $V_0 = I$, by

$$V_{k+1} = V_k L - a_k I, \qquad k \geq 0.$$

This gives

$$\begin{aligned} V_0 &= I \\ V_1 &= L - a_0 I \\ V_2 &= L^2 - a_0 L - a_1 I, \\ &\vdots \\ V_n &= L^n - a_0 L^{n-1} - a_1 L^{n-2} + \dots - a_{n-1} I \\ &\vdots \end{aligned}$$

that is

$$V_n = L^n - \sum_{l=0}^{n-1} a_l L^{n-l-1}, \qquad n \geq 0.$$

In [62] (see also [60]) it has been proved that these $(1,1)$-type tensor fields define an integrable hierarchy of hydrodynamic type. Remarkably, this construction does not require that L is diagonalizable.

4.2.1 Examples

Generalized ε-system

The system of hydrodynamic type

$$\begin{bmatrix} u^1_{t_1} \\ u^2_{t_1} \\ \vdots \\ u^n_{t_1} \end{bmatrix} = \begin{bmatrix} u^1 - \sum_{k=1}^n \varepsilon_k u^k & 0 & \dots & 0 \\ 0 & u^2 - \sum_{k=1}^n \varepsilon_k u^k & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^n - \sum_{k=1}^n \varepsilon_k u^k \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^n_x \end{bmatrix} \tag{4.19}$$

has been obtained in [73] as finite component reduction of an infinite hydrodynamic chain. It can be written as $u_{t_1} = (L - a_0 I) u_x$ with $a_0 = \sum_{k=1}^n \varepsilon_k u^k$ and

$$L = \begin{bmatrix} u^1 & 0 & \dots & 0 \\ 0 & u^2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^n \end{bmatrix}. \tag{4.20}$$

For specific values of the constants $\varepsilon_1, \dots, \varepsilon_n$, it provides well-known examples of integrable systems of hydrodynamic type. The above hierarchy is related to the principal hierarchy associated with Lauricella bi-flat F-manifolds [63, 61].

Kodama-Konopelchenko system

The system of hydrodynamic type [53]

$$\begin{bmatrix} u^1_{t_1} \\ u^2_{t_1} \\ \vdots \\ u^{n-1}_{t_1} \\ u^n_{t_1} \end{bmatrix} = \begin{bmatrix} u^1 & 1 & 0 & \dots & 0 \\ 0 & u^1 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^1 & 1 \\ 0 & \dots & 0 & 0 & u^1 \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^{n-1}_x \\ u^n_x \end{bmatrix} \tag{4.21}$$

can be written as $u_{t_1} = (L - a_0 I)u_x$ with $a_0 = -u^1$ and

$$L = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{bmatrix}.$$

Clearly L has vanishing Nijenhuis torsion and $dd_L a_0 = 0$. By applying the first step of the recursive procedure, we have

$$\begin{aligned} \partial_1 a_1 &= -a_0 \partial_1 a_0 = -u^1 \\ \partial_2 a_1 &= \partial_1 a_0 - a_0 \partial_2 a_0 = -1 \\ \partial_3 a_1 &= 0 \\ &\vdots \\ \partial_n a_1 &= 0. \end{aligned}$$

This implies that, up to an negligible constant, $a_1 = -u^2 - \frac{(u^1)^2}{2}$. Therefore the first commuting flow $u_t = (L^2 - a_0 L - a_1 I)u_x$ is given by

$$\begin{bmatrix} u^1_{t_2} \\ u^2_{t_2} \\ \vdots \\ u^{n-1}_{t_2} \\ u^n_{t_2} \end{bmatrix} = \begin{bmatrix} u^2 + \frac{(u^1)^2}{2} & u^1 & 1 & \dots & 0 \\ 0 & u^2 + \frac{(u^1)^2}{2} & u^1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & u^2 + \frac{(u^1)^2}{2} & u^1 \\ 0 & \dots & 0 & 0 & u^2 + \frac{(u^1)^2}{2} \end{bmatrix} \begin{bmatrix} u^1_x \\ u^2_x \\ \vdots \\ u^{n-1}_x \\ u^n_x \end{bmatrix} \tag{4.22}$$

Higher flows can be obtained in a similar way. The system (4.21) is related to the theory of confluent Lauricella functions. Further details can be found in [53].

4.3 From integrable hierarchies to Lauricella bi-flat F-manifolds

Let $(M, \circ, e, E)$ be an F-manifold with Euler vector field. By using the Hertling-Manin condition and the properties of the Euler vector field, it is easy to check that the operator $E\circ$ has vanishing Nijenhuis torsion (see for instance [3]). Among Tsarev's connections of the associated integrable system we consider those satisfying the additional conditions

$$\nabla_j e^i = 0, \qquad \nabla_k c^i_{jl} = \nabla_j c^i_{kl}, \qquad i, j, k, l \in \{1, \dots, n\}.$$

In [63] such connections are called *natural connections*. In the next two subsections we will show that for special choices of the function a_0 the natural connections defined in this way are flat. Moreover, it is possible to define a second compatible flat structure.

4.3.1 Semisimple Lauricella bi-flat structure

Let us recall the following theorem of [61] (see also [60, 63] for the special case where $\varepsilon_1 = \varepsilon_2 = ... = \varepsilon_n$).

Theorem 4.4 *For any choice of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ there exists a unique semisimple bi-flat structure $(\nabla, \nabla^*, \circ, *, e, E)$ with canonical coordinates $\{u^1, ..., u^n\}$ such that $L = E\circ$ and*

$$d_\nabla(L - a_0 I) = 0, \tag{4.23}$$

where $a_0 = \sum_{k=1}^n \varepsilon_k u^k$. Moreover, in canonical coordinates this structure is given by

$$\begin{aligned}
e &= \sum_{k=1}^n \partial_k, \quad E = \sum_{k=1}^n u^k \partial_k,\\
c^i_{jk} &= \delta^i_j \delta^i_k, \qquad c^{*i}_{jk} = \frac{1}{u^i}\delta^i_j \delta^i_k, \qquad i, j, k,\\
\Gamma^i_{jk} &= 0, \quad \Gamma^{*i}_{jk} = 0, \qquad i \neq j \neq k \neq i,\\
\Gamma^i_{jj} &= -\Gamma^i_{ij}, \quad \Gamma^{*i}_{jj} = -\frac{u^i}{u^j}\Gamma^{*i}_{ij}, \qquad i \neq j,\\
\Gamma^i_{ij} &= \frac{\varepsilon_j}{u^i - u^j}, \quad \Gamma^{*i}_{ij} = \frac{\varepsilon_j}{u^i - u^j}, \qquad i \neq j,\\
\Gamma^i_{ii} &= -\sum_{l \neq i} \Gamma^i_{li}, \quad \Gamma^{*i}_{ii} = -\sum_{l \neq i} \frac{u^l}{u^i}\Gamma^{*i}_{li} - \frac{1}{u^i}.
\end{aligned}$$

4.3.2 Regular non-semisimple Lauricella bi-flat F-manifolds

We consider a generalization of Lauricella bi-flat structures to the regular non-semisimple case. Let $(M, \circ, e, E)$ be an n-dimensional regular F-manifold around a point $m \in M$ where the operator $E\circ$ has r Jordan blocks, of sizes $m_1, \dots, m_r$. Recalling Theorem 1.6 from the above Chapter 1, for the proof of which we remind to [18], let

$$\{u^{j(\alpha)} \,|\, \alpha \in \{1, \dots, r\},\, j \in \{1, \dots, m_\alpha\}\}$$

denote canonical coordinates, realizing

$$c^{i(\alpha)}_{j(\beta)k(\gamma)} = \delta^\alpha_\beta \delta^\alpha_\gamma \delta^i_{j+k-1}, \qquad e = \sum_{\alpha=1}^{r} \partial_{u^{1(\alpha)}}, \qquad E = \sum_{\alpha=1}^{r}\sum_{s=1}^{m_\alpha} u^{s(\alpha)}\,\partial_{u^{s(\alpha)}}$$

for all suitable indices. We start from the integrable hierarchy associated with the tensor field $L = E\circ$ and with $a_0 = \sum_{\alpha=1}^{r} \varepsilon_\alpha \mathrm{Tr}(L_\alpha)$. By construction, L contains r blocks $L_1, ..., L_r$ of size $m_1, ..., m_r$ respectively. Each block has the form

$$L_\alpha = \begin{bmatrix} u^{1(\alpha)} & 0 & \dots & 0 \\ u^{2(\alpha)} & u^{1(\alpha)} & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ u^{m_\alpha(\alpha)} & \dots & u^{2(\alpha)} & u^{1(\alpha)} \end{bmatrix} \tag{4.24}$$

where $u^{j(\alpha)} = u^{m_1+\dots+m_{\alpha-1}+j}$ for each $\alpha \in \{2, \dots, r\}$ and for each $j \in \{1, \dots, m_\alpha\}$, while $u^{j(\alpha)=u^j}$ for each $j \in \{1, \dots, m_\alpha\}$ for $\alpha = 1$.

The case where $m_1 = \dots = m_r = 1$ corresponds to the usual generalized ε-system. In the next section we will prove that for any choice of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$ and $m_1, ..., m_r$, up to dimension $n = 5$, there exists a unique bi-flat F-manifold structure such that $d_\nabla(L - a_0 I) = 0$. Thereafter, we will consider the case corresponding to a single Jordan block ($r = 1$) of arbitrary size. Finally, we will show that for any choice of $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$ there exists a unique regular bi-flat F-manifold structure such that $L = E\circ$ and

$$d_\nabla(L - a_0 I) = 0. \tag{4.25}$$

Remark 4.5 *By straightforward computation, one gets*

$$(d_\nabla(L - a_0 I))^i_{jk}\,\partial_i a_0 = L^i_j \nabla_i (da_0)_k - L^i_k \nabla_i (da_0)_j, \quad j, k \in \{1, \dots, n\}.$$

Therefore condition (4.25) *implies*

$$L^i_j \nabla_i (da_0)_k - L^i_k \nabla_i (da_0)_j = 0, \qquad j, k \in \{1, \dots, n\}.$$

4.4 Bi-flat Lauricella structures in dimension $2, 3, 4, 5$

In this section we provide a complete classification of regular non-semisimple bi-flat F-manifold structures in 2, 3, 4 and 5 dimensions.

4.4.1 2-dimensional case

2×2 **Jordan block**

$$L = \begin{bmatrix} u^1 & 0 \\ u^2 & u^1 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1}. \tag{4.26}$$

The non-vanishing Christoffel symbol of ∇ is $\Gamma^2_{22} = -\frac{2\varepsilon_1}{u^2}$.

4.4.2 3-dimensional case

3×3 **Jordan block**

$$L = \begin{bmatrix} u^1 & 0 & 0 \\ u^2 & u^1 & 0 \\ u^3 & u^2 & u^1 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1}, \quad a_0 = 3\varepsilon_1 u^1. \tag{4.27}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\Gamma^2_{22} = \Gamma^3_{23} = -\frac{3\varepsilon_1}{u^2}, \, \Gamma^3_{22} = \frac{3\varepsilon_1 u^3}{(u^2)^2}.$$

$2 \times 2 + 1 \times 1$ **Jordan blocks**

$$L = \begin{bmatrix} u^1 & 0 & 0 \\ u^2 & u^1 & 0 \\ 0 & 0 & u^3 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^3}, \quad a_0 = 2\varepsilon_1 u^1 + \varepsilon_3 u^3. \tag{4.28}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\Gamma^2_{22} = -\frac{2\varepsilon_1}{u^2}, \, \Gamma^1_{13} = \Gamma^2_{23} = -\Gamma^1_{11} = -\Gamma^1_{33} = -\Gamma^2_{12} = \frac{\varepsilon_3}{u^1 - u^3},$$
$$\Gamma^3_{11} = \Gamma^3_{33} = -\Gamma^3_{13} = \frac{2\varepsilon_1}{u^1 - u^3}, \, \Gamma^2_{11} = \Gamma^2_{33} = -\Gamma^2_{13} = \frac{\varepsilon_3 u^2}{(u^1 - u^3)^2}.$$

4.4.3 4-dimensional case

4×4 **Jordan block**

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0\\ u^3 & u^2 & u^1 & 0\\ u^4 & u^3 & u^2 & u^1 \end{bmatrix},\quad e=\frac{\partial}{\partial u^1},\quad a_0=4\varepsilon_1 u^1. \tag{4.29}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\Gamma^2_{22}=\Gamma^3_{23}=\Gamma^4_{33}=\Gamma^4_{24}=-\frac{4\varepsilon_1}{u^2},\ \Gamma^3_{22}=\Gamma^4_{23}=\frac{4\varepsilon_1 u^3}{(u^2)^2},\ \Gamma^4_{22}=\frac{4\varepsilon_1(u^2u^4-(u^3)^2)}{(u^2)^3}.$$

$3\times 3+1\times 1$ **Jordan blocks**

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0\\ u^3 & u^2 & u^1 & 0\\ 0 & 0 & 0 & u^4 \end{bmatrix},\quad e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^4},\quad a_0=3\varepsilon_1 u^1+\varepsilon_4 u^4. \tag{4.30}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\begin{aligned}
&\Gamma^2_{22}=\Gamma^3_{23}=-\frac{3\varepsilon_1}{u^2},\ \Gamma^3_{11}=\Gamma^3_{44}=-\Gamma^3_{14}=\frac{\varepsilon_4 u^3}{(u^1-u^4)^2}-\frac{\varepsilon_4 (u^2)^2}{(u^1-u^4)^3},\\
&\Gamma^4_{11}=\Gamma^4_{44}=-\Gamma^4_{14}=\frac{3\varepsilon_1}{u^1-u^4},\\
&\Gamma^3_{12}=-\Gamma^3_{24}=-\Gamma^2_{14}=\Gamma^2_{11}=\Gamma^2_{44}=\frac{\varepsilon_4 u^2}{(u^1-u^4)^2},\\
&\Gamma^3_{34}=-\Gamma^3_{13}=-\Gamma^2_{12}=-\Gamma^1_{11}=\Gamma^1_{14}=-\Gamma^1_{44}=\frac{\varepsilon_4}{u^1-u^4},\\
&\Gamma^3_{22}=\frac{3\varepsilon_1 u^3}{(u^2)^2}-\frac{\varepsilon_4}{u^1-u^4}.
\end{aligned}$$

$2\times 2+2\times 2$ **Jordan blocks**

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0\\ 0 & 0 & u^3 & 0\\ 0 & 0 & u^4 & u^3 \end{bmatrix},\quad e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^3},\quad a_0=2\varepsilon_1 u^1+2\varepsilon_3 u^3. \tag{4.31}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\begin{aligned}
&\Gamma^2_{22}=-\frac{2\varepsilon_1}{u^2},\ \Gamma^4_{44}=-\frac{2\varepsilon_3}{u^4},\ \Gamma^1_{13}=-\Gamma^2_{12}=-\Gamma^1_{11}=-\Gamma^1_{33}=\frac{2\varepsilon_3}{u^1-u^3},\\
&\Gamma^3_{11}=\Gamma^4_{34}=\Gamma^3_{33}=-\Gamma^3_{13}=-\Gamma^4_{14}=\frac{2\varepsilon_1}{u^1-u^3},
\end{aligned}$$

$$\Gamma^2_{11}=\Gamma^2_{33}=-\Gamma^2_{13}=\frac{2\varepsilon_3u^2}{(u^1-u^3)^2},\ \Gamma^4_{11}=\Gamma^4_{33}=-\Gamma^4_{13}=\frac{2\varepsilon_1u^4}{(u^1-u^3)^2}.$$

$2\times 2+1\times 1+1\times 1$ **Jordan blocks**

$$L=\begin{bmatrix}u^1&0&0&0\\u^2&u^1&0&0\\0&0&u^3&0\\0&0&0&u^4\end{bmatrix},\quad e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^3}+\frac{\partial}{\partial u^4},\quad a_0=2\varepsilon_1u^1+\varepsilon_3u^3+\varepsilon_4u^4. \tag{4.32}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\begin{aligned}
&\Gamma^2_{22}=-\frac{2\varepsilon_1}{u^2},\ \Gamma^1_{13}=\Gamma^2_{23}=-\Gamma^1_{33}=\frac{\varepsilon_3}{u^1-u^3},\ \Gamma^1_{14}=\Gamma^2_{24}=-\Gamma^1_{44}=\frac{\varepsilon_4}{u^1-u^4},\\
&\Gamma^3_{34}=-\Gamma^3_{44}=\frac{\varepsilon_4}{u^3-u^4},\ \Gamma^4_{34}=-\frac{\varepsilon_3}{u^3-u^4},\ \Gamma^3_{13}=-\Gamma^3_{11}=\frac{-2\varepsilon_1}{u^1-u^3},\\
&\Gamma^4_{14}=\frac{-2\varepsilon_1}{u^1-u^4},\ \Gamma^1_{11}=\Gamma^2_{12}=\Gamma^1_{33}+\Gamma^1_{44},\ \Gamma^2_{11}=\Gamma^2_{33}+\Gamma^2_{44},\\
&\Gamma^2_{33}=-\Gamma^2_{13}=\frac{\varepsilon_3u^2}{(u^1-u^3)^2},\ \Gamma^2_{44}=-\Gamma^2_{14}=\frac{\varepsilon_4u^2}{(u^1-u^4)^2},\\
&\Gamma^3_{33}=\frac{2\varepsilon_1}{u^1-u^3}-\frac{\varepsilon_4}{u^3-u^4},\ \Gamma^4_{44}=\frac{2\varepsilon_1}{u^1-u^4}+\frac{\varepsilon_3}{u^3-u^4}.
\end{aligned}$$

4.4.4 5-dimensional case

5×5 **Jordan block**

$$L=\begin{bmatrix}u^1&0&0&0&0\\u^2&u^1&0&0&0\\u^3&u^2&u^1&0&0\\u^4&u^3&u^2&u^1&0\\u^5&u^4&u^3&u^2&u^1\end{bmatrix},\quad e=\frac{\partial}{\partial u^1},\quad a_0=5\varepsilon_1u^1. \tag{4.33}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\begin{aligned}
&\Gamma^2_{22}=\Gamma^3_{23}=\Gamma^4_{33}=\Gamma^4_{24}=\Gamma^5_{25}=\Gamma^5_{34}=-\frac{5\varepsilon_1}{u^2},\\
&\Gamma^3_{22}=\Gamma^4_{23}=\Gamma^5_{24}=\Gamma^5_{33}=\frac{5\varepsilon_1u^3}{(u^2)^2},\ \Gamma^4_{22}=\Gamma^5_{23}=\frac{5\varepsilon_1(u^2u^4-(u^3)^2)}{(u^2)^3},\\
&\Gamma^5_{22}=\frac{5\varepsilon_1((u^2)^2u^5-2u^2u^3u^4+(u^3)^3)}{(u^2)^4}.
\end{aligned}$$

$4 \times 4 + 1 \times 1$ **Jordan blocks**

$$L = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 \\ u^3 & u^2 & u^1 & 0 & 0 \\ u^4 & u^3 & u^2 & u^1 & 0 \\ 0 & 0 & 0 & 0 & u^5 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^5}, \quad a_0 = 4\varepsilon_1 u^1 + \varepsilon_5 u^5. \tag{4.34}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\begin{aligned}
&\Gamma^2_{22} = \Gamma^3_{23} = \Gamma^4_{24} = \Gamma^4_{33} = -4\frac{\varepsilon_1}{u^2}, \\
&\Gamma^1_{15} = \Gamma^2_{25} = \Gamma^3_{35} = \Gamma^4_{45} = \frac{\varepsilon_5}{u^1 - u^5} \\
&\Gamma^1_{55} = \Gamma^2_{12} = \Gamma^1_{11} = \Gamma^3_{13} = \Gamma^4_{14} = \Gamma^4_{41} = -\frac{\varepsilon_5}{u^1 - u^5}, \\
&\Gamma^5_{11} = \Gamma^5_{55} = -\Gamma^5_{15} = \frac{4\varepsilon_1}{u^1 - u^5}, \\
&\Gamma^2_{11} = \Gamma^2_{55} = \Gamma^3_{12} = \Gamma^4_{13} = -\Gamma^2_{15} = -\Gamma^3_{25} = -\Gamma^4_{35} = -\Gamma^4_{53} = \frac{\varepsilon_5 u^2}{(u^1 - u^5)^2}, \\
&\Gamma^3_{11} = \Gamma^3_{55} = \Gamma^4_{12} = -\Gamma^3_{15} = -\Gamma^4_{25} = \frac{\varepsilon_5 u^3}{(u^1 - u^5)^2} - \frac{\varepsilon_5 (u^2)^2}{(u^1 - u^5)^3}, \\
&\Gamma^3_{22} = \Gamma^4_{23} = \frac{4\varepsilon_1 u^3}{(u^2)^2} - \frac{\varepsilon_5}{u^1 - u^5}, \\
&\Gamma^4_{11} = \Gamma^4_{55} = -\Gamma^4_{15} = \frac{\varepsilon_5 (u^2)^3}{(u^1 - u^5)^4} - \frac{2\varepsilon_5 u^2 u^3}{(u^1 - u^5)^3} + \frac{\varepsilon_5 u^4}{(u^1 - u^5)^2}, \\
&\Gamma^4_{22} = \frac{4\varepsilon_1 (u^2 u^4 - (u^3)^2)}{(u^2)^3} + \frac{\varepsilon_5 u^2}{(u^1 - u^5)^2}.
\end{aligned}$$

$3 \times 3 + 2 \times 2$ **Jordan blocks**

$$L = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 \\ u^3 & u^2 & u^1 & 0 & 0 \\ 0 & 0 & 0 & u^4 & 0 \\ 0 & 0 & 0 & u^5 & u^4 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^4} \quad a_0 = 3\varepsilon_1 u^1 + 2\varepsilon_4 u^4. \tag{4.35}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\begin{aligned}
&\Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = -\Gamma^1_{11} = -\Gamma^1_{44} = -\Gamma^2_{12} = -\Gamma^3_{13} = \frac{2\varepsilon_4}{u^1 - u^4}, \\
&\Gamma^2_{11} = \Gamma^2_{44} = \Gamma^3_{12} = -\Gamma^2_{14} = -\Gamma^3_{24} = \frac{2u^2\varepsilon_4}{(u^1 - u^4)^2}, \ \Gamma^2_{22} = \Gamma^3_{23} = -\frac{3\varepsilon_1}{u^2},
\end{aligned}$$

$$\Gamma^5_{55} = -\frac{2\varepsilon_4}{u^5},\ \Gamma^3_{22} = \frac{3\varepsilon_1 u^3}{(u^2)^2} - \frac{2\varepsilon_4}{u^1 - u^4},$$
$$\Gamma^3_{11} = \Gamma^3_{44} = -\Gamma^3_{14} = \frac{2\varepsilon_4(u^1u^3 - (u^2)^2 - u^3u^4)}{(u^1 - u^4)^3},$$
$$\Gamma^4_{11} = \Gamma^4_{44} = -\Gamma^4_{14} = -\Gamma^5_{15} = \Gamma^5_{45} = \frac{3\varepsilon_1}{u^1 - u^4},$$
$$\Gamma^5_{11} = \Gamma^5_{44} = -\Gamma^5_{14} = \frac{3u^5\varepsilon_1}{(u^1 - u^4)^2}.$$

$3 \times 3 + 1 \times 1 + 1 \times 1$ **Jordan blocks**

$$L = \begin{bmatrix} u^1 & 0 & 0 & 0 & 0 \\ u^2 & u^1 & 0 & 0 & 0 \\ u^3 & u^2 & u^1 & 0 & 0 \\ 0 & 0 & 0 & u^4 & 0 \\ 0 & 0 & 0 & 0 & u^5 \end{bmatrix}, \quad e = \frac{\partial}{\partial u^1} + \frac{\partial}{\partial u^4} + \frac{\partial}{\partial u^5}, \quad a_0 = 3\varepsilon_1 u^1 + \varepsilon_4 u^4 + \varepsilon_5 u^5. \tag{4.36}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\Gamma^3_{23} = \Gamma^2_{22} = -\frac{3\varepsilon_1}{u^2},\ \Gamma^1_{14} = \Gamma^2_{24} = \Gamma^3_{34} = -\Gamma^1_{44} = \frac{\varepsilon_4}{u^1 - u^4},$$
$$\Gamma^1_{15} = \Gamma^2_{25} = \Gamma^3_{35} = -\Gamma^1_{55} = \frac{\varepsilon_5}{u^1 - u^5},\ \Gamma^4_{45} = -\Gamma^4_{55} = \frac{\varepsilon_5}{u^4 - u^5},$$
$$\Gamma^5_{44} = -\Gamma^5_{45} = \frac{\varepsilon_4}{u^4 - u^5},\ \Gamma^4_{11} = -\Gamma^4_{14} = \frac{3\varepsilon_1}{u^1 - u^4},\ \Gamma^5_{11} = -\Gamma^5_{15} = \frac{3\varepsilon_1}{u^1 - u^5},$$
$$\Gamma^2_{11} = \Gamma^3_{12} = \frac{\varepsilon_4 u^2}{(u^1 - u^4)^2} + \frac{\varepsilon_5 u^2}{(u^1 - u^5)^2},$$
$$\Gamma^2_{44} = -\Gamma^2_{14} = -\Gamma^3_{24} = \frac{u^2\varepsilon_4}{(u^1 - u^4)^2},\ \Gamma^2_{55} = -\Gamma^2_{15} = -\Gamma^3_{25} = \frac{u^2\varepsilon_5}{(u^1 - u^5)^2},$$
$$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = -\frac{\varepsilon_4}{u^1 - u^4} - \frac{\varepsilon_5}{u^1 - u^5},$$
$$\Gamma^3_{11} = \frac{\varepsilon_4(-(u^2)^2 + (u^1 - u^4)u^3)}{(u^1 - u^4)^3} + \frac{\varepsilon_5(-(u^2)^2 + (u^1 - u^5)u^3)}{(u^1 - u^5)^3},$$
$$\Gamma^3_{44} = -\Gamma^3_{14} = \frac{\varepsilon_4(u^1u^3 - (u^2)^2 - u^3u^4)}{(u^1 - u^4)^3},\ \Gamma^3_{55} = -\Gamma^3_{15} = \frac{\varepsilon_5(u^1u^3 - (u^2)^2 - u^3u^5)}{(u^1 - u^5)^3},$$
$$\Gamma^4_{44} = \frac{3\varepsilon_1}{u^1 - u^4} - \frac{\varepsilon_5}{u^4 - u^5},\ \Gamma^5_{55} = \frac{3\varepsilon_1}{u^1 - u^5} + \frac{\varepsilon_4}{u^4 - u^5},$$
$$\Gamma^3_{22} = \frac{3u^3\varepsilon_1}{(u^2)^2} - \frac{\varepsilon_4}{u^1 - u^4} - \frac{\varepsilon_5}{u^1 - u^5}.$$

$2\times 2+2\times 2+1\times 1$ **Jordan blocks**

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0 & 0\\ 0 & 0 & u^3 & 0 & 0\\ 0 & 0 & u^4 & u^3 & 0\\ 0 & 0 & 0 & 0 & u^5 \end{bmatrix}, \quad e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^3}+\frac{\partial}{\partial u^5}, \quad a_0=2\varepsilon_1 u^1+2\varepsilon_3 u^3+\varepsilon_5 u^5. \tag{4.37}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\begin{aligned}
&\Gamma^1_{11}=\Gamma^2_{12}=-\frac{2\varepsilon_3}{u^1-u^3}-\frac{\varepsilon_5}{u^1-u^5},\ \Gamma^3_{33}=\Gamma^4_{34}=\frac{2\varepsilon_1}{u^1-u^3}-\frac{\varepsilon_5}{u^3-u^5},\\
&-\Gamma^1_{13}=\Gamma^1_{33}=-\frac{2\varepsilon_3}{u^1-u^3},\ \Gamma^1_{15}=\Gamma^2_{25}=-\Gamma^1_{55}=\frac{\varepsilon_5}{u^1-u^5},\\
&\Gamma^2_{11}=\frac{2u^2\varepsilon_3}{(u^1-u^3)^2}+\frac{u^2\varepsilon_5}{(u^1-u^5)^2},\ \Gamma^2_{33}=-\Gamma^2_{13}=\frac{2u^2\varepsilon_3}{(u^1-u^3)^2},\\
&\Gamma^2_{55}=-\Gamma^2_{15}=\frac{u^2\varepsilon_5}{(u^1-u^5)^2},\ \Gamma^2_{22}=-\frac{2\varepsilon_1}{u^2},\ \Gamma^4_{44}=-\frac{2\varepsilon_3}{u^4},\ \Gamma^2_{23}=\frac{2\varepsilon_3}{u^1-u^3},\\
&\Gamma^3_{11}=-\Gamma^3_{13}=-\Gamma^4_{14}=\frac{2\varepsilon_1}{u^1-u^3},\ \Gamma^3_{35}=\Gamma^4_{45}=-\Gamma^3_{55}=\frac{\varepsilon_5}{u^3-u^5},\\
&\Gamma^4_{11}=-\Gamma^4_{13}=\frac{2\varepsilon_1 u^4}{(u^1-u^3)^2},\ \Gamma^4_{33}=\frac{2u^4\varepsilon_1}{(u^1-u^3)^2}+\frac{u^4\varepsilon_5}{(u^3-u^5)^2},\\
&\Gamma^4_{55}=-\Gamma^4_{35}=\frac{u^4\varepsilon_5}{(u^3-u^5)^2},\ \Gamma^5_{11}=-\Gamma^5_{15}=\frac{2\varepsilon_1}{u^1-u^5},\ \Gamma^5_{33}=-\Gamma^5_{35}=\frac{2\varepsilon_3}{u^3-u^5},\\
&\Gamma^5_{55}=\frac{2\varepsilon_1}{u^1-u^5}+\frac{\varepsilon_3}{u^3-u^5}.
\end{aligned}$$

$2\times 2+1\times 1+1\times 1+1\times 1$ **Jordan blocks**

$$L=\begin{bmatrix} u^1 & 0 & 0 & 0 & 0\\ u^2 & u^1 & 0 & 0 & 0\\ 0 & 0 & u^3 & 0 & 0\\ 0 & 0 & 0 & u^4 & 0\\ 0 & 0 & 0 & 0 & u^5 \end{bmatrix}, \ e=\frac{\partial}{\partial u^1}+\frac{\partial}{\partial u^3}+\frac{\partial}{\partial u^4}+\frac{\partial}{\partial u^5},\ a_0=2\varepsilon_1 u^1+\varepsilon_3 u^3+\varepsilon_4 u^4+\varepsilon_5 u^5. \tag{4.38}$$

The non-vanishing Christoffel symbols Γ^i_{jk}, up to exchanging j with k, are

$$\begin{aligned}
&\Gamma^1_{11}=\Gamma^2_{12}=-\frac{\varepsilon_3}{u^1-u^3}-\frac{\varepsilon_4}{u^1-u^4}-\frac{\varepsilon_5}{u^1-u^5},\\
&\Gamma^1_{13}=-\Gamma^1_{33}=\frac{\varepsilon_3}{u^1-u^3},\ \Gamma^1_{14}=\frac{\varepsilon_4}{u^1-u^4},\ \Gamma^1_{15}=-\Gamma^1_{55}=\frac{\varepsilon_5}{u^1-u^5},\ \Gamma^1_{44}=-\frac{\varepsilon_4}{u^1-u^4},\\
&\Gamma^2_{11}=\frac{\varepsilon_3 u^2}{(u^1-u^3)^2}+\frac{\varepsilon_4 u^2}{(u^1-u^4)^2}+\frac{\varepsilon_5 u^2}{(u^1-u^5)^2},
\end{aligned}$$

$$\Gamma^2_{13} = -\frac{u^2\varepsilon_3}{(u^1-u^3)^2},\ \Gamma^2_{14} = -\frac{u^2\varepsilon_4}{(u^1-u^4)^2},\ \Gamma^2_{15} = -\frac{u^2\varepsilon_5}{(u^1-u^5)^2},$$
$$\Gamma^2_{22} = -\frac{2\varepsilon_1}{u^2},\ \Gamma^2_{23} = \frac{\varepsilon_3}{u^1-u^3},\ \Gamma^2_{24} = \frac{\varepsilon_4}{u^1-u^4},\ \Gamma^2_{25} = \frac{\varepsilon_5}{u^1-u^5},$$
$$\Gamma^2_{33} = \frac{u^2\varepsilon_3}{(u^1-u^3)^2},\ \Gamma^2_{44} = \frac{u^2\varepsilon_4}{(u^1-u^4)^2},\ \Gamma^2_{55} = \frac{u^2\varepsilon_5}{(u^1-u^5)^2},\ \Gamma^3_{11} = -\Gamma^3_{13} = \frac{2\varepsilon_1}{u^1-u^3},$$
$$\Gamma^3_{33} = \frac{2\varepsilon_1}{u^1-u^3} - \frac{\varepsilon_4}{u^3-u^4} - \frac{\varepsilon_5}{u^3-u^5},$$
$$\Gamma^3_{34} = -\Gamma^3_{44} = \frac{\varepsilon_4}{u^3-u^4},\ \Gamma^3_{35} = -\Gamma^3_{55} = \frac{\varepsilon_5}{u^3-u^5},$$
$$\Gamma^4_{11} = -\Gamma^4_{14} = \frac{2\varepsilon_1}{u^1-u^4},\ \Gamma^4_{33} = -\Gamma^4_{34} = \frac{\varepsilon_3}{u^3-u^4},$$
$$\Gamma^4_{44} = \frac{2\varepsilon_1}{u^1-u^4} + \frac{\varepsilon_3}{u^3-u^4} - \frac{\varepsilon_5}{u^4-u^5},$$
$$\Gamma^4_{45} = -\Gamma^4_{55} = \frac{\varepsilon_5}{u^4-u^5},\ \Gamma^5_{11} = -\Gamma^5_{15} = \frac{2\varepsilon_1}{u^1-u^5},$$
$$\Gamma^5_{33} = -\Gamma^5_{35} = \frac{\varepsilon_3}{u^3-u^5},\ \Gamma^5_{44} = -\Gamma^5_{45} = \frac{\varepsilon_4}{u^4-u^5},$$
$$\Gamma^5_{55} = \frac{2\varepsilon_1}{u^1-u^5} + \frac{\varepsilon_3}{u^3-u^5} + \frac{\varepsilon_4}{u^4-u^5}.$$

Remark 4.6 *Starting from the above formulas, by using the expression of the Euler vector field in canonical coordinates, the definition of the dual product*

$$X * Y = (E\circ)^{-1} X \circ Y$$

and the formula (1.60) *for the dual connection, one can reconstruct all the data defining a bi-flat structure.*

4.5 The case of a Jordan block of arbitrary size

This section is devoted to the proof of the following Theorem.

Theorem 4.7 *For any choice of a parameter ε, there exists a unique non-semisimple regular bi-flat structure $(\nabla, \nabla^*, \circ, *, e, E)$ with canonical coordinates $\{u^1, \dots, u^n\}$ such that $d_\nabla(E \circ - a_0 I) = 0$, where $E\circ$ has a single Jordan block of size n and $a_0 = \varepsilon u^1$.*

Let us start with some preliminary observations. In the first place, the formulas for the case of a single Jordan block in the above examples up to dimension 5 suggest a simple rule in order to define the Christoffel symbols of the $(n+1)\times(n+1)$ Jordan block starting from the ones of the $n \times n$ Jordan block. More precisely, we define the new Christoffel symbols for the original range of the indices to have the same expression as the old ones, up to rescaling the corresponding weight from ε_1 to $\frac{n+1}{n}\,\varepsilon_1$. The remaining Christoffel symbols (the ones where $n+1$ appears at least once among the indices) are defined as follows. The Christoffel symbol Γ^{n+1}_{22} is given by the formula

$$\Gamma^{n+1}_{22} = -\frac{1}{u^2}\sum_{s=1}^{n-1}\Gamma^{n-s+1}_{22}\,u^{s+2} \tag{4.39}$$

while the Christoffel symbols Γ^{n+1}_{ij} with $(i,j)\neq(2,2)$ are given by $\Gamma^{n+1}_{ij}=0$ when $i=1$ or $j=1$ and are determined in terms of the Christoffel symbols associated with the $n\times n$ Jordan block via the relations

$$\Gamma^{n+1}_{ij} = \Gamma^{n}_{i-1,j} = \Gamma^{n}_{i,j-1} \tag{4.40}$$

when both i and j are greater or equal than 2 (provided that $i-1\neq1$, if using the second term of (4.40), or $j-1\neq1$, if using the third term of (4.40)).

The new non-vanishing Christoffel symbols are then Γ^{n+1}_{ij} with $i,j\neq1$ and $n-i-j\geq-3$. The above definition immediately implies that all the non-vanishing Christoffel symbols can be recursively obtained starting from

$$\Gamma^{2}_{22} = -\frac{\varepsilon}{u^2}. \tag{4.41}$$

Indeed, by applying $i+j-4$ times ($i-2$ times on the i-th side and $j-2$ times on the j-th side) the relation (4.40) we obtain

$$\Gamma^{n+1}_{ij} = \Gamma^{n-i-j+5}_{22}. \tag{4.42}$$

Since the above property holds for all n, more in general we have

$$\Gamma^{k}_{ij} = \Gamma^{k-i-j+4}_{22} \qquad \text{if } k-i-j\geq-2 \text{ and } i,j\neq1 \tag{4.43}$$
$$\Gamma^{k}_{ij} = 0 \qquad \text{if } k-i-j\leq-3. \tag{4.44}$$

4.5.1 Technical lemmas

By virtue of the above remarks, one can prove the following lemmas.

Lemma 4.8 *The Christoffel symbols $\{\Gamma^k_{ij}\}_{i,j,k\in\{1,\dots,n\}}$ associated with the $n\times n$ Jordan block, recursively defined as explained above, starting from the 2×2 Jordan block, satisfy the following identity:*

$$\frac{\partial\Gamma^k_{ij}}{\partial u^l} = \frac{\partial\Gamma^{k-1}_{ij}}{\partial u^{l-1}}, \qquad l>2, \tag{4.45}$$

for each $i,j,k\in\{1,\dots,n\}$.

Proof. It is sufficient to prove the lemma in the case $i = j = 2$. In fact, if at least one index among i and j is equal to 1 then both Γ^k_{ij} and Γ^{k-1}_{ij} vanish, while if $i + j \geq 5$ then (4.45) reduces to

$$\frac{\partial\Gamma^{k-i-j+4}_{22}}{\partial u^l} = \frac{\partial\Gamma^{k-i-j+3}_{22}}{\partial u^{l-1}}, \qquad l > 2,$$

by means of (4.43). Let us then handle the case where $i = j = 2$. For $k = 2$ we have to prove that

$$\frac{\partial\Gamma^2_{22}}{\partial u^l} = \frac{\partial\Gamma^1_{22}}{\partial u^{l-1}}, \qquad l > 2. \tag{4.46}$$

The left hand side term vanishes since Γ^2_{22} only depends on u^2, while the right hand side term vanishes since $\Gamma^1_{22} = 0$. For $k > 2$ we have to prove that

$$\frac{\partial\Gamma^k_{22}}{\partial u^l} = \frac{\partial\Gamma^{k-1}_{22}}{\partial u^{l-1}}, \qquad l > 2. \tag{4.47}$$

For $k = 3$ the left hand side term reads

$$\frac{\partial\Gamma^3_{22}}{\partial u^l} = \frac{\partial}{\partial u^l}\left(-\frac{1}{u^2}\,\Gamma^2_{22}\,u^3\right) = -\frac{1}{u^2}\,\Gamma^2_{22}\,\delta^3_l = \frac{\varepsilon}{(u^2)^2}\,\delta^3_l$$

and the right hand side term reads

$$\frac{\partial\Gamma^2_{22}}{\partial u^{l-1}} = \frac{\partial}{\partial u^{l-1}}\left(-\frac{\varepsilon}{u^2}\right) = \frac{\varepsilon}{(u^2)^2}\,\delta^2_{l-1} = \frac{\varepsilon}{(u^2)^2}\,\delta^3_l.$$

Let us now fix $h \in \{3, \dots, n-1\}$ and let us assume that (4.45) holds for each $k \in \{3, \dots, h\}$. Then, by means of this inductive assumption and of (4.39), we get

$$\frac{\partial\Gamma^{h+1}_{22}}{\partial u^l} = -\frac{1}{u^2}\sum_{s=1}^{h-1}\left(\frac{\partial\Gamma^{h-s+1}_{22}}{\partial u^l}\right)u^{s+2} - \frac{1}{u^2}\Gamma^{h-l+3}_{22} = -\frac{1}{u^2}\sum_{s=1}^{h-1}\left(\frac{\partial\Gamma^{h-s}_{22}}{\partial u^{l-1}}\right)u^{s+2} - \frac{1}{u^2}\Gamma^{h-l+3}_{22},$$

$$\frac{\partial\Gamma^{h}_{22}}{\partial u^{l-1}} = \frac{\delta^3_l}{(u^2)^2}\sum_{s=1}^{h-2}\Gamma^{h-s}_{22}u^{s+2} - \frac{1}{u^2}\sum_{s=1}^{h-2}\left(\frac{\partial\Gamma^{h-s}_{22}}{\partial u^{l-1}}\right)u^{s+2} - \frac{1-\delta^l_3}{u^2}\Gamma^{h-l+3}_{22}$$

thus

$$\begin{aligned}\frac{\partial\Gamma^{h+1}_{22}}{\partial u^l} - \frac{\partial\Gamma^{h}_{22}}{\partial u^{l-1}} &= -\frac{\delta^3_l}{(u^2)^2}\sum_{s=1}^{h-2}\Gamma^{h-s}_{22}u^{s+2} - \frac{1}{u^2}\left(\frac{\partial\Gamma^1_{22}}{\partial u^{l-1}}\right)u^{h+1} - \frac{\delta^3_l}{u^2}\Gamma^{h-l+3}_{22}\\ &= -\frac{\delta^3_l}{(u^2)^2}\sum_{s=1}^{h-2}\Gamma^{h-s}_{22}u^{s+2} + \frac{\delta^3_l}{(u^2)^2}\sum_{s=1}^{h-2}\Gamma^{h-s}_{22}u^{s+2} = 0.\end{aligned}$$

■

Lemma 4.9 *The Christoffel symbols $\{\Gamma^k_{ij}\}_{i,j,k\in\{1,\dots,n\}}$ associated with the $n\times n$ Jordan block, recursively defined as explained above, starting from the 2×2 Jordan block, satisfy the following identities:*

$$\sum_{k=1}^{n}\Gamma^i_{jk}u^k=\sum_{k=2}^{n}\Gamma^i_{jk}u^k=0,\qquad i\neq j\quad\text{or}\quad i=1\quad\text{or}\quad j=1,\tag{4.48}$$
$$\sum_{k=1}^{n}\Gamma^i_{jk}u^k=\sum_{k=2}^{n}\Gamma^i_{jk}u^k=-\varepsilon,\qquad i=j\neq 1.\tag{4.49}$$

Proof. For $i=1$ or $j=1$ the first identity is trivially satisfied since all the summands vanish. Thus we can assume both indices are different from 1. Let us first consider the case of $i=j\neq 1$. For $i=j=2$ we have

$$\sum_{k=2}^{n}\Gamma^2_{2k}u^k\overset{(4.44)}{=}\Gamma^2_{22}u^2\overset{(4.41)}{=}-\varepsilon.\tag{4.50}$$

Let us fix $h\in\{2,\dots,n-1\}$ and assume

$$\sum_{k=2}^{n}\Gamma^i_{ik}u^k=-\varepsilon,\qquad i\in\{2,\dots,h\}.\tag{4.51}$$

For $i=j=h+1$ we have

$$\sum_{k=2}^{n}\Gamma^{h+1}_{h+1,k}u^k\overset{(4.40)}{=}\sum_{k=2}^{n}\Gamma^h_{h,k}u^k\overset{(4.51)}{=}-\varepsilon.$$

Therefore

$$\sum_{k=1}^{n}\Gamma^i_{jk}u^k=\sum_{k=2}^{n}\Gamma^i_{jk}u^k=-\varepsilon,\qquad i=j\neq 1.$$

Let us now consider the case where $i\neq j$. For $j=2$, which here implies $i\geq 3$, we have

$$\begin{aligned}\sum_{k=2}^{n}\Gamma^i_{2k}u^k&\overset{(4.44)}{=}\sum_{k=2}^{i}\Gamma^i_{2k}u^k=\Gamma^i_{22}u^2+\sum_{k=3}^{i}\Gamma^i_{2k}u^k\\&\underset{(4.43)}{\overset{(4.39)}{=}}-\sum_{s=1}^{i-2}\Gamma^{i-s}_{22}\,u^{s+2}+\sum_{k=3}^{i}\Gamma^{i-k+2}_{22}u^k=0\end{aligned}$$

as an index shift in the sums shows. Let us fix $h\in\{2,\dots,n-1\}$ and assume

$$\sum_{k=2}^{n}\Gamma^i_{jk}u^k=0,\qquad i\neq j,\quad j\in\{2,\dots,h\}.\tag{4.52}$$

For $j = h+1$ we have

$$\sum_{k=2}^{n} \Gamma^i_{h+1,k} u^k \overset{(4.40)}{=} \sum_{k=2}^{n} \Gamma^{i-1}_{h,k} u^k \overset{(4.52)}{=} 0.$$

Therefore

$$\sum_{k=1}^{n} \Gamma^i_{jk} u^k = \sum_{k=2}^{n} \Gamma^i_{jk} u^k = 0, \qquad i \neq j.$$

■

Lemma 4.10 *The connection ∇ associated with the $n \times n$ Jordan block satisfies the condition*

$$\nabla_j E^i = (1-\varepsilon)\delta^i_j + \varepsilon e^i e^j \tag{4.53}$$

for each $i, j \in \{1, \ldots, n\}$.

Proof. For each $i, j \in \{1, \ldots, n\}$ we have

$$\nabla_j E^i = \partial_j E^i + \Gamma^i_{jk} E^k = \delta^i_j + \Gamma^i_{jk} u^k$$

which, by means of the previous lemma, becomes

$$\nabla_j E^i = \delta^i_j - \varepsilon\, \delta^i_j\, (1 - \delta^i_1) = (1-\varepsilon)\, \delta^i_j + \varepsilon\, \delta^i_1 \delta^j_1 = (1-\varepsilon)\delta^i_j + \varepsilon e^i e^j.$$

■

Lemma 4.11 *The components of E^{-1} are defined recursively by*

$$(E^{-1})^1 = \frac{1}{u^1},$$
$$(E^{-1})^{m+1} = -\frac{1}{u^1} \sum_{s=1}^{m} (E^{-1})^{m-s+1}\, u^{s+1}, \qquad m \in \{1, \ldots, n-1\}.$$

Proof. By spelling out $E^{-1} \circ E = e$ in canonical coordinates, we obtain

$$\sum_{k=1}^{i} (E^{-1})^{i-k+1} u^k = \delta^i_1, \qquad i \in \{1, \ldots, n\}.$$

For $i = 1$ we clearly get

$$(E^{-1})^1 = \frac{1}{u^1}.$$

For $i = m+1$ we get

$$(E^{-1})^1 u^{m+1} + \sum_{k=1}^{m} (E^{-1})^{i-k+1} u^k = 0$$

that is

$$(E^{-1})^{m+1} = -\frac{1}{u^1} \sum_{s=1}^{m} (E^{-1})^{m-s+1}\, u^{s+1}.$$

■

Lemma 4.12 *The dual connection ∇^* is defined by*

$$\Gamma^{*k}_{ij} = \Gamma^k_{ij} + (\varepsilon - 1)(E^{-1})^{k-i-j+2} - \varepsilon\,(E^{-1})^1\,\delta^k_1\delta^1_i\delta^1_j \tag{4.54}$$

for each $i, j, k \in \{1, \dots, n\}$, where it is understood that $(E^{-1})^{k-i-j+2} = 0$ if $k-i-j < -1$.

Proof. For each $i, j, k \in \{1, \dots, n\}$ we have

$$\begin{aligned}\Gamma^{*k}_{ij} &\overset{(1.60)}{=} \Gamma^k_{ij} - c^{*l}_{ji}\,\nabla_l E^k \overset{(4.53)}{=} \Gamma^k_{ij} - c^{*l}_{ji}\left((1-\varepsilon)\delta^k_l + \varepsilon e^k e^l\right)\\ &= \Gamma^k_{ij} - (1-\varepsilon)\,c^{*k}_{ji} - \varepsilon\,c^{*1}_{ji}\,\delta^k_1\end{aligned} \tag{4.55}$$

where, by definition, the structure constants of the dual product are of the form

$$c^{*i}_{jk} = c^i_{jl}c^l_{km}(E^{-1})^m = \delta^i_{j+l-1}\delta^l_{k+m-1}(E^{-1})^m = \delta^i_{j+l-1}(E^{-1})^{l-k+1} = (E^{-1})^{i-j-k+2}.$$

Then

$$\begin{aligned}\Gamma^{*k}_{ij} &= \Gamma^k_{ij} - (1-\varepsilon)\,(E^{-1})^{k-i-j+2} - \varepsilon\,(E^{-1})^{3-i-j}\,\delta^k_1\\ &= \Gamma^k_{ij} + (\varepsilon - 1)\,(E^{-1})^{k-i-j+2} - \varepsilon\,(E^{-1})^1\,\delta^1_i\delta^1_j\delta^k_1, \qquad \in \{1, \dots, n\}.\end{aligned} \tag{4.56}$$

■

Remark 28 *The condition $\nabla e = 0$ is equivalent to $\Gamma^i_{j1} = 0$ for each $i, j \in \{1, \dots, n\}$.*

By virtue of the above lemmas we can prove Theorem 4.7. The proof can be divided in the following steps:

1. Flatness of ∇.
2. Compatibility of ∇ and $\circ$.
3. Linearity of the Euler vector field.
4. The condition $d_\nabla(E \circ -a_0 I) = 0$.
5. Uniqueness.

4.5.2 Flatness of ∇

We already know that the connection ∇ is flat for $n \in \{2, 3, 4, 5\}$. We need to prove that if the connection ∇ associated with the $n \times n$ Jordan block is flat, that is

$$\partial_k\Gamma^i_{hj} - \partial_h\Gamma^i_{kj} - \sum_{l=1}^n(\Gamma^i_{hl}\Gamma^l_{kj} - \Gamma^i_{kl}\Gamma^l_{hj}) = 0, \qquad i, j, h, k \in \{1, \dots, n\}, \tag{4.57}$$

then also the connection associated with the $(n+1)\times(n+1)$ Jordan block is flat:

$$R^i_{hkj}=\partial_k\Gamma^i_{hj}-\partial_h\Gamma^i_{kj}-\sum_{l=1}^{n+1}(\Gamma^i_{hl}\Gamma^l_{kj}-\Gamma^i_{kl}\Gamma^l_{hj})=0,\qquad i,j,h,k\in\{1,\dots,n+1\}. \quad (4.58)$$

Let us first consider the case where $i\leq n$. For each $h,k,j\in\{1,\dots,n+1\}$ we have

$$R^i_{hkj}=\partial_k\Gamma^i_{hj}-\partial_h\Gamma^i_{kj}-\sum_{l=1}^{n}(\Gamma^i_{hl}\Gamma^l_{kj}-\Gamma^i_{kl}\Gamma^l_{hj})-(\Gamma^i_{h,n+1}\Gamma^{n+1}_{kj}-\Gamma^i_{k,n+1}\Gamma^{n+1}_{hj})$$

where both $\Gamma^i_{h,n+1}$ and $\Gamma^i_{k,n+1}$ vanish due to (4.44). Then R^i_{hkj} is the same as the one for the connection associated with the $n\times n$ Jordan block, which by hypothesis is flat, yielding $R^i_{hkj}=0$. Let us now fix $i=n+1$. If $h=1$ (or equivalently $k=1$) then R^i_{hkj} trivially vanishes, as $\Gamma^m_{hj}=0$ for each $m,j\in\{1,\dots,n+1\}$ and $\partial_h\Gamma^m_{kj}=0$ for each $m,k,j\in\{1,\dots,n+1\}$, since there is no dependence of the Christoffel symbols on u^1. If $j=1$ or $h=k$ then R^i_{hkj} clearly vanishes as well. We therefore have 3 interesting subcases:

a. $h>2,\ k>2,\ j>2$

b. $h=2,\ k>2,\ j>2$ (this covers $h>2,\ k=2,\ j>2$ as well)

c. $h=j=2$, $k>2$ (this covers $h>2,\ k=j=2$ as well)

d. $j=2,\ h>2,\ k>2.$

Subcase a: $h>2,\ k>2,\ j>2$**.** We have

$$R^{n+1}_{hkj}\overset{(4.45)}{=}\partial_{k-1}\Gamma^n_{hj}-\partial_{h-1}\Gamma^n_{kj}-\sum_{l=2}^{n+1}(\Gamma^{n+1}_{hl}\Gamma^l_{kj}-\Gamma^{n+1}_{kl}\Gamma^l_{hj})$$

where in the sum both the terms for $l=2$ and $l=n+1$ vanish due to (4.44). Then

$$\begin{aligned}
R^{n+1}_{hkj} &= \partial_{k-1}\Gamma^n_{hj}-\partial_{h-1}\Gamma^n_{kj}-\sum_{l=3}^{n}(\Gamma^{n+1}_{hl}\Gamma^l_{kj}-\Gamma^{n+1}_{kl}\Gamma^l_{hj})\\
&\overset{(4.40)}{=} \partial_{k-1}\Gamma^{n-1}_{h-1,j}-\partial_{h-1}\Gamma^{n-1}_{k-1,j}-\sum_{l=3}^{n}(\Gamma^{n-1}_{h-1,l-1}\Gamma^{l-1}_{k-1,j}-\Gamma^{n-1}_{k-1,l-1}\Gamma^{l-1}_{h-1,j})\\
&= \partial_{k-1}\Gamma^{n-1}_{h-1,j}-\partial_{h-1}\Gamma^{n-1}_{k-1,j}-\sum_{l=2}^{n-1}(\Gamma^{n-1}_{h-1,l}\Gamma^{l}_{k-1,j}-\Gamma^{n-1}_{k-1,l}\Gamma^{l}_{h-1,j})
\end{aligned}$$

where the sum can be enlarged as to include the terms for $l=1$ and $l=n$, as they vanish. Then

$$R^{n+1}_{hkj} = \partial_{k-1}\Gamma^{n-1}_{h-1,j}-\partial_{h-1}\Gamma^{n-1}_{k-1,j}-\sum_{l=1}^{n}(\Gamma^{n-1}_{h-1,l}\Gamma^{l}_{k-1,j}-\Gamma^{n-1}_{k-1,l}\Gamma^{l}_{h-1,j})$$

$$= R^{n-1}_{h-1,k-1,j}.$$

The quantity $R^{n-1}_{h-1,k-1,j}$ vanishes by hypothesis if $j \leq n$. For $j = n+1$, all of the terms appearing in R^{n+1}_{hkj} vanish.

Subcase b: $h = 2,\ k > 2,\ j > 2$. We have

$$R^{n+1}_{2kj} \overset{(4.45)}{=} \partial_{k-1}\Gamma^n_{2j} - \partial_2\Gamma^{n+1}_{kj} - \sum_{l=2}^{n+1}(\Gamma^{n+1}_{2l}\Gamma^l_{kj} - \Gamma^{n+1}_{kl}\Gamma^l_{2j})$$

where in the sum the term for $l = 2$ vanishes due to (4.44). Then

$$\begin{aligned} R^{n+1}_{2kj} &= \partial_{k-1}\Gamma^n_{2j} - \partial_2\Gamma^{n+1}_{kj} - \sum_{l=3}^{n+1}(\Gamma^{n+1}_{2l}\Gamma^l_{kj} - \Gamma^{n+1}_{kl}\Gamma^l_{2j}) \\ &\overset{(4.40)}{=} \partial_{k-1}\Gamma^n_{2j} - \partial_2\Gamma^n_{k-1,j} - \sum_{l=3}^{n+1}(\Gamma^n_{2,l-1}\Gamma^{l-1}_{k-1,j} - \Gamma^n_{k-1,l}\Gamma^l_{2j}) \\ &= \partial_{k-1}\Gamma^n_{2j} - \partial_2\Gamma^n_{k-1,j} - \sum_{l=2}^{n}\Gamma^n_{2l}\Gamma^l_{k-1,j} + \sum_{l=3}^{n+1}\Gamma^n_{k-1,l}\Gamma^l_{2j} \end{aligned}$$

where the ranges of the summation indices can be modified by adding vanishing terms in order to obtain

$$R^{n+1}_{2kj} = \partial_{k-1}\Gamma^n_{2j} - \partial_2\Gamma^n_{k-1,j} - \sum_{l=1}^{n}(\Gamma^n_{2l}\Gamma^l_{k-1,j} - \Gamma^n_{k-1,l}\Gamma^l_{2j}) = R^n_{2,k-1,j}$$

which trivially vanishes for $j = n+1$ and which vanishes by the induction assumption on the connection associated with the $n \times n$ Jordan block for $j \leq n$.

Subcase c: $h = j = 2$**,** $k > 2$. Since Γ^l_{k2} vanishes for $l = 2$, we have

$$\begin{aligned} R^{n+1}_{2k2} &= \partial_k\Gamma^{n+1}_{22} - \partial_2\Gamma^{n+1}_{k2} - \sum_{l=3}^{n+1}\Gamma^{n+1}_{2l}\Gamma^l_{k2} + \sum_{l=2}^{n+1}\Gamma^{n+1}_{kl}\Gamma^l_{22} \\ &\underset{(4.40)}{\overset{(4.45)}{=}} \partial_{k-1}\Gamma^n_{22} - \partial_2\Gamma^n_{k-1,2} - \sum_{l=3}^{n+1}\Gamma^n_{2,l-1}\Gamma^{l-1}_{k-1,2} + \sum_{l=2}^{n}\Gamma^n_{k-1,l}\Gamma^l_{22} \\ &= \partial_{k-1}\Gamma^n_{22} - \partial_2\Gamma^n_{k-1,2} - \sum_{l=1}^{n}\Gamma^n_{2l}\Gamma^l_{k-1,2} + \sum_{l=1}^{n}\Gamma^n_{k-1,l}\Gamma^l_{22} = R^n_{2,k-1,2} = 0. \end{aligned}$$

Subcase d: $j = 2,\ h > 2,\ k > 2$. We have

$$\begin{aligned} R^{n+1}_{hk2} &\overset{(4.45)}{=} \partial_{k-1}\Gamma^n_{h2} - \partial_{h-1}\Gamma^n_{k2} - \sum_{l=1}^{n+1}(\Gamma^{n+1}_{hl}\Gamma^l_{k2} - \Gamma^{n+1}_{kl}\Gamma^l_{h2}) \\ &\overset{(4.40)}{=} \partial_{k-1}\Gamma^n_{h-1,3} - \partial_{h-1}\Gamma^n_{k-1,3} - \sum_{l=1}^{n+1}(\Gamma^{n+1}_{hl}\Gamma^l_{k-1,3} - \Gamma^{n+1}_{kl}\Gamma^l_{h-1,3}) = R^n_{h-1,k-1,3} \end{aligned}$$

which vanishes by the inductive assumption on the connection associated with the $n \times n$ Jordan block.

4.5.3 Compatibility of ∇ and $\circ$

The compatibility between the connection ∇ and the product $\circ$ is expressed as the symmetry of the tensor ∇c with respect to the exchange of lower indices. In canonical coordinates this means that

$$\Gamma^{k}_{i+j-1,l} - \Gamma^{k-i+1}_{lj} = \Gamma^{k}_{l+j-1,i} - \Gamma^{k-l+1}_{ij}$$

for each $i, j, k, l \in \{1, \dots, n\}$, where it is understood that a Christoffel symbol vanishes whenever at least one of its indices exceed the range $\{1, \dots, n\}$. Let us prove this condition by induction. We already know that it is satisfied up to $n = 5$. Let us fix $m \geq 5$ and assume

$$\Gamma^{k}_{i+j-1,l} - \Gamma^{k-i+1}_{lj} = \Gamma^{k}_{l+j-1,i} - \Gamma^{k-l+1}_{ij} \tag{4.59}$$

for each $i, j, k, l \in \{1, \dots, m\}$. We prove that (4.59) holds whenever at least one index among i, j, k, l is equal to $m + 1$.

If $i = m + 1$ (by the symmetry of (4.59) with respect to the exchange of i and l, this covers the case where $l = m + 1$ as well) then (4.59) becomes

$$\Gamma^{k}_{m+j,l} - \Gamma^{k-m}_{lj} = \Gamma^{k}_{l+j-1,m+1} - \Gamma^{k-l+1}_{m+1,j}$$

which, by means of (4.40), becomes

$$\Gamma^{k-1}_{m+j-1,l} - \Gamma^{k-m}_{lj} = \Gamma^{k-1}_{l+j-1,m} - \Gamma^{k-l}_{mj}$$

that is true by the induction assumption for the indices being chosen as $m, j, k-1, l$.

If $j = m + 1$ then (4.59) becomes

$$\Gamma^{k}_{i+m,l} - \Gamma^{k-i+1}_{m+1,j} = \Gamma^{k}_{l+m,i} - \Gamma^{k-l+1}_{i,m+1}$$

which, by means of (4.40), becomes

$$\Gamma^{k-1}_{i+m-1,l} - \Gamma^{k-i}_{mj} = \Gamma^{k-1}_{l+m-1,i} - \Gamma^{k-l}_{im}$$

that is true by the induction assumption for the indices being chosen as $i, m, k-1, l$.

If $k = m + 1$ then (4.59) becomes

$$\Gamma^{m+1}_{i+j-1,l} - \Gamma^{m-i+2}_{lj} = \Gamma^{m+1}_{l+j-1,i} - \Gamma^{m-l+2}_{ij}$$

which trivially holds when at least one index among i, j, l is equal to 1. When all of the indices i, j, l are greater or equal than 2, by means of (4.43), it becomes

$$\Gamma^{m-i-j-l+6}_{22} - \Gamma^{m-i-j-l+6}_{22} = \Gamma^{m-i-j-l+6}_{22} - \Gamma^{m-i-j-l+6}_{22}$$

that is clearly true.

4.5.4 Linearity of the Euler vector field

Asking for the Euler vector field to be linear in flat coordinates amounts to the condition

$$\nabla\nabla E = 0.$$

In canonical coordinates such condition is expressed as

$$\begin{aligned}\nabla_i\nabla_j E^k &= \partial_i\big(\nabla_j E^k\big) + \Gamma^k_{is}\big(\nabla_j E^s\big) - \Gamma^s_{ij}\big(\nabla_s E^k\big)\\ &\overset{(4.53)}{=} \partial_i\big((1-\varepsilon)\delta^k_j + \varepsilon e^k e^j\big) + \Gamma^k_{is}\big((1-\varepsilon)\delta^s_j + \varepsilon e^s e^j\big) - \Gamma^s_{ij}\big((1-\varepsilon)\delta^k_s + \varepsilon e^k e^s\big)\\ &= (1-\varepsilon)\Gamma^k_{ij} + \varepsilon\Gamma^k_{i1}\delta^1_j - (1-\varepsilon)\Gamma^k_{ij} - \varepsilon\Gamma^1_{ij}\delta^k_1 = -\varepsilon\Gamma^1_{ij}\delta^k_1\end{aligned}$$

which vanishes trivially when either i or j is equal to 1 and by means of (4.43) when both i and j are greater or equal than 2.

The flatness of ∇^* and the additional properties follows from the linearity of E (here we are using the already mentioned result of [50] which holds true also in the non-semisimple setting) and from the definition of ∇^* and $*$.

4.5.5 The condition $d_\nabla(E\circ - a_0 I) = 0$

We prove by induction that the Christoffel symbols obtained by virtue of (4.39) and (4.40) satisfy the condition $d_\nabla(L - a_0 I) = 0$ with $L = E\circ$. Let us denote by $V_{(n)}$ the tensor field $V = E\circ - a_0 I$ in the n-dimensional case and by $\nabla_{(n)}$ the connection associated with the $n\times n$ block. For $n=2$ the tensor field $V_{(2)}$ is represented by the matrix

$$\begin{bmatrix}(1-\varepsilon)\,u^1 & 0\\ u^2 & (1-\varepsilon)\,u^1\end{bmatrix}$$

thus the quantity

$$(d_{\nabla_{(2)}}V_{(2)})^i_{jk} = \partial_j V_{(2)}{}^i_k - \partial_k V_{(2)}{}^i_j + \Gamma^i_{js}V_{(2)}{}^s_k - \Gamma^i_{ks}V_{(2)}{}^s_j$$

vanishes for each $i,j,k\in\{1,2\}$ as a simple computation shows. Let us assume that the Christoffel symbols $\{\Gamma^i_{jk}\,|\,1\le i,j,k\le n\}$ of the connection $\nabla_{(n)}$ associated with the $n\times n$ block obtained applying the formulas (4.39) and (4.40) satisfy the condition $(d_{\nabla_{(n)}}V_{(n)})^k_{ij} = 0$. We show that the Christoffel symbols for the $(n+1)\times(n+1)$ Jordan block obtained applying the formulas (4.39) and (4.40) satisfy the condition $(d_{\nabla_{(n+1)}}V_{(n+1)})^k_{ij} = 0$. The components of $V_{(n+1)}$ are

$$\begin{aligned}V_{(n+1)}{}^i_i &= (1-\varepsilon)\,u^1 &&\text{for } i\in\{1,\dots,n+1\},\\ V_{(n+1)}{}^i_j &= u^{i-j+1} &&\text{for } i\in\{1,\dots,n+1\},\ j\in\{1,\dots,i-1\}.\end{aligned}$$

Four cases are possible:

$a.\ 1 \leq i, j \leq n$, $k = n+1$

$b.\ 1 \leq i, j, k \leq n$

$c.\ 1 \leq i, k \leq n, j = n+1$

$d.\ 1 \leq i \leq n, k = j = n+1.$

Subcase a: $1 \leq i, j \leq n$**,** $k = n+1$**.** We have

$$(d_{\nabla_{(n+1)}} V_{(n+1)})^{n+1}_{ij} = \frac{\partial V^{n+1}_j}{\partial u^i} + \Gamma^{n+1}_{il} V^l_j - \frac{\partial V^{n+1}_i}{\partial u^j} - \Gamma^{n+1}_{jl} V^l_i = \sum_{l=j}^{n} \Gamma^{n+1}_{il} V^l_j - \sum_{l=i}^{n} \Gamma^{n+1}_{jl} V^l_i$$

as

$$\frac{\partial V^{n+1}_j}{\partial u^i} - \frac{\partial V^{n+1}_i}{\partial u^j} = \delta^{n-j+2}_i - \delta^{n-i+2}_j = 0.$$

If $i = 1$ (or equivalently $j = 1$) we have

$$(d_{\nabla_{(n+1)}} V_{(n+1)})^{n+1}_{ij} = -\sum_{l=2}^{n} \Gamma^{n+1}_{jl} V^l_1 = -\sum_{l=2}^{n} \Gamma^{n+1}_{jl} u^l \overset{(4.48)}{=} 0.$$

If both i and j are greater or equal than 2 we have

$$\begin{aligned}(d_{\nabla_{(n+1)}} V_{(n+1)})^{n+1}_{ij} &= \sum_{l=j}^{n} \Gamma^{n+1}_{il} V^l_j - \sum_{l=i}^{n} \Gamma^{n+1}_{jl} V^l_i = \sum_{l=j+1}^{n} \Gamma^{n+1}_{il} V^l_j - \sum_{l=i+1}^{n} \Gamma^{n+1}_{jl} V^l_i \\ &= \sum_{l=j+1}^{n} \Gamma^{n+1}_{il} u^{l-j+1} - \sum_{l=i+1}^{n} \Gamma^{n+1}_{jl} u^{l-i+1} \\ &\overset{(4.43)}{=} \sum_{l=j+1}^{n} \Gamma^{n-i-l+5}_{22} u^{l-j+1} - \sum_{l=i+1}^{n} \Gamma^{n-j-l+5}_{22} u^{l-i+1}\end{aligned}$$

which vanishes after replacing l with $k = l - j + 1$ in the first sum and l with $k = l - i + 1$ in the second sum.

Subcase b: $1 \leq i, j, k \leq n$**.** The quantity

$$(d_{\nabla_{(n+1)}} V_{(n+1)})^k_{ij} = \frac{\partial V^k_j}{\partial u^i} + \Gamma^k_{il} V^l_j - \frac{\partial V^k_i}{\partial u^j} - \Gamma^k_{jl} V^l_i = (d_{\nabla_{(n)}} V_{(n)})^k_{ij}$$

vanishes by means of the inductive assumption for each $i, j, k \leq n$.

Subcase c: $1 \leq i, k \leq n, j = n+1$**.** We have

$$(d_{\nabla_{(n+1)}} V_{(n+1)})^k_{i,n+1} = \frac{\partial V^k_{n+1}}{\partial u^i} + \Gamma^k_{il} V^l_{n+1} - \frac{\partial V^k_i}{\partial u^{n+1}} - \Gamma^k_{n+1,l} V^l_i$$

where the terms of the form V^r_{n+1} with $r \leq n$ vanish. Then

$$(d_{\nabla_{(n+1)}} V_{(n+1)})^k_{i,n+1} = \Gamma^k_{i,n+1} V^{n+1}_{n+1} - \frac{\partial V^k_i}{\partial u^{n+1}} - \sum_{l=i}^{n+1} \Gamma^k_{n+1,l} V^l_i$$

which vanishes term by term.
Subcase d: $1 \le i \le n, k = j = n+1$ We have

$$(d_{\nabla_{(n+1)}} V_{(n+1)})^{n+1,k}_{i,n+1} = \frac{\partial V^{n+1}_{n+1}}{\partial u^i} + \Gamma^{n+1}_{il} V^l_{n+1} - \frac{\partial V^{n+1}_i}{\partial u^{n+1}} - \Gamma^{n+1}_{n+1,l} V^l_i$$

which becomes

$$\begin{aligned}(d_{\nabla_{(n+1)}} V_{(n+1)})^{n+1,k}_{1,n+1} &= \frac{\partial V^{n+1}_{n+1}}{\partial u^1} + \Gamma^{n+1}_{1l} V^l_{n+1} - \frac{\partial V^{n+1}_1}{\partial u^{n+1}} - \Gamma^{n+1}_{n+1,l} V^l_1 \\ &= 1 - \varepsilon - \delta^{n+1}_{n+1} - \Gamma^{n+1}_{n+1,l} u^l \overset{(4.49)}{=} 0\end{aligned}$$

when $i = 1$,

$$\begin{aligned}(d_{\nabla_{(n+1)}} V_{(n+1)})^{n+1,k}_{2,n+1} &= \frac{\partial V^{n+1}_{n+1}}{\partial u^2} + \Gamma^{n+1}_{2l} V^l_{n+1} - \frac{\partial V^{n+1}_2}{\partial u^{n+1}} - \Gamma^{n+1}_{n+1,l} V^l_2 \\ &= \Gamma^{n+1}_{2,n+1} V^{n+1}_{n+1} - \Gamma^{n+1}_{n+1,2} V^2_2 = 0\end{aligned}$$

when $i = 2$ and

$$(d_{\nabla_{(n+1)}} V_{(n+1)})^{n+1,k}_{i,n+1} = \frac{\partial V^{n+1}_{n+1}}{\partial u^i} + \Gamma^{n+1}_{il} V^l_{n+1} - \frac{\partial V^{n+1}_i}{\partial u^{n+1}} - \Gamma^{n+1}_{n+1,l} V^l_i \overset{(4.44)}{=} 0$$

when $i > 2$.

4.5.6 Uniqueness

The connection ∇ is uniquely determined by the conditions

$$\begin{aligned}\nabla_j e^i &= \partial_j e^i + \Gamma^i_{jl} e^l = 0 \\ (d_\nabla V)^k_{ij} &= \frac{\partial V^k_j}{\partial u^i} + \Gamma^k_{il} V^l_j - \frac{\partial V^k_i}{\partial u^j} - \Gamma^k_{jl} V^l_i = 0\end{aligned}$$

where $V = L - a_0 I$. Indeed, in the case of a single Jordan block in David-Hertling coordinates the $(1,1)$-tensor field V has the form

$$V = \begin{bmatrix} (1-\varepsilon)\, u^1 & 0 & \dots & 0 \\ u^2 & (1-\varepsilon)\, u^1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ u^n & \dots & u^2 & (1-\varepsilon)\, u^1 \end{bmatrix}. \tag{4.60}$$

The vanishing of $(d_\nabla V)^i_{n,n-1}$ uniquely defines Γ^i_{nn}. Together with this condition, the vanishing of $(d_\nabla V)^i_{n-2,n}$ uniquely defines $\Gamma^i_{n-1,n}$. More in general, by using the previous conditions, for each $k \in \{1, \dots, n-1\}$ the vanishing of $(d_\nabla V)^i_{k,n}$ uniquely defines $\Gamma^i_{k+1,n}$.

Similarly, the vanishing of $(d_\nabla V)^i_{n-2,n-1}$ uniquely defines $\Gamma^i_{n-1,n-1}$ and the vanishing of $(d_\nabla V)^i_{k,n-1}$ uniquely defines $\Gamma^i_{k+1,n-1}$.

In general, the vanishing of $(d_\nabla V)^i_{n-j-1,n-j}$ uniquely defines $\Gamma^i_{n-j,n-j}$ and the vanishing of $(d_\nabla V)^i_{k,n-j}$ uniquely defines $\Gamma^i_{k+1,n-j}$, once all of the the previous conditions have been taken into account, starting from $j = n-1, k = n$.

We are then able to recursively determine all of the Christoffel symbols apart from $\{\Gamma^i_{1j} = \Gamma^i_{j1} \,|\, j \in \{1, \ldots, n\}\}$, which vanish due to the condition $\nabla e = 0$. This means that the connection constructed above is unique and thus it coincides with the connection obtained by using conditions (4.39) and (4.40).

Remark 29 *Alternatively, one may prove uniqueness also by observing that conditions* (4.39) *and* (4.40) *can be recovered by the properties of* ∇ *and by means of the condition* $d_\nabla(E\circ -a_0 I) = 0$*.*

4.6 The case of an arbitrary number of Jordan blocks

Theorem 4.7 can be extended to the general case where the operator $L = E\circ$ has a block diagonal form.

Theorem 4.13 *For any choice of* $\varepsilon_1, \ldots, \varepsilon_r$ *there exists a unique regular bi-flat structure* $(\nabla, \nabla^*, \circ, *, e, E)$ *with canonical coordinates* $\{u^1, \ldots, u^n\}$ *such that* $d_\nabla(E\circ -a_0 I) = 0$*, where* r *is the number of the Jordan blocks (of sizes* $m_1, \ldots, m_r$*) of* $E\circ$ *and, set* $m_0 = 0$*,*

$$a_0 = \sum_{\alpha=1}^{r} m_\alpha \varepsilon_\alpha u^{1(\alpha)} = \sum_{\alpha=1}^{r} m_\alpha \varepsilon_\alpha u^{m_0+m_1+\cdots+m_{\alpha-1}+1}.$$

In order to prove this theorem, the crucial Lemmas 4.8-4.12 must also be suitably extended and some new preliminary results must be taken into account.

Let $(M, \circ, e, E)$ be a regular F-manifold of dimension $n \geq 2$ with an Euler vector field E. Around a point $m \in M$, let the canonical form of the operator $L = E\circ$ have r Jordan blocks $L_1, \ldots, L_r$ of sizes $m_1, \ldots, m_r$ with distinct eigenvalues. Let us consider the function

$$a_0 = \sum_{\alpha=1}^{r} \varepsilon_\alpha \operatorname{Tr}(L_\alpha).$$

Our final goal is to prove that for any choice of $\varepsilon_1, \ldots, \varepsilon_r$ there exists a unique regular bi-flat F-structure with canonical coordinates such that the operator $L = E\circ$ satisfies the condition $d_\nabla(L - a_0\, I) = 0$. We recall that, in canonical coordinates, the structure constants of the product $\circ$ are given by

$$c^{i(\alpha)}_{j(\beta)k(\gamma)} = \delta^\alpha_\beta \delta^\alpha_\gamma \delta^i_{j+k-1}$$

for all suitable indices. The unit and the Euler vector fields are given respectively by

$$e = \sum_{\alpha=1}^{r} \partial_{1(\alpha)}, \qquad E = \sum_{\alpha=1}^{r} \sum_{s=1}^{m_\alpha} u^{s(\alpha)}\, \partial_{s(\alpha)}.$$

Remark 4.14 *Due to the regularity condition, we are implicitly assuming that* $u^{2(\alpha)} \neq 0$ *for each* $\alpha \in \{1, \dots, r\}$ *and* $u^{1(\alpha)} \neq u^{1(\beta)}$ *if* $\alpha \neq \beta$ *for each* $\alpha, \beta \in \{1, \dots, r\}$.

4.6.1 The Christoffel symbols

The following proposition plays the role of conditions (4.39) and (4.40) in the case of a single block of arbitrary size.

Proposition 4.15 *Let* α, β, γ *be pairwise distinct. Then there exists a unique torsionless connection* ∇ *satisfying the conditions listed in the following.*

1. *For each value of* i, j, k

$$\Gamma^{k(\gamma)}_{i(\alpha)j(\beta)} = 0. \tag{4.61}$$

2. *For every* j, k *when* $i \geq 2$

$$\Gamma^{k(\alpha)}_{i(\beta)j(\alpha)} = 0, \tag{4.62}$$

and when $i = 1$

$$\Gamma^{k(\alpha)}_{1(\beta)j(\alpha)} = \Gamma^{(k-j+1)(\alpha)}_{1(\beta)1(\alpha)} = \begin{cases} 0 & \text{if } k < j, \\ \frac{m_\beta \varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}} & \text{if } k = j, \\ -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{k-j+1} \Gamma^{(k-j-s+2)(\alpha)}_{1(\beta)1(\alpha)}\, u^{s(\alpha)} & \text{if } k > j. \end{cases} \tag{4.63}$$

3. *For each* k *when* $i + j \geq 3$

$$\Gamma^{k(\alpha)}_{i(\beta)j(\beta)} = 0, \tag{4.64}$$

and when $i = j = 1$

$$\Gamma^{k(\alpha)}_{1(\beta)1(\beta)} = -\Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}. \tag{4.65}$$

4. *The Christoffel symbols* $\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}$ *are defined by the following formulas*

$$\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} = -\frac{m_\alpha \varepsilon_\alpha}{u^{2(\alpha)}}, \tag{4.66}$$

and

$$\Gamma^{k(\alpha)}_{1(\alpha)j(\alpha)} = -\sum_{\sigma\neq\alpha}\Gamma^{k(\alpha)}_{1(\sigma)j(\alpha)} = \begin{cases} 0 & \text{if } k<j, \\ -\sum_{\sigma\neq\alpha}\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}} & \text{if } k=j, \\ \sum_{\sigma\neq\alpha}\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=2}^{k-j+1}\Gamma^{(k-j-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)} & \text{if } k>j, \end{cases} \tag{4.67}$$

and (for $k\geq 3$*)*

$$\Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)} = \Gamma^{(k-2)(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{k(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}}\sum_{l=1}^{k-3}\big(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{(k-l)(\alpha)}, \tag{4.68}$$

and (for $i,j\geq 2$*)*

$$\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)} = \begin{cases} 0 & \text{if } k-i-j\leq -3, \\ \Gamma^{(k-i-j+4)(\alpha)}_{2(\alpha)2(\alpha)} & \text{if } k-i-j\geq -2. \end{cases} \tag{4.69}$$

Proof: The above formulas uniquely determine the expressions for all of the Christoffel symbols. By (4.61) all of those Christoffel symbols whose indices correspond to pairwise distinct Jordan blocks vanish. One therefore only needs expressions for the ones whose indices correspond to at most two different Jordan blocks. Let us first explain how to construct Christoffel symbols whose indices correspond to two distinct Jordan blocks, which we label by α and β. By (4.63) we determine $\Gamma^{k(\alpha)}_{1(\beta)j(\alpha)}$ for $k\leq j$ and starting from these functions we determine the Christoffel symbols

$$\{\Gamma^{k(\alpha)}_{1(\beta)j(\alpha)}\,|\,j\in\{1,\dots,m_\alpha\},\ k\in\{1,\dots,m_\alpha\}\}.$$

By (4.62) we determine

$$\Gamma^{k(\alpha)}_{i(\beta)j(\alpha)} = 0$$

for $i\geq 2$ for each $j,k\in\{1,\dots,m_\alpha\}$. By (4.64) we determine

$$\Gamma^{k(\alpha)}_{i(\beta)j(\beta)} = 0$$

when $i+j\geq 3$ for each $k\in\{1,\dots,m_\alpha\}$. By (4.65) we determine

$$\Gamma^{k(\alpha)}_{1(\beta)1(\beta)} = -\Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}$$

for each $k\in\{1,\dots,m_\alpha\}$. Let us now explain how to construct Christoffel symbols whose indices correspond to a single Jordan block, which we label by α. By

(4.66) we determine $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}$. By (4.67), for each $j,k \in \{1,\ldots,m_\alpha\}$ we determine $\Gamma^{k(\alpha)}_{1(\alpha)j(\alpha)}$. By (4.68), for each $k \geq 3$ we can determine recursively $\Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)}$ (since the formula for $\Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)}$ involves the Christoffel symbols $\{\Gamma^{t(\alpha)}_{1(\alpha)1(\alpha)}\,|\,t \in \{1,\ldots,k-2\}\}$ that we already know from above and $\{\Gamma^{t(\alpha)}_{2(\alpha)2(\alpha)}\,|\,t \in \{2,\ldots,k-1\}\}$). By (4.69), for each $i,j \geq 2$ and for each $k \in \{1,\ldots,m_\alpha\}$ we determine $\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}$ in terms of the Christoffel symbols $\{\Gamma^{t(\alpha)}_{2(\alpha)2(\alpha)}\,|\,t \in \{2,\ldots,m_\alpha\}\}$ that we know from above. ■

Example 4.16 *Let us reconstruct the Christoffel symbols in the* 5*-dimensional case of* $3 \times 3 + 2 \times 2$ *Jordan blocks by means of the above formulas. In this case we have*

$$u^1 = u^{1(1)},\; u^2 = u^{2(1)},\; u^3 = u^{3(1)},\; u^4 = u^{1(2)},\; u^5 = u^{2(2)}.$$

By (4.63) *we get*

$$\begin{aligned}
\Gamma^1_{14} &= \frac{2\varepsilon_4}{u^1 - u^4} = \Gamma^2_{24} = \Gamma^3_{34}\\
\Gamma^4_{14} &= -\frac{3\varepsilon_1}{u^1 - u^4} = \Gamma^5_{15}\\
\Gamma^2_{14} &= -\frac{1}{u^1 - u^4}\Gamma^1_{14}u^2 = -\frac{2\varepsilon_4}{(u^1 - u^4)^2}u^2 = \Gamma^3_{24}\\
\Gamma^3_{14} &= -\frac{1}{u^1 - u^4}\left(\Gamma^2_{14}u^2 + \Gamma^1_{14}u^3\right) = \frac{2\varepsilon_4}{(u^1 - u^4)^3}\left((u^2)^2 - u^1u^3 + u^3u^4\right)\\
\Gamma^5_{14} &= \frac{1}{u^1 - u^4}\Gamma^4_{14}u^5 = -\frac{3\varepsilon_1}{(u^1 - u^4)^2}u^5\\
\Gamma^4_{15} &= 0\\
\Gamma^1_{24} &= 0 = \Gamma^1_{34} = \Gamma^2_{34}\\
\Gamma^1_{25} &= 0 = \Gamma^1_{35} = \Gamma^2_{35}.
\end{aligned}$$

By (4.62) *we get*

$$\begin{aligned}
\Gamma^k_{i5} &= 0 && \textit{for } i,k \in \{1,2,3\}\\
\Gamma^k_{2j} &= \Gamma^k_{3j} = 0 && \textit{for } j,k \in \{4,5\}.
\end{aligned}$$

By (4.64) *we get*

$$\begin{aligned}
\Gamma^4_{22} &= 0 = \Gamma^4_{23} = \Gamma^4_{33} = \Gamma^5_{22} = \Gamma^5_{23} = \Gamma^5_{33}\\
\Gamma^1_{55} &= 0 = \Gamma^2_{55} = \Gamma^3_{55}.
\end{aligned}$$

By (4.65) *and* (4.67) *we get*

$$\begin{aligned}
\Gamma^1_{12} &= -\Gamma^1_{24} = 0\\
\Gamma^2_{12} &= -\Gamma^2_{24} = -\frac{2\varepsilon_4}{u^1 - u^4}
\end{aligned}$$

$$\Gamma^3_{12} = -\Gamma^3_{24} = \frac{2\varepsilon_4}{(u^1-u^4)^2}u^2$$
$$\Gamma^4_{12} = -\Gamma^4_{24} = 0$$
$$\Gamma^5_{12} = -\Gamma^5_{24} = 0$$
$$\Gamma^1_{13} = -\Gamma^1_{34} = 0$$
$$\Gamma^2_{13} = -\Gamma^2_{34} = 0$$
$$\Gamma^3_{13} = -\Gamma^3_{34} = -\frac{2\varepsilon_4}{u^1-u^4}$$
$$\Gamma^4_{13} = -\Gamma^4_{34} = 0$$
$$\Gamma^5_{13} = -\Gamma^5_{34} = 0$$
$$\Gamma^1_{45} = -\Gamma^1_{15} = 0$$
$$\Gamma^2_{45} = -\Gamma^2_{15} = 0$$
$$\Gamma^3_{45} = -\Gamma^3_{15} = 0$$
$$\Gamma^4_{45} = -\Gamma^4_{15} = 0$$
$$\Gamma^5_{45} = -\Gamma^5_{15} = \frac{3\varepsilon_1}{u^1-u^4}$$
$$\Gamma^1_{11} = -\Gamma^1_{14} = -\frac{2\varepsilon_4}{u^1-u^4}$$
$$\Gamma^2_{11} = -\Gamma^2_{14} = \frac{2\varepsilon_4}{(u^1-u^4)^2}u^2$$
$$\Gamma^3_{11} = -\Gamma^3_{14} = -\frac{2\varepsilon_4\left((u^2)^2-u^1u^3+u^3u^4\right)}{(u^1-u^4)^3}$$
$$\Gamma^4_{11} = -\Gamma^4_{14} = \frac{3\varepsilon_1}{u^1-u^4}$$
$$\Gamma^5_{11} = -\Gamma^5_{14} = \frac{3\varepsilon_1}{(u^1-u^4)^2}u^5$$
$$\Gamma^1_{44} = -\Gamma^1_{14} = -\frac{2\varepsilon_4}{u^1-u^4}$$
$$\Gamma^2_{44} = -\Gamma^2_{14} = \frac{2\varepsilon_4}{(u^1-u^4)^2}u^2$$
$$\Gamma^3_{44} = -\Gamma^3_{14} = -\frac{2\varepsilon_4\left((u^2)^2-u^1u^3+u^3u^4\right)}{(u^1-u^4)^3}$$
$$\Gamma^4_{44} = -\Gamma^4_{14} = \frac{3\varepsilon_1}{u^1-u^4}$$
$$\Gamma^5_{44} = -\Gamma^5_{14} = \frac{3\varepsilon_1}{(u^1-u^4)^2}u^5.$$

By (4.66) *we get*

$$\Gamma^2_{22} = -\frac{3\varepsilon_1}{u^2}$$

$$\Gamma^5_{55} = -\frac{2\varepsilon_4}{u^5}.$$

By (4.68) *we get*

$$\Gamma^3_{22} = \Gamma^1_{11} - \Gamma^2_{22}\frac{u^3}{u^2} = -\frac{2\varepsilon_4}{u^1 - u^4} + \frac{3\varepsilon_1}{(u^2)^2}u^3.$$

By (4.69) *we get*

$$\begin{aligned}
&\Gamma^1_{22} = 0 = \Gamma^1_{23} = \Gamma^1_{33} = \Gamma^2_{23} = \Gamma^2_{33} = \Gamma^3_{33}\\
&\Gamma^3_{23} = \Gamma^2_{22} = -\frac{3\varepsilon_1}{u^2}\\
&\Gamma^4_{55} = 0.
\end{aligned}$$

Thus, we have obtained the same expressions as the ones in the examples above.

Example 4.17 *Let us reconstruct the Christoffel symbols in the* 5*-dimensional case of* $3 \times 3 + 1 \times 1 + 1 \times 1$ *Jordan blocks by means of the above formulas. In this case we have*

$$u^1 = u^{1(1)},\ u^2 = u^{2(1)},\ u^3 = u^{3(1)},\ u^4 = u^{1(2)},\ u^5 = u^{1(3)}.$$

By (4.61) *we get*

$$\Gamma^i_{45} = \Gamma^4_{i5} = \Gamma^5_{i4} = 0, \qquad i \in \{1,2,3\}.$$

By (4.63) *we get*

$$\Gamma^1_{2i} = \Gamma^1_{3i} = \Gamma^2_{3i} = 0, \qquad i \in \{4,5\},$$

$$\begin{aligned}
\Gamma^1_{14} &= \frac{\varepsilon_4}{u^1 - u^4} = \Gamma^2_{24} = \Gamma^3_{34}\\
\Gamma^1_{15} &= \frac{\varepsilon_5}{u^1 - u^5} = \Gamma^2_{25} = \Gamma^3_{35}\\
\Gamma^4_{45} &= \frac{\varepsilon_5}{u^4 - u^5}\\
\Gamma^4_{14} &= -\frac{3\varepsilon_1}{u^1 - u^4}\\
\Gamma^5_{15} &= -\frac{3\varepsilon_1}{u^1 - u^5}\\
\Gamma^5_{45} &= -\frac{\varepsilon_4}{u^4 - u^5}
\end{aligned} \tag{4.70}$$

and

$$\begin{aligned}
\Gamma^2_{14} &= -\frac{1}{u^1 - u^4}\Gamma^1_{14}u^2 = -\frac{\varepsilon_4}{(u^1 - u^4)^2}u^2 = \Gamma^3_{24}\\
\Gamma^3_{14} &= -\frac{1}{u^1 - u^4}\left(\Gamma^2_{14}u^2 + \Gamma^1_{14}u^3\right) = \frac{\varepsilon_4}{(u^1 - u^4)^3}\left((u^2)^2 - u^1u^3 + u^3u^4\right)
\end{aligned}$$

$$\Gamma^2_{15} = -\frac{1}{u^1-u^5}\Gamma^1_{15}u^2 = -\frac{\varepsilon_5}{(u^1-u^5)^2}u^2 = \Gamma^3_{25}$$
$$\Gamma^3_{15} = -\frac{1}{u^1-u^5}\big(\Gamma^2_{15}u^2 + \Gamma^1_{15}u^3\big) = \frac{\varepsilon_5}{(u^1-u^5)^3}\big((u^2)^2 - u^1u^3 + u^3u^5\big).$$

By (4.62) *we get*

$$\Gamma^i_{2i} = \Gamma^i_{3i} = 0, \qquad i \in \{4,5\}.$$

By (4.64) *we get*

$$\Gamma^i_{12} = \Gamma^i_{13} = \Gamma^i_{22} = \Gamma^i_{23} = \Gamma^i_{33} = 0, \qquad i \in \{4,5\}.$$

By (4.65) *and* (4.67) *we get*

$$\Gamma^1_{11} = -\Gamma^1_{14} - \Gamma^1_{15} = -\frac{\varepsilon_4}{u^1-u^4} - \frac{\varepsilon_5}{u^1-u^5}$$
$$\Gamma^2_{11} = -\Gamma^2_{14} - \Gamma^2_{15} = \frac{\varepsilon_4}{(u^1-u^4)^2}u^2 + \frac{\varepsilon_5}{(u^1-u^5)^2}u^2$$
$$\Gamma^3_{11} = -\Gamma^3_{14} - \Gamma^3_{15} = -\frac{\varepsilon_4\left((u^2)^2 - u^1u^3 + u^3u^4\right)}{(u^1-u^4)^3} - \frac{\varepsilon_5\left((u^2)^2 - u^1u^3 + u^3u^5\right)}{(u^1-u^5)^3}$$
$$\Gamma^4_{11} = -\Gamma^4_{14} - \Gamma^4_{15} = \frac{3\varepsilon_1}{u^1-u^4}$$
$$\Gamma^5_{11} = -\Gamma^5_{14} - \Gamma^5_{15} = \frac{3\varepsilon_1}{u^1-u^5}$$
$$\Gamma^1_{12} = -\Gamma^1_{24} - \Gamma^1_{25} = 0$$
$$\Gamma^2_{12} = -\Gamma^2_{24} - \Gamma^2_{25} = -\frac{\varepsilon_4}{u^1-u^4} - \frac{\varepsilon_5}{u^1-u^5}$$
$$\Gamma^3_{12} = -\Gamma^3_{24} - \Gamma^3_{25} = \frac{\varepsilon_4}{(u^1-u^4)^2}u^2 + \frac{\varepsilon_5}{(u^1-u^5)^2}u^2$$
$$\Gamma^4_{12} = -\Gamma^4_{24} - \Gamma^4_{25} = 0$$
$$\Gamma^5_{12} = -\Gamma^5_{24} - \Gamma^5_{25} = 0$$
$$\Gamma^1_{13} = -\Gamma^1_{34} - \Gamma^1_{35} = 0$$
$$\Gamma^2_{13} = -\Gamma^2_{34} - \Gamma^2_{35} = 0$$
$$\Gamma^3_{13} = -\Gamma^3_{34} - \Gamma^3_{35} = -\frac{\varepsilon_4}{u^1-u^4} - \frac{\varepsilon_5}{u^1-u^5}$$
$$\Gamma^4_{13} = -\Gamma^4_{34} - \Gamma^4_{35} = 0$$
$$\Gamma^5_{13} = -\Gamma^5_{34} - \Gamma^5_{35} = 0$$
$$\Gamma^1_{44} = -\Gamma^1_{14} - \Gamma^1_{45} = -\frac{\varepsilon_4}{u^1-u^4}$$
$$\Gamma^2_{44} = -\Gamma^2_{14} - \Gamma^2_{45} = \frac{\varepsilon_4}{(u^1-u^4)^2}u^2$$
$$\Gamma^3_{44} = -\Gamma^3_{14} - \Gamma^3_{45} = -\frac{\varepsilon_4}{(u^1-u^4)^3}\big((u^2)^2 - u^1u^3 + u^3u^4\big)$$

$$\begin{aligned}
\Gamma^4_{44} &= -\Gamma^4_{14} - \Gamma^4_{45} = \frac{3\varepsilon_1}{u^1 - u^4} - \frac{\varepsilon_5}{u^4 - u^5} \\
\Gamma^5_{44} &= -\Gamma^5_{14} - \Gamma^5_{45} = \frac{\varepsilon_4}{u^4 - u^5} \\
\Gamma^1_{55} &= -\Gamma^1_{15} - \Gamma^1_{45} = -\frac{\varepsilon_5}{u^1 - u^5} \\
\Gamma^2_{55} &= -\Gamma^2_{15} - \Gamma^2_{45} = \frac{\varepsilon_5}{(u^1 - u^5)^2} u^2 \\
\Gamma^3_{55} &= -\Gamma^3_{15} - \Gamma^3_{45} = -\frac{\varepsilon_5}{(u^1 - u^5)^3}\left((u^2)^2 - u^1 u^3 + u^3 u^5\right) \\
\Gamma^4_{55} &= -\Gamma^4_{15} - \Gamma^4_{45} = -\frac{\varepsilon_5}{u^4 - u^5} \\
\Gamma^5_{55} &= -\Gamma^5_{15} - \Gamma^5_{45} = \frac{3\varepsilon_1}{u^1 - u^5} + \frac{\varepsilon_4}{u^4 - u^5}.
\end{aligned}$$

By (4.66) *we get*

$$\Gamma^2_{22} = -\frac{3\varepsilon_1}{u^2}.$$

By (4.68) *we get*

$$\Gamma^3_{22} = \Gamma^1_{11} - \Gamma^2_{22}\frac{u^3}{u^2} = -\frac{\varepsilon_4}{u^1 - u^4} - \frac{\varepsilon_5}{u^1 - u^5} + \frac{3\varepsilon_1}{(u^2)^2} u^3.$$

By (4.69) *we get*

$$\begin{aligned}
&\Gamma^1_{22} = \Gamma^1_{23} = \Gamma^1_{33} = \Gamma^2_{23} = \Gamma^2_{33} = \Gamma^3_{33} = 0 \\
&\Gamma^3_{23} = \Gamma^2_{22} = -\frac{3\varepsilon_1}{u^2}.
\end{aligned}$$

Thus, we have obtained the same expressions as the ones in the examples above.

Remark 4.18 *All of the Christoffel symbols can be obtained starting from the functions*

$$\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} = -\frac{m_\alpha \varepsilon_\alpha}{u^{2(\alpha)}}, \qquad \alpha \in \{1, \dots, r\}, \quad m_\alpha \geq 2,$$

and

$$\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} = \frac{m_\beta \varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}}, \qquad \alpha, \beta \in \{1, \dots, r\}, \quad \alpha \neq \beta.$$

The last functions appear only in the case of multiple Jordan blocks and they are at the origin of the additional difficulties that one meets in the proof of the general case. However, exactly as in the case of a single block, increasing the size of a block $m_\alpha = N \to m_\alpha = N+1$ *and rescaling the corresponding weight* $\varepsilon_\alpha \to \frac{N+1}{N}\varepsilon_\alpha$ *does not affect the definition of the Christoffels symbols* $\Gamma^{k(\beta)}_{i(\sigma)j(\tau)}$ *for the original range of the indices.*

Remark 4.19 *It is easy to observe that a_0 is a flat coordinate for ∇, namely $\nabla(da_0) = 0$. In fact, the i-th component of da_0 is*

$$(da_0)_i = \partial_i(a_0) = \partial_i\left(\sum_{\sigma=1}^{r} m_\sigma \varepsilon_\sigma \, u^{1(\sigma)}\right) = \sum_{\sigma=1}^{r} m_\sigma \varepsilon_\sigma \, \delta_i^{1(\sigma)}$$

thus

$$\nabla_i(da_0)_j = \partial_i(da_0)_j - \Gamma_{ij}^k(da_0)_k = -\Gamma_{ij}^k \sum_{\sigma=1}^{r} m_\sigma \varepsilon_\sigma \, \delta_k^{1(\sigma)}.$$

Given $\alpha, \beta \in \{1, \ldots, r\}$ this reads

$$\nabla_{i(\alpha)}(da_0)_{j(\beta)} = -\sum_{\gamma=1}^{r} \Gamma_{i(\alpha)j(\beta)}^{k(\gamma)} \sum_{\sigma=1}^{r} m_\sigma \varepsilon_\sigma \, \delta_{k(\gamma)}^{1(\sigma)} = -\sum_{\sigma=1}^{r} \Gamma_{i(\alpha)j(\beta)}^{1(\sigma)} \, m_\sigma \varepsilon_\sigma. \tag{4.71}$$

Let us first consider the case where $\alpha = \beta$. We get

$$\nabla_{i(\alpha)}(da_0)_{j(\alpha)} = -\sum_{\sigma=1}^{r} \Gamma_{i(\alpha)j(\alpha)}^{1(\sigma)} \, m_\sigma \varepsilon_\sigma = -\Gamma_{i(\alpha)j(\alpha)}^{1(\alpha)} \, m_\alpha \varepsilon_\alpha - \sum_{\sigma \neq \alpha} \Gamma_{i(\alpha)j(\alpha)}^{1(\sigma)} \, m_\sigma \varepsilon_\sigma$$

where (by (4.62), (4.63), (4.64), (4.69)*) the terms $\Gamma_{i(\alpha)j(\alpha)}^{1(\alpha)}$ and $\Gamma_{i(\alpha)j(\alpha)}^{1(\sigma)}$ (for each $\sigma \neq \alpha$) only survive for $i = j = 1$. Therefore $\nabla_{i(\alpha)}(da_0)_{j(\alpha)} = 0$ trivially whenever at least one among i and j is greater or equal than 2. When $i = j = 1$ we get*

$$\begin{aligned}
\nabla_{1(\alpha)}(da_0)_{1(\alpha)} &= -\Gamma_{1(\alpha)1(\alpha)}^{1(\alpha)} \, m_\alpha \varepsilon_\alpha - \sum_{\sigma \neq \alpha} \Gamma_{1(\alpha)1(\alpha)}^{1(\sigma)} \, m_\sigma \varepsilon_\sigma \\
&\overset{(4.79)}{=} \sum_{\sigma \neq \alpha} \Gamma_{1(\sigma)1(\alpha)}^{1(\alpha)} \, m_\alpha \varepsilon_\alpha + \sum_{\sigma \neq \alpha} \Gamma_{1(\alpha)1(\sigma)}^{1(\sigma)} \, m_\sigma \varepsilon_\sigma \\
&\overset{(4.63)}{=} \sum_{\sigma \neq \alpha} \left(\frac{m_\sigma \varepsilon_\sigma}{u^{1(\alpha)} - u^{1(\sigma)}} \, m_\alpha \varepsilon_\alpha - \frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\sigma)}} \, m_\sigma \varepsilon_\sigma \right) = 0.
\end{aligned}$$

Let us now consider the case of $\alpha \neq \beta$, where (4.71) *becomes*

$$\nabla_{i(\alpha)}(da_0)_{j(\beta)} = -\sum_{\sigma=1}^{r} \Gamma_{i(\alpha)j(\beta)}^{1(\sigma)} \, m_\sigma \varepsilon_\sigma \overset{(4.61)}{=} -\Gamma_{i(\alpha)j(\beta)}^{1(\alpha)} \, m_\alpha \varepsilon_\alpha - \Gamma_{i(\alpha)j(\beta)}^{1(\beta)} \, m_\beta \varepsilon_\beta$$

which trivially vanishes whenever at least one among i and j is greater or equal than 2, as (by (4.62), (4.63)*) both $\Gamma_{i(\alpha)j(\beta)}^{1(\alpha)}$ and $\Gamma_{i(\alpha)j(\beta)}^{1(\beta)}$ only survive for $i = j = 1$. In this latter case we get*

$$\begin{aligned}
\nabla_{1(\alpha)}(da_0)_{1(\beta)} &= -\Gamma_{1(\alpha)1(\beta)}^{1(\alpha)} \, m_\alpha \varepsilon_\alpha - \Gamma_{1(\alpha)1(\beta)}^{1(\beta)} \, m_\beta \varepsilon_\beta \\
&\overset{(4.63)}{=} -\frac{m_\beta \varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}} \, m_\alpha \varepsilon_\alpha + \frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}} \, m_\beta \varepsilon_\beta = 0.
\end{aligned}$$

Remark 4.20 *It is likewise easy to check that $dd_L a_0 = 0$. In fact, given $\alpha, \beta \in \{1, \dots, r\}$ the $i(\alpha)$-th component of $d_L a_0$ is*

$$(d_L a_0)_{i(\alpha)} = L^k_{i(\alpha)}\, \partial_k(a_0) = L^k_{i(\alpha)}\, \partial_k\left(\sum_{\sigma=1}^{r} m_\sigma \varepsilon_\sigma\, u^{1(\sigma)}\right) = L^k_{i(\alpha)} \sum_{\sigma=1}^{r} m_\sigma \varepsilon_\sigma\, \delta^{1(\sigma)}_k$$
$$= \sum_{\sigma=1}^{r} L^{1(\sigma)}_{i(\alpha)}\ m_\sigma \varepsilon_\sigma = \sum_{\sigma=1}^{r} \delta^\sigma_\alpha\, u^{1(\alpha)}\, \delta^1_i\ m_\sigma \varepsilon_\sigma = u^{1(\alpha)}\, \delta^1_i\ m_\alpha \varepsilon_\alpha.$$

Thus given $\alpha, \beta \in \{1, \dots, r\}$ and i, j such that $i(\alpha) \neq j(\beta)$ we have

$$(dd_L a_0)_{j(\beta) i(\alpha)} = \partial_{j(\beta)} (d_L a_0)_{i(\alpha)} = \partial_{j(\beta)} \left(u^{1(\alpha)}\, \delta^1_i\ m_\alpha \varepsilon_\alpha\right)$$
$$= \delta^1_i\ m_\alpha \varepsilon_\alpha \delta^\alpha_\beta \delta^1_j = \delta^1_i\ m_\alpha \varepsilon_\alpha \delta^\alpha_\beta \delta^1_j\, \delta_{ij} = 0.$$

4.6.2 Technical lemmas

The results of this subsection follow from the above expressions for the Christoffel symbols and play a crucial role in the proof of the main theorem. We refer to Appendix B for their demonstrations.

Lemma 4.21 *For every choice of α, β, γ, $\delta \in \{1, \dots, r\}$ we have*

$$\frac{\partial \Gamma^{k(\gamma)}_{i(\alpha) j(\beta)}}{\partial u^{l(\delta)}} = \frac{\partial \Gamma^{(k-1)(\gamma)}_{i(\alpha) j(\beta)}}{\partial u^{(l-1)(\delta)}} \tag{4.72}$$

for all $k \in \{2, \dots, m_\gamma\}$ and $l \in \{3, \dots, m_\delta\}$. Moreover, if $\beta \neq \alpha = \gamma = \delta$ then (4.72) *holds for $l = 2$ as well.*

Lemma 4.22 *For each α, $\beta \in \{1, \dots, r\}$, $i \in \{1, \dots, m_\alpha\}$, $j \in \{1, \dots, m_\beta\}$ we have*

$$\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\beta) k}\, u^k = \begin{cases} 0 & \textit{if } i \neq j, \\ -\delta^\alpha_\beta \sum\limits_{\sigma \neq \alpha} m_\sigma \varepsilon_\sigma + (1 - \delta^\alpha_\beta)\, m_\beta \varepsilon_\beta & \textit{if } i = j = 1, \\ -\delta^\alpha_\beta \sum\limits_{\tau=1}^{r} m_\tau \varepsilon_\tau & \textit{if } i = j \neq 1. \end{cases} \tag{4.73}$$

Lemma 4.23 *For each α, $\beta \in \{1, \dots, r\}$, $i \in \{1, \dots, m_\alpha\}$, $j \in \{1, \dots, m_\beta\}$ we have*

$$\nabla_{j(\beta)} E^{i(\alpha)} = \begin{cases} 0 & \textit{if } i \neq j, \\ \delta^\alpha_\beta \left(1 - \sum\limits_{\sigma \neq \alpha} m_\sigma \varepsilon_\sigma\right) + (1 - \delta^\alpha_\beta)\, m_\beta \varepsilon_\beta & \textit{if } i = j = 1, \\ \delta^\alpha_\beta \left(1 - \sum\limits_{\tau=1}^{r} m_\tau \varepsilon_\tau\right) & \textit{if } i = j \neq 1. \end{cases} \tag{4.74}$$

Lemma 4.24 *For each $\alpha \in \{1, \ldots, r\}$ and $l \in \{3, \ldots, m_\alpha - 1\}$, we have*

$$A^{l(\alpha)} := \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\, u^{l(\alpha)} - u^{(l+1)(\alpha)} \right) - \sum_{s=2}^{l-1} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(l-s+1)(\alpha)} = 0. \tag{4.75}$$

Lemma 4.25 *For each $\alpha, \sigma \in \{1, \ldots, r\}$ with $\alpha \neq \sigma$ we have*

$$\partial_{1(\sigma)} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} \right) = 0 \tag{4.76}$$

for every $l \in \{1, \ldots, m_\alpha - 2\}$.

Lemma 4.26 *Given $\alpha, \beta, \epsilon \in \{1, \ldots, r\}$ with $\alpha \neq \beta \neq \epsilon \neq \alpha$ we have*

$$B^{s(\alpha)}_{\beta\epsilon} := -\sum_{t=1}^{s+1} \Gamma^{(s-t+2)(\alpha)}_{1(\epsilon)1(\alpha)}\, \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)}\, \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} + \Gamma^{(s+1)(\alpha)}_{1(\epsilon)1(\alpha)}\, \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} = 0 \tag{4.77}$$

for every $s \in \{1, \ldots, m_\alpha - 1\}$.

Lemma 4.27 *Given $\alpha, \beta \in \{1, \ldots, r\}$ with $\alpha \neq \beta$ we have*

$$\begin{aligned} C^{s(\alpha)}_{\beta} := &\sum_{l=2}^{s+1} \left(\Gamma^{(s-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(s-l+2)(\alpha)}_{1(\alpha)1(\alpha)} \right) \Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} \\ &+ \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\, \Gamma^{(s+2)(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)}\, \Gamma^{1(\beta)}_{1(\alpha)1(\beta)} = 0 \end{aligned} \tag{4.78}$$

for every $s \in \{0, \ldots, m_\alpha - 2\}$[1].

We now have all the ingredients to prove Theorem 4.13. The proof is divided in the following steps:

1. The condition $\nabla e = 0$.
2. The condition $d_\nabla(E \circ -a_0 I) = 0$.
3. Compatibility between ∇ and $\circ$.
4. Linearity of the Euler vector field.
5. Flatness of ∇.
6. Uniqueness.

[1] The summation is intended to be non-zero when $s \geq 1$.

4.6.3 The condition $\nabla e = 0$

The condition $\nabla e = 0$ is equivalent to the request that for every $\alpha, \beta \in \{1, \dots, r\}$ and $i \in \{1, \dots, m_\alpha\}$, $j \in \{1, \dots, m_\beta\}$

$$\sum_{\sigma=1}^{r} \Gamma^{i(\alpha)}_{1(\sigma)j(\beta)} = 0. \tag{4.79}$$

This condition is verified by the Christoffel symbols defined above.

4.6.4 The condition $d_\nabla(E \circ -a_0 I) = 0$

Let us now consider the condition

$$d_\nabla(L - a_0\, I) = 0 \tag{4.80}$$

for $L = E\circ$. For each $\alpha, \beta, \gamma \in \{1, \dots, r\}$ and $i \in \{1, \dots, m_\alpha\}$, $j \in \{1, \dots, m_\beta\}$, $k \in \{1, \dots, m_\gamma\}$ we have

$$\begin{aligned}
\left(d_\nabla(L - a_0\, I)\right)^{i(\alpha)}_{j(\beta)k(\gamma)} &= \partial_{j(\beta)}(L - a_0\, I)^{i(\alpha)}_{k(\gamma)} - \partial_{k(\gamma)}(L - a_0\, I)^{i(\alpha)}_{j(\beta)} \\
&\quad + \Gamma^{i(\alpha)}_{j(\beta)l(\delta)}(L - a_0\, I)^{l(\delta)}_{k(\gamma)} - \Gamma^{i(\alpha)}_{k(\gamma)l(\delta)}(L - a_0\, I)^{l(\delta)}_{j(\beta)} \\
&= \delta^\alpha_\gamma \delta^\alpha_\beta \delta^{i-k+1}_j - \delta^\alpha_\gamma \delta^i_k m_\beta \varepsilon_\beta \delta^1_j - \delta^\alpha_\beta \delta^\alpha_\gamma \delta^{i-j+1}_k + \delta^\alpha_\beta \delta^i_j m_\gamma \varepsilon_\gamma \delta^1_k \\
&\quad + \sum_{\delta=1}^{r}\sum_{l=1}^{m_\delta} \Gamma^{i(\alpha)}_{j(\beta)l(\delta)} \delta_{\delta\gamma} u^{(l-k+1)(\gamma)} \mathbb{1}_{\{l \geq k\}} - \sum_{\delta=1}^{r}\sum_{l=1}^{m_\delta} \Gamma^{i(\alpha)}_{k(\gamma)l(\delta)} \delta_{\delta\beta} u^{(l-j+1)(\beta)} \mathbb{1}_{\{l \geq j\}} \\
&= \delta^\alpha_\beta \delta^i_j m_\gamma \varepsilon_\gamma \delta^1_k - \delta^\alpha_\gamma \delta^i_k m_\beta \varepsilon_\beta \delta^1_j + \sum_{l=k}^{m_\gamma} \Gamma^{i(\alpha)}_{j(\beta)l(\gamma)} u^{(l-k+1)(\gamma)} - \sum_{l=j}^{m_\beta} \Gamma^{i(\alpha)}_{k(\gamma)l(\beta)} u^{(l-j+1)(\beta)}
\end{aligned}$$

as

$$(L - a_0\, I)^{a(\eta)}_{b(\mu)} = L^{a(\eta)}_{b(\mu)} - a_0\, \delta^{a(\eta)}_{b(\mu)} = \delta^\eta_\mu \sum_{s=1}^{m_\eta} u^{s(\eta)}\, \delta^{a+b-1}_s - \delta^\eta_\mu\, \delta^a_b \sum_{\alpha=1}^{r} m_\alpha \varepsilon_\alpha u^{1(\alpha)}$$

for each $\eta, \mu \in \{1, \dots, r\}$ and $a \in \{1, \dots, m_\eta\}$, $b \in \{1, \dots, m_\beta\}$. Therefore (4.80) amounts to

$$\delta^\alpha_\beta \delta^i_j m_\gamma \varepsilon_\gamma \delta^1_k - \delta^\alpha_\gamma \delta^i_k m_\beta \varepsilon_\beta \delta^1_j + \sum_{l=k}^{m_\gamma} \Gamma^{i(\alpha)}_{j(\beta)l(\gamma)} u^{(l-k+1)(\gamma)} - \sum_{l=j}^{m_\beta} \Gamma^{i(\alpha)}_{k(\gamma)l(\beta)} u^{(l-j+1)(\beta)} = 0 \tag{4.81}$$

for each $\alpha, \beta, \gamma \in \{1, \dots, r\}$ and $i \in \{1, \dots, m_\alpha\}$, $j \in \{1, \dots, m_\beta\}$, $k \in \{1, \dots, m_\gamma\}$. We split the proof in the following cases:

1. $\alpha = \beta = \gamma$

2. $\alpha = \beta \neq \gamma$ (this also covers $\alpha = \gamma \neq \beta$)

3. $\alpha \neq \beta = \gamma$

4. α, β, γ are pairwise distinct.

Case 1: $\alpha = \beta = \gamma$**.** Condition (4.81) becomes

$$m_\alpha \varepsilon_\alpha (\delta^i_j \delta^1_k - \delta^i_k \delta^1_j) + \sum_{l=k}^{m_\alpha} \Gamma^{i(\alpha)}_{j(\alpha)l(\alpha)} u^{(l-k+1)(\alpha)} - \sum_{l=j}^{m_\alpha} \Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} u^{(l-j+1)(\alpha)} = 0 \tag{4.82}$$

which is trivially satisfied if $j = k = 1$ due to the symmetry between the indices j, k. If both j and k are greater or equal than 2, the left hand side term of (4.82) reads

$$\begin{aligned} &\sum_{l=k}^{m_\alpha} \Gamma^{i(\alpha)}_{j(\alpha)l(\alpha)} u^{(l-k+1)(\alpha)} - \sum_{l=j}^{m_\alpha} \Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} u^{(l-j+1)(\alpha)} \\ \overset{(4.69)}{=} &\sum_{l=k}^{i-j+2} \Gamma^{(i-j-l+4)(\alpha)}_{2(\alpha)2(\alpha)} u^{(l-k+1)(\alpha)} - \sum_{l=j}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} u^{(l-j+1)(\alpha)} \end{aligned}$$

which vanishes by changing the variables in the two summations. Let us then consider the case where $j = 1$ and $k \geq 2$ (this covers the case where $j \geq 2$ and $k = 1$ as well). The left hand side term of (4.82) reads

$$\begin{aligned} &- \delta^i_k m_\alpha \varepsilon_\alpha + \sum_{l=k}^{m_\alpha} \Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)} u^{(l-k+1)(\alpha)} - \sum_{l=1}^{m_\alpha} \Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} u^{l(\alpha)} \\ \underset{(4.69)}{\overset{(4.67)}{=}} &-\delta^i_k m_\alpha \varepsilon_\alpha + \sum_{l=k}^{i} \Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)} u^{(l-k+1)(\alpha)} - \Gamma^{i(\alpha)}_{1(\alpha)k(\alpha)} u^{1(\alpha)} - \sum_{l=2}^{i-k+2} \Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} u^{l(\alpha)} \\ \underset{(4.69)}{\overset{(4.67)}{=}} &-\delta^i_k m_\alpha \varepsilon_\alpha + \sum_{l=k+1}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)} u^{(l-k+1)(\alpha)} - \sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} u^{l(\alpha)} \end{aligned} \tag{4.83}$$

which trivially vanishes if $i < k$ and which vanishes by means of (4.66) if $i = k$. Let us then fix $i > k$. (4.83) becomes

$$\begin{aligned} &\sum_{l=k+1}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)} u^{(l-k+1)(\alpha)} - \sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} u^{l(\alpha)} \\ = &\sum_{s=2}^{i-k+1} \Gamma^{(i-k-s+2)(\alpha)}_{1(\alpha)1(\alpha)} u^{s(\alpha)} - \sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} u^{l(\alpha)} \\ = &-\sum_{l=2}^{i-k+1} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{l(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} u^{(i-k+2)(\alpha)} \end{aligned}$$

$$= -\sum_{s=1}^{i-k}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{(i-k-s+2)(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}u^{(i-k+2)(\alpha)}$$

$$\overset{(4.75)}{=} -\big(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{(i-k+1)(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{3(\alpha)}}{u^{2(\alpha)}}u^{(i-k+1)(\alpha)} \overset{(4.68)}{=} 0.$$

Case 2: $\alpha = \beta \neq \gamma$**.** Condition (4.81) becomes

$$\delta^i_j m_\gamma \varepsilon_\gamma \delta^1_k + \sum_{l=k}^{m_\gamma}\Gamma^{i(\alpha)}_{j(\alpha)l(\gamma)}u^{(l-k+1)(\gamma)} - \sum_{l=j}^{m_\alpha}\Gamma^{i(\alpha)}_{k(\gamma)l(\alpha)}u^{(l-j+1)(\alpha)} = 0$$

that, by means of (4.62) and (4.63), is

$$\delta^1_k\left(\delta^i_j m_\gamma \varepsilon_\gamma + \Gamma^{(i-j+1)(\alpha)}_{1(\alpha)1(\gamma)}u^{1(\gamma)} - \sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\gamma)1(\alpha)}u^{(l-j+1)(\alpha)}\right) = 0 \qquad (4.84)$$

which is trivially satisfied for $k \geq 2$. Let us then fix $k = 1$. If $i < j$ then (4.84) is satisfied by means of (4.63). If $i = j$ then the left hand side of (4.84) reads

$$m_\gamma \varepsilon_\gamma + \Gamma^{1(\alpha)}_{1(\alpha)1(\gamma)}u^{1(\gamma)} - \Gamma^{1(\alpha)}_{1(\gamma)1(\alpha)}u^{1(\alpha)} \overset{(4.63)}{=} 0.$$

If $i > j$ then the left hand side of (4.84) reads

$$\Gamma^{(i-j+1)(\alpha)}_{1(\alpha)1(\gamma)}u^{1(\gamma)} - \sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\gamma)1(\alpha)}u^{(l-j+1)(\alpha)}$$

$$= -\Gamma^{(i-j+1)(\alpha)}_{1(\gamma)1(\alpha)}\big(u^{1(\alpha)} - u^{1(\gamma)}\big) - \sum_{l=j+1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\gamma)1(\alpha)}u^{(l-j+1)(\alpha)}$$

$$= -\Gamma^{(i-j+1)(\alpha)}_{1(\gamma)1(\alpha)}\big(u^{1(\alpha)} - u^{1(\gamma)}\big) - \sum_{s=2}^{i-j+1}\Gamma^{(i-j-s+2)(\alpha)}_{1(\gamma)1(\alpha)}u^{s(\alpha)} \overset{(4.63)}{=} 0.$$

Case 3: $\alpha \neq \beta = \gamma$**.** Condition (4.81) becomes

$$\sum_{l=k}^{m_\beta}\Gamma^{i(\alpha)}_{j(\beta)l(\beta)}u^{(l-k+1)(\beta)} - \sum_{l=j}^{m_\beta}\Gamma^{i(\alpha)}_{k(\beta)l(\beta)}u^{(l-j+1)(\beta)} = 0 \qquad (4.85)$$

where, by means of (4.64), the two sums survive only if $j = k = 1$ (and with the only $l = 1$ term), in which case they mutually cancel out.
Case 4: $\alpha \neq \beta \neq \gamma \neq \alpha$**.** Condition (4.81) is trivially satisfied by means of (4.61).

4.6.5 Compatibility between ∇ and $\circ$

We are now going to prove that

$$\nabla_{i(\alpha)}\, c^{l(\epsilon)}_{j(\beta)k(\gamma)} = \nabla_{j(\beta)}\, c^{l(\epsilon)}_{i(\alpha)k(\gamma)}$$

which is equivalent to

$$\Gamma^{l(\epsilon)}_{i(\alpha)s(\sigma)}c^{s(\sigma)}_{j(\beta)k(\gamma)} - \Gamma^{s(\sigma)}_{i(\alpha)k(\gamma)}c^{l(\epsilon)}_{j(\beta)s(\sigma)} = \Gamma^{l(\epsilon)}_{j(\beta)s(\sigma)}c^{s(\sigma)}_{i(\alpha)k(\gamma)} - \Gamma^{s(\sigma)}_{j(\beta)k(\gamma)}c^{l(\epsilon)}_{i(\alpha)s(\sigma)}$$

and to

$$\Gamma^{l(\epsilon)}_{i(\alpha)(j+k-1)(\beta)}\delta_{\beta\gamma} - \Gamma^{(l-j+1)(\beta)}_{i(\alpha)k(\gamma)}\delta^{\epsilon}_{\beta} = \Gamma^{l(\epsilon)}_{j(\beta)(i+k-1)(\alpha)}\delta_{\alpha\gamma} - \Gamma^{(l-i+1)(\alpha)}_{j(\beta)k(\gamma)}\delta^{\varepsilon}_{\alpha} \tag{4.86}$$

for all α, β, γ, $\epsilon \in \{1, \ldots, r\}$ and any suitable choice of the indices i, j, k, l. The possible cases are the following ones:

1. $\alpha = \beta = \gamma = \epsilon$
2. $\alpha = \beta = \gamma \neq \epsilon$
3. $\alpha = \beta = \epsilon \neq \gamma$
4. $\alpha = \gamma = \epsilon \neq \beta$
5. $\beta = \gamma = \epsilon \neq \alpha$
6. $\alpha = \beta \neq \gamma = \epsilon$
7. $\alpha = \gamma \neq \beta = \epsilon$
8. $\alpha = \epsilon \neq \beta = \gamma$
9. otherwise.

Case 1: $\alpha = \beta = \gamma = \epsilon$**.** (4.86) becomes

$$\Gamma^{l(\alpha)}_{i(\alpha)(j+k-1)(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{i(\alpha)k(\alpha)} = \Gamma^{l(\alpha)}_{j(\alpha)(i+k-1)(\alpha)} - \Gamma^{(l-i+1)(\alpha)}_{j(\alpha)k(\alpha)} \tag{4.87}$$

If $i = 1$ (or equivalently $j = 1$) then this is

$$\Gamma^{l(\alpha)}_{1(\alpha)(j+k-1)(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{1(\alpha)k(\alpha)} = \Gamma^{l(\alpha)}_{j(\alpha)k(\alpha)} - \Gamma^{l(\alpha)}_{j(\alpha)k(\alpha)}$$

where both the left and the right-hand sides vanish, as

$$\begin{aligned}\Gamma^{l(\alpha)}_{1(\alpha)(j+k-1)(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{1(\alpha)k(\alpha)} &\overset{(4.79)}{=} -\sum_{\sigma\neq\alpha}\left(\Gamma^{l(\alpha)}_{1(\sigma)(j+k-1)(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{1(\sigma)k(\alpha)}\right)\\ &\overset{(4.63)}{=} -\sum_{\sigma\neq\alpha}\left(\Gamma^{(l-j+1)(\alpha)}_{1(\sigma)k(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{1(\sigma)k(\alpha)}\right) = 0.\end{aligned}$$

If $k = 1$ then (4.87) reads

$$\Gamma^{l(\alpha)}_{i(\alpha)j(\alpha)} - \Gamma^{(l-j+1)(\alpha)}_{i(\alpha)1(\alpha)} = \Gamma^{l(\alpha)}_{j(\alpha)i(\alpha)} - \Gamma^{(l-i+1)(\alpha)}_{j(\alpha)1(\alpha)}$$

that holds true, as

$$\Gamma^{(l-j+1)(\alpha)}_{i(\alpha)1(\alpha)} \overset{(4.79)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{(l-j+1)(\alpha)}_{i(\alpha)1(\sigma)} \overset{(4.63)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{(l-i-j+2)(\alpha)}_{1(\alpha)1(\sigma)}$$
$$\overset{(4.63)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{(l-i+1)(\alpha)}_{j(\alpha)1(\sigma)} \overset{(4.79)}{=} \Gamma^{(l-i+1)(\alpha)}_{j(\alpha)1(\alpha)}.$$

If all of i, j and k are greater or equal then 2 then, by (4.69), (4.87) reads

$$\Gamma^{(l-i-j-k+5)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(l-i-j-k+5)(\alpha)}_{2(\alpha)2(\alpha)} = \Gamma^{(l-i-j-k+5)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(l-i-j-k+5)(\alpha)}_{2(\alpha)2(\alpha)}$$

which is trivially verified.
Case 2: $\alpha = \beta = \gamma \neq \epsilon$. (4.86) becomes

$$\Gamma^{l(\epsilon)}_{i(\alpha)(j+k-1)(\alpha)} = \Gamma^{l(\epsilon)}_{j(\alpha)(i+k-1)(\alpha)}$$

which is true by means of (4.64) and (4.65).
Case 3: $\alpha = \beta = \epsilon \neq \gamma$. (4.86) becomes

$$-\Gamma^{(l-j+1)(\alpha)}_{i(\alpha)k(\gamma)} = -\Gamma^{(l-i+1)(\alpha)}_{j(\alpha)k(\gamma)}$$

which is true by means of (4.62) and (4.63).
Case 4: $\alpha = \gamma = \epsilon \neq \beta$. (4.86) becomes

$$0 = \Gamma^{l(\alpha)}_{j(\beta)(i+k-1)(\alpha)} - \Gamma^{(l-i+1)(\alpha)}_{j(\beta)k(\alpha)}$$

which is true by means of (4.62) and (4.63).
Case 5: $\beta = \gamma = \epsilon \neq \alpha$. (4.86) becomes

$$\Gamma^{l(\beta)}_{i(\alpha)(j+k-1)(\beta)} - \Gamma^{(l-j+1)(\beta)}_{i(\alpha)k(\beta)} = 0$$

which is true by means of (4.62) and (4.63).
Case 6: $\alpha = \beta \neq \gamma = \epsilon$. (4.86) becomes $0 = 0$.
Case 7: $\alpha = \gamma \neq \beta = \epsilon$. (4.86) becomes

$$-\Gamma^{(l-j+1)(\beta)}_{i(\alpha)k(\alpha)} = \Gamma^{l(\beta)}_{j(\beta)(i+k-1)(\alpha)}$$

which is true by means of (4.62), (4.63), (4.64) and (4.65).
Case 8: $\alpha = \epsilon \neq \beta = \gamma$. (4.86) becomes

$$\Gamma^{l(\alpha)}_{i(\alpha)(j+k-1)(\beta)} = -\Gamma^{(l-i+1)(\alpha)}_{j(\beta)k(\beta)}$$

which is true by means of (4.62), (4.63), (4.64) and (4.65).
Case 9: at least three among α, β, γ, ϵ are pairwise distinct. (4.86) becomes trivially $0 = 0$.
This proves the compatibility between ∇ and $\circ$.

4.6.6 Linearity of the Euler vector field

We are now going to show that ∇, ∇^* are flat connections. We know that if we take the flatness of ∇ as already verified and assume $\nabla\nabla E = 0$, then we deduce that $(\nabla, \nabla^*, \circ, *, e, E)$ define a bi-flat structure on M. It is then enough for us to only prove the flatness of ∇ and to verify the condition $\nabla\nabla E = 0$.

Let us start by proving $\nabla\nabla E = 0$. We have

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} = \partial_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} + \Gamma^{k(\gamma)}_{i(\alpha)l(\sigma)}\nabla_{j(\beta)}E^{l(\sigma)} - \Gamma^{l(\sigma)}_{i(\alpha)j(\beta)}\nabla_{l(\sigma)}E^{k(\gamma)}$$

where, by means of (4.74), $\nabla_{j(\beta)}E^{k(\gamma)}$ is constant and $\nabla_{j(\beta)}E^{l(\sigma)}$, $\nabla_{l(\sigma)}E^{k(\gamma)}$ vanish respectively whenever $l \neq j$, $l \neq k$. Thus

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} = \Gamma^{k(\gamma)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\gamma)}. \tag{4.88}$$

The possible cases are:

1. $\alpha = \beta = \gamma$
2. $\alpha = \beta \neq \gamma$
3. $\alpha = \gamma \neq \beta$
4. $\beta = \gamma \neq \alpha$
5. $\alpha \neq \beta \neq \gamma \neq \alpha$.

Case 1: $\alpha = \beta = \gamma$**.**

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\alpha)}\\ &= \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\big(\nabla_{j(\alpha)}E^{j(\alpha)} - \nabla_{k(\alpha)}E^{k(\alpha)}\big)\\ &+ \sum_{\sigma\neq\alpha}\left(\Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\alpha)}\right).\end{aligned} \tag{4.89}$$

If $j = k = 1$ then

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\alpha)}_{i(\alpha)1(\alpha)}\big(\nabla_{1(\alpha)}E^{1(\alpha)} - \nabla_{1(\alpha)}E^{1(\alpha)}\big)\\ &+ \sum_{\sigma\neq\alpha}\left(\Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}\nabla_{1(\alpha)}E^{1(\sigma)} - \Gamma^{1(\sigma)}_{i(\alpha)1(\alpha)}\nabla_{1(\sigma)}E^{1(\alpha)}\right)\end{aligned}$$

which trivially vanishes if $i \geq 2$ (by (4.62) and (4.64)) and becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\big(\nabla_{1(\alpha)}E^{1(\alpha)} - \nabla_{1(\alpha)}E^{1(\alpha)}\big)\\ &+ \sum_{\sigma\neq\alpha}\left(\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\nabla_{1(\alpha)}E^{1(\sigma)} - \Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)}\nabla_{1(\sigma)}E^{1(\alpha)}\right)\end{aligned}$$

$$
\overset{(4.79)}{\underset{(4.61)}{=}} \sum_{\sigma\neq\alpha}\left(\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\nabla_{1(\alpha)}E^{1(\sigma)} + \Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)}\nabla_{1(\sigma)}E^{1(\alpha)}\right)
$$

$$
\overset{(4.63)}{\underset{(4.74)}{=}} \sum_{\sigma\neq\alpha}\left(\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}}\,m_\alpha\varepsilon_\alpha - \frac{m_\alpha\varepsilon_\alpha}{u^{1(\alpha)}-u^{1(\sigma)}}\,m_\sigma\varepsilon_\sigma\right) = 0
$$

if $i = 1$. If both j and k are greater or equal then 2 then (4.89) becomes

$$
\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\big(\nabla_{j(\alpha)}E^{j(\alpha)} - \nabla_{k(\alpha)}E^{k(\alpha)}\big)\\
&\quad + \sum_{\sigma\neq\alpha}\left(\Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\alpha)}\right)\\
&\overset{(4.74)}{=} \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\left(1 - \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau - 1 + \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau\right)\\
&\quad + \sum_{\sigma\neq\alpha}\left(\Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\alpha)}\right) = 0.
\end{aligned}
$$

If $j = 1$ and $k \geq 2$ then (4.89) becomes

$$
\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\alpha)}_{i(\alpha)1(\alpha)}\big(\nabla_{1(\alpha)}E^{1(\alpha)} - \nabla_{k(\alpha)}E^{k(\alpha)}\big)\\
&\quad + \sum_{\sigma\neq\alpha}\left(\Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}\nabla_{1(\alpha)}E^{1(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)1(\alpha)}\nabla_{k(\sigma)}E^{k(\alpha)}\right)\\
&\overset{(4.74)}{=} \Gamma^{k(\alpha)}_{i(\alpha)1(\alpha)}\left(1 - \sum_{\sigma\neq\alpha} m_\sigma\varepsilon_\sigma - 1 + \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau\right)\\
&\quad + \sum_{\sigma\neq\alpha}\left(\Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}\,m_\alpha\varepsilon_\alpha - \Gamma^{k(\sigma)}_{i(\alpha)1(\alpha)}\nabla_{k(\sigma)}E^{k(\alpha)}\right)\\
&\overset{(4.74)}{=} \Gamma^{k(\alpha)}_{i(\alpha)1(\alpha)}\,m_\alpha\varepsilon_\alpha + \sum_{\sigma\neq\alpha}\Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}\,m_\alpha\varepsilon_\alpha\\
&\overset{(4.79)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}\,m_\alpha\varepsilon_\alpha + \sum_{\sigma\neq\alpha}\Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}\,m_\alpha\varepsilon_\alpha = 0.
\end{aligned}
$$

If $j \geq 2$ and $k = 1$ then (4.89) becomes

$$
\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\alpha)}_{i(\alpha)j(\alpha)}\big(\nabla_{j(\alpha)}E^{j(\alpha)} - \nabla_{1(\alpha)}E^{1(\alpha)}\big)\\
&\quad + \sum_{\sigma\neq\alpha}\left(\Gamma^{1(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{1(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{1(\sigma)}E^{1(\alpha)}\right)\\
&\overset{(4.74)}{=} -\Gamma^{1(\alpha)}_{i(\alpha)j(\alpha)}\,m_\alpha\varepsilon_\alpha + \sum_{\sigma\neq\alpha}\left(\Gamma^{1(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{1(\sigma)}_{i(\alpha)j(\alpha)}\,m_\sigma\varepsilon_\sigma\right)\\
&\overset{(4.74)}{=} -\Gamma^{1(\alpha)}_{i(\alpha)j(\alpha)}\,m_\alpha\varepsilon_\alpha - \sum_{\sigma\neq\alpha}\Gamma^{1(\sigma)}_{i(\alpha)j(\alpha)}\,m_\sigma\varepsilon_\sigma
\end{aligned}
$$

which is

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} = -\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)}\, m_\alpha\varepsilon_\alpha - \sum_{\sigma\neq\alpha}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\, m_\sigma\varepsilon_\sigma$$

$$\overset{(4.79)}{\underset{(4.61)}{=}} \sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\sigma)j(\alpha)}\, m_\alpha\varepsilon_\alpha + \sum_{\sigma\neq\alpha}\Gamma^{1(\sigma)}_{1(\sigma)j(\alpha)}\, m_\sigma\varepsilon_\sigma = 0$$

if $i = 1$ and

$$\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} = -\Gamma^{1(\alpha)}_{i(\alpha)j(\alpha)}\, m_\alpha\varepsilon_\alpha - \sum_{\sigma\neq\alpha}\Gamma^{1(\sigma)}_{i(\alpha)j(\alpha)}\, m_\sigma\varepsilon_\sigma \overset{(4.64)}{\underset{(4.69)}{=}} 0$$

(as $1 - i - j \leq 1 - 2 - 2 = -3$) if $i \geq 2$.

Case 2: $\alpha = \beta \neq \gamma$**.**

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\alpha)}E^{k(\gamma)} &= \Gamma^{k(\gamma)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\gamma)} \\ &= \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{j(\alpha)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\nabla_{k(\alpha)}E^{k(\gamma)} \\ &+ \Gamma^{k(\gamma)}_{i(\alpha)j(\gamma)}\nabla_{j(\alpha)}E^{j(\gamma)} - \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{k(\gamma)}E^{k(\gamma)} \\ &+ \sum_{\sigma\neq\alpha,\gamma}\left(\Gamma^{k(\gamma)}_{i(\alpha)j(\sigma)}\nabla_{j(\alpha)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\gamma)}\right)\end{aligned} \tag{4.90}$$

where $\Gamma^{k(\gamma)}_{i(\alpha)j(\sigma)}$ vanishes due to (4.61). If both j and k are greater or equal than 2 then it becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\alpha)}E^{k(\gamma)} &= \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{j(\alpha)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\nabla_{k(\alpha)}E^{k(\gamma)} \\ &+ \Gamma^{k(\gamma)}_{i(\alpha)j(\gamma)}\nabla_{j(\alpha)}E^{j(\gamma)} - \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{k(\gamma)}E^{k(\gamma)} \\ &- \sum_{\sigma\neq\alpha,\gamma}\Gamma^{k(\sigma)}_{i(\alpha)j(\alpha)}\nabla_{k(\sigma)}E^{k(\gamma)} \\ &\overset{(4.74)}{=} \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\big(\nabla_{j(\alpha)}E^{j(\alpha)} - \nabla_{k(\gamma)}E^{k(\gamma)}\big) \overset{(4.74)}{=} 0.\end{aligned}$$

If $j = k = 1$ then (4.90) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\alpha)}E^{k(\gamma)} &= \Gamma^{1(\gamma)}_{i(\alpha)1(\alpha)}\nabla_{1(\alpha)}E^{1(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)1(\alpha)}\nabla_{1(\alpha)}E^{1(\gamma)} \\ &+ \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\alpha)}E^{1(\gamma)} - \Gamma^{1(\gamma)}_{i(\alpha)1(\alpha)}\nabla_{1(\gamma)}E^{1(\gamma)} \\ &- \sum_{\sigma\neq\alpha,\gamma}\Gamma^{1(\sigma)}_{i(\alpha)1(\alpha)}\nabla_{1(\sigma)}E^{1(\gamma)} \\ &\overset{(4.79)}{\underset{(4.61)}{=}} -\Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\alpha)}E^{1(\alpha)} \\ &+ \Gamma^{1(\alpha)}_{i(\alpha)1(\gamma)}\nabla_{1(\alpha)}E^{1(\gamma)} + \sum_{\sigma\neq\alpha,\gamma}\Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}\nabla_{1(\alpha)}E^{1(\gamma)} \\ &+ \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\alpha)}E^{1(\gamma)} + \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\gamma)}E^{1(\gamma)}\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\sigma\neq\alpha,\gamma} \Gamma^{1(\sigma)}_{i(\alpha)1(\sigma)} \nabla_{1(\sigma)} E^{1(\gamma)} \\
&\overset{(4.74)}{=} -\Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)} \left(1 - \sum_{\sigma\neq\alpha} m_\sigma \varepsilon_\sigma\right) \\
&+ \Gamma^{1(\alpha)}_{i(\alpha)1(\gamma)}\, m_\alpha \varepsilon_\alpha + \sum_{\sigma\neq\alpha,\gamma} \Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}\, m_\alpha \varepsilon_\alpha \\
&+ \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\, m_\alpha \varepsilon_\alpha + \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)} \left(1 - \sum_{\sigma\neq\gamma} m_\sigma \varepsilon_\sigma\right) \\
&+ \sum_{\sigma\neq\alpha,\gamma} \Gamma^{1(\sigma)}_{i(\alpha)1(\sigma)}\, m_\sigma \varepsilon_\sigma \\
&= \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\, m_\gamma \varepsilon_\gamma + \Gamma^{1(\alpha)}_{i(\alpha)1(\gamma)}\, m_\alpha \varepsilon_\alpha + \sum_{\sigma\neq\alpha,\gamma} \Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}\, m_\alpha \varepsilon_\alpha \\
&+ \sum_{\sigma\neq\alpha,\gamma} \Gamma^{1(\sigma)}_{i(\alpha)1(\sigma)}\, m_\sigma \varepsilon_\sigma
\end{aligned}
$$

which trivially vanishes if $i \geq 2$ (by (4.62) and (4.63)) and becomes

$$
\begin{aligned}
\nabla_{i(\alpha)} \nabla_{j(\alpha)} E^{k(\gamma)} &= \Gamma^{1(\gamma)}_{1(\alpha)1(\gamma)}\, m_\gamma \varepsilon_\gamma + \Gamma^{1(\alpha)}_{1(\alpha)1(\gamma)}\, m_\alpha \varepsilon_\alpha \\
&+ \sum_{\sigma\neq\alpha,\gamma} \left(\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\, m_\alpha \varepsilon_\alpha + \Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)}\, m_\sigma \varepsilon_\sigma\right) \\
&\overset{(4.63)}{=} -\frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\gamma)}}\, m_\gamma \varepsilon_\gamma + \frac{m_\gamma \varepsilon_\gamma}{u^{1(\alpha)} - u^{1(\gamma)}}\, m_\alpha \varepsilon_\alpha \\
&+ \sum_{\sigma\neq\alpha,\gamma} \left(\frac{m_\sigma \varepsilon_\sigma}{u^{1(\alpha)} - u^{1(\sigma)}}\, m_\alpha \varepsilon_\alpha - \frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\sigma)}}\, m_\sigma \varepsilon_\sigma\right) = 0
\end{aligned}
$$

if $i = 1$. If $j = 1$ and $k \geq 2$ then (4.90) becomes

$$
\begin{aligned}
\nabla_{i(\alpha)} \nabla_{j(\alpha)} E^{k(\gamma)} &= \Gamma^{k(\gamma)}_{i(\alpha)1(\alpha)} \nabla_{1(\alpha)} E^{1(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)1(\alpha)} \nabla_{k(\alpha)} E^{k(\gamma)} \\
&+ \Gamma^{k(\gamma)}_{i(\alpha)1(\gamma)} \nabla_{1(\alpha)} E^{1(\gamma)} - \Gamma^{k(\gamma)}_{i(\alpha)1(\alpha)} \nabla_{k(\gamma)} E^{k(\gamma)} \\
&- \sum_{\sigma\neq\alpha,\gamma} \Gamma^{k(\sigma)}_{i(\alpha)1(\alpha)} \nabla_{k(\sigma)} E^{k(\gamma)} \\
&\underset{(4.79),(4.61)}{\overset{(4.74)}{=}} -\Gamma^{k(\gamma)}_{i(\alpha)1(\gamma)} \left(1 - \sum_{\sigma\neq\alpha} m_\sigma \varepsilon_\sigma\right) \\
&+ \Gamma^{k(\gamma)}_{i(\alpha)1(\gamma)}\, m_\alpha \varepsilon_\alpha + \Gamma^{k(\gamma)}_{i(\alpha)1(\gamma)} \left(1 - \sum_{\tau=1}^{r} m_\tau \varepsilon_\tau\right) = 0.
\end{aligned}
$$

If $j \geq 2$ and $k = 1$ then (4.90) becomes

$$
\begin{aligned}
\nabla_{i(\alpha)} \nabla_{j(\alpha)} E^{k(\gamma)} &= \Gamma^{1(\gamma)}_{i(\alpha)j(\alpha)} \nabla_{j(\alpha)} E^{j(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)j(\alpha)} \nabla_{1(\alpha)} E^{1(\gamma)} \\
&+ \Gamma^{1(\gamma)}_{i(\alpha)j(\gamma)} \nabla_{j(\alpha)} E^{j(\gamma)} - \Gamma^{1(\gamma)}_{i(\alpha)j(\alpha)} \nabla_{1(\gamma)} E^{1(\gamma)}
\end{aligned}
$$

$$- \sum_{\sigma\neq\alpha,\gamma} \Gamma^{1(\sigma)}_{i(\alpha)j(\alpha)} \nabla_{1(\sigma)} E^{1(\gamma)} \overset{(4.63),(4.64)}{\underset{(4.69),(4.67)}{=}} 0.$$

Case 3: $\alpha = \gamma \neq \beta$**.**

$$\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\alpha)} \\
&= \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\alpha)} \\
&+ \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\alpha)} \\
&+ \sum_{\sigma\neq\alpha,\beta}\left(\Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\alpha)}\right)
\end{aligned} \tag{4.91}$$

where $\Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}$ vanishes due to (4.61). If both j and k are greater or equal than 2 then it becomes

$$\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\alpha)} \\
&+ \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\alpha)} \\
&+ \sum_{\sigma\neq\alpha,\beta}\Gamma^{k(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} \\
&\overset{(4.74)}{=} \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\big(\nabla_{j(\beta)}E^{j(\beta)} - \nabla_{k(\alpha)}E^{k(\alpha)}\big) \overset{(4.74)}{=} 0.
\end{aligned}$$

If $j = k = 1$ then (4.91) becomes

$$\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\alpha)}_{i(\alpha)1(\alpha)}\nabla_{1(\beta)}E^{1(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}\nabla_{1(\alpha)}E^{1(\alpha)} \\
&+ \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\beta)} - \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\alpha)} \\
&+ \sum_{\sigma\neq\alpha,\beta}\Gamma^{1(\alpha)}_{i(\alpha)1(\sigma)}\nabla_{1(\beta)}E^{1(\sigma)}
\end{aligned}$$

which trivially vanishes if $i \geq 2$ (by (4.62), (4.63) and (4.67)) and is

$$\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\nabla_{1(\beta)}E^{1(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}\nabla_{1(\alpha)}E^{1(\alpha)} \\
&+ \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\beta)} - \Gamma^{1(\beta)}_{1(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\alpha)} \\
&+ \sum_{\sigma\neq\alpha,\beta}\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\nabla_{1(\beta)}E^{1(\sigma)} \\
&\overset{(4.74)}{\underset{(4.79)}{=}} -\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\,m_\beta\varepsilon_\beta - \sum_{\sigma\neq\alpha,\beta}\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}\,m_\beta\varepsilon_\beta \\
&- \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}\left(1 - \sum_{\sigma\neq\alpha} m_\sigma\varepsilon_\sigma\right) \\
&+ \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}\left(1 - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma\right) - \Gamma^{1(\beta)}_{1(\alpha)1(\beta)}\,m_\beta\varepsilon_\beta
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{\sigma\neq\alpha,\beta} \Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\, m_\beta\varepsilon_\beta \\
&= \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\Bigg(- m_\beta\varepsilon_\beta + \sum_{\sigma\neq\alpha} m_\sigma\varepsilon_\sigma - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma\Bigg) - \Gamma^{1(\beta)}_{1(\alpha)1(\beta)}\, m_\beta\varepsilon_\beta \\
&= -\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\, m_\alpha\varepsilon_\alpha - \Gamma^{1(\beta)}_{1(\alpha)1(\beta)}\, m_\beta\varepsilon_\beta \\
&\overset{(4.74)}{=} -\frac{m_\beta\varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}}\, m_\alpha\varepsilon_\alpha + \frac{m_\alpha\varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}}\, m_\beta\varepsilon_\beta = 0
\end{aligned}
$$

if $i = 1$. If $j = 1$ and $k \geq 2$ then (4.91) becomes

$$
\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\alpha)}_{i(\alpha)1(\alpha)}\nabla_{1(\beta)}E^{1(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\nabla_{k(\alpha)}E^{k(\alpha)} \\
&+ \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)1(\beta)}\nabla_{k(\beta)}E^{k(\alpha)} \\
&+ \sum_{\sigma\neq\alpha,\beta} \Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}\nabla_{1(\beta)}E^{1(\sigma)} \\
&\overset{(4.74)}{=} -\Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\, m_\beta\varepsilon_\beta - \sum_{\sigma\neq\alpha,\beta} \Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}\, m_\beta\varepsilon_\beta \\
&- \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\Bigg(1 - \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau\Bigg) + \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\Bigg(1 - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma\Bigg) \\
&+ \sum_{\sigma\neq\alpha,\beta} \Gamma^{k(\alpha)}_{i(\alpha)1(\sigma)}\, m_\beta\varepsilon_\beta \\
&= \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\Bigg(- m_\beta\varepsilon_\beta + \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma\Bigg) = 0.
\end{aligned}
$$

If $j \geq 2$ and $k = 1$ then (4.91) becomes

$$
\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\alpha)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)j(\beta)}\nabla_{1(\alpha)}E^{1(\alpha)} \\
&+ \Gamma^{1(\alpha)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{1(\beta)}_{i(\alpha)j(\beta)}\nabla_{1(\beta)}E^{1(\alpha)} \\
&+ \sum_{\sigma\neq\alpha,\beta} \Gamma^{1(\alpha)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} \underset{(4.62),(4.63)}{\overset{(4.74)}{=}} 0.
\end{aligned}
$$

Case 4: $\beta = \gamma \neq \alpha$**.**

$$
\begin{aligned}
\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\beta)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\beta)} \\
&= \Gamma^{k(\beta)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\beta)} \\
&+ \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\beta)} \\
&+ \sum_{\sigma\neq\alpha,\beta} \left(\Gamma^{k(\beta)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\beta)}\right) \qquad (4.92)
\end{aligned}
$$

where $\Gamma^{k(\beta)}_{i(\alpha)j(\sigma)}$ and $\Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}$ vanish due to (4.61). If both j and k are greater or equal than 2 then it becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\beta)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\beta)} \\ &+ \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\beta)} \\ &\overset{(4.74)}{=} \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\left(1 - \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau - 1 + \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau\right) \overset{(4.74)}{=} 0.\end{aligned}$$

If $j = k = 1$ then (4.92) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\beta)}_{i(\alpha)1(\alpha)}\nabla_{1(\beta)}E^{1(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}\nabla_{1(\alpha)}E^{1(\beta)} \\ &+ \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\beta)} - \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\beta)} \\ &\underset{(4.79),(4.61)}{\overset{(4.74)}{=}} -\Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\, m_\beta\varepsilon_\beta - \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}\, m_\alpha\varepsilon_\alpha \\ &+ \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\left(1 - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma\right) - \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\left(1 - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma\right) \\ &\overset{(4.63)}{=} \frac{m_\alpha\varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}}\, m_\beta\varepsilon_\beta - \frac{m_\beta\varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}}\, m_\alpha\varepsilon_\alpha = 0.\end{aligned}$$

If $j = 1$ and $k \geq 2$ then (4.92) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\beta)}_{i(\alpha)1(\alpha)}\nabla_{1(\beta)}E^{1(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\nabla_{k(\alpha)}E^{k(\beta)} \\ &+ \Gamma^{k(\beta)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)1(\beta)}\nabla_{k(\beta)}E^{k(\beta)} \\ &\overset{(4.74)}{=} \Gamma^{k(\beta)}_{i(\alpha)1(\beta)}\left(- m_\beta\varepsilon_\beta + 1 - \sum_{\sigma\neq\beta} m_\sigma\varepsilon_\sigma - 1 + \sum_{\tau=1}^{r} m_\tau\varepsilon_\tau\right) = 0.\end{aligned}$$

If $j \geq 2$ and $k = 1$ then (4.92) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\beta)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)j(\beta)}\nabla_{1(\alpha)}E^{1(\beta)} \\ &+ \Gamma^{1(\beta)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{1(\beta)}_{i(\alpha)j(\beta)}\nabla_{1(\beta)}E^{1(\beta)} \underset{(4.62),(4.63)}{\overset{(4.74)}{=}} 0.\end{aligned}$$

Case 5: $\alpha \neq \beta \neq \gamma \neq \alpha$**.**

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\gamma)}_{i(\alpha)j(\sigma)}\nabla_{j(\beta)}E^{j(\sigma)} - \Gamma^{k(\sigma)}_{i(\alpha)j(\beta)}\nabla_{k(\sigma)}E^{k(\gamma)} \\ &\overset{(4.61)}{=} \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\gamma)} \\ &+ \Gamma^{k(\gamma)}_{i(\alpha)j(\beta)}\nabla_{j(\beta)}E^{j(\beta)} - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\gamma)} \\ &+ \Gamma^{k(\gamma)}_{i(\alpha)j(\gamma)}\nabla_{j(\beta)}E^{j(\gamma)} - \Gamma^{k(\gamma)}_{i(\alpha)j(\beta)}\nabla_{k(\gamma)}E^{k(\gamma)}\end{aligned} \tag{4.93}$$

where $\Gamma^{k(\gamma)}_{i(\alpha)j(\beta)}$ and $\Gamma^{k(\gamma)}_{i(\alpha)j(\beta)}$ vanish due to (4.61). If both j and k are greater or equal than 2 then it becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}\nabla_{k(\alpha)}E^{k(\gamma)}\\ &\quad - \Gamma^{k(\beta)}_{i(\alpha)j(\beta)}\nabla_{k(\beta)}E^{k(\gamma)} + \Gamma^{k(\gamma)}_{i(\alpha)j(\gamma)}\nabla_{j(\beta)}E^{j(\gamma)} \overset{(4.74)}{=} 0.\end{aligned}$$

If $j = k = 1$ then (4.93) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\gamma)}_{i(\alpha)1(\alpha)}\nabla_{1(\beta)}E^{1(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}\nabla_{1(\alpha)}E^{1(\gamma)}\\ &\quad - \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\nabla_{1(\beta)}E^{1(\gamma)} + \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\beta)}E^{1(\gamma)}\\ &\underset{(4.79),(4.61)}{\overset{(4.74)}{=}} -\Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\, m_\beta\varepsilon_\beta - \Gamma^{1(\alpha)}_{i(\alpha)1(\beta)}\, m_\alpha\varepsilon_\alpha\\ &\quad - \Gamma^{1(\beta)}_{i(\alpha)1(\beta)}\, m_\beta\varepsilon_\beta + \Gamma^{1(\gamma)}_{i(\alpha)1(\gamma)}\, m_\beta\varepsilon_\beta\\ &\overset{(4.63)}{=} -\frac{m_\beta\varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}}\, m_\alpha\varepsilon_\alpha + \frac{m_\alpha\varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}}\, m_\beta\varepsilon_\beta = 0.\end{aligned}$$

If $j = 1$ and $k \geq 2$ then (4.93) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{k(\gamma)}_{i(\alpha)1(\alpha)}\nabla_{1(\beta)}E^{1(\alpha)} - \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}\nabla_{k(\alpha)}E^{k(\gamma)}\\ &\quad - \Gamma^{k(\beta)}_{i(\alpha)1(\beta)}\nabla_{k(\beta)}E^{k(\gamma)} + \Gamma^{k(\gamma)}_{i(\alpha)1(\gamma)}\nabla_{1(\beta)}E^{1(\gamma)}\\ &\overset{(4.74)}{=} -\Gamma^{k(\gamma)}_{i(\alpha)1(\gamma)}\, m_\beta\varepsilon_\beta + \Gamma^{k(\gamma)}_{i(\alpha)1(\gamma)}\, m_\beta\varepsilon_\beta = 0.\end{aligned}$$

If $j \geq 2$ and $k = 1$ then (4.93) becomes

$$\begin{aligned}\nabla_{i(\alpha)}\nabla_{j(\beta)}E^{k(\gamma)} &= \Gamma^{1(\gamma)}_{i(\alpha)j(\alpha)}\nabla_{j(\beta)}E^{j(\alpha)} - \Gamma^{1(\alpha)}_{i(\alpha)j(\beta)}\nabla_{1(\alpha)}E^{1(\gamma)}\\ &\quad - \Gamma^{1(\beta)}_{i(\alpha)j(\beta)}\nabla_{1(\beta)}E^{1(\gamma)} + \Gamma^{1(\gamma)}_{i(\alpha)j(\gamma)}\nabla_{j(\beta)}E^{j(\gamma)} \underset{(4.62)}{\overset{(4.74)}{=}} 0.\end{aligned}$$

This proves that $\nabla\nabla E = 0$.

4.6.7 Flatness of ∇

We are now left with proving the flatness of ∇, that is $R = 0$. Due to the symmetries of

$$\begin{aligned}R^{i(\alpha)}_{h(\epsilon)k(\gamma)j(\beta)} &= \partial_{k(\gamma)}\Gamma^{i(\alpha)}_{h(\epsilon)j(\beta)} - \partial_{h(\epsilon)}\Gamma^{i(\alpha)}_{k(\gamma)j(\beta)}\\ &\quad + \sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\sigma)}\Gamma^{l(\sigma)}_{h(\epsilon)j(\beta)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\gamma)j(\beta)}\right)\end{aligned} \tag{4.94}$$

the cases to be considered are the following ones:

1. $\alpha = \beta = \gamma = \epsilon$

2. $\alpha = \beta = \gamma \neq \epsilon$

3. $\alpha = \gamma = \epsilon \neq \beta$

4. $\beta = \gamma = \epsilon \neq \alpha$

5. $\alpha = \beta \neq \gamma = \epsilon$

6. $\alpha = \gamma \neq \beta = \epsilon$

7. $\alpha = \beta \notin \{\gamma, \epsilon\}$, $\gamma \neq \epsilon$

8. $\alpha = \gamma \notin \{\beta, \epsilon\}$, $\beta \neq \epsilon$

9. $\beta = \gamma \notin \{\alpha, \epsilon\}$, $\alpha \neq \epsilon$

10. $\gamma = \epsilon \notin \{\alpha, \beta\}$, $\alpha \neq \beta$

11. α, β, γ and ϵ are pairwise distinct.

Case 1: $\alpha = \beta = \gamma = \epsilon$**.** Our goal is to prove that

$$\begin{aligned}
R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} &= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} \\
&\quad + \sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right) \\
&= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} \\
&\quad + \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) \\
&\quad + \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right) \qquad (4.95)
\end{aligned}$$

vanishes. Let us first note that for each integer $N \geq 2$ it is possible to recover part of the Christoffel symbols for the case where $m_\alpha = N+1$ starting from the ones for the case where $m_\alpha = N$. More precisely, let us denote by $(\Gamma^{N+1})^k_{ij}$ the Christoffel symbols in the case where $m_\alpha = N+1$ and by $(\Gamma^N)^k_{ij}$ the Christoffel symbols in the case where $m_\alpha = N$, where the sizes m_σ of the remaining blocks $\sigma \neq \alpha$ are the same and where the constant ε_α has been replaced by $\frac{N+1}{N}\varepsilon_\alpha$. Then

$$(\Gamma^{N+1})^{k(\beta)}_{i(\sigma)j(\tau)} = (\Gamma^N)^{k(\beta)}_{i(\sigma)j(\tau)} \qquad (4.96)$$

for any possible choice of the indices in the right hand side (see Remark 4.18). In the wake of this property, we will proceed by induction over m_α. Let us first consider the case[2] where $m_\alpha = 2$, so that the indices i, j, k and h run from 1 to 2. In particular, since $R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)}$ automatically vanishes when $k = h$, the only relevant cases are the one where $k = 1$, $h = 2$ and the one where $k = 2$, $h = 1$. By using the symmetries of R, we only consider the case where $k = 1$ and $h = 2$, hence obtaining

$$\begin{aligned} R^{i(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{i(\alpha)}_{2(\alpha)j(\alpha)} - \partial_{2(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \sum_{l=1}^{2}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\right) \\ &+ \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{2(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{2(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{1(\alpha)j(\alpha)}\right) \end{aligned} \tag{4.97}$$

where both $\Gamma^{i(\alpha)}_{1(\alpha)l(\sigma)}$ and $\Gamma^{i(\alpha)}_{2(\alpha)l(\sigma)}$ survive only for $l = 1$ by (4.62). This yields

$$\begin{aligned} R^{i(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{i(\alpha)}_{2(\alpha)j(\alpha)} - \partial_{2(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \Gamma^{i(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \Gamma^{i(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \sum_{\sigma\neq\alpha}\left(\Gamma^{i(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right). \end{aligned} \tag{4.98}$$

If $i = 1$ we get

$$\begin{aligned} R^{1(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)} - \partial_{2(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \sum_{\sigma\neq\alpha}\left(\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right), \end{aligned}$$

where $\Gamma^{1(\alpha)}_{2(\alpha)1(\sigma)}$ vanishes due to (4.63), that becomes

$$\begin{aligned} R^{1(\alpha)}_{2(\alpha)1(\alpha)1(\alpha)} &= \partial_{1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} \\ &+ \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} \\ &+ \Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \\ &+ \sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)1(\alpha)} \end{aligned}$$

[2] The case where $m_\alpha = 1$ is trivial, as $k = h = 1$ directly implies $R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = 0$.

$$\overset{(4.79)}{=} -\sum_{\sigma\neq\alpha}\partial_{1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)1(\sigma)} - \partial_{2(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}$$

$$-\sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)1(\sigma)} + \sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}$$

$$-\sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\sigma)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)}$$

$$-\sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)1(\sigma)}$$

$$\underset{(4.79)}{\overset{(4.63),(4.69)}{=}} \sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)} \overset{(4.63)}{=} \sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\left(\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}}\right) = 0$$

when $j=1$ and

$$\begin{aligned} R^{1(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)} &= \partial_{1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)} - \partial_{2(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)} \\ &+ \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)} \\ &+ \Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)2(\alpha)} \\ &+ \sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)2(\alpha)} \underset{(4.67)}{\overset{(4.64),(4.69)}{=}} 0 \end{aligned}$$

when $j=2$. If $i=2$ then (4.98) reads

$$\begin{aligned} R^{2(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)j(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \Gamma^{2(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)j(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)j(\alpha)} \\ &+ \sum_{\sigma\neq\alpha}\left(\Gamma^{2(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right) \end{aligned}$$

that becomes

$$\begin{aligned} R^{2(\alpha)}_{2(\alpha)1(\alpha)1(\alpha)} &= \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \\ &+ \Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} \\ &+ \Gamma^{2(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \\ &+ \sum_{\sigma\neq\alpha}\left(\Gamma^{2(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)}\right) \\ &\underset{(4.79)}{\overset{(4.64),(4.67)}{=}} \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \\ &+ \sum_{\sigma\neq\alpha}\left(\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)} + \Gamma^{2(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)}\right) \end{aligned}$$

$$
\begin{aligned}
&\overset{(4.63)}{\underset{(4.66)}{=}} \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \\
&+ \sum_{\sigma\neq\alpha}\Bigg(\frac{m_\alpha\varepsilon_\alpha}{u^{2(\alpha)}}\,\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}\,u^{2(\alpha)} \\
&- \Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\,\frac{m_\alpha\varepsilon_\alpha}{u^{1(\alpha)}-u^{1(\sigma)}}\Bigg) \\
&= \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \\
&\overset{(4.79)}{=} \sum_{\sigma\neq\alpha}\Big(- \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)1(\sigma)} + \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)}\Big) \\
&\overset{(4.63)}{=} \sum_{\sigma\neq\alpha}\left[- \partial_{1(\alpha)}\left(\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}}\right) + \partial_{2(\alpha)}\left(- \frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}\,u^{2(\alpha)}\right)\right] \\
&= \sum_{\sigma\neq\alpha}\left[\frac{m_\sigma\varepsilon_\sigma}{(u^{1(\alpha)}-u^{1(\sigma)})^2} - \frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}\right] \\
&\overset{(4.63)}{=} \sum_{\sigma\neq\alpha}\left[\frac{m_\sigma\varepsilon_\sigma}{(u^{1(\alpha)}-u^{1(\sigma)})^2} - \frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\,\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}}\right] = 0
\end{aligned}
$$

when $j = 1$ and

$$
\begin{aligned}
R^{2(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)} &= \partial_{1(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} - \partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)2(\alpha)} \\
&+ \Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)} \\
&+ \Gamma^{2(\alpha)}_{1(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{2(\alpha)}_{1(\alpha)2(\alpha)} \\
&+ \sum_{\sigma\neq\alpha}\left(\Gamma^{2(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)2(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)2(\alpha)}\right) \\
&\overset{(4.64),(4.69)}{\underset{(4.67),(4.79)}{=}} \partial_{1(\alpha)}\left(- \frac{m_\alpha\varepsilon_\alpha}{u^{2(\alpha)}}\right) + \sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\Gamma^{2(\alpha)}_{1(\sigma)2(\alpha)} \\
&\overset{(4.63)}{=} \sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\left(\frac{m_\sigma\varepsilon_\sigma}{u^{1(\alpha)}-u^{1(\sigma)}}\right) = 0
\end{aligned}
$$

when $j = 2$. Therefore we proved that (4.95) vanishes when $m_\alpha = 2$. Given an integer $N \geq 2$, let us now suppose that (4.95) vanishes for $m_\alpha = N$ and show it vanishes for $m_\alpha = N+1$ as well. In other words, we are supposing that

$$
\begin{aligned}
(R^N)^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} &:= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} \\
&+ \sum_{l=1}^{N}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) \\
&+ \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right) = 0
\end{aligned}
\tag{4.99}
$$

for every i, j, k, $h \in \{1, \ldots, m_\alpha\}$ for each $m_\alpha \le N$ and we want to prove that

$$\begin{aligned}(R^{N+1})^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} := {} & \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} \\ & + \sum_{l=1}^{N+1}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) \\ & + \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right) = 0 \qquad (4.100)\end{aligned}$$

for every i, j, k, $h \in \{1, \ldots, N+1\}$ for $m_\alpha = N+1$. Notice that, due to the property (4.96) and by replacing ε_α with $\frac{N+1}{N}\varepsilon_\alpha$, we can use in both cases the same notation for the Christoffel symbols. Let us start by considering the case where $i \le N$ and observe that

$$\Gamma^{i(\alpha)}_{j(\alpha)(N+1)(\alpha)} = 0, \qquad j \in \{1, \ldots, N+1\}, \qquad (4.101)$$

as $i - j - (N+1) \le N - 2 - N - 1 = -3$ for $j \ge 2$ (we recall (4.69)) and

$$\Gamma^{i(\alpha)}_{1(\alpha)(N+1)(\alpha)} \overset{(4.79)}{=} -\sum_{\sigma\neq\alpha}\Gamma^{i(\alpha)}_{1(\sigma)(N+1)(\alpha)} \overset{(4.63)}{=} 0$$

($i \le N < N+1$) for $j = 1$. If all of j, k, h are less or equal than N then

$$\begin{aligned}(R^{N+1})^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = {} & \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} \\ & + \sum_{l=1}^{N+1}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) \\ & + \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)\end{aligned}$$

where in the first summation only the terms for $l \le N$ survive, as both $\Gamma^{i(\alpha)}_{k(\alpha)(N+1)(\alpha)}$ and $\Gamma^{i(\alpha)}_{h(\alpha)(N+1)(\alpha)}$ vanish due to (4.101). This yields

$$\begin{aligned}(R^{N+1})^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = {} & \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} \\ & + \sum_{l=1}^{N}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right) \\ & + \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right) \\ = {} & (R^N)^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} \overset{(4.99)}{=} 0.\end{aligned}$$

If k, $h \le N$ and $j = N+1$ then

$$(R^{N+1})^{i(\alpha)}_{h(\alpha)k(\alpha)(N+1)(\alpha)} = \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)(N+1)(\alpha)} - \partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)(N+1)(\alpha)}$$

$$
\begin{aligned}
&+\sum_{l=1}^{N}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)(N+1)(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)(N+1)(\alpha)}\right)\\
&+\Gamma^{i(\alpha)}_{k(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{h(\alpha)(N+1)(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{k(\alpha)(N+1)(\alpha)}\\
&+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)(N+1)(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)(N+1)(\alpha)}\right)
\end{aligned}
$$

where, due to (4.101), only the last summation survives. This yields

$$
(R^{N+1})^{i(\alpha)}_{h(\alpha)k(\alpha)(N+1)(\alpha)}=\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)(N+1)(\alpha)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)(N+1)(\alpha)}\right)\overset{(4.64)}{=}0.
$$

If k, $j\leq N$ and $h=N+1$ (due to the symmetries of R, this covers the case where h, $j\leq N$ and $k=N+1$ as well) then

$$
\begin{aligned}
(R^{N+1})^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)j(\alpha)}&=\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{(N+1)(\alpha)j(\alpha)}-\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}\\
&+\sum_{l=1}^{N}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{(N+1)(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{(N+1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\
&+\Gamma^{i(\alpha)}_{k(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{(N+1)(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{(N+1)(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{k(\alpha)j(\alpha)}\\
&+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{(N+1)(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{(N+1)(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right).
\end{aligned}
$$

By means of (4.101), this yields

$$
\begin{aligned}
(R^{N+1})^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)j(\alpha)}&=-\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}\\
&+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{(N+1)(\alpha)j(\alpha)}-\Gamma^{i(\alpha)}_{(N+1)(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)\\
&\underset{(4.64)}{\overset{(4.63)}{=}}-\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}
\end{aligned}
$$

that becomes

$$
(R^{N+1})^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)1(\alpha)}=-\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\alpha)}=0
$$

($\Gamma^{i(\alpha)}_{k(\alpha)1(\alpha)}\overset{(4.79)}{=}-\sum_{\sigma\neq\alpha}\Gamma^{i(\alpha)}_{k(\alpha)1(\sigma)}$ only depends on $\{u^{1(\alpha)}-u^{1(\sigma)}\,|\,\sigma\neq\alpha\}$ and $\{u^{s(\alpha)}\,|\,2\leq s\leq i-k+1\}$ by (4.63), thus it does not depend on $u^{(N+1)(\alpha)}$ as $i-k+1\leq N-1+1=N<N+1$) when $j=1$,

$$
(R^{N+1})^{i(\alpha)}_{(N+1)(\alpha)1(\alpha)j(\alpha)}=-\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)}=0
$$

(analogously) when $k=1$ and

$$
(R^{N+1})^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)j(\alpha)}=-\partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}=0
$$

when both j and k are greater or equal than 2, as[3]

$$\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} \overset{(4.69)}{=} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} \overset{(4.68)}{=} \Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{(i-j-k+4)(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}} \sum_{l=1}^{i-j-k+1} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(i-j-k+4-l)(\alpha)}$$

does not depend on $u^{(N+1)(\alpha)}$ ($\Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\alpha)}$ only depends on $\{u^{1(\alpha)} - u^{1(\sigma)} \,|\, \sigma \neq \alpha\}$ and $\{u^{s(\alpha)} \,|\, 2 \leq s \leq i-j-k+2\}$ by (4.67) where $i-j-k+2 \leq N-2-2+2 = N-2 < N+1$, $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}$ only depends on $u^{2(\alpha)}$, $i-j-k+4 \leq N-2-2+4 = N < N+1$ and for every $1 \leq l \leq i-j-k+1$ we have $i-j-k+4-l < i-j-k+4 < N+1$ and $\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}$ only depends on quantities that correspond to lower indices so it does not depend on $u^{(N+1)(\alpha)}$ a fortiori). If $k = h = N+1$ then $(R^{N+1})^{i(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = 0$ for every value of j due to the symmetries of R. If $k \leq N$ and $j = h = N+1$ (due to the symmetries of R, this covers the case where $h \leq N$ and $j = k = N+1$ as well) then

$$\begin{aligned}(R^{N+1})^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)(N+1)(\alpha)} &= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{(N+1)(\alpha)(N+1)(\alpha)} - \partial_{(N+1)(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)(N+1)(\alpha)} \\ &+ \sum_{l=1}^{N}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{(N+1)(\alpha)(N+1)(\alpha)} - \Gamma^{i(\alpha)}_{(N+1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)(N+1)(\alpha)}\right) \\ &+ \Gamma^{i(\alpha)}_{k(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{(N+1)(\alpha)(N+1)(\alpha)} - \Gamma^{i(\alpha)}_{(N+1)(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{k(\alpha)(N+1)(\alpha)} \\ &+ \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{(N+1)(\alpha)(N+1)(\alpha)} - \Gamma^{i(\alpha)}_{(N+1)(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)(N+1)(\alpha)}\right).\end{aligned}$$

By means of (4.101), this yields

$$\begin{aligned}(R^{N+1})^{i(\alpha)}_{(N+1)(\alpha)k(\alpha)(N+1)(\alpha)} &= \sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{(N+1)(\alpha)(N+1)(\alpha)} - \Gamma^{i(\alpha)}_{(N+1)(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)(N+1)(\alpha)}\right) \\ &\overset{(4.64)}{=} 0.\end{aligned}$$

We have therefore proved (4.100) under the assumption that $i \leq N$. Let us now fix $i = N+1$. We have

$$\begin{aligned}(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} &= \partial_{k(\alpha)}\Gamma^{(N+1)(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{(N+1)(\alpha)}_{k(\alpha)j(\alpha)} \\ &+ \sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\end{aligned}$$

[3]Without loss of generality we assume $i-j-k \geq -2$, as $\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} = 0$ automatically when $i-j-k \leq -3$.

$$+\sum_{\sigma\neq\alpha}\sum_{l=1}^{m_\sigma}\left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)$$

where in the last summation only the terms for $l = 1$ survive by (4.62), yielding

$$\begin{aligned}(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} &= \partial_{k(\alpha)}\Gamma^{(N+1)(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{(N+1)(\alpha)}_{k(\alpha)j(\alpha)}\\ &+\sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\ &+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{k(\alpha)j(\alpha)}\right).\end{aligned}\tag{4.102}$$

We distinguish between the following subcases:

a. both k and h are greater or equal than 3

b. $k = 1$, $h \geq 3$ (this covers $h = 1$, $k \geq 3$ as well)

c. $k = 2$, $h \geq 3$ (this covers $h = 2$, $k \geq 3$ as well)

d. $k = 1$, $h = 2$ (this covers $h = 1$, $k = 2$ as well)

observing that $(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = 0$ automatically whenever $k = h$.
Subcase a: both k and h are greater or equal than 3. We have

$$\begin{aligned}(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} &= \partial_{k(\alpha)}\Gamma^{(N+1)(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{(N+1)(\alpha)}_{k(\alpha)j(\alpha)}\\ &+\sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\ &+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{k(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{k(\alpha)j(\alpha)}\right)\\ &\overset{(4.72)}{\underset{(4.63)}{=}} \partial_{(k-1)(\alpha)}\Gamma^{N(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{k(\alpha)j(\alpha)}\\ &+\Gamma^{N(\alpha)}_{(k-1)(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{k(\alpha)j(\alpha)}\\ &+\sum_{l=2}^{N+1}\left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\ &+\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{k(\alpha)j(\alpha)}\right)\end{aligned}\tag{4.103}$$

where $\Gamma^{1(\alpha)}_{h(\alpha)j(\alpha)}$ and $\Gamma^{1(\alpha)}_{k(\alpha)j(\alpha)}$ vanish due to (4.63) if $j = 1$ and due to (4.69) if $j \geq 2$.
If $j \leq N-1$ we get

$$(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)j(\alpha)} = \partial_{(k-1)(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)(j+1)(\alpha)} - \partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{(k-1)(\alpha)(j+1)(\alpha)}$$

$$
\begin{aligned}
&+ \sum_{l=2}^{N+1} \left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{(h-1)(\alpha)(j+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{(k-1)(\alpha)(j+1)(\alpha)} \right) \\
&+ \sum_{\sigma \neq \alpha} \left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{(h-1)(\alpha)(j+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{(k-1)(\alpha)(j+1)(\alpha)} \right) \\
&\overset{(4.69)}{=} (R^N)^{N(\alpha)}_{(h-1)(\alpha)(k-1)(\alpha)(j+1)(\alpha)} \\
&+ \Gamma^{N(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{(h-1)(\alpha)(j+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{(k-1)(\alpha)(j+1)(\alpha)} = 0
\end{aligned}
$$

as $(R^N)^{N(\alpha)}_{(h-1)(\alpha)(k-1)(\alpha)(j+1)(\alpha)}$ vanishes because of (4.99) and $\Gamma^{N(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)}$, $\Gamma^{N(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)}$ vanish because of (4.101). If $j = N$ then (4.103) becomes

$$
\begin{aligned}
(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)N(\alpha)} &= \partial_{(k-1)(\alpha)} \Gamma^{N(\alpha)}_{h(\alpha)N(\alpha)} - \partial_{(h-1)(\alpha)} \Gamma^{N(\alpha)}_{k(\alpha)N(\alpha)} \\
&+ \sum_{l=2}^{N} \left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)} \right) \\
&+ \Gamma^{N(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{h(\alpha)N(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{k(\alpha)N(\alpha)} \\
&+ \sum_{\sigma \neq \alpha} \left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{h(\alpha)N(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{k(\alpha)N(\alpha)} \right) \\
&\underset{(4.69)}{\overset{(4.64)}{=}} \sum_{l=2}^{N} \left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)} \right) \\
&+ \Gamma^{N(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{h(\alpha)N(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{k(\alpha)N(\alpha)} \overset{(4.101)}{=} 0.
\end{aligned}
$$

If $j = N + 1$ then (4.103) becomes

$$
\begin{aligned}
(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)k(\alpha)(N+1)(\alpha)} &= \partial_{(k-1)(\alpha)} \Gamma^{N(\alpha)}_{h(\alpha)(N+1)(\alpha)} - \partial_{(h-1)(\alpha)} \Gamma^{N(\alpha)}_{k(\alpha)(N+1)(\alpha)} \\
&+ \sum_{l=2}^{N} \left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{h(\alpha)(N+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)(N+1)(\alpha)} \right) \\
&+ \Gamma^{N(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{h(\alpha)(N+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{k(\alpha)(N+1)(\alpha)} \\
&+ \sum_{\sigma \neq \alpha} \left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{h(\alpha)(N+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)} \Gamma^{1(\sigma)}_{k(\alpha)(N+1)(\alpha)} \right) \\
&\overset{(4.64)}{=} \partial_{(k-1)(\alpha)} \Gamma^{N(\alpha)}_{h(\alpha)(N+1)(\alpha)} - \partial_{(h-1)(\alpha)} \Gamma^{N(\alpha)}_{k(\alpha)(N+1)(\alpha)} \\
&+ \sum_{l=2}^{N} \left(\Gamma^{N(\alpha)}_{(k-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{h(\alpha)(N+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)(N+1)(\alpha)} \right) \\
&+ \Gamma^{N(\alpha)}_{(k-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{h(\alpha)(N+1)(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)(N+1)(\alpha)} \Gamma^{(N+1)(\alpha)}_{k(\alpha)(N+1)(\alpha)} \overset{(4.101)}{=} 0.
\end{aligned}
$$

Subcase b: $k = 1$, $h \geq 3$. We have

$$
\begin{aligned}
(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{h(\alpha)j(\alpha)} - \partial_{h(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)j(\alpha)} \\
&+ \sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\right) \\
&+ \sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right) \\
&\overset{(4.72)}{=} \partial_{1(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)j(\alpha)} - \partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)j(\alpha)} \\
&+ \sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} - \sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)} \\
&+ \sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right) \\
&\underline{+\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)} \\
&\underline{-\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)}
\end{aligned}
$$

where the underlined terms cancel out and where the terms

$$\partial_{1(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)j(\alpha)},\ -\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)j(\alpha)},$$

$$
\begin{aligned}
\sum_{l=2}^{N+1}\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} &\underset{(4.67)}{\overset{(4.69)}{=}} \sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{1(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{(h-1)(\alpha)j(\alpha)} \\
&= \sum_{l=1}^{N}\Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{(h-1)(\alpha)j(\alpha)},
\end{aligned}
$$

$$-\sum_{l=1}^{N}\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)} \underset{(4.67)}{\overset{(4.69)}{=}} -\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}$$

and

$$\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)} - \Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)$$

combine to form $(R^N)^{N(\alpha)}_{(h-1)(\alpha)1(\alpha)j(\alpha)}$, which[4] vanishes by (4.99). Thus

$$(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)1(\alpha)j(\alpha)} = \Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{h(\alpha)j(\alpha)} - \Gamma^{(N+1)(\alpha)}_{h(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)j(\alpha)}$$

[4]This only holds for $j \leq N$. Still, for $j = N+1$ each of these addends vanishes by itself, by means of (4.101), (4.69) and (4.64).

$$
\begin{aligned}
&+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)\\
&-\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)\\
&\overset{(4.63),(4.64)}{\underset{(4.69),(4.67)}{=}}\sum_{\sigma\neq\alpha}\left(-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}+\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)=0.
\end{aligned}
$$

Subcase c: $k=2$, $h\geq 3$. We have

$$
\begin{aligned}
(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)2(\alpha)j(\alpha)} &= \partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{h(\alpha)j(\alpha)}-\partial_{h(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)j(\alpha)}\\
&+\sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}\right)\\
&+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)\\
&\overset{(4.69),(4.67)}{\underset{(4.72)}{=}}\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)j(\alpha)}-\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{2(\alpha)j(\alpha)}\\
&+\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)}-\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}\\
&+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)\\
&\underline{+\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)}\\
&\underline{-\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)}
\end{aligned}
$$

where the underlined terms cancel out and where the terms

$$
\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{(h-1)(\alpha)j(\alpha)},\ -\partial_{(h-1)(\alpha)}\Gamma^{N(\alpha)}_{2(\alpha)j(\alpha)},
$$

$$
\begin{aligned}
\sum_{l=2}^{N+1}\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\alpha)} &\overset{(4.69),(4.67)}{=}\sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{2(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{(h-1)(\alpha)j(\alpha)}\\
&=\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{(h-1)(\alpha)j(\alpha)},
\end{aligned}
$$

$$
-\sum_{l=1}^{N}\Gamma^{(N+1)(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}\overset{(4.69),(4.67)}{=}-\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{(h-1)(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}
$$

and

$$\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)$$

combine to form $(R^N)^{N(\alpha)}_{(h-1)(\alpha)2(\alpha)j(\alpha)}$, which[5] vanishes by (4.99). Thus

$$\begin{aligned}(R^{N+1})^{(N+1)(\alpha)}_{h(\alpha)2(\alpha)j(\alpha)} &= \Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)(N+1)(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)j(\alpha)}\\ &+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{h(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{h(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)\\ &-\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{(h-1)(\alpha)j(\alpha)}-\Gamma^{N(\alpha)}_{(h-1)(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}\right)\underset{(4.64)}{\overset{(4.69),(4.67)}{=}}0.\end{aligned}$$

Subcase d: $k=1$, $h=2$. We have

$$\begin{aligned}(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)j(\alpha)}-\partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)j(\alpha)}\\ &+\sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\right)\\ &+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right).\end{aligned}\tag{4.104}$$

If $j\geq 3$ we get

$$\begin{aligned}(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)j(\alpha)}-\partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)j(\alpha)}\\ &+\sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\right)\\ &+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)\\ &\underset{(4.67)}{\overset{(4.69)}{=}}\partial_{1(\alpha)}\Gamma^{N(\alpha)}_{2(\alpha)(j-1)(\alpha)}-\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)(j-1)(\alpha)}\\ &+\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}-\sum_{l=1}^{N+1}\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\\ &+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)\\ &\underline{+\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)(j-1)(\alpha)}-\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)(j-1)(\alpha)}\right)}\end{aligned}$$

[5]This only holds for $j\leq N$. Still, for $j=N+1$ each of these addends vanishes by itself, by means of (4.101), (4.69) and (4.64).

$$\underline{-\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)(j-1)(\alpha)}-\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)(j-1)(\alpha)}\right)}$$

where the underlined terms cancel out and where the terms

$$\partial_{1(\alpha)}\Gamma^{N(\alpha)}_{2(\alpha)(j-1)(\alpha)},\ -\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)(j-1)(\alpha)},$$

$$\sum_{l=2}^{N+1}\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)j(\alpha)}\overset{(4.69)}{\underset{(4.67)}{=}}\sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{1(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{2(\alpha)(j-1)(\alpha)}=\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)(j-1)(\alpha)},$$

$$-\sum_{l=2}^{N+1}\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)}\overset{(4.69)}{\underset{(4.67)}{=}}-\sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{2(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{1(\alpha)(j-1)(\alpha)}=-\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)(j-1)(\alpha)}$$

and

$$\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)(j-1)(\alpha)}-\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)(j-1)(\alpha)}\right)$$

combine to form $(R^N)^{N(\alpha)}_{2(\alpha)1(\alpha)(j-1)(\alpha)}$, which vanishes by (4.99). Thus

$$\begin{aligned}(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)j(\alpha)}&=\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)}\\&+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)j(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)j(\alpha)}\right)\\&-\sum_{\sigma\neq\alpha}\left(\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)(j-1)(\alpha)}-\Gamma^{N(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)(j-1)(\alpha)}\right)\overset{(4.69),(4.67)}{\underset{(4.64)}{=}}0.\end{aligned}$$

If $j=2$ then (4.104) becomes

$$\begin{aligned}(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)}&=\partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)}-\partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)2(\alpha)}\\&+\sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)2(\alpha)}\right)\\&+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)2(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)2(\alpha)}\right)\end{aligned}$$

where in the first summation only the terms for $l\geq 2$ survive, as $\Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)}=0$ by (4.69) and $\Gamma^{1(\alpha)}_{1(\alpha)2(\alpha)}\overset{(4.79)}{=}-\sum_{\sigma\neq\alpha}\Gamma^{1(\alpha)}_{1(\sigma)2(\alpha)}\overset{(4.63)}{=}0$. This yields

$$(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)}=\partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)}-\partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)2(\alpha)}$$

$$
\begin{aligned}
&+\sum_{l=2}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)2(\alpha)}\right)\\
&+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)2(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)2(\alpha)}\right)\\
&\overset{(4.64)}{\underset{(4.79)}{=}}\partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)}-\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}\\
&-\sum_{\sigma\neq\alpha}\sum_{l=2}^{N+1}\Gamma^{(N-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)2(\alpha)}-\sum_{l=2}^{N+1}\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)2(\alpha)}
\end{aligned}
$$

where

$$
\begin{aligned}
-\sum_{l=2}^{N+1}\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)2(\alpha)}&\overset{(4.79)}{=}\sum_{\sigma\neq\alpha}\sum_{l=2}^{N+1}\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(l-1)(\alpha)}_{1(\sigma)1(\alpha)}\\
&\overset{t:=N-l+3}{=}\sum_{\sigma\neq\alpha}\sum_{t=2}^{N+1}\Gamma^{t(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(N-t+2)(\alpha)}_{1(\sigma)1(\alpha)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)}&=\partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)}-\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}\\
&-\sum_{\sigma\neq\alpha}\sum_{l=2}^{N+1}\Gamma^{(N-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)2(\alpha)}+\sum_{\sigma\neq\alpha}\sum_{t=2}^{N+1}\Gamma^{t(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(N-t+2)(\alpha)}_{1(\sigma)1(\alpha)}\\
&=\partial_{1(\alpha)}\left[\underline{\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}}+\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)}\underline{-\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}}\right)\right]-\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}
\end{aligned}
$$

where the underlined terms cancel out and where

$$
\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}\overset{(4.79)}{=}-\sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\sigma)1(\alpha)}\overset{(4.72)}{=}-\sum_{\sigma\neq\alpha}\partial_{1(\alpha)}\Gamma^{(N-1)(\alpha)}_{1(\sigma)1(\alpha)}\overset{(4.79)}{=}\partial_{1(\alpha)}\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}.
$$

This implies

$$
\begin{aligned}
(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)2(\alpha)}&=\partial_{1(\alpha)}\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}+\partial_{1(\alpha)}\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}\right)-\partial_{2(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}\\
&=\partial_{1(\alpha)}\left(\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}\right)=0
\end{aligned}
$$

because

$$
\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}\overset{(4.68)}{=}-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{(N+1)(\alpha)}}{u^{2(\alpha)}}-\frac{1}{u^{2(\alpha)}}\sum_{l=1}^{N-2}\left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(N-l+1)(\alpha)}
$$

does not depend on $u^{1(\alpha)}$ (it only depends on $\{u^{1(\sigma)}\,|\,\sigma\neq\alpha\}$ and $\{u^{2(\alpha)}\,|\,2\leq s\leq n\}$).
If $j=1$ then (4.104) becomes

$$
(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)1(\alpha)}=\partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)}-\partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)}
$$

$$+\sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)1(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right)$$
$$+\sum_{\sigma\neq\alpha}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{2(\alpha)1(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)}\right)$$

where $\Gamma^{1(\sigma)}_{2(\alpha)1(\alpha)}$ vanishes due to (4.64) and where

$$\partial_{1(\alpha)}\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)}\overset{(4.67)}{=}\partial_{1(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}$$

and

$$\partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)}\overset{(4.79)}{=}-\sum_{\sigma\neq\alpha}\partial_{2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\sigma)1(\alpha)}\overset{(4.72)}{=}-\sum_{\sigma\neq\alpha}\partial_{1(\alpha)}\Gamma^{N(\alpha)}_{1(\sigma)1(\alpha)}\overset{(4.79)}{=}\partial_{1(\alpha)}\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}$$

mutually cancel out. This yields

$$\begin{aligned}
(R^{N+1})^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)1(\alpha)} &= \sum_{l=1}^{N+1}\left(\Gamma^{(N+1)(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{2(\alpha)1(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right)\\
&-\sum_{\sigma\neq\alpha}\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)}\\
&\underset{(4.67)}{\overset{(4.69)}{=}}\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{2(\alpha)1(\alpha)}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\\
&+\sum_{l=2}^{N+1}\Gamma^{N(\alpha)}_{1(\alpha)(l-1)(\alpha)}\Gamma^{(l-1)(\alpha)}_{1(\alpha)1(\alpha)}-\sum_{l=2}^{N+1}\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\\
&-\sum_{\sigma\neq\alpha}\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)}\\
&\overset{(4.67)}{=}-\Gamma^{(N+1)(\alpha)}_{2(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}+\sum_{l=1}^{N}\Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\\
&-\sum_{l=2}^{N+1}\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}-\sum_{\sigma\neq\alpha}\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)}\\
&\overset{(4.67)}{=}-\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}+\Gamma^{N(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}+\sum_{l=2}^{N}\Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\\
&-\sum_{l=2}^{N}\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)}-\sum_{\sigma\neq\alpha}\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\alpha)}\\
&\underset{(4.61)}{\overset{(4.79)}{=}}\sum_{l=2}^{N}\left(\Gamma^{N(\alpha)}_{1(\alpha)l(\alpha)}-\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)}\right)\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\\
&-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\alpha)1(\alpha)}+\sum_{\sigma\neq\alpha}\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)}
\end{aligned}$$

$$\overset{(4.79)}{=} \sum_{\sigma\neq\alpha}\Big[\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)} + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\sigma)1(\alpha)}$$

$$+ \sum_{l=2}^{N}\big(\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\big)\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}\Big] = 0$$

because for each $\sigma \neq \alpha$ we have

$$\Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\alpha)1(\sigma)} + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(N+1)(\alpha)}_{1(\sigma)1(\alpha)} + \sum_{l=2}^{N}\big(\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\big)\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}$$

$$\overset{(4.63)}{=} \Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\left[-\frac{m_\alpha\varepsilon_\alpha}{u^{1(\alpha)}-u^{1(\sigma)}}\right] + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\left[-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=2}^{N+1}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)}\right]$$

$$+ \big(\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}\big)\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)} + \sum_{l=3}^{N}\big(\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\big)\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}$$

$$\overset{(4.66)}{=} \Gamma^{N(\alpha)}_{1(\alpha)1(\sigma)}\left[\frac{u^{2(\alpha)}\,\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\right] + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Bigg[-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{N(\alpha)}_{1(\sigma)1(\alpha)}\,u^{2(\alpha)}$$

$$-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=3}^{N}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)} - \frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}\,u^{(N+1)(\alpha)}\Bigg]$$

$$+ \big(\Gamma^{(N+1)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-1)(\alpha)}_{1(\alpha)1(\alpha)}\big)\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)} + \sum_{l=3}^{N}\big(\Gamma^{(N-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(N-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\big)\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}$$

$$\underset{(4.68)}{\overset{(4.63)}{=}} -\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=3}^{N}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}\,u^{(N+1)(\alpha)}$$

$$+ \left(-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{(N+1)(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}}\sum_{l=1}^{N-2}\big(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{(N-l+1)(\alpha)}\right)\left[-\frac{\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}\,u^{2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\right]$$

$$+ \sum_{s=1}^{N-2}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)\Gamma^{(N-s+1)(\alpha)}_{1(\sigma)1(\alpha)}$$

$$= -\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=3}^{N-1}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)}\,u^{N(\alpha)}$$

$$+ \sum_{l=1}^{N-2}\big(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{(N-l+1)(\alpha)}\left[\frac{\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\right]$$

$$+ \sum_{s=1}^{N-2}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)\Gamma^{(N-s+1)(\alpha)}_{1(\sigma)1(\alpha)}$$

$$= -\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=3}^{N-1}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)}\,u^{N(\alpha)}$$

$$+\sum_{s=1}^{N-3}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)\Bigg[\Gamma^{(N-s+1)(\alpha)}_{1(\sigma)1(\alpha)}+\frac{\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,u^{(N-s+1)(\alpha)}\Bigg]$$

$$+\big(\Gamma^{N(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(N-2)(\alpha)}_{1(\alpha)1(\alpha)}\big)\Bigg[\Gamma^{3(\alpha)}_{1(\sigma)1(\alpha)}+\frac{\Gamma^{1(\alpha)}_{1(\sigma)1(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,u^{3(\alpha)}\Bigg]$$

$$\overset{(4.63)}{\underset{(4.68)}{=}}-\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=3}^{N-1}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)}-\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)}\,u^{N(\alpha)}$$

$$+\sum_{s=1}^{N-3}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)\Bigg[-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{l=2}^{N-s}\Gamma^{(N-s-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{l(\alpha)}\Bigg]$$

$$+\Bigg(-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{N(\alpha)}}{u^{2(\alpha)}}-\frac{1}{u^{2(\alpha)}}\sum_{l=1}^{N-3}\big(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{(N-l)(\alpha)}\Bigg)\Bigg[-\frac{\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,u^{2(\alpha)}\Bigg]$$

$$=-\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=3}^{N-1}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)}$$

$$+\sum_{s=1}^{N-3}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)\Bigg[-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{l=2}^{N-s-1}\Gamma^{(N-s-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{l(\alpha)}-\frac{\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)}\,u^{(N-s)(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\Bigg]$$

$$+\Bigg(\sum_{l=1}^{N-3}\big(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{(N-l)(\alpha)}\Bigg)\Bigg[\frac{\Gamma^{2(\alpha)}_{1(\sigma)1(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\Bigg]$$

$$=-\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=3}^{N-1}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)}$$

$$-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=1}^{N-3}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)\sum_{l=2}^{N-s-1}\Gamma^{(N-s-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{l(\alpha)}$$

$$=-\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{(N-1)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{3(\alpha)}-\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=4}^{N-1}\Gamma^{(N-s+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{s(\alpha)}$$

$$-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\big(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\big)\sum_{l=2}^{N-2}\Gamma^{(N-l+1)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{l(\alpha)}$$

$$-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=2}^{N-3}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)\sum_{l=2}^{N-s-1}\Gamma^{(N-s-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{l(\alpha)}$$

$$=-\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\,\Gamma^{(N-1)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{3(\alpha)}-\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{l=3}^{N-2}\Gamma^{(N-l+1)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{(l+1)(\alpha)}$$

$$+\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\Bigg(\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\Bigg)\Bigg[\Gamma^{(N-1)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{2(\alpha)}+\sum_{l=3}^{N-2}\Gamma^{(N-l+1)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{l(\alpha)}\Bigg]$$

$$-\frac{1}{u^{1(\alpha)}-u^{1(\sigma)}}\sum_{s=2}^{N-3}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)\sum_{l=2}^{N-s-1}\Gamma^{(N-s-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\,u^{l(\alpha)}$$

$$= \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\sigma)}} \sum_{l=3}^{N-2} \Gamma^{(N-l+1)(\alpha)}_{1(\sigma)1(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{l(\alpha)} - u^{(l+1)(\alpha)} \right)$$
$$- \frac{1}{u^{1(\alpha)} - u^{1(\sigma)}} \sum_{s=2}^{N-3} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)} \right) \sum_{l=2}^{N-s-1} \Gamma^{(N-s-l+2)(\alpha)}_{1(\sigma)1(\alpha)} u^{l(\alpha)}$$
$$= \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\sigma)}} \sum_{l=3}^{N-2} \Gamma^{(N-l+1)(\alpha)}_{1(\sigma)1(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{l(\alpha)} - u^{(l+1)(\alpha)} \right)$$
$$- \frac{1}{u^{1(\alpha)} - u^{1(\sigma)}} \sum_{s=2}^{N-3} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)} \right) \sum_{t=s+1}^{N-2} \Gamma^{(N-t+1)(\alpha)}_{1(\sigma)1(\alpha)} u^{(t-s+1)(\alpha)}$$
$$= \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\sigma)}} \sum_{l=3}^{N-2} \Gamma^{(N-l+1)(\alpha)}_{1(\sigma)1(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{l(\alpha)} - u^{(l+1)(\alpha)} \right)$$
$$- \frac{1}{u^{1(\alpha)} - u^{1(\sigma)}} \sum_{t=3}^{N-2} \sum_{s=2}^{t-1} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)} \right) \Gamma^{(N-t+1)(\alpha)}_{1(\sigma)1(\alpha)} u^{(t-s+1)(\alpha)}$$
$$= \frac{1}{u^{1(\alpha)} - u^{1(\sigma)}} \sum_{l=3}^{N-2} \Gamma^{(N-l+1)(\alpha)}_{1(\sigma)1(\alpha)} \left[\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{l(\alpha)} - u^{(l+1)(\alpha)} \right) \right.$$
$$\left. - \sum_{s=2}^{l-1} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(l-s+1)(\alpha)} \right] \overset{(4.75)}{=} 0.$$

Case 2: $\alpha = \beta = \gamma \neq \epsilon$. Our goal is to prove that

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\alpha)} = \partial_{k(\alpha)} \Gamma^{i(\alpha)}_{h(\epsilon)j(\alpha)} - \partial_{h(\epsilon)} \Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{\sigma=1}^{r} \sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)} \Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)} \right) \tag{4.105}$$

vanishes. Let us first consider the case where $h \geq 2$. We have

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\alpha)} = \partial_{k(\alpha)} \Gamma^{i(\alpha)}_{h(\epsilon)j(\alpha)} - \partial_{h(\epsilon)} \Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} + \sum_{\sigma=1}^{r} \sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)} \Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)} \right)$$

where $\Gamma^{i(\alpha)}_{h(\epsilon)j(\alpha)}$ vanishes due to (4.62) and where $\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}$ only depends on $\{u^{s(\alpha)} \,|\, 1 \leq s \leq m_\alpha\}$ and $\{u^{1(\sigma)} \,|\, \sigma \neq \alpha\}$ (thus it does not depend on $u^{h(\epsilon)}$ as $h \geq 2$, that is $\partial_{h(\epsilon)} \Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} = 0$). This yields

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\alpha)} = \sum_{\sigma=1}^{r} \sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)} \Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)} \right)$$

where the terms $\Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)}$ and $\Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}$ trivially vanish for $\sigma \notin \{\alpha, \epsilon\}$ by (4.61). Thus

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\alpha)} = \sum_{l=1}^{m_\alpha} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{h(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)} \right)$$

$$+\sum_{l=1}^{m_\epsilon}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\epsilon)}\Gamma^{l(\epsilon)}_{h(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\epsilon)}\Gamma^{l(\epsilon)}_{k(\alpha)j(\alpha)}\right)$$
$$\overset{(4.62)}{\underset{(4.64)}{=}}\sum_{l=h}^{m_\epsilon}\Gamma^{i(\alpha)}_{k(\alpha)l(\epsilon)}\Gamma^{l(\epsilon)}_{h(\epsilon)j(\alpha)}=0$$

as $\Gamma^{i(\alpha)}_{k(\alpha)l(\epsilon)}=0$ by (4.63) for every $l\geq h$ ($h\geq 2$ implies $l\geq 2$). Let us now fix $h=1$. We have (the terms $\Gamma^{l(\sigma)}_{1(\epsilon)j(\alpha)}$ and $\Gamma^{i(\alpha)}_{1(\epsilon)l(\sigma)}$ trivially vanish for $\sigma\notin\{\alpha,\epsilon\}$ by (4.61))

$$\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)j(\alpha)}&=\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}\\
&+\sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{1(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\alpha)}\right)\\
&=\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}\\
&+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\
&+\sum_{l=1}^{m_\epsilon}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\epsilon)}\Gamma^{l(\epsilon)}_{1(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\epsilon)}\Gamma^{l(\epsilon)}_{k(\alpha)j(\alpha)}\right)\\
&\overset{(4.63)}{\underset{(4.64)}{=}}\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}\\
&+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)}\right)\\
&+\Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)}\Gamma^{1(\epsilon)}_{k(\alpha)j(\alpha)}.
\end{aligned}\tag{4.106}$$

We distinguish between the following subcases:

a. both j and k are greater or equal than 2

b. $j=k=1$

c. $j=1$, $k\geq 2$

d. $j\geq 2$, $k=1$.

Subcase a: both j and k are greater or equal than 2. Let us first claim that in this case

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}=0.\tag{4.107}$$

Indeed, if $i<j$ then both $\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}$ and $\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}$ trivially vanish by (4.62) and (4.69) respectively. If $i=j$ then

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)}=\partial_{k(\alpha)}\Gamma^{j(\alpha)}_{1(\epsilon)j(\alpha)}-\partial_{1(\epsilon)}\Gamma^{j(\alpha)}_{k(\alpha)j(\alpha)}$$

$$\overset{(4.63)}{\underset{(4.69)}{=}} \partial_{k(\alpha)}\left[\frac{m_\epsilon \varepsilon_\epsilon}{u^{1(\alpha)}-u^{1(\epsilon)}}\right] - \partial_{1(\epsilon)}\Gamma^{(4-k)(\alpha)}_{2(\alpha)2(\alpha)}$$
$$\overset{(4.69)}{=} -\partial_{1(\epsilon)}\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\delta^2_k$$
$$\overset{(4.66)}{=} \delta^2_k\,\partial_{1(\epsilon)}\left[\frac{m_\alpha \varepsilon_\alpha}{u^{2(\alpha)}}\right] = 0.$$

We are going to prove (4.107) when $i > j$ by induction over i (starting from the case $i = j$ that we just proved). Given an integer $s \geq 1$, let us suppose that (4.107) holds true when $i = j + t$ for each $t \leq s - 1$ that is

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} = 0 \quad \text{for } i = j+t, \quad t \leq s-1, \tag{4.108}$$

and show it holds for $t = s$ as well. We are thus considering $i = j + s$, so that

$$\begin{aligned}
\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} &= \partial_{k(\alpha)}\Gamma^{(j+s)(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)}\Gamma^{(j+s)(\alpha)}_{k(\alpha)j(\alpha)}\\
&\overset{(4.63)}{\underset{(4.69)}{=}} \partial_{k(\alpha)}\Gamma^{(s+1)(\alpha)}_{1(\epsilon)1(\alpha)} - \partial_{1(\epsilon)}\Gamma^{(s-k+4)(\alpha)}_{2(\alpha)2(\alpha)}\\
&\overset{(4.76)}{=} \partial_{k(\alpha)}\left[-\frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\sum_{l=2}^{s+1}\Gamma^{(s-l+2)(\alpha)}_{1(\epsilon)1(\alpha)}\,u^{l(\alpha)}\right]\\
&\quad - \partial_{1(\epsilon)}\Gamma^{(s-k+2)(\alpha)}_{1(\alpha)1(\alpha)}\\
&\overset{(4.79)}{=} -\frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\sum_{l=2}^{s+1}\left(\partial_{k(\alpha)}\Gamma^{(s-l+2)(\alpha)}_{1(\epsilon)1(\alpha)}\right)u^{l(\alpha)}\\
&\quad - \frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)} + \partial_{1(\epsilon)}\Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)}\\
&\overset{(4.108)}{\underset{(4.69)}{=}} -\frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\sum_{l=2}^{s+1}\left(\partial_{1(\epsilon)}\Gamma^{(s-l-k+5)(\alpha)}_{2(\alpha)2(\alpha)}\right)u^{l(\alpha)}\\
&\quad - \frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)} + \partial_{1(\epsilon)}\Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)}\\
&\overset{(4.76)}{=} -\frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\sum_{l=2}^{s+1}\left(\partial_{1(\epsilon)}\Gamma^{(s-l-k+3)(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{l(\alpha)}\\
&\quad - \frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)} + \partial_{1(\epsilon)}\Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)}
\end{aligned}$$

where $\Gamma^{(s-l-k+3)(\alpha)}_{1(\alpha)1(\alpha)}$ vanishes for each $l \geq s - k + 3$ by (4.67). It follows that

$$\begin{aligned}
\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\alpha)} &\overset{(4.79)}{\underset{(4.63)}{=}} \frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\sum_{l=2}^{s-k+2}\left(\partial_{1(\epsilon)}\Gamma^{(s-l-k+3)(\alpha)}_{1(\epsilon)1(\alpha)}\right)u^{l(\alpha)}\\
&\quad - \frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)}\\
&\quad + \partial_{1(\epsilon)}\left[-\frac{1}{u^{1(\alpha)}-u^{1(\epsilon)}}\sum_{l=2}^{s-k+2}\Gamma^{(s-l-k+3)(\alpha)}_{1(\epsilon)1(\alpha)}\,u^{l(\alpha)}\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=2}^{s-k+2} \left(\partial_{1(\epsilon)} \Gamma^{(s-l-k+3)(\alpha)}_{1(\epsilon)1(\alpha)} \right) u^{l(\alpha)} \\
&- \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)} \\
&- \frac{1}{(u^{1(\alpha)} - u^{1(\epsilon)})^2} \sum_{l=2}^{s-k+2} \Gamma^{(s-l-k+3)(\alpha)}_{1(\epsilon)1(\alpha)} u^{l(\alpha)} \\
&- \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=2}^{s-k+2} \left(\partial_{1(\epsilon)} \Gamma^{(s-l-k+3)(\alpha)}_{1(\epsilon)1(\alpha)} \right) u^{l(\alpha)} \\
&\overset{(4.63)}{=} \frac{1}{(u^{1(\alpha)} - u^{1(\epsilon)})^2} \sum_{l=2}^{s-k+2} \Gamma^{(s-l-k+3)(\alpha)}_{1(\epsilon)1(\alpha)} u^{l(\alpha)} \\
&- \frac{1}{(u^{1(\alpha)} - u^{1(\epsilon)})^2} \sum_{l=2}^{s-k+2} \Gamma^{(s-l-k+3)(\alpha)}_{1(\epsilon)1(\alpha)} u^{l(\alpha)} = 0.
\end{aligned}
$$

This proves (4.107), thus (4.106) becomes

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)j(\alpha)} = \sum_{l=1}^{m_\alpha} & \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)} \right) \\
& + \Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)} \Gamma^{1(\epsilon)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)} \Gamma^{1(\epsilon)}_{k(\alpha)j(\alpha)}
\end{aligned}
$$

where in the first summation only the terms for $j \leq l \leq i$ survive by (4.63) and where $\Gamma^{1(\epsilon)}_{1(\epsilon)j(\alpha)}$ and $\Gamma^{1(\epsilon)}_{k(\alpha)j(\alpha)}$ vanish due to (4.62) and (4.64). We get

$$
R^{i(\alpha)}_{1(\epsilon)k(\alpha)j(\alpha)} = \sum_{l=j}^{i} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)j(\alpha)} \right)
$$

which trivially vanishes for $i < j$. For $i \geq j$ we have

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)j(\alpha)} &\underset{(4.69)}{\overset{(4.63)}{=}} \sum_{l=j}^{i} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} - \Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{(l-k-j+4)(\alpha)}_{2(\alpha)2(\alpha)} \right) \\
&= \sum_{l=j}^{i} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} - \sum_{l=j}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{(l-k-j+4)(\alpha)}_{2(\alpha)2(\alpha)} \\
&= \sum_{l=j}^{i} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} \Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} - \sum_{t=j}^{i} \Gamma^{(t-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{(i-k-t+4)(\alpha)}_{2(\alpha)2(\alpha)} = 0.
\end{aligned}
$$

Subcase b: $j = k = 1$. We have

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)1(\alpha)1(\alpha)} &= \partial_{1(\alpha)} \Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} - \partial_{1(\epsilon)} \Gamma^{i(\alpha)}_{1(\alpha)1(\alpha)} \\
&+ \sum_{l=1}^{m_\alpha} \left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)} \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} \right)
\end{aligned}
$$

$$+\Gamma^{i(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\alpha)1(\alpha)}$$

where in the summation only the terms for $l\leq i$ survive (by (4.63) and (4.67)) and

$$\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\alpha)1(\alpha)}\overset{(4.79)}{=}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}=\partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}$$

as $\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}$ only depends on both $u^{1(\epsilon)}$ and $u^{1(\alpha)}$ by means of the term $u^{1(\alpha)}-u^{1(\epsilon)}$. It follows that

$$\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)1(\alpha)1(\alpha)}&=\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right)\\
&\quad+\Gamma^{i(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}\\
&\overset{(4.79)}{=}-\sum_{\sigma\neq\alpha}\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{1(\sigma)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}\right)\\
&=-\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}\right)\\
&\quad-\sum_{\sigma\notin\{\alpha,\epsilon\}}\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{1(\sigma)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}\right)\\
&\overset{(4.63)}{=}=-\sum_{\sigma\notin\{\alpha,\epsilon\}}\left(\sum_{l=1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\sum_{l=1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{l(\alpha)}_{1(\sigma)1(\alpha)}\right)\\
&=-\sum_{\sigma\notin\{\alpha,\epsilon\}}\left(\sum_{l=1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\sum_{t=1}^{i}\Gamma^{t(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(i-t+1)(\alpha)}_{1(\sigma)1(\alpha)}\right)=0.
\end{aligned}$$

Subcase c: $j=1$, $k\geq 2$. We have

$$\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)}&=\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}-\partial_{1(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)1(\alpha)}\\
&\quad+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)}\right)\\
&\quad+\Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)}\Gamma^{1(\epsilon)}_{k(\alpha)1(\alpha)}\\
&\underset{(4.63),(4.69)}{\overset{(4.79)}{=}}\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}+\partial_{1(\epsilon)}\Gamma^{(i-k+1)(\alpha)}_{1(\epsilon)1(\alpha)}\\
&\quad+\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)}\right)\\
&\quad+\Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{1(\epsilon)}_{k(\alpha)1(\epsilon)}
\end{aligned}$$

where $\Gamma^{1(\epsilon)}_{k(\alpha)1(\epsilon)}$ vanishes due to (4.62). Let us first claim that in this case

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)}+\partial_{1(\epsilon)}\Gamma^{(i-k+1)(\alpha)}_{1(\epsilon)1(\alpha)}=0. \tag{4.109}$$

Indeed, if $i = 1$ then

$$\partial_{k(\alpha)}\Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)} + \partial_{1(\epsilon)}\Gamma^{1(\alpha)}_{1(\epsilon)k(\alpha)} \overset{(4.63)}{=} \partial_{k(\alpha)}\left[\frac{m_\epsilon \varepsilon_\epsilon}{u^{1(\alpha)} - u^{1(\epsilon)}}\right] = 0.$$

We are going to prove (4.109) when $i \geq 1$ by induction over i (starting from the case $i = 1$ that we just proved). Given an integer $s \geq 1$, let us suppose that (4.109) holds true when $i = 1 + t$ for each $t \leq s - 1$ that is

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} + \partial_{1(\epsilon)}\Gamma^{(i-k+1)(\alpha)}_{1(\epsilon)1(\alpha)} = 0 \quad \text{for } i = 1 + t, \quad t \leq s - 1, \tag{4.110}$$

and show it holds for $t = s$ as well. We are thus considering $i = 1 + s$, so that

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} = \partial_{k(\alpha)}\Gamma^{(1+s)(\alpha)}_{1(\epsilon)1(\alpha)} \overset{(4.72)}{=} \partial_{(k-1)(\alpha)}\Gamma^{s(\alpha)}_{1(\epsilon)1(\alpha)}. \tag{4.111}$$

If $k = 2$ then (4.111) becomes

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} = \partial_{1(\alpha)}\Gamma^{s(\alpha)}_{1(\epsilon)1(\alpha)} = -\partial_{1(\epsilon)}\Gamma^{s(\alpha)}_{1(\epsilon)1(\alpha)} = -\partial_{1(\epsilon)}\Gamma^{(i-k+1)(\alpha)}_{1(\epsilon)1(\alpha)}$$

as $\Gamma^{s(\alpha)}_{1(\epsilon)1(\alpha)}$ only depends on both $u^{1(\epsilon)}$ and $u^{1(\alpha)}$ by means of the term $u^{1(\alpha)} - u^{1(\epsilon)}$. If $k \geq 3$ then (4.111) becomes

$$\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} = \partial_{(k-1)(\alpha)}\Gamma^{s(\alpha)}_{1(\epsilon)1(\alpha)} \overset{(4.110)}{=} -\partial_{1(\epsilon)}\Gamma^{(s-k+2)(\alpha)}_{1(\epsilon)1(\alpha)} = -\partial_{1(\epsilon)}\Gamma^{(i-k+1)(\alpha)}_{1(\epsilon)1(\alpha)}.$$

Thus (4.109) holds, yielding

$$\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)} &= \sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)}\right) \\
&\quad + \Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)} \\
&\underset{(4.67)}{\overset{(4.69)}{=}} \sum_{l=1}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} - \sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)} \\
&\quad + \Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}
\end{aligned}$$

which trivially vanishes for $i < k$ by (4.63) and (4.67). For $i \geq k$ we get

$$\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)} &= \sum_{l=1}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} - \sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)} \\
&\quad + \Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)} \\
&\underset{(4.69)}{\overset{(4.63)}{=}} \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)} + \sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \\
&\quad - \sum_{l=k}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\alpha)} + \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}
\end{aligned}$$

$$
\begin{aligned}
&= \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)} + \sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \\
&- \sum_{t=2}^{i-k+1} \Gamma^{t(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(i-t+1)(\alpha)}_{k(\alpha)1(\alpha)} + \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)} \\
&= \sum_{l=2}^{i-k+1} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(i-l+1)(\alpha)}_{k(\alpha)1(\alpha)}\right) \\
&+ \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\epsilon)1(\alpha)} + \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)} \\
&\overset{(4.67)}{=} \sum_{l=2}^{i-k+1} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right) \\
&+ \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\epsilon)1(\alpha)} + \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)}
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\epsilon)1(\alpha)} + \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)} &\underset{(4.66)}{\overset{(4.63)}{=}} \frac{m_\alpha \varepsilon_\alpha}{u^{2(\alpha)}} \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{s=2}^{i-k+2} \Gamma^{(i-k-s+3)(\alpha)}_{1(\epsilon)1(\alpha)} u^{s(\alpha)} \\
&+ \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)} \\
&\overset{(4.63)}{=} -\frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}} \left[\Gamma^{(i-k+1)(\alpha)}_{1(\epsilon)1(\alpha)} u^{2(\alpha)}\right. \\
&\left. + \sum_{s=3}^{i-k+2} \Gamma^{(i-k-s+3)(\alpha)}_{1(\epsilon)1(\alpha)} u^{s(\alpha)}\right] \\
&+ \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\alpha)} \\
&= -\frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}} \sum_{s=3}^{i-k+2} \Gamma^{(i-k-s+3)(\alpha)}_{1(\epsilon)1(\alpha)} u^{s(\alpha)}
\end{aligned}
$$

thus

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)} &= \sum_{l=2}^{i-k+1} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right) \\
&- \frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}} \sum_{s=3}^{i-k+2} \Gamma^{(i-k-s+3)(\alpha)}_{1(\epsilon)1(\alpha)} u^{s(\alpha)}. \qquad (4.112)
\end{aligned}
$$

We are going to prove that (4.112) vanishes for each $i \geq k$ by induction over i (starting from the case $i = k$, where (4.112) vanishes trivially). Given an integer $s \geq 1$, let us suppose that (4.112) vanishes when $i = k + t$ for each $t \leq s - 1$ that is

$$
\sum_{l=2}^{t+1} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{(t-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(t-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)
$$

$$
- \frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}} \sum_{s=3}^{t+2} \Gamma^{(t-s+3)(\alpha)}_{1(\epsilon)1(\alpha)} \, u^{s(\alpha)} = 0, \qquad t \leq s-1, \tag{4.113}
$$

and show that (4.112) vanishes for $t = s$ as well. We are thus considering $i = k + s$, so that

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)} &= \sum_{l=2}^{s+1} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{(s-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(s-l+2)(\alpha)}_{1(\alpha)1(\alpha)} \right) \\
&\quad - \frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}} \sum_{l=3}^{s+2} \Gamma^{(s-l+3)(\alpha)}_{1(\epsilon)1(\alpha)} \, u^{l(\alpha)}
\end{aligned}
$$

where

$$
\begin{aligned}
&\sum_{l=2}^{s+1} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{(s-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(s-l+2)(\alpha)}_{1(\alpha)1(\alpha)} \right) \\
&= \sum_{t=1}^{s} \Gamma^{(s-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{(t+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{t(\alpha)}_{1(\alpha)1(\alpha)} \right) \\
&\overset{(4.63)}{=} \sum_{t=1}^{s} \left[- \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{r=2}^{s-t+2} \Gamma^{(s-t-r+3)(\alpha)}_{1(\epsilon)1(\alpha)} \, u^{r(\alpha)} \right] \left(\Gamma^{(t+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{t(\alpha)}_{1(\alpha)1(\alpha)} \right) \\
&= - \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{t=1}^{s} \left[\sum_{l=1}^{s-t+1} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \, u^{(s-t-l+3)(\alpha)} \right] \left(\Gamma^{(t+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{t(\alpha)}_{1(\alpha)1(\alpha)} \right) \\
&= - \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=1}^{s} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \sum_{t=1}^{s-l+1} \left(\Gamma^{(t+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{t(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(s-t-l+3)(\alpha)} \\
&= - \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=1}^{s} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \left[\left(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(s-l+2)(\alpha)} \right. \\
&+ \left. \sum_{t=2}^{s-l+1} \left(\Gamma^{(t+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{t(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(s-t-l+3)(\alpha)} \right] \\
&\underset{(4.68)}{\overset{(4.75)}{=}} - \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=1}^{s} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \left[- \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{(s-l+2)(\alpha)} \right. \\
&+ \left. \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{(s-l+2)(\alpha)} - u^{(s-l+3)(\alpha)} \right) \right] \\
&= \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=1}^{s} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \, u^{(s-l+3)(\alpha)}
\end{aligned}
$$

thus

$$
R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\alpha)} = \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=1}^{s} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} \, u^{(s-l+3)(\alpha)}
$$

$$
\begin{aligned}
&- \frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}} \sum_{l=3}^{s+2} \Gamma^{(s-l+3)(\alpha)}_{1(\epsilon)1(\alpha)} u^{l(\alpha)} \\
&= \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=1}^{s} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} u^{(s-l+3)(\alpha)} \\
&- \frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}} \sum_{t=1}^{s} \Gamma^{t(\alpha)}_{1(\epsilon)1(\alpha)} u^{(s-t+3)(\alpha)} \\
&= \sum_{l=1}^{s} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} u^{(s-l+3)(\alpha)} \left(\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\epsilon)}} - \frac{\Gamma^{1(\epsilon)}_{1(\alpha)1(\epsilon)}}{u^{2(\alpha)}} \right) \\
&\overset{(4.63)}{\underset{(4.66)}{=}} \sum_{l=1}^{s} \Gamma^{l(\alpha)}_{1(\epsilon)1(\alpha)} u^{(s-l+3)(\alpha)} \left(-\frac{m_\alpha \varepsilon_\alpha}{u^{2(\alpha)}} \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} + \frac{1}{u^{2(\alpha)}} \frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\epsilon)}} \right) = 0.
\end{aligned}
$$

Subcase d: $j \geq 2$, $k = 1$. We have

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)1(\alpha)j(\alpha)} &= \partial_{1(\alpha)} \Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)} - \partial_{1(\epsilon)} \Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)} \\
&+ \sum_{l=1}^{m_\alpha} \left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)} \Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)} \right) \\
&+ \Gamma^{i(\alpha)}_{1(\alpha)1(\epsilon)} \Gamma^{1(\epsilon)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)} \Gamma^{1(\epsilon)}_{1(\alpha)j(\alpha)}
\end{aligned}
$$

where $\Gamma^{1(\epsilon)}_{1(\epsilon)j(\alpha)}$ and $\Gamma^{1(\epsilon)}_{1(\alpha)j(\alpha)}$ vanish due to (4.62) and (4.64) and where

$$
\begin{aligned}
\partial_{1(\epsilon)} \Gamma^{i(\alpha)}_{1(\alpha)j(\alpha)} &\overset{(4.79)}{=} -\sum_{\sigma \neq \alpha} \partial_{1(\epsilon)} \Gamma^{i(\alpha)}_{1(\sigma)j(\alpha)} = -\partial_{1(\epsilon)} \Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)} - \sum_{\sigma \notin \{\alpha, \epsilon\}} \partial_{1(\epsilon)} \Gamma^{i(\alpha)}_{1(\sigma)j(\alpha)} \\
&= \partial_{1(\alpha)} \Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}
\end{aligned}
$$

as $\Gamma^{i(\alpha)}_{1(\epsilon)j(\alpha)}$ only depends on both $u^{1(\epsilon)}$ and $u^{1(\alpha)}$ by means of the term $u^{1(\alpha)} - u^{1(\epsilon)}$. It follows that

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)1(\alpha)j(\alpha)} &= \sum_{l=1}^{m_\alpha} \left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)} - \Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)} \Gamma^{l(\alpha)}_{1(\alpha)j(\alpha)} \right) \\
&\overset{(4.63)}{\underset{(4.67)}{=}} \sum_{l=j}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)} \Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} - \sum_{l=j}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{(l-j+1)(\alpha)}_{1(\alpha)1(\alpha)} \\
&= \sum_{l=j}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)} \Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} - \sum_{t=j}^{i} \Gamma^{(t-j+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{(i-t+1)(\alpha)}_{1(\alpha)1(\alpha)} = 0.
\end{aligned}
$$

Case 3: $\alpha = \gamma = \epsilon \neq \beta$**.** Out goal is to prove that

$$
R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\beta)} = \partial_{k(\alpha)} \Gamma^{i(\alpha)}_{h(\alpha)j(\beta)} - \partial_{h(\alpha)} \Gamma^{i(\alpha)}_{k(\alpha)j(\beta)}
$$

$$
\begin{aligned}
&+\sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\alpha)j(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\beta)}\right)\\
&\overset{(4.61)}{=}\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\beta)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\beta)}\\
&+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\beta)}\right)\\
&+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\alpha)j(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)j(\beta)}\right) \qquad (4.114)
\end{aligned}
$$

vanishes. If $j \geq 2$ we get

$$
\begin{aligned}
R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\beta)} &= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)j(\beta)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)j(\beta)}\\
&+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)j(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\beta)}\right)\\
&+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\alpha)j(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)j(\beta)}\right)
\end{aligned}
$$

where, by means of (4.62), only the last summation survives, in which only the terms for $l = 1$ survive. Thus

$$
R^{i(\alpha)}_{h(\alpha)k(\alpha)j(\beta)} = \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\alpha)j(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)1(\beta)}\Gamma^{1(\beta)}_{k(\alpha)j(\beta)}\overset{(4.63)}{=}0
$$

for $j \geq 2$. Let us then fix $j = 1$. We have

$$
\begin{aligned}
R^{i(\alpha)}_{h(\alpha)k(\alpha)1(\beta)} &= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)1(\beta)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\\
&+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)1(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}\right)\\
&+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\alpha)1(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}\right). \qquad (4.115)
\end{aligned}
$$

Let us recall that, as we have already seen in (4.109),

$$
\partial_{t(\alpha)}\Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} = -\partial_{1(\epsilon)}\Gamma^{(i-t+1)(\alpha)}_{1(\epsilon)1(\alpha)} \qquad (4.116)
$$

for every $\epsilon \neq \alpha$ and $t \geq 2$. We distinguish between the following subcases:

$a.$ both h and k are greater or equal than 2

$b.$ $h = 1$, $k \geq 2$ (this covers $h \geq 2$, $k = 1$ as well)

observing that $R^{i(\alpha)}_{h(\epsilon)k(\gamma)1(\beta)} = 0$ automatically whenever $k = h$.
Subcase a: both k and h are greater or equal than 2. We have

$$
R^{i(\alpha)}_{h(\alpha)k(\alpha)1(\beta)} \underset{(4.69)}{\overset{(4.63)}{=}} \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)1(\beta)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}
$$

$$
\begin{aligned}
&+\sum_{l=h}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)1(\beta)}-\sum_{l=k}^{i-h+2}\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}\\
&+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\alpha)1(\beta)}-\Gamma^{i(\alpha)}_{h(\alpha)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}\right)
\end{aligned}
$$

where $\Gamma^{l(\beta)}_{h(\alpha)1(\beta)}$ and $\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}$ vanish due to (4.62) and where

$$
\begin{aligned}
\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\alpha)1(\beta)}-\partial_{h(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} &\overset{(4.63)}{=} \partial_{k(\alpha)}\Gamma^{(i-h+1)(\alpha)}_{1(\alpha)1(\beta)}-\partial_{h(\alpha)}\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\\
&\overset{(4.116)}{=} -\partial_{1(\beta)}\Gamma^{(i-h-k+2)(\alpha)}_{1(\alpha)1(\beta)}+\partial_{1(\beta)}\Gamma^{(i-k-h+2)(\alpha)}_{1(\alpha)1(\beta)}=0.
\end{aligned}
$$

This yields

$$
R^{i(\alpha)}_{h(\alpha)k(\alpha)1(\beta)}=\sum_{l=h}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\alpha)1(\beta)}-\sum_{l=k}^{i-h+2}\Gamma^{i(\alpha)}_{h(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)} \tag{4.117}
$$

which automatically vanishes for $i<k$. For $i\geq k$ (4.117) becomes

$$
\begin{aligned}
R^{i(\alpha)}_{h(\alpha)k(\alpha)1(\beta)} &\underset{(4.69)}{\overset{(4.63)}{=}} \sum_{l=h}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(l-h+1)(\alpha)}_{1(\alpha)1(\beta)}-\sum_{l=k}^{i-h+2}\Gamma^{(i-h-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(l-k+1)(\alpha)}_{1(\alpha)1(\beta)}\\
&=\sum_{l=h}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(l-h+1)(\alpha)}_{1(\alpha)1(\beta)}-\sum_{t=h}^{i-k+2}\Gamma^{(i-k-t+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(t-h+1)(\alpha)}_{1(\alpha)1(\beta)}=0.
\end{aligned}
$$

Subcase b: $h=1$, $k\geq 2$. We have

$$
\begin{aligned}
R^{i(\alpha)}_{1(\alpha)k(\alpha)1(\beta)} &\underset{(4.67)}{\overset{(4.69)}{=}} \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}-\partial_{1(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\\
&+\sum_{l=1}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}-\sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}\\
&+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{1(\alpha)1(\beta)}-\Gamma^{i(\alpha)}_{1(\alpha)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}\right)
\end{aligned}
$$

where, by means of (4.62), $\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}$ vanishes and in the last summation only the term for $l=1$ survives and where

$$
\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}-\partial_{1(\alpha)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} \overset{(4.116)}{=} -\partial_{1(\beta)}\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}+\partial_{1(\beta)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}=0.
$$

This yields

$$
R^{i(\alpha)}_{1(\alpha)k(\alpha)1(\beta)}=\sum_{l=1}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}-\sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}
$$

$$+ \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$

$$\overset{(4.63),(4.69)}{\underset{(4.67)}{=}} \Gamma^{i(\alpha)}_{k(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)} + \sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)} - \sum_{l=k}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{(l-k+1)(\alpha)}_{1(\alpha)1(\beta)}$$

$$+ \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$

$$= \Gamma^{i(\alpha)}_{k(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)} + \sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)} - \sum_{t=1}^{i-k+1} \Gamma^{(i-k-t+2)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{t(\alpha)}_{1(\alpha)1(\beta)}$$

$$+ \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$

$$\overset{(4.67)}{=} \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)} + \sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}$$

$$- \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)} - \sum_{l=2}^{i-k+1} \Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}$$

$$+ \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$

$$\overset{(4.63)}{=} \sum_{l=2}^{i-k+1} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)} + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\alpha)1(\beta)}$$

$$+ \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \overset{(4.78)}{=} 0. \tag{4.118}$$

Case 4: $\beta = \gamma = \epsilon \neq \alpha$**.** Our goal is to prove that

$$\begin{aligned} R^{i(\alpha)}_{h(\beta)k(\beta)j(\beta)} &= \partial_{k(\beta)}\Gamma^{i(\alpha)}_{h(\beta)j(\beta)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\beta)j(\beta)} \\ &+ \sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\beta)l(\sigma)}\Gamma^{l(\sigma)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\sigma)}\Gamma^{l(\sigma)}_{k(\beta)j(\beta)}\right) \\ &\overset{(4.61)}{=} \partial_{k(\beta)}\Gamma^{i(\alpha)}_{h(\beta)j(\beta)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\beta)j(\beta)} \\ &+ \sum_{l=1}^{m_\alpha} \left(\Gamma^{i(\alpha)}_{k(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\beta)j(\beta)}\right) \\ &+ \sum_{l=1}^{m_\beta} \left(\Gamma^{i(\alpha)}_{k(\beta)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)}\Gamma^{l(\beta)}_{k(\beta)j(\beta)}\right) \end{aligned}$$

vanishes. Without loss of generality, by the symmetries of R, we can set $h > k$. In particular, $h \geq 2$. We get

$$\begin{aligned} R^{i(\alpha)}_{h(\beta)k(\beta)j(\beta)} &= \partial_{k(\beta)}\Gamma^{i(\alpha)}_{h(\beta)j(\beta)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\beta)j(\beta)} \\ &+ \sum_{l=1}^{m_\alpha} \left(\Gamma^{i(\alpha)}_{k(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\beta)j(\beta)}\right) \\ &+ \sum_{l=1}^{m_\beta} \left(\Gamma^{i(\alpha)}_{k(\beta)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)}\Gamma^{l(\beta)}_{k(\beta)j(\beta)}\right) \end{aligned} \tag{4.119}$$

which vanishes due to (4.64) for $k \geq 2$. For $k = 1$ (4.119) becomes

$$R^{i(\alpha)}_{h(\beta)1(\beta)j(\beta)} = -\partial_{h(\beta)}\Gamma^{i(\alpha)}_{1(\beta)j(\beta)} + \sum_{l=1}^{m_\beta} \Gamma^{i(\alpha)}_{1(\beta)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)j(\beta)}$$
$$\overset{(4.79),(4.61)}{\underset{(4.62)}{=}} \partial_{h(\beta)}\Gamma^{i(\alpha)}_{1(\alpha)j(\beta)} - \Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\beta)j(\beta)}$$

which is

$$R^{i(\alpha)}_{h(\beta)1(\beta)j(\beta)} = \partial_{h(\beta)}\Gamma^{i(\alpha)}_{1(\alpha)j(\beta)} - \Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\beta)j(\beta)} \overset{(4.62)}{\underset{(4.69)}{=}} 0$$

for $j \geq 2$ and

$$R^{i(\alpha)}_{h(\beta)1(\beta)1(\beta)} = \partial_{h(\beta)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)} - \Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\beta)1(\beta)} = 0$$

for $j = 1$, as $\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}$ does not depend on any of the $u^{h(\beta)}$s for $h \geq 2$ and $(1 < 2 \leq h)$

$$\Gamma^{1(\beta)}_{h(\beta)1(\beta)} \overset{(4.79)}{=} -\sum_{\sigma\neq\beta}\Gamma^{1(\beta)}_{h(\beta)1(\sigma)} \overset{(4.63)}{=} 0.$$

Case 5: $\alpha = \beta \neq \gamma = \epsilon$**.** Our goal is to prove that

$$\begin{aligned} R^{i(\alpha)}_{h(\gamma)k(\gamma)j(\alpha)} &= \partial_{k(\gamma)}\Gamma^{i(\alpha)}_{h(\gamma)j(\alpha)} - \partial_{h(\gamma)}\Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)} \\ &+ \sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\sigma)}\Gamma^{l(\sigma)}_{h(\gamma)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\sigma)}\Gamma^{l(\sigma)}_{k(\gamma)j(\alpha)}\right) \\ &\overset{(4.61)}{=} \partial_{k(\gamma)}\Gamma^{i(\alpha)}_{h(\gamma)j(\alpha)} - \partial_{h(\gamma)}\Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)} \\ &+ \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\alpha)}\Gamma^{l(\alpha)}_{h(\gamma)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\alpha)}\Gamma^{l(\alpha)}_{k(\gamma)j(\alpha)}\right) \\ &+ \sum_{l=1}^{m_\gamma}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{h(\gamma)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{k(\gamma)j(\alpha)}\right) \end{aligned} \tag{4.120}$$

vanishes. Without loss of generality, by the symmetries of R, we can set $h > k$. In particular, $h \geq 2$. We get

$$\begin{aligned} R^{i(\alpha)}_{h(\gamma)k(\gamma)j(\alpha)} &= \partial_{k(\gamma)}\Gamma^{i(\alpha)}_{h(\gamma)j(\alpha)} - \partial_{h(\gamma)}\Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)} \\ &+ \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\alpha)}\Gamma^{l(\alpha)}_{h(\gamma)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\alpha)}\Gamma^{l(\alpha)}_{k(\gamma)j(\alpha)}\right) \\ &+ \sum_{l=1}^{m_\gamma}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{h(\gamma)j(\alpha)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\gamma)}\Gamma^{l(\gamma)}_{k(\gamma)j(\alpha)}\right) \end{aligned}$$

where $\Gamma^{i(\alpha)}_{h(\gamma)j(\alpha)}$, $\Gamma^{l(\alpha)}_{h(\gamma)j(\alpha)}$, $\Gamma^{i(\alpha)}_{h(\gamma)l(\alpha)}$ and $\Gamma^{i(\alpha)}_{h(\gamma)l(\gamma)}$ vanish due to (4.62) and (4.64) and where

$$\sum_{l=1}^{m_\gamma} \Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)} \Gamma^{l(\gamma)}_{h(\gamma)j(\alpha)} \underset{(4.63)}{\overset{(4.62)}{=}} \sum_{l=h}^{m_\gamma} \Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)} \Gamma^{l(\gamma)}_{h(\gamma)j(\alpha)} \overset{(4.64)}{=} 0.$$

It follows that

$$R^{i(\alpha)}_{h(\gamma)k(\gamma)j(\alpha)} = -\partial_{h(\gamma)} \Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)} = 0$$

as $\Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)}$ does not depend on any of the $u^{h(\gamma)}$s for $h \geq 2$.
Case 6: $\alpha = \gamma \neq \beta = \epsilon$**.** Our goal is to prove that

$$\begin{aligned}
R^{i(\alpha)}_{h(\beta)k(\alpha)j(\beta)} &= \partial_{k(\alpha)} \Gamma^{i(\alpha)}_{h(\beta)j(\beta)} - \partial_{h(\beta)} \Gamma^{i(\alpha)}_{k(\alpha)j(\beta)} \\
&+ \sum_{\sigma=1}^{r} \sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\sigma)} \Gamma^{l(\sigma)}_{k(\alpha)j(\beta)} \right) \\
&= \partial_{k(\alpha)} \Gamma^{i(\alpha)}_{h(\beta)j(\beta)} - \partial_{h(\beta)} \Gamma^{i(\alpha)}_{k(\alpha)j(\beta)} \\
&+ \sum_{l=1}^{m_\alpha} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)j(\beta)} \right) \\
&+ \sum_{l=1}^{m_\beta} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)} \Gamma^{l(\beta)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)} \Gamma^{l(\beta)}_{k(\alpha)j(\beta)} \right) \\
&+ \sum_{\sigma \notin \{\alpha,\beta\}} \sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\sigma)} \Gamma^{l(\sigma)}_{k(\alpha)j(\beta)} \right)
\end{aligned} \tag{4.121}$$

(where $\Gamma^{i(\alpha)}_{h(\beta)l(\sigma)}$ vanishes due to (4.61)) vanishes. For $j \geq 2$ (4.121) vanishes trivially, as

$$\begin{aligned}
R^{i(\alpha)}_{h(\beta)k(\alpha)j(\beta)} &= \partial_{k(\alpha)} \Gamma^{i(\alpha)}_{h(\beta)j(\beta)} - \partial_{h(\beta)} \Gamma^{i(\alpha)}_{k(\alpha)j(\beta)} \\
&+ \sum_{l=1}^{m_\alpha} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)} \Gamma^{l(\alpha)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)} \Gamma^{l(\alpha)}_{k(\alpha)j(\beta)} \right) \\
&+ \sum_{l=1}^{m_\beta} \left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)} \Gamma^{l(\beta)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)} \Gamma^{l(\beta)}_{k(\alpha)j(\beta)} \right) \\
&+ \sum_{\sigma \notin \{\alpha,\beta\}} \sum_{l=1}^{m_\sigma} \Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)} \Gamma^{l(\sigma)}_{h(\beta)j(\beta)} \\
&\underset{(4.64)}{\overset{(4.62)}{=}} \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} \Gamma^{1(\beta)}_{h(\beta)j(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)1(\beta)} \Gamma^{1(\beta)}_{k(\alpha)j(\beta)} \\
&\underset{(4.63)}{\overset{(4.62)}{=}} \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} \Gamma^{1(\beta)}_{h(\beta)j(\beta)}
\end{aligned}$$

$$= \begin{cases} \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\beta)j(\beta)} \overset{(4.79)}{=} -\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\sum_{\sigma\neq\beta}\Gamma^{1(\beta)}_{1(\sigma)j(\beta)} \overset{(4.63)}{=} 0 & \text{if } h=1 \\ \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\beta)j(\beta)} \underset{(4.69)}{\overset{1-h-j\leq -3}{=}} 0 & \text{if } h\geq 2. \end{cases}$$

Let us then fix $j=1$. (4.121) becomes

$$\begin{aligned} R^{i(\alpha)}_{h(\beta)k(\alpha)1(\beta)} &= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\beta)1(\beta)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} \\ &+ \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}\right) \\ &+ \sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}\right) \\ &+ \sum_{\sigma\notin\{\alpha,\beta\}}\sum_{l=1}^{m_\sigma}\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\beta)1(\beta)}. \end{aligned} \tag{4.122}$$

We distinguish between the following subcases:

$a.$ both h and k are greater or equal than 2

$b.$ $h=k=1$

$c.$ $h\geq 2$, $k=1$

$d.$ $h=1$, $k\geq 2$.

Subcase a: both k and h are greater or equal than 2. We have

$$\begin{aligned} R^{i(\alpha)}_{h(\beta)k(\alpha)1(\beta)} &= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\beta)1(\beta)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} \\ &+ \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}\right) \\ &+ \sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{h(\beta)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}\right) \\ &+ \sum_{\sigma\notin\{\alpha,\beta\}}\sum_{l=1}^{m_\sigma}\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\beta)1(\beta)} \\ &\underset{(4.64)}{\overset{(4.62)}{=}} -\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\beta)1(\alpha)} - \partial_{h(\beta)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} + \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\beta)1(\beta)} \end{aligned}$$

where $\Gamma^{i(\alpha)}_{h(\beta)1(\alpha)}$ and $\Gamma^{1(\beta)}_{h(\beta)1(\beta)}$ vanish due to (4.62) and (4.67) and where $\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}$ does not depend on $u^{h(\beta)}$ for $h\geq 2$. Thus $R^{i(\alpha)}_{h(\beta)k(\alpha)1(\beta)}=0$.

Subcase b: $h=k=1$. (4.122) reads

$$R^{i(\alpha)}_{1(\beta)1(\alpha)1(\beta)} = \partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\beta)1(\beta)} - \partial_{1(\beta)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}$$

$$+\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)}-\Gamma^{i(\alpha)}_{1(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}\right)$$
$$+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\beta)}\Gamma^{l(\beta)}_{1(\beta)1(\beta)}-\Gamma^{i(\alpha)}_{1(\beta)l(\beta)}\Gamma^{l(\beta)}_{1(\alpha)1(\beta)}\right)$$
$$+\sum_{\sigma\notin\{\alpha,\beta\}}\sum_{l=1}^{m_\sigma}\Gamma^{i(\alpha)}_{1(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{1(\beta)1(\beta)}$$

where in the second and in the third summation only the terms for $l=1$ survive (by (4.62) and (4.64)) and

$$\partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\beta)1(\beta)}-\partial_{1(\beta)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\underset{(4.61)}{\overset{(4.79)}{=}}-\partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}+\partial_{1(\alpha)}\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}=0.$$

We get

$$R^{i(\alpha)}_{1(\beta)1(\alpha)1(\beta)}=\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)}-\Gamma^{i(\alpha)}_{1(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}\right)$$
$$+\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\beta)1(\beta)}-\Gamma^{i(\alpha)}_{1(\beta)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$
$$+\sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{i(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\beta)}$$
$$\underset{(4.67)}{\overset{(4.63)}{=}}\sum_{l=1}^{i}\left(\Gamma^{i(\alpha)}_{1(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)}-\Gamma^{i(\alpha)}_{1(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{1(\alpha)1(\beta)}\right)$$
$$+\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\beta)1(\beta)}-\Gamma^{i(\alpha)}_{1(\beta)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$
$$+\sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{i(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\beta)}$$
$$\underset{(4.61)}{\overset{(4.79)}{=}}\sum_{l=1}^{i}\left(-\Gamma^{i(\alpha)}_{1(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)}-\sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{i(\alpha)}_{1(\sigma)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)}+\Gamma^{i(\alpha)}_{1(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)}\right)$$
$$-\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\left(\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}+\sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)}\right)+\Gamma^{i(\alpha)}_{1(\beta)1(\alpha)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$
$$+\sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{i(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\beta)}$$
$$\overset{(4.63)}{=}\sum_{\sigma\notin\{\alpha,\beta\}}\left(-\sum_{l=1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)}-\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)}+\Gamma^{i(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\beta)}\right)$$
$$\underset{(4.61)}{\overset{(4.79)}{=}}\sum_{\sigma\notin\{\alpha,\beta\}}\left(\sum_{l=1}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)}-\Gamma^{i(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)}-\Gamma^{i(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)}\right)$$

which vanishes by (4.77).

Subcase c: $h\geq 2$, $k=1$. The argument of *subcase a* applies here as well.

Subcase d: $h = 1$, $k \geq 2$. (4.122) reads

$$\begin{aligned}R^{i(\alpha)}_{1(\beta)k(\alpha)1(\beta)} &= \partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\beta)1(\beta)} - \partial_{1(\beta)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} \\ &+ \sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{1(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}\right) \\ &+ \sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{1(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{1(\beta)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)1(\beta)}\right) \\ &+ \sum_{\sigma\notin\{\alpha,\beta\}}\sum_{l=1}^{m_\sigma}\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{1(\beta)1(\beta)}\end{aligned}$$

where in the second and in the third summation only the terms for $l = 1$ survive (by (4.62) and (4.64)) and

$$\begin{aligned}\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\beta)1(\beta)} - \partial_{1(\beta)}\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)} &\overset{(4.79),(4.61)}{\underset{(4.63)}{=}} -\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{1(\beta)1(\alpha)} - \partial_{1(\beta)}\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)} \\ &\overset{(4.109)}{=} \partial_{1(\beta)}\Gamma^{(i-k+1)(\alpha)}_{1(\beta)1(\alpha)} - \partial_{1(\beta)}\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)} = 0.\end{aligned}$$

We get

$$\begin{aligned}R^{i(\alpha)}_{1(\beta)k(\alpha)1(\beta)} &\overset{(4.63)}{\underset{(4.69)}{=}} \sum_{l=1}^{i-k+2}\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\beta)} - \sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)} \\ &+ \Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\beta)1(\beta)} - \Gamma^{i(\alpha)}_{1(\beta)1(\beta)}\Gamma^{1(\beta)}_{k(\alpha)1(\beta)} \\ &+ \sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{i(\alpha)}_{k(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\beta)} \\ &\overset{(4.79),(4.61),(4.62)}{\underset{(4.63),(4.69),(4.67)}{=}} -\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} - \sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} \\ &- \sum_{l=k}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\beta)1(\alpha)}\Gamma^{(l-k+1)(\alpha)}_{1(\alpha)1(\beta)} \\ &- \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} - \sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)} \\ &- \sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)} \\ &\overset{(4.79)}{=} \Gamma^{(i-k+1)(\alpha)}_{1(\beta)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} + \sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} \\ &- \sum_{l=2}^{i-k+2}\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} - \sum_{l=k}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\beta)1(\alpha)}\Gamma^{(l-k+1)(\alpha)}_{1(\alpha)1(\beta)} \\ &- \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} - \sum_{\sigma\notin\{\alpha,\beta\}}\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)}\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\sigma\notin\{\alpha,\beta\}} \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)} \\
&= \Gamma^{(i-k+1)(\alpha)}_{1(\beta)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} - \sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} \\
&- \Gamma^{(i-k+1)(\alpha)}_{1(\beta)1(\alpha)}\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)} - \sum_{l=k+1}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\beta)1(\alpha)}\Gamma^{(l-k+1)(\alpha)}_{1(\alpha)1(\beta)} \\
&- \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&+ \sum_{\sigma\notin\{\alpha,\beta\}} \left(\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)}\right) \\
&= -\sum_{l=2}^{i-k+2} \Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} - \sum_{t=2}^{i-k+1} \Gamma^{(i-k-t+2)(\alpha)}_{1(\beta)1(\alpha)}\Gamma^{t(\alpha)}_{1(\alpha)1(\beta)} \\
&- \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&+ \sum_{\sigma\notin\{\alpha,\beta\}} \left(\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)}\right) \\
&= -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\beta)1(\alpha)} - \sum_{l=2}^{i-k+1} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} + \Gamma^{(i-k-l+2)(\alpha)}_{1(\beta)1(\alpha)}\right)\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} \\
&- \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&+ \sum_{\sigma\notin\{\alpha,\beta\}} \left(\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)}\right) \\
&\overset{(4.79)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\beta)1(\alpha)} - \sum_{l=2}^{i-k+1} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} \\
&+ \sum_{\sigma\notin\{\alpha,\beta\}} \sum_{l=2}^{i-k+1} \Gamma^{(i-k-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&+ \sum_{\sigma\notin\{\alpha,\beta\}} \left(\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)}\right)
\end{aligned}
$$

where

$$
-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\Gamma^{(i-k+2)(\alpha)}_{1(\beta)1(\alpha)} - \Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} = \sum_{l=2}^{i-k+1} \left(\Gamma^{(i-k-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(i-k-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)}
$$

by means of (4.78). This yields

$$
R^{i(\alpha)}_{1(\beta)k(\alpha)1(\beta)} = \sum_{\sigma\notin\{\alpha,\beta\}} \sum_{l=2}^{i-k+1} \Gamma^{(i-k-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)}
$$

$$+\sum_{\sigma\notin\{\alpha,\beta\}}\left(\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}-\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)}-\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)}\right)$$

where for each $\sigma\notin\{\alpha,\beta\}$ we have

$$-\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\sigma)1(\beta)}-\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\sigma)}\Gamma^{1(\sigma)}_{1(\beta)1(\sigma)}=-\sum_{t=1}^{i-k+1}\Gamma^{(i-k-t+2)(\alpha)}_{1(\sigma)1(\alpha)}\,\Gamma^{t(\alpha)}_{1(\beta)1(\alpha)}$$

by means of (4.77). Thus

$$\begin{aligned}
R^{i(\alpha)}_{1(\beta)k(\alpha)1(\beta)}&=\sum_{\sigma\notin\{\alpha,\beta\}}\sum_{l=2}^{i-k+1}\Gamma^{(i-k-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)}\\
&\quad+\sum_{\sigma\notin\{\alpha,\beta\}}\left(\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}-\sum_{t=1}^{i-k+1}\Gamma^{(i-k-t+2)(\alpha)}_{1(\sigma)1(\alpha)}\,\Gamma^{t(\alpha)}_{1(\beta)1(\alpha)}\right)\\
&=\sum_{\sigma\notin\{\alpha,\beta\}}\left(\sum_{l=2}^{i-k+1}\Gamma^{(i-k-l+2)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)}+\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\right.\\
&\quad\left.-\Gamma^{(i-k+1)(\alpha)}_{1(\sigma)1(\alpha)}\,\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}-\sum_{t=2}^{i-k+1}\Gamma^{(i-k-t+2)(\alpha)}_{1(\sigma)1(\alpha)}\,\Gamma^{t(\alpha)}_{1(\beta)1(\alpha)}\right)=0.
\end{aligned}$$

Case 7: $\alpha=\beta\notin\{\gamma,\epsilon\}$**,** $\gamma\neq\epsilon$**.** Our goal is to prove that

$$\begin{aligned}
R^{i(\alpha)}_{h(\epsilon)k(\gamma)j(\alpha)}&=\partial_{k(\gamma)}\Gamma^{i(\alpha)}_{h(\epsilon)j(\alpha)}-\partial_{h(\epsilon)}\Gamma^{i(\alpha)}_{k(\gamma)j(\alpha)}\\
&\quad+\sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\sigma)}\Gamma^{l(\sigma)}_{h(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\gamma)j(\alpha)}\right)\\
&\overset{(4.79)}{\underset{(4.61)}{=}}\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\gamma)l(\alpha)}\Gamma^{l(\alpha)}_{h(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\gamma)j(\alpha)}\right)
\end{aligned}\tag{4.123}$$

vanishes. This trivially holds when $h\geq 2$ or $k\geq 2$, by (4.62). Let us then fix $h=k=1$. We have

$$\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)1(\gamma)j(\alpha)}&=\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{1(\gamma)l(\alpha)}\Gamma^{l(\alpha)}_{1(\epsilon)j(\alpha)}-\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{1(\gamma)j(\alpha)}\right)\\
&\overset{(4.63)}{=}\sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\gamma)1(\alpha)}\Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)}-\sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(l-j+1)(\alpha)}_{1(\gamma)1(\alpha)}\\
&=\sum_{l=j}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\gamma)1(\alpha)}\Gamma^{(l-j+1)(\alpha)}_{1(\epsilon)1(\alpha)}-\sum_{t=j}^{i}\Gamma^{(t-j+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(i-t+1)(\alpha)}_{1(\gamma)1(\alpha)}=0.
\end{aligned}$$

Case 8: $\alpha=\gamma\notin\{\beta,\epsilon\}$**,** $\beta\neq\epsilon$**.** Our goal is to prove that

$$R^{i(\alpha)}_{h(\epsilon)k(\alpha)j(\beta)}=\partial_{k(\alpha)}\Gamma^{i(\alpha)}_{h(\epsilon)j(\beta)}-\partial_{h(\epsilon)}\Gamma^{i(\alpha)}_{k(\alpha)j(\beta)}$$

$$
\begin{aligned}
&+\sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\sigma)}\Gamma^{l(\sigma)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\alpha)j(\beta)}\right)\\
&\overset{(4.61)}{=}\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\alpha)}\Gamma^{l(\alpha)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\beta)}\right)\\
&+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\beta)}\Gamma^{l(\beta)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\beta)}\Gamma^{l(\beta)}_{k(\alpha)j(\beta)}\right)\\
&+\sum_{l=1}^{m_\epsilon}\left(\Gamma^{i(\alpha)}_{k(\alpha)l(\epsilon)}\Gamma^{l(\epsilon)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\epsilon)}\Gamma^{l(\epsilon)}_{k(\alpha)j(\beta)}\right)\\
&\underset{(4.63)}{\overset{(4.61),(4.62)}{=}}-\sum_{l=k}^{i}\Gamma^{i(\alpha)}_{h(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)j(\beta)}+\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{h(\epsilon)j(\beta)}+\Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{h(\epsilon)j(\beta)}
\end{aligned}
\tag{4.124}
$$

vanishes. If $j \geq 2$ or $h \geq 2$ then (4.124) trivially vanishes, by (4.62) and (4.63). Let us then fix $j = 1$ and $h = 1$. We have

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)k(\alpha)1(\beta)} &= -\sum_{l=k}^{i}\Gamma^{i(\alpha)}_{1(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\alpha)1(\beta)}+\Gamma^{i(\alpha)}_{k(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)}+\Gamma^{i(\alpha)}_{k(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\beta)}\\
&\overset{(4.63)}{=}-\sum_{l=k}^{i}\Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{(l-k+1)(\alpha)}_{1(\alpha)1(\beta)}+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)}+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\beta)}\\
&=-\sum_{t=1}^{i-k+1}\Gamma^{(i-t-k+2)(\alpha)}_{1(\epsilon)1(\alpha)}\Gamma^{t(\alpha)}_{1(\alpha)1(\beta)}+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\beta)}\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)}+\Gamma^{(i-k+1)(\alpha)}_{1(\alpha)1(\epsilon)}\Gamma^{1(\epsilon)}_{1(\epsilon)1(\beta)}
\end{aligned}
$$

which vanishes by means of (4.77).

Case 9: $\beta = \gamma \notin \{\alpha, \epsilon\}$**,** $\alpha \neq \epsilon$**.** Our goal is to prove that

$$
\begin{aligned}
R^{i(\alpha)}_{h(\epsilon)k(\beta)j(\beta)} &= \partial_{k(\beta)}\Gamma^{i(\alpha)}_{h(\epsilon)j(\beta)}-\partial_{h(\epsilon)}\Gamma^{i(\alpha)}_{k(\beta)j(\beta)}\\
&+\sum_{\sigma=1}^{r}\sum_{l=1}^{m_\sigma}\left(\Gamma^{i(\alpha)}_{k(\beta)l(\sigma)}\Gamma^{l(\sigma)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)}\Gamma^{l(\sigma)}_{k(\beta)j(\beta)}\right)\\
&\overset{(4.61)}{=}\sum_{l=1}^{m_\alpha}\left(\Gamma^{i(\alpha)}_{k(\beta)l(\alpha)}\Gamma^{l(\alpha)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\beta)j(\beta)}\right)\\
&+\sum_{l=1}^{m_\beta}\left(\Gamma^{i(\alpha)}_{k(\beta)l(\beta)}\Gamma^{l(\beta)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\beta)}\Gamma^{l(\beta)}_{k(\beta)j(\beta)}\right)\\
&+\sum_{l=1}^{m_\epsilon}\left(\Gamma^{i(\alpha)}_{k(\beta)l(\epsilon)}\Gamma^{l(\epsilon)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)l(\epsilon)}\Gamma^{l(\epsilon)}_{k(\beta)j(\beta)}\right)\\
&\underset{(4.64)}{\overset{(4.61),(4.63)}{=}}-\sum_{l=1}^{i}\Gamma^{i(\alpha)}_{h(\epsilon)l(\alpha)}\Gamma^{l(\alpha)}_{k(\beta)j(\beta)}+\Gamma^{i(\alpha)}_{k(\beta)1(\beta)}\Gamma^{1(\beta)}_{h(\epsilon)j(\beta)}-\Gamma^{i(\alpha)}_{h(\epsilon)1(\epsilon)}\Gamma^{1(\epsilon)}_{k(\beta)j(\beta)}
\end{aligned}
\tag{4.125}
$$

vanishes. If $j \geq 2$ or $h \geq 2$ or $k \geq 2$ then (4.125) trivially vanishes, by (4.62), (4.63) and (4.64). Let us then fix $j = h = k = 1$. We have

$$
\begin{aligned}
R^{i(\alpha)}_{1(\epsilon)1(\beta)1(\beta)} &= -\sum_{l=1}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{l(\alpha)}_{1(\beta)1(\beta)} + \Gamma^{i(\alpha)}_{1(\beta)1(\beta)} \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{i(\alpha)}_{1(\epsilon)1(\epsilon)} \Gamma^{1(\epsilon)}_{1(\beta)1(\beta)} \\
&\overset{(4.79)}{\underset{(4.61)}{=}} \sum_{l=1}^{i} \Gamma^{(i-l+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} - \Gamma^{i(\alpha)}_{1(\beta)1(\alpha)} \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{i(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)}
\end{aligned}
$$

which vanishes by means of (4.77).

Case 10: $\gamma = \epsilon \notin \{\alpha, \beta\}$**,** $\alpha \neq \beta$**.** Our goal is to prove that

$$
\begin{aligned}
R^{i(\alpha)}_{h(\gamma)k(\gamma)j(\beta)} &= \partial_{k(\gamma)} \Gamma^{i(\alpha)}_{h(\gamma)j(\beta)} - \partial_{h(\gamma)} \Gamma^{i(\alpha)}_{k(\gamma)j(\beta)} \\
&\quad + \sum_{\sigma=1}^{r} \sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\gamma)l(\sigma)} \Gamma^{l(\sigma)}_{h(\gamma)j(\beta)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\sigma)} \Gamma^{l(\sigma)}_{k(\gamma)j(\beta)} \right) \\
&\overset{(4.61)}{=} \sum_{l=1}^{m_\gamma} \left(\Gamma^{i(\alpha)}_{k(\gamma)l(\gamma)} \Gamma^{l(\gamma)}_{h(\gamma)j(\beta)} - \Gamma^{i(\alpha)}_{h(\gamma)l(\gamma)} \Gamma^{l(\gamma)}_{k(\gamma)j(\beta)} \right) \\
&\overset{(4.64)}{=} \Gamma^{i(\alpha)}_{k(\gamma)1(\gamma)} \Gamma^{1(\gamma)}_{h(\gamma)j(\beta)} - \Gamma^{i(\alpha)}_{h(\gamma)1(\gamma)} \Gamma^{1(\gamma)}_{k(\gamma)j(\beta)}
\end{aligned}
\tag{4.126}
$$

vanishes. If $j \geq 2$ or $h \geq 2$ or $k \geq 2$ then (4.126) trivially vanishes, by (4.62), (4.63) and (4.64). Let us then fix $j = h = k = 1$. We have

$$
R^{i(\alpha)}_{1(\gamma)1(\gamma)1(\beta)} = \Gamma^{i(\alpha)}_{1(\gamma)1(\gamma)} \Gamma^{1(\gamma)}_{1(\gamma)1(\beta)} - \Gamma^{i(\alpha)}_{1(\gamma)1(\gamma)} \Gamma^{1(\gamma)}_{1(\gamma)1(\beta)} = 0.
$$

Case 11: α, β, γ and ϵ are pairwise distinct. Our goal is to prove that

$$
\begin{aligned}
R^{i(\alpha)}_{h(\epsilon)k(\gamma)j(\beta)} &= \partial_{k(\gamma)} \Gamma^{i(\alpha)}_{h(\epsilon)j(\beta)} - \partial_{h(\epsilon)} \Gamma^{i(\alpha)}_{k(\gamma)j(\beta)} \\
&\quad + \sum_{\sigma=1}^{r} \sum_{l=1}^{m_\sigma} \left(\Gamma^{i(\alpha)}_{k(\gamma)l(\sigma)} \Gamma^{l(\sigma)}_{h(\epsilon)j(\beta)} - \Gamma^{i(\alpha)}_{h(\epsilon)l(\sigma)} \Gamma^{l(\sigma)}_{k(\gamma)j(\beta)} \right)
\end{aligned}
\tag{4.127}
$$

vanishes. Since the only Christoffel symbols appearing have each of the three indices belonging to a different block, (4.127) trivially vanishes by (4.61).

This concludes the proof about the flatness of ∇.

4.6.8 Uniqueness

In order to prove uniqueness we have to prove that (part of the) conditions (1), (2), (3), (4) and (5) force ∇ to be of the form given in Proposition 4.15. We have seen that the condition $\nabla e = 0$ is equivalent to (4.79). Let α, β, γ be pairwise distinct. By means of the condition

$$
\nabla_i c^l_{jk} = \nabla_j c^l_{ik}, \qquad i, j, k, l \in \{1, \dots, n\},
$$

after a long but straightforward computation one obtains the following relations:

(i) $\Gamma^{l(\beta)}_{i(\alpha)(j+k)(\alpha)} = \Gamma^{l(\beta)}_{(i+k)(\alpha)j(\alpha)}$,

(ii) $\Gamma^{k(\beta)}_{i(\alpha)j(\beta)} = \Gamma^{l(\beta)}_{i(\alpha)m(\beta)}$ when $k - j = l - m$,

(iii) $\Gamma^{k(\alpha)}_{(i-1)(\alpha)j(\alpha)} = \Gamma^{k(\alpha)}_{i(\alpha)(j-1)(\alpha)}$ when the lower indices are both different from 1 and not simultaneously equal to 2,

(iv) $\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)} = \Gamma^{(k+1)(\alpha)}_{i(\alpha)(j+1)(\alpha)}$ when $j \neq 1$,

(v) $\Gamma^{k(\alpha)}_{i(\alpha)j(\beta)} = -\Gamma^{(k-i+1)(\alpha)}_{1(\beta)j(\beta)}$,

(vi) $\Gamma^{k(\gamma)}_{i(\alpha)j(\beta)} = 0$.

The above quantities must be considered non-null when the indices do not exceed the size of the corresponding block. By virtue of the condition (4.81), a straightforward computation leads to the following relations:

1. $\Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)} = \begin{cases} \Gamma^{(k-i-j+4)(\alpha)}_{2(\alpha)2(\alpha)} & \text{if } k-i-j \geq -2, \\ 0 & \text{if } k-i-j \leq -3, \end{cases} \qquad \text{when } i, j > 1,$

2. $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} = -\frac{m_\alpha \varepsilon_\alpha}{u^{2(\alpha)}}$,

3. $\Gamma^{1(\alpha)}_{1(\alpha)j(\alpha)} = 0 \quad$ when or $j \geq 2$,

4. $\Gamma^{k(\alpha)}_{i(\alpha)j(\beta)} = 0 \quad$ when $k < i$ or $j \geq 2$,

5. $\Gamma^{i(\alpha)}_{i(\alpha)1(\beta)} = \frac{m_\beta \varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}}$,

6. $\Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)} = -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h+1} \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)} \, u^{s(\alpha)} \quad$ for $h \geq 1$,

7. $\Gamma^{n(\alpha)}_{2(\alpha)2(\alpha)} = \Gamma^{(n-2)(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{n(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}} \sum_{l=1}^{n-3} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(n-l)(\alpha)}$ for $n \geq 3$[6].

By collecting all of the conditions above, one obtains Christoffel symbols of the form given in Proposition 4.15.

[6]The last summation is not to be considered for $n \in \{2, 3\}$.

4.7 The dual structure

In this section we study in more detail the dual structure $(\nabla^*, *, E)$, where

- E is the Euler vector field,
- $*$ is the dual product defined by the formula

$$X * Y = E^{-1} \circ X \circ Y$$

  for arbitrary vector fields X and Y,
- ∇^* is the connection defined by the Christoffel symbols

$$\Gamma^{*k}_{ij} = \Gamma^{k}_{ij} - c^{*l}_{ji} \nabla_l E^k, \qquad i, j, k \in \{1, \dots, n\}.$$

Proposition 4.28 *For each $\alpha \in \{1, \dots, r\}$ the components of the inverse of the Euler vector field E are given by*

$$(E^{-1})^{1(\alpha)} = \frac{1}{u^{1(\alpha)}} \tag{4.128}$$

$$(E^{-1})^{(k+1)(\alpha)} = -\frac{1}{u^{1(\alpha)}} \sum_{s=1}^{k} (E^{-1})^{(k-s+1)(\alpha)}\, u^{(s+1)(\alpha)} \qquad \textit{for } 1 \le k \le m_\alpha - 1. \tag{4.129}$$

Proof: By definition we have

$$E^{-1} \circ E = e = \sum_{\tau=1}^{r} \partial_{1(\tau)}$$

which (by taking the $k(\alpha)$-th component) yields

$$\begin{aligned}
\sum_{\tau=1}^{r} \delta^{k(\alpha)}_{1(\tau)} = (E^{-1} \circ E)^{k(\alpha)} &= \sum_{i,j=1}^{n} (E^{-1})^i\, c^{k(\alpha)}_{ij}\, E^j \\
&= \sum_{\sigma,\tau=1}^{r} \sum_{i=1}^{m_\sigma} \sum_{j=1}^{m_\tau} (E^{-1})^{i(\sigma)}\, c^{k(\alpha)}_{i(\sigma)j(\tau)}\, E^{j(\tau)} \\
&= \sum_{i,j=1}^{m_\alpha} (E^{-1})^{i(\alpha)}\, c^{k(\alpha)}_{i(\alpha)j(\alpha)}\, E^{j(\alpha)} = \sum_{i,j=1}^{m_\alpha} (E^{-1})^{i(\alpha)}\, \delta^{k}_{i+j-1}\, E^{j(\alpha)} \\
&= \sum_{j=1}^{m_\alpha} (E^{-1})^{(k-j+1)(\alpha)}\, E^{j(\alpha)}.
\end{aligned}$$

Since

$$\sum_{\tau=1}^{r} \delta^{k(\alpha)}_{1(\tau)} = \delta^k_1$$

we obtain

$$\delta_1^k = \sum_{j=1}^{m_\alpha} (E^{-1})^{(k-j+1)(\alpha)} \, E^{j(\alpha)}. \tag{4.130}$$

By taking $k = 1$ we get

$$1 = \sum_{j=1}^{m_\alpha} (E^{-1})^{(2-j)(\alpha)} \, E^{j(\alpha)}$$

where the quantity $(E^{-1})^{(2-j)(\alpha)}$ only makes sense for $j = 1$, thus

$$1 = (E^{-1})^{1(\alpha)} \, E^{1(\alpha)} = (E^{-1})^{1(\alpha)} \, u^{1(\alpha)}$$

which yields

$$(E^{-1})^{1(\alpha)} = \frac{1}{u^1(\alpha)}.$$

By taking $k \geq 2$ in (4.130) we get

$$0 = \sum_{j=1}^{m_\alpha} (E^{-1})^{(k-j+1)(\alpha)} \, E^{j(\alpha)}$$

where the quantity $(E^{-1})^{(k-j+1)(\alpha)}$ only makes sense for $j \leq k$, thus

$$\begin{aligned} 0 &= \sum_{j=1}^{k} (E^{-1})^{(k-j+1)(\alpha)} \, E^{j(\alpha)} \\ &= (E^{-1})^{k(\alpha)} \, u^{1(\alpha)} + \sum_{j=2}^{k} (E^{-1})^{(k-j+1)(\alpha)} \, u^{j(\alpha)} \\ &= (E^{-1})^{k(\alpha)} \, u^{1(\alpha)} + \sum_{s=1}^{k-1} (E^{-1})^{(k-s)(\alpha)} \, u^{(s+1)(\alpha)} \end{aligned}$$

which yields

$$(E^{-1})^{k(\alpha)} = -\frac{1}{u^{1(\alpha)}} \sum_{s=1}^{k-1} (E^{-1})^{(k-s)(\alpha)} \, u^{(s+1)(\alpha)}.$$

In particular, by relabelling $k = h + 1$ this becomes

$$(E^{-1})^{(h+1)(\alpha)} = -\frac{1}{u^{1(\alpha)}} \sum_{s=1}^{h} (E^{-1})^{(h-s+1)(\alpha)} \, u^{(s+1)(\alpha)}.$$

■

Remark 4.29 *By definition, the dual product $*$ must verify the following relation:*

$$X * Y = E^{-1} \circ X \circ Y \tag{4.131}$$

for X, Y arbitrary vector fields. This means that

$$X^j {c^*}^i_{jk} Y^k = (E^{-1})^a c^i_{ab} X^j c^b_{jk} Y^k, \qquad X, Y \in \mathfrak{X}(M),$$

namely

$${c^*}^i_{jk} = (E^{-1})^a c^i_{ab} c^b_{jk}, \qquad i, j, k \in \{1, \dots, n\}.$$

Therefore

$$\begin{aligned} {c^*}^{i(\alpha)}_{j(\beta)k(\gamma)} &= \sum_{\sigma,\tau=1}^{r} \sum_{a=1}^{m_\sigma} \sum_{b=1}^{m_\tau} (E^{-1})^{a(\sigma)} c^{i(\alpha)}_{a(\sigma)b(\tau)} c^{b(\tau)}_{j(\beta)k(\gamma)} = \sum_{a,b=1}^{m_\alpha} (E^{-1})^{a(\alpha)} c^{i(\alpha)}_{a(\alpha)b(\alpha)} c^{b(\alpha)}_{j(\beta)k(\gamma)} \\ &= \sum_{a,b=1}^{m_\alpha} (E^{-1})^{a(\alpha)} \delta^i_{a+b-1}\, \delta^\alpha_\beta \delta^\alpha_\gamma \delta^b_{j+k-1} = \delta^\alpha_\beta \delta^\alpha_\gamma \sum_{b=1}^{m_\alpha} (E^{-1})^{(i-b+1)(\alpha)} \delta^b_{j+k-1} \\ &= \delta^\alpha_\beta \delta^\alpha_\gamma\, (E^{-1})^{(i-j-k+2)(\alpha)} \end{aligned} \tag{4.132}$$

for all suitable indices.

Proposition 4.30 *The Christoffel symbols of the dual connection ∇^* are given by*

$$\begin{aligned} {\Gamma^*}^{k(\alpha)}_{i(\beta)j(\gamma)} = \Gamma^{k(\alpha)}_{i(\beta)j(\gamma)} &- \delta^\alpha_\beta \delta^\alpha_\gamma\, (E^{-1})^{(k-i-j+2)(\alpha)} \Bigg[\delta^k_1 \Bigg(1 - \sum_{\sigma \neq \alpha} m_\sigma \varepsilon_\sigma \Bigg) \\ &+ (1 - \delta^k_1) \Bigg(1 - \sum_{\tau=1}^{r} m_\tau \varepsilon_\tau \Bigg) \Bigg] - (1 - \delta^\alpha_\beta)\, \delta_{\beta\gamma}\, \delta^1_i \delta^1_j \delta^k_1\, \frac{m_\beta \varepsilon_\beta}{u^{1(\beta)}} \end{aligned} \tag{4.133}$$

for every choice of α, β, $\gamma \in \{1, \dots, r\}$ and every $k \in \{1, \dots, m_\alpha\}$, $i \in \{1, \dots, m_\beta\}$, $j \in \{1, \dots, m_\gamma\}$.

Proof: By means of (1.60) we have

$$\begin{aligned} {\Gamma^*}^{k(\alpha)}_{i(\beta)j(\gamma)} &= \Gamma^{k(\alpha)}_{i(\beta)j(\gamma)} - \sum_{l=1}^{n} {c^*}^l_{i(\beta)j(\gamma)}\, \nabla_l E^{k(\alpha)} \\ &= \Gamma^{k(\alpha)}_{i(\beta)j(\gamma)} - \sum_{\tau=1}^{r} \sum_{l=1}^{m_\tau} {c^*}^{l(\tau)}_{i(\beta)j(\gamma)}\, \nabla_{l(\tau)} E^{k(\alpha)} \end{aligned}$$

where $\nabla_{l(\tau)} E^{k(\alpha)} \neq 0$ only for $\tau = \alpha$ and for $\tau \neq \alpha$ and $l = k = 1$ by (4.74). It follows that

$${\Gamma^*}^{k(\alpha)}_{i(\beta)j(\gamma)} = \Gamma^{k(\alpha)}_{i(\beta)j(\gamma)} - \sum_{l=1}^{m_\alpha} {c^*}^{l(\alpha)}_{i(\beta)j(\gamma)}\, \nabla_{l(\alpha)} E^{k(\alpha)}$$

$$
\begin{aligned}
&- \sum_{\tau \neq \alpha} c^{*\,1(\tau)}_{\;i(\beta)j(\gamma)} \, \nabla_{1(\tau)} E^{1(\alpha)} \, \delta^k_1 \\
&= \Gamma^{k(\alpha)}_{i(\beta)j(\gamma)} - \sum_{l=1}^{m_\alpha} \delta^\alpha_\beta \delta^\alpha_\gamma \, (E^{-1})^{(l-i-j+2)(\alpha)} \, \nabla_{l(\alpha)} E^{k(\alpha)} \\
&- \sum_{\tau \neq \alpha} \delta^\tau_\beta \delta^\tau_\gamma \, (E^{-1})^{(3-i-j)(\tau)} \, \nabla_{1(\tau)} E^{1(\alpha)} \, \delta^k_1
\end{aligned}
$$

which immediately implies

$$
\Gamma^{*\,k(\alpha)}_{\;i(\beta)j(\gamma)} = \Gamma^{k(\alpha)}_{i(\beta)j(\gamma)}
$$

whenever $\beta \neq \gamma$. If $\alpha = \beta = \gamma$ then we get

$$
\Gamma^{*\,k(\alpha)}_{\;i(\alpha)j(\alpha)} = \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)} - \sum_{l=1}^{m_\alpha} (E^{-1})^{(l-i-j+2)(\alpha)} \, \nabla_{l(\alpha)} E^{k(\alpha)}
$$

as $\delta^\tau_\beta = \delta^\tau_\alpha = 0$ for every $\tau \neq \alpha$, where

$$
\nabla_{l(\alpha)} E^{k(\alpha)} = \delta^k_l \left[\delta^k_1 \left(1 - \sum_{\sigma \neq \alpha} m_\sigma \varepsilon_\sigma \right) + (1 - \delta^k_1) \left(1 - \sum_{\tau=1}^{r} m_\tau \varepsilon_\tau \right) \right]
$$

by (4.74). It follows that

$$
\begin{aligned}
\Gamma^{*\,k(\alpha)}_{\;i(\alpha)j(\alpha)} = \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)} - (E^{-1})^{(k-i-j+2)(\alpha)} & \left[\delta^k_1 \left(1 - \sum_{\sigma \neq \alpha} m_\sigma \varepsilon_\sigma \right) \right. \\
& \left. + (1 - \delta^k_1) \left(1 - \sum_{\tau=1}^{r} m_\tau \varepsilon_\tau \right) \right].
\end{aligned}
$$

If $\alpha \neq \beta = \gamma$ then we have

$$
\begin{aligned}
\Gamma^{*\,k(\alpha)}_{\;i(\beta)j(\beta)} &= \Gamma^{k(\alpha)}_{i(\beta)j(\beta)} - \sum_{\tau \neq \alpha} \delta^\tau_\beta \, (E^{-1})^{(3-i-j)(\tau)} \, \nabla_{1(\tau)} E^{1(\alpha)} \, \delta^k_1 \\
&= \Gamma^{k(\alpha)}_{i(\beta)j(\beta)} - (E^{-1})^{(3-i-j)(\beta)} \, \nabla_{1(\beta)} E^{1(\alpha)} \, \delta^k_1 \\
&= \Gamma^{k(\alpha)}_{i(\beta)j(\beta)} - (E^{-1})^{(3-i-j)(\beta)} \, m_\beta \varepsilon_\beta \, \delta^k_1
\end{aligned}
$$

by (4.74), where $(E^{-1})^{(3-i-j)(\beta)}$ only makes sense for $3 - i - j \geq 1$ that is $i + j \leq 2$ namely $i = j = 1$. Thus $(E^{-1})^{(3-i-j)(\beta)} = \delta^1_i \delta^1_j \, (E^{-1})^{1(\beta)} = \delta^1_i \delta^1_j \, \frac{1}{u^{1(\beta)}}$ yielding

$$
\Gamma^{*\,k(\alpha)}_{\;i(\beta)j(\beta)} = \Gamma^{k(\alpha)}_{i(\beta)j(\beta)} - \delta^1_i \delta^1_j \delta^k_1 \, \frac{m_\beta \varepsilon_\beta}{u^{1(\beta)}}.
$$

■

4.8 Generalized Lenard-Magri chains and the principal hierarchy

The main idea at the origin of our work was to identify the integrable hierarchy obtained by applying the construction of [62] starting from a $(1,1)$-type tensor field L with vanishing Nijenhuis torsion with a set of symmetries of the principal hierarchy associated with a bi-flat F-manifold, for each of the canonical forms found by David and Hertling in [18] for $L = E\circ$ in the case of regular F-manifolds with Euler vector field. This amounts to requiring that all of the tensor fields

$$\begin{aligned}
V_0 &= X_{(0)}\circ = e\circ = I\\
V_1 &= X_{(1)}\circ = (E - a_0\, e)\circ = L - a_0 I\\
V_2 &= X_{(2)}\circ = (E\circ E - a_0\, E - a_1\, e)\circ = L^2 - a_0 L - a_1 I\\
&\vdots\\
V_{k+1} &= X_{(k+1)}\circ = LV_k - a_k I = (E^{k+1} - a_0 E^k - a_1 E^{k-1} + \cdots - a_k e)\circ\\
&= L^{k+1} - a_0 L^k - a_1 L^{k-1} + \cdots - a_k I\\
&\vdots
\end{aligned}$$

defined recursively by

$$da_{k+1} = d_L a_k - a_k da_0, \qquad k \geq 0,$$

starting from

$$a_0 = \sum_{\alpha=1}^{r} m_\alpha \varepsilon_\alpha\, u^{1(\alpha)}$$

satisfy the condition $d_\nabla V_k = 0$. Actually, in order to get the connection ∇, we needed to impose only the first non-trivial condition

$$d_\nabla (L - a_0 I) = 0.$$

In this section we will prove that the same condition is satisfied by all tensor fields V_k. In other words, as it is natural to expect, the connection ∇ is associated to the full hierarchy and not only to a single special flow. In order to prove this fact, we will use the commutativity of the associated flows [62]. According to the results of [64], the commutativity of the flows associated with V_α and V_β can be written as

$$\begin{aligned}
&c^r_{is}\left[\left(\mathcal{L}_{X_{(\alpha)}} c\right)^i_{jk} X^k_{(\beta)} - \left(\mathcal{L}_{X_{(\beta)}} c\right)^i_{jk} X^k_{(\alpha)} + c^i_{jk}[X_{(\alpha)}, X_{(\beta)}]^k\right] +\\
&c^r_{ij}\left[\left(\mathcal{L}_{X_{(\alpha)}} c\right)^i_{sk} X^k_{(\beta)} - \left(\mathcal{L}_{X_{(\beta)}} c\right)^i_{sk} X^k_{(\alpha)} + c^i_{sk}[X_{(\alpha)}, X_{(\beta)}]^k\right] = 0
\end{aligned}$$

where, as above, $\mathcal{L}_X$ denotes the Lie derivative along a vector field X. We have the following lemma.

Lemma 4.31 *The commutativity condition can be written as*

$$V_i^s(d_\nabla W)^l_{js} + V_j^s(d_\nabla W)^l_{is} + W_i^s(d_\nabla V)^l_{js} + W_j^s(d_\nabla V)^l_{is} = 0 \tag{4.134}$$

for all indices, where $V = X_{(\alpha)}\circ$ *and* $W = X_{(\beta)}\circ$.

The proof is a straightforward computation. This lemma leads to the following proposition.

Proposition 4.32 *The tensor fields* V_β *satisfy the condition*

$$d_\nabla V_\beta = 0, \qquad \beta \in \{2, 3, 4, \dots\}.$$

Proof: Due to the previous lemma and taking into account that $d_\nabla V_1 = 0$, we can assume the validity of the equation

$$V_i^s(d_\nabla W)^l_{js} + V_j^s(d_\nabla W)^l_{is} = 0, \tag{4.135}$$

with $V = L - a_0 I$ and $W = V_\beta$ for some fixed $\beta \geq 2$. We recall that

$$V^{a(\alpha)}_{b(\beta)} = \delta^\alpha_\beta\big(u^{(a-b+1)(\alpha)}\mathbb{1}_{\{a\geq b\}} - \delta^a_b a_0\big).$$

In particular, $V^{a(\alpha)}_{b(\beta)} = 0$ whenever $\alpha \neq \beta$. Using these facts, it is immediate to check that the condition (4.135) in David-Hertling canonical coordinates reads

$$V^{s(\sigma)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\sigma)} + V^{s(\sigma)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\sigma)} = 0.$$

Let us study its consequences. We consider the following cases:

1. $\alpha = \beta$
2. $\alpha \neq \beta$.

Case 1: $\alpha = \beta$**.** We fix the index i. In this case we have

$$\begin{aligned} 0 &= V^{s(\sigma)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\alpha)s(\sigma)} + V^{s(\sigma)}_{l(\alpha)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\sigma)} \\ &= V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\alpha)s(\alpha)} + V^{s(\alpha)}_{l(\alpha)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\alpha)}. \end{aligned}$$

We show that $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-h)(\alpha)} = 0$ by a double procedure of induction, over q and h. By taking $j = m_\alpha$ we get

$$0 = V^{s(\alpha)}_{m_\alpha(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\alpha)s(\alpha)} + V^{s(\alpha)}_{l(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)}$$

which gives

$$
\begin{aligned}
0 &= V^{s(\alpha)}_{m_\alpha(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-1)(\alpha)s(\alpha)} + V^{s(\alpha)}_{(m_\alpha-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)} \\
&= V^{m_\alpha(\alpha)}_{m_\alpha(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-1)(\alpha)m_\alpha(\alpha)} + V^{(m_\alpha-1)(\alpha)}_{(m_\alpha-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-1)(\alpha)} \\
&+ V^{m_\alpha(\alpha)}_{(m_\alpha-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)m_\alpha(\alpha)} = u^{2(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)m_\alpha(\alpha)}
\end{aligned}
$$

thus $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)m_\alpha(\alpha)} = 0$ for $l = m_\alpha - 1$ (we already knew it, due to the antisymmetry of $d_\nabla W$ in the lower indices),

$$
\begin{aligned}
0 &= V^{s(\alpha)}_{m_\alpha(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-2)(\alpha)s(\alpha)} + V^{s(\alpha)}_{(m_\alpha-2)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)} \\
&= V^{m_\alpha(\alpha)}_{m_\alpha(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-2)(\alpha)m_\alpha(\alpha)} + V^{(m_\alpha-2)(\alpha)}_{(m_\alpha-2)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-2)(\alpha)} \\
&+ V^{(m_\alpha-1)(\alpha)}_{(m_\alpha-2)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-1)(\alpha)} + V^{m_\alpha(\alpha)}_{(m_\alpha-2)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)m_\alpha(\alpha)} \\
&= u^{2(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-1)(\alpha)}
\end{aligned}
$$

thus $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-1)(\alpha)} = 0$ for $l = m_\alpha - 2$ and

$$
\begin{aligned}
0 &= V^{s(\alpha)}_{m_\alpha(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-h-1)(\alpha)s(\alpha)} + V^{s(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)} \\
&= V^{m_\alpha(\alpha)(\alpha)}_{m_\alpha(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-h-1)(\alpha)m_\alpha(\alpha)(\alpha)} + V^{(m_\alpha-h-1)(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-h-1)(\alpha)} \\
&+ \sum_{s=m_\alpha-h}^{m_\alpha} V^{s(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)} \\
&= V^{(m_\alpha-h)(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-h)(\alpha)} + \sum_{s=m_\alpha-h+1}^{m_\alpha} V^{s(\alpha)}_{(m_\alpha-h-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)s(\alpha)}
\end{aligned}
$$

for $l = m_\alpha - h - 1$ (for a given $h \geq 1$). This last condition, if we inductively assume that $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-r)(\alpha)} = 0$ for each $r \leq h-1$, yields $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)(m_\alpha-h)(\alpha)} = 0$. We have just proved that $(d_\nabla W)^{i(\gamma)}_{m_\alpha(\alpha)j(\alpha)} = 0$ for every choice of j. We want to prove now that $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)j(\alpha)} = 0$ for each choice of j for a given $q \geq 1$. We inductively assume that $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-r)(\alpha)j(\alpha)} = 0$ for each choice of j and $r \leq q-1$. By taking $l = m_\alpha - q$ we get

$$
0 = V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)} + V^{s(\alpha)}_{(m_\alpha-q)(\alpha)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\alpha)}
$$

where $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)} = 0$ and $(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\alpha)} = 0$ for each $s \geq m_\alpha - q + 1$ thus

$$
\begin{aligned}
0 &= \sum_{s=j}^{m_\alpha-q} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)} + \sum_{s=m_\alpha-q}^{m_\alpha-q} V^{s(\alpha)}_{(m_\alpha-q)(\alpha)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\alpha)} \\
&= \sum_{s=j}^{m_\alpha-q} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)} + V^{(m_\alpha-q)(\alpha)}_{(m_\alpha-q)(\alpha)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)(m_\alpha-q)(\alpha)}
\end{aligned}
$$

$$= V^{j(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)j(\alpha)} + \sum_{s=j+1}^{m_\alpha-q} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)}$$

$$+ V^{(m_\alpha-q)(\alpha)}_{(m_\alpha-q)(\alpha)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)(m_\alpha-q)(\alpha)} = \sum_{s=j+1}^{m_\alpha-q} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)}$$

which is trivially verified whenever $j \geq m_\alpha - q$ and gives

$$0 = V^{(m_\alpha-q)(\alpha)}_{(m_\alpha-q-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-q)(\alpha)}$$

thus $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-q)(\alpha)} = 0$ for $j = m_\alpha - q - 1$ and

$$0 = \sum_{s=m_\alpha-t}^{m_\alpha-q} V^{s(\alpha)}_{(m_\alpha-t-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)}$$

$$= V^{(m_\alpha-t)(\alpha)}_{(m_\alpha-t-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-t)(\alpha)} + \sum_{s=m_\alpha-t+1}^{m_\alpha-q} V^{s(\alpha)}_{(m_\alpha-t-1)(\alpha)}(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)s(\alpha)}$$

for $j = m_\alpha - t - 1$ (given some $t \geq 1$). This last condition, together with the inductive assumption of $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-r)(\alpha)} = 0$ for each choice of $r \leq t-1$, yields $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)(m_\alpha-t)(\alpha)} = 0$. This proves that $(d_\nabla W)^{i(\gamma)}_{(m_\alpha-q)(\alpha)j(\alpha)} = 0$ for each choice of j and in turn that $(d_\nabla W)^{i(\gamma)}_{j(\alpha)l(\alpha)} = 0$ for each choice of j and l.

Case 2: $\alpha \neq \beta$**.** We fix the index i. In this case we have

$$0 = V^{s(\sigma)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\sigma)} + V^{s(\sigma)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\sigma)} = \sum_{s=j}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\alpha)}$$

$$+ \sum_{s=l}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\beta)} = \left(u^{1(\alpha)} - a_0\right)(d_\nabla W)^{i(\gamma)}_{l(\beta)j(\alpha)} + \sum_{s=j+1}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\alpha)}$$

$$+ \left(u^{1(\beta)} - a_0\right)(d_\nabla W)^{i(\gamma)}_{j(\alpha)l(\beta)} + \sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\beta)}$$

$$= \left(u^{1(\alpha)} - u^{1(\beta)}\right)(d_\nabla W)^{i(\gamma)}_{l(\beta)j(\alpha)} + \sum_{s=j+1}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\gamma)}_{l(\beta)s(\alpha)} + \sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{i(\gamma)}_{j(\alpha)s(\beta)}$$

which is trivially verified when $\gamma \neq \alpha$, $\gamma \neq \beta$ since

$$(d_\nabla W)^{i(\gamma)}_{j(\alpha)l(\beta)} = \partial_{j(\alpha)} W^{i(\gamma)}_{l(\beta)} + \Gamma^{i(\gamma)}_{j(\alpha)s(\sigma)} W^{s(\sigma)}_{l(\beta)} - \partial_{l(\beta)} W^{i(\gamma)}_{j(\alpha)} - \Gamma^{i(\gamma)}_{l(\beta)s(\sigma)} W^{s(\sigma)}_{j(\alpha)}$$

where $W^{a(\mu)}_{b(\nu)} = 0$ whenever $\mu \neq \nu$ (because $W = V_k$ is a polynomial in L and $L^{a(\mu)}_{b(\nu)} = 0$ whenever $\mu \neq \nu$) and $\Gamma^{a(\mu)}_{b(\nu)c(\tau)} = 0$ whenever μ, ν and τ are pairwise distinct. We are then left to consider the case where $\gamma = \alpha \neq \beta$ (due to the antisymmetry of $d_\nabla W$ in the lower indices, this covers the case $\gamma = \beta \neq \alpha$ as well). We have

$$0 = \left(u^{1(\alpha)} - u^{1(\beta)}\right)(d_\nabla W)^{i(\alpha)}_{l(\beta)j(\alpha)}$$

$$+\sum_{s=j+1}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{i(\alpha)}_{l(\beta)s(\alpha)}+\sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{i(\alpha)}_{j(\alpha)s(\beta)}$$

where

$$\begin{aligned}(d_\nabla W)^{i(\alpha)}_{j(\alpha)l(\beta)}&=\partial_{j(\alpha)}W^{i(\alpha)}_{l(\beta)}+\Gamma^{i(\alpha)}_{j(\alpha)s(\beta)}W^{s(\beta)}_{l(\beta)}-\partial_{l(\beta)}W^{i(\alpha)}_{j(\alpha)}-\Gamma^{i(\alpha)}_{l(\beta)s(\alpha)}W^{s(\alpha)}_{j(\alpha)}\\&=\Gamma^{i(\alpha)}_{j(\alpha)1(\beta)}W^{1(\beta)}_{l(\beta)}-\partial_{l(\beta)}W^{i(\alpha)}_{j(\alpha)}-\Gamma^{i(\alpha)}_{l(\beta)s(\alpha)}W^{s(\alpha)}_{j(\alpha)}\end{aligned}$$

trivially vanishes whenever $i<j$ ($W^{i(\alpha)}_{j(\alpha)}=0$ for $i<j$ because $W=V_k$ is a polynomial in L and $L^{i(\alpha)}_{j(\alpha)}=0$ for $i<j$). We are then left to consider $i\geq j$. For $i=j$ we get

$$\begin{aligned}0&=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{j(\alpha)}_{l(\beta)j(\alpha)}\\&+\sum_{s=j+1}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{j(\alpha)}_{l(\beta)s(\alpha)}+\sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{j(\alpha)}_{j(\alpha)s(\beta)}\\&=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{j(\alpha)}_{l(\beta)j(\alpha)}+\sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{j(\alpha)}_{j(\alpha)s(\beta)}\end{aligned}$$

which gives

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{j(\alpha)}_{m_\beta(\beta)j(\alpha)}$$

thus $(d_\nabla W)^{j(\alpha)}_{m_\beta(\beta)j(\alpha)}=0$ for $l=m_\beta$ and

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{j(\alpha)}_{(m_\beta-h)(\beta)j(\alpha)}+\sum_{s=m_\beta-h+1}^{m_\beta} V^{s(\beta)}_{(m_\beta-h)(\beta)}(d_\nabla W)^{j(\alpha)}_{j(\alpha)s(\beta)}$$

for $l=m_\beta-h$ (for a given $h\geq 1$). This last condition, together with the inductive assumption of $(d_\nabla W)^{j(\alpha)}_{j(\alpha)(m_\beta-r)(\beta)}=0$ for each choice of $r\leq h-1$, yields $(d_\nabla W)^{j(\alpha)}_{j(\alpha)(m_\beta-h)(\beta)}=0$. This proves $(d_\nabla W)^{j(\alpha)}_{j(\alpha)l(\beta)}=0$ for every choice of l. We inductively assume that $(d_\nabla W)^{(j+t)(\alpha)}_{j(\alpha)l(\beta)}=0$ for every l and for every $t\leq p-1$ (for a fixed $p\geq 1$). We want to show that $(d_\nabla W)^{(j+p)(\alpha)}_{j(\alpha)l(\beta)}=0$ for every l. For $i=j+p$ we get

$$\begin{aligned}0&=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{(j+p)(\alpha)}_{l(\beta)j(\alpha)}\\&+\sum_{s=j+1}^{m_\alpha} V^{s(\alpha)}_{j(\alpha)}(d_\nabla W)^{(j+p)(\alpha)}_{l(\beta)s(\alpha)}+\sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{(j+p)(\alpha)}_{j(\alpha)s(\beta)}\end{aligned}$$

where $(d_\nabla W)^{(j+p)(\alpha)}_{l(\beta)s(\alpha)}=0$ for every $s\geq j+1$ by the inductive hypothesis, so

$$0=\left(u^{1(\alpha)}-u^{1(\beta)}\right)(d_\nabla W)^{(j+p)(\alpha)}_{l(\beta)j(\alpha)}+\sum_{s=l+1}^{m_\beta} V^{s(\beta)}_{l(\beta)}(d_\nabla W)^{(j+p)(\alpha)}_{j(\alpha)s(\beta)}$$

which gives

$$0 = \left(u^{1(\alpha)} - u^{1(\beta)}\right)(d_\nabla W)^{(j+p)(\alpha)}_{m_\beta(\beta)j(\alpha)}$$

thus $(d_\nabla W)^{(j+p)(\alpha)}_{m_\beta(\beta)j(\alpha)} = 0$ for $l = m_\beta$ and

$$0 = \left(u^{1(\alpha)} - u^{1(\beta)}\right)(d_\nabla W)^{(j+p)(\alpha)}_{(m_\beta-h)(\beta)j(\alpha)} + \sum_{s=m_\beta-h+1}^{m_\beta} V^{s(\beta)}_{(m_\beta-h)(\beta)}(d_\nabla W)^{(j+p)(\alpha)}_{j(\alpha)s(\beta)}$$

for $l = m_\beta - h$ (for a fixed $h \geq 1$). This last condition, together with the inductive assumption of $(d_\nabla W)^{(j+p)(\alpha)}_{(m_\beta-r)(\beta)j(\alpha)} = 0$ for each choice of $r \leq h-1$, yields $(d_\nabla W)^{(j+p)(\alpha)}_{(m_\beta-h)(\beta)j(\alpha)} = 0$. This proves $(d_\nabla W)^{(j+p)(\alpha)}_{j(\alpha)l(\beta)} = 0$ for every choice of l and in turn $(d_\nabla W)^{i(\alpha)}_{j(\alpha)l(\beta)} = 0$ for every choice of i, j and l.

This concludes the proof of the fact that (4.135) implies $d_\nabla W = 0$ for the choice of $V = L - a_0 I$ and $W = V_k$. ■

Open perspectives

As seen above, Frobenius manifolds provide a geometric reformulation of the WDVV equations and a way to investigate two-dimensional topological field theories. The correlators of the fields of these theories can be generated by the so-called (full) free energy, a part of which, the primary free energy, satisfies the WDVV equations. The full free energy can be written as a genus expansion

$$\mathcal{F} = \sum_{g \geq 0} \mathcal{F}_g,$$

the term $\mathcal{F}_0$ providing the primary free energy when suitably restricted to the so-called small phase space. One may wonder how to reconstruct the full free energy starting from a solution to the WDVV equations.

From an integrable system point of view, this problem can be formulated in terms of perturbations of an integrable hierarchy, the times of the hierarchy being the arguments of the free energy.

In the semisimple case, Dubrovin and Zhang [34] showed that, for any solution to the WDVV equations, the genus one approximation of the integrable hierarchy, also known as one-loop deformation of the genus zero hierarchy, exists and it is uniquely determined by properties of the genus one correlators, proved by Dijkgraaf and Witten [26] and E. Getzler [41]. More precisely, the genus one part of the free energy is expressed in terms of a function

$$G = \log \frac{\tau_I}{J^{1/24}}$$

which takes the name of G-function. Here τ_I is a function related to a system of equations, equations of isomonodromy deformations of some linear differential operator with rational coefficients, to which the WDVV equations can be reduced in the semisimple case [27], while J denotes the Jacobian of the transformation from the canonical coordinates to the flat ones. As a byproduct, this result proves conjectures formulated by A. Givental in [42].

Thus, such a G-function provides a tool for the reconstruction of genus one information starting from genus zero data. In turn, it influences different areas of mathematics. For instance, in enumerative geometry it governs Gromov-Witten invariants, while in the theory of integrable systems it appears in the first order corrections of bi-Hamiltonian structures.

When the semisimplicity assumption is dropped, little is understood. The knowledge about dispersive integrable deformations may be broadened by tackling the problem of reconstructing, starting from a solution to the WDVV equations, the genus one contribution to the full free energy in the non-semisimple case.

Just as relevant are integrable deformations of integrable systems. In this area, a good deal of progress has been recently made. For instance, in the semisimple bi-Hamiltonian case we know that bi-Hamiltonian deformations are parametrized by functions of a single variable and in the special case related to Frobenius manifolds we know that there exists a special deformation of topological type, uniquely determined by the dispersionless limit (mirroring the Givental reconstruction procedure of higher genus information starting from the genus zero information encoded in the underlying Frobenius manifold).

Again, few results are available in the non-semisimple case, other than some preliminary work about deformations of non-semisimple bi-Hamiltonian structures of hydrodynamic type, carried out in [23]. In the wake of the contents presented in this thesis, it is natural to wonder whether it is possible to construct integrable deformations of the integrable systems of hydrodynamic type associated with Lauricella functions [66], with particular interest in deformations of topological type.

More in general, non-semisimple structures are related to integrable systems of hydrodynamic type without Riemann invariants, thus not reducible to a diagonal form. In this context, first results have been obtained in [39, 38]. In particular, in [39], the authors introduced the notion of quasilinear systems of Jordan block type and studied their connection with the mKP hierarchy. Our works [65, 66] also fit in this context and may provide the starting point for future developments.

Bibliography

[1] A. Arsie, A. Buryak, P. Lorenzoni, P. Rossi. *Flat F-manifolds, F-CohFTs, and integrable hierarchies*. Comm. Math. Phys. 388, 291–328 (2021).

[2] A. Arsie, A. Buryak, P. Lorenzoni, P. Rossi. *Semisimple flat F-manifolds in higher genus*, Comm. Math. Phys. 397, 141–197 (2023).

[3] A. Arsie, P. Lorenzoni, *F-manifolds with eventual identities, bidifferential calculus and twisted Lenard-Magri chains*, International Mathematics Research Notices, Vol. 2013, No. 17, 3931–3976 (2013).

[4] A. Arsie, P. Lorenzoni, *From Darboux-Egorov system to bi-flat F-manifolds*, Journal of Geometry and Physics, Vol 70, (2013), 98–116.

[5] A. Arsie, P. Lorenzoni *Complex reflection groups, logarithmic connections and bi-flat F-manifolds*, Lett. Math. Phys. (2017) 107, 1919–1961.

[6] A. Arsie, P. Lorenzoni, *F-manifolds, multi-flat structures and and Painlevé trascendents*, Asian J. Math, Vol. 23, No. 5, pp. 877–904, (2019).

[7] M.F. Atiyah, *Topological quantum field theories*, Publ. Math. I.H.E.S. **68** (1988) 175.

[8] A. Basalaev, C. Hertling, *3-dimensional F-manifolds,* Lett. Math. Phys. (2021) 111:90.

[9] A.V. Bolsinov, A.Y. Konyaev, V.S. Matveev, *Nijenhuis geometry IV: conservation laws, symmetries and integration of certain non-diagonalisable systems of hydrodynamic type in quadratures*, arXiv:2304.10626.

[10] A. Buryak, *Double ramification cycles and integrable hierarchies*, Commun. Math. Phys. **336**, 1085–1107 (2015).

[11] A. Buryak, *New Approaches to integrable hierarchies of topological type*, Russian Mathematical Surveys **72** (2017), no. 5, 841-887

[12] A. Buryak, P. Rossi, *Extended r-spin theory in all genera and the discrete KdV hierarchy*, Advances in Mathematics **386** (2021), Paper No. 107794.

[13] P. Candelas, X.C. de la Ossa, P.S. Green, L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B, **359** (1991), 21-74.

[14] N. Combe, Y.I. Manin, *F-manifolds and geometry of information*, Bulletin, London Mathematical Society, 52(5), 777-792.

[15] R.M. Coreless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, D.E. Knuth, *On the Lambert W Function*. Advances in Computational Mathematics, Vol. 5, (1996): 329-359.

[16] G. Cotti, B. Dubrovin, D. Guzzetti, *Local Moduli of Semisimple Frobenius Coalescent Structures*, SIGMA **16** (2020), 040.

[17] G. Darboux, *Leçons sur les systèmes ortogonaux et les cordonnées curvilignes*, Paris, 1897.

[18] L. David, C. Hertling, *Regular F-manifolds: initial conditions and Frobenius metrics*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. XVII (2017), 1121-1152.

[19] L. David, C. Hertling, *Meromorphic connections over F-manifolds*, in *Integrability, Quantization, and Geometry. I. Integrable systems*, Proceedings of Symposia in Pure Mathematics, Vol. 103, Editors S. Novikov, I. Krichever, O. Ogievetsky, S. Shlosman, Amer. Math. Soc., Providence, RI (2021), 171–216.

[20] L. David, I. A. B. Strachan, *Dubrovin's duality for F-manifolds with eventual identities*, Advances in Mathematics, Volume 226, Issue 5 (2011), 4031–4060.

[21] L. David, I. A. B. Strachan, *Symmetries of F-manifolds with eventual identities and special families of connections*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) Vol. XIII (2014), 641-674.

[22] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. Inst. Hautes ´ Etudes Sci. 36.1 (1969), pp. 75–109.

[23] A. Della Vedova, P. Lorenzoni, A. Savoldi, *Deformations of non-semisimple Poisson pencils of hydrodynamic type*, Nonlinearity **29** (2016), 2715–2754.

[24] R. Dijkgraaf, *A Geometrical Approach to Two-Dimensional Conformal Field Theory*, Ph.D. Thesis, Utrecht (1989).

[25] R. Dijkgraaf, H. Verlinde, E. Verlinde, *Topological strings in $d < 1$*, Nucl. Phys. B **352** (1991), 59–86.

[26] R. Dijkgraaf, E. Witten, *Mean field theory, topological field theory, and multimatrix models*, Nucl. Phys. B **342** (1990), 486–522.

[27] B. Dubrovin, *Integrable systems in topological field theory*, Nucl. Phys. B **379** (1992), 627–689.

[28] B. Dubrovin, *Geometry of 2D topological field theories*, in "Integrable systems and quantum groups" (Montecatini Terme, 1993), Lecture Notes in Math. **1620** (1994), 120–348.

[29] B. Dubrovin, *Flat Pencils of Metrics and Frobenius Manifolds*, Integrable Systems and Algebraic Geometry (Kobe/Kyoto), (1997), 47–72. World Sci. Publishing, River Edge (1998).

[30] B. Dubrovin, *Painlevé transcendents and two-dimensional topological field theory*, The Painlevé property, 287–412, CRM Ser. Math. Phys., Springer, New York (1999).

[31] B. Dubrovin, *Differential geometry of the space of orbits of a Coxeter group*, Surv. Diff. Geom. **4**, 181–211 (1999).

[32] B. Dubrovin, *On almost duality for Frobenius manifolds*, in: Geometry, topology, and mathematical physics (V.M. Buchstaber, I.M. Krichever, eds.). AMS Translations: Series 2, vol. 212. Providence, RI: AMS (2004), 75–132.

[33] B. Dubrovin, *Hamiltonian PDEs and Frobenius manifolds*, Russ. Math. Surv. **63** (2008), 999.

[34] B. Dubrovin, Y. Zhang, *Bi-Hamilatonian hierarchies in 2D topological field theory at one-loop approximation*, Comm. Math. Phys. **198** (1998), no. 2, 311-361.

[35] B. Dubrovin, Y. Zhang, *Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov–Witten invariants*, 187 pp. (2001).

[36] D. T. Egorov, *Collected papers on differential geometry*, Nauka, Moscow (1970) (in Russian).

[37] B. Eynard, N. Orantin, *Invariants of algebraic curves and topological expansion*, Commun. Number Theory Phys. **1** (2007), no. 2, 347–452.

[38] E. Ferapontov, P. Vergallo, *Hamiltonian systems of Jordan block type: delta-functional reductions of the kinetic equation for soliton gas*, arXiv:2212.01413 (2023).

[39] E. Ferapontov, L. Xue, *Quasilinear systems of Jordan block type and the mKP hierarchy*, J. Phys. A: Math. Theor. **53** (2020), 205202.

[40] A. Frölicher, A. Nijenhuis, *Theory of vector-valued differential forms*, Proc. Ned. Acad. Wetensch. Ser. A **59** (1956), 338–359.

[41] E. Getzler, *Intersection theory on $\overline{\mathcal{M}}_{1,4}$ and elliptic Gromov-Witten invariants*, J. Am. Math. Soc., **10** (1997), 973-998.

[42] A. Givental, *Elliptic Gromov-Witten invariants and the generalized mirror conjecture*, in Integrable Systems and Algebraic Geometry. River Edge: World Sci Publishing (1998), 107–155.

[43] A. Givental, *Semisimple Frobenius structures at higher genus*, Int. Math. Res. Not. **2001** (2001), no. 23, 1265–1286.

[44] A. Givental, *Gromov–Witten invariants and quantization of quadratic Hamiltonians*, Moscow Math. J. **1** (2001), no. 4, 551–568.

[45] A. Givental, *Symplectic geometry of Frobenius structures*, In: Frobenius Manifolds, 91-112, Aspects Mathematics, E36, Friedr. Vieweg, Wiesbaden (2004).

[46] A. Haantjes, *On X_{n-1}-forming sets of eigenvectors*, Indagationes Mathematicae 17 (1955), no. 2, 158–162.

[47] C. Hertling, *Frobenius manifolds and moduli spaces for singularities*, Cambridge Tracts in Mathematics, vol. 151, Cambridge University Press (2002).

[48] C. Hertling, Y. Manin, *Weak Frobenius manifolds*, Int. Math. Res. Notices (1999), no. 6, 277–286.

[49] M. Kato, T. Mano, J. Sekiguchi. *Flat structure on the space of isomonodromic deformations*. SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), paper no. 110.

[50] Y. Konishi, S. Minabe, Y. Shiraishi, *Almost duality for Saito structure and complex reflection groups*, J. Integrable Syst. **3** (2018), no. 1.

[51] H. Kawakami, T. Mano, *Regular flat structure and generalized Okubo system*, Commun. Math. Phys. 369 (2019), 403–431.

[52] J. Kock, *Frobenius algebras*, In: Frobenius Algebras and 2D Topological Quantum Field Theories. London Mathematical Society Student Texts. Cambridge: Cambridge University Press (2003), 78-137.

[53] Y. Kodama, B. G. Konopelchenko, *Confluence of hypergeometric functions and integrable hydrodynamic-type systems*, Theor Math Phys 188, No. 3, 429–455 (2016).

[54] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Commun. Math. Phys. 147.1 (1992).

[55] M. Kontsevich, Y. Manin, *Gromov–Witten classes, quantum cohomology, and enumerative geometry*, Communications in Mathematical Physics **164** (1994), 525–562.

[56] G. Lauricella, *Sulle funzioni ipergeometriche a più variabili*, Rendiconti del Circolo Mat. Palermo 7 (1893) 111-158.

[57] J. Lee, *Manifolds and Differential Geometry*, Graduate Studies in Mathematics. Providence, RI: American Mathematical Society.

[58] S. Liu, Y. Ruan, Y. Zhang, *BCFG Drinfeld–Sokolov hierarchies and FJRW-theory*, Invent. Math. **201** (2015), no. 2, 711–772.

[59] E. Looijenga, *Uniformization by Lauricella functions: an overview of the theory of Deligne-Mostow*, in Arithmetic and Geometry Around Hypergeometric Functions, Prog. Math. 260 Birkhäuser, Basel (2007) 207-244.

[60] P. Lorenzoni, *Flat bidifferential ideals and semihamiltonian PDEs*, J. Phys. A: Math. Gen. 39 (2006) 13701-13715.

[61] P. Lorenzoni, *Darboux-Egorov system, bi-flat F-manifolds and Painlevé VI*, International Mathematics Research Notices, Vol. 2014 (2014), no. 12, 3279–3302.

[62] P. Lorenzoni, F. Magri, *A cohomological construction of integrable hierarchies of hydrodynamic type*, Int Math Res Notices (2005) 2005 (34): 2087-2100.

[63] P. Lorenzoni, M. Pedroni, *Natural connections for semi-Hamiltonian systems: the case of the ϵ-system*, Lett. Math. Phys. **97** (2011), no. 1, 85–108.

[64] P. Lorenzoni, M. Pedroni, A. Raimondo, *F-manifolds and integrable systems of hydrodynamic type*, Archivum Mathematicum **47** (2011), 163-180.

[65] P. Lorenzoni, S. Perletti, *Regular non-semisimple Dubrovin-Frobenius manifolds*, J. Math. Phys. **63** 102301 (2022).

[66] P. Lorenzoni, S. Perletti, *Integrable hierarchies, Frölicher–Nijenhuis bicomplexes and Lauricella bi-flat F-manifolds*, Nonlinearity **36** 6925 (2023).

[67] A. Losev, Y. Manin, *Extended modular operad*, In: Frobenius manifolds, quantum cohomology and singularities (C. Hertling and M. Marcolli, Eds.), Aspects of Math., E36, 181–211 (2004).

[68] F. Magri, *Lenard chains for classical integrable systems*, Theoretical and Mathematical Physics, **137** (2003), no. 3, 1716–1722.

[69] Yu. Manin, *F-manifolds with flat structure and Dubrovin's duality*, Adv. Math. **198** (2005), no. 1, 5–26.

[70] K. Okamoto, *Studies on the Painlevé equations*. Annali di Matematica pura ed applicata 146, 337–381.

[71] R. Pandharipande, *Cohomological field theory calculations*, Proceedings of the International Congress of Mathematicians (ICM 2018). World Scientific, River Edge, NJ, 2019, pp. 869–898.

[72] R. Pandharipande, A. Pixton, D. Zvonkine, *Relations on $\overline{\mathcal{M}}_{g,n}$ via 3-spin structures*, J. Am. Math. Soc. **28** (2015), no. 1, 279–309.

[73] M.V. Pavlov, *Integrable hydrodynamic chains*, J. Math. Phys. **44** (2003), no. 9, 4134–4156.

[74] M.V. Pavlov, R.A. Sharipov, S.I. Svinolupov, *Invariant Integrability Criterion for the Equations of Hydrodynamical Type*, Funktsional. Anal. i Prilozhen 30 (1996), no. 1, 18–29.

[75] G. Ricci, *Dei sistemi di congruenze ortogonali in una varietà qualunque*, Memorie della Reale Accademia dei Lincei, Classe di Scienze (5), vol. 2 (1896).

[76] S. Romano, *4-dimensional Frobenius manifolds and Painlevé VI.* Mathematische Annalen 360.3 (2014): 715-751.

[77] C. Sabbah *Isomonodromic Deformations and Frobenius Manifolds. An Introduction.*, Universitext. Springer, Berlin (2002).

[78] K. Saito, *On a linear structure of the quotient variety by a finite reflexion group*, Preprint RIMS-288 (1979), vol. 29, pp. 535–579. Publications RIMS, Kyoto University (1993).

[79] K. Saito, T. Yano, J. Sekiguchi, *On a certain generator system of the ring of invariants of a finite reflection group*, Comm. Algebra **8**, 373–408 (1980).

[80] G.B. Segal, *The definition of conformal field theory*, In: Bleuler, K., Werner, M. (eds) Differential Geometrical Methods in Theoretical Physics. NATO ASI Series, vol 250. Springer, Dordrecht (1988).

[81] B. Sevennec, *Géométrie des systèmes hyperboliques de lois de conservation*, Mémoires de la Société Mathématique de France, **56**, 1–125 (1994).

[82] C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math. **188** no. 3 (2012), pp. 525–588.

[83] S.P. Tsarev, *The geometry of Hamiltonian systems of hydrodynamic type. The generalised hodograph transform*, USSR Izv. **37** (1991) 397–419.

[84] E. Witten, *On the structure of the topological phase of two dimensional gravity*, Nucl. Phys. B, **340** (1990), 281–332.

[85] E. Witten, *Two-dimensional gravity and intersection theory on moduli space*, Surv. Differ. Geom. 1.1 (1990), pp. 243–310.

[86] D. Zvonkine, *An introduction to moduli spaces of curves and their intersection theory*, Handbook of Teichmüller theory. Ed. by A. Papadopoulos. Vol. 3. Eur. Math. Soc., Zürich (2012), Chap. 11, pp. 667–716.

Appendix A

Four-dimensional regular non-semisimple Frobenius manifolds

This appendix integrates Chapter 3 by completing the description of the four-dimensional regular non-semisimple cases corresponding to a Jordan canonical form of the operator of multiplication by the Euler vector field having at least one Jordan block of size 2.

In the first place, we give the formulas expressing the flatness conditions. In the case of two blocks of size 2, the flatness conditions amount to the system of PDEs

$$\begin{aligned}
\partial_z^3 f = {} & \frac{1}{2z^3(\partial_w f)^2(w\partial_w f + z\partial_z f + (d-1)f - C_2)^2}(((2(d-1)\,f + z\,C_1 \\
& - 2\,C_2)\,w^2\,(\partial_w f)^2 - 2\,(-(d-1)\,f + C_2)\,(z\,\partial_z f + (d-1)\,f - C_2)\,w\,\partial_w f \\
& - z\,\partial_z f\,(z\,\partial_z f + (d-1)\,f - C_2)^2)\,w^2\,(\partial_w^2 f)^2 + 4\,((((2(d-1)\,f + z\,C_1 \\
& - 2\,C_2)\,w^2\,(\partial_w f)^2 - 2\,(-(d-1)\,f + C_2)\,(z\,\partial_z f + (d-1)\,f - C_2)\,w\,\partial_w f \\
& - z\,\partial_z f\,(z\,\partial_z f + (d-1)\,f - C_2)^2)\,z\,\partial_z\partial_w f)\tfrac{1}{2} \\
& + (z^2\,(z\,\partial_z f + (d-1)\,f - C_2)\,(w\,\partial_w f + z\,\partial_z f + (d-1)\,f - C_2)\,\partial_z^2 f\,\tfrac{1}{2} \\
& + w\,(-w^2\,(d-1)\,(\partial_w f)^2 + 2\,w\,(-z\,(d-1)\,\partial_z f - ((-(d-1)\,f \\
& + C_2)\,(2-d))\tfrac{1}{2})\,\partial_w f + ((-z\,(d-2)\,\partial_z f + 2(d-1)\,f - 2\,C_2)\,(z\,\partial_z f \\
& + (d-1)\,f - C_2))\tfrac{1}{2})\,\partial_w f)\,\partial_w f)\,w\,\partial_w^2 f + z^3\,(w^2\,(C_1 - 2\,\partial_z f)\,(\partial_w f)^2 \\
& - 2\,\partial_z f\,(z\,\partial_z f + (d-1)\,f - C_2)\,w\,\partial_w f - \partial_z f\,(z\,\partial_z f + (d-1)\,f \\
& - C_2)^2)\,(\partial_z\partial_w f)^2 + 4\,z\,(z^2\,(w\,\partial_w f + z\,\partial_z f\,\tfrac{1}{2} + \tfrac{d-1}{2}\,f \\
& - \tfrac{C_2}{2})\,(w\,\partial_w f + z\,\partial_z f + (d-1)\,f - C_2)\,\partial_z^2 f + (w^2\,(1-d)\,(\partial_w f)^2 \\
& + (z\,\partial_z f + (d-1)\,f - C_2)\,w\,(2-d)\,\partial_w f - ((z\,\partial_z f + (d-1)\,f \\
& - C_2)\,(-z\,(d+2)\,\partial_z f - 2(d-1)\,f + 2\,C_2))\tfrac{1}{2})\,w\,\partial_w f)\,\partial_w f\,\partial_z\partial_w f \\
& - 4\,(-z^4\,(w\,\partial_w f + z\,\partial_z f + (d-1)\,f - C_2)\,(\partial_z^2 f)^2\,\tfrac{1}{2} - z^2\,(-w\,(d+2)\,\partial_w f
\end{aligned} \tag{A.1}$$

$$
\begin{aligned}
&+ 2\,z\,(d-1)\,\partial_z f - 2(d-1)\,f + 2\,C_2)\,(w\,\partial_w f + z\,\partial_z f + (d-1)\,f \\
&- C_2)\,\partial_z^2 f\,\tfrac{1}{2} - d\,(-w^3\,(d-1)\,(\partial_w f)^3 + 2\,(-z\,(d-\tfrac{3}{4})\,\partial_z f \\
&- 3\,(-(d-1)\,f + C_2)\,(\tfrac{4}{3}-d)\tfrac{1}{4})\,w^2\,(\partial_w f)^2 - (-(d-1)\,f \\
&+ C_2)\,(-z\,(d-1)\,\partial_z f - ((-(d-1)\,f + C_2)\,(2-d))\tfrac{1}{2})\,w\,\partial_w f \\
&- (3\,z\,\partial_z f\,(z\,(\tfrac{2}{3}-d)\,\partial_z f - ((-(d-1)\,f + C_2)\,(2-d))\tfrac{1}{3})\,(z\,\partial_z f \\
&+ (d-1)\,f - C_2))\tfrac{1}{4}))\,(\partial_w f)^2)
\end{aligned}
$$

$$
\partial_z^2\partial_w f = \frac{1}{2z^2\partial_w f(w\partial_w f + z\partial_z f + (d-1)f - C_2)^2}(2\,z^2\,(w^2\,(\partial_w f)^2 - ((-4\,z\,\partial_z f \tag{A.2}
$$

$$
\begin{aligned}
&- 3(d-1)\,f + z\,C_1 + 3\,C_2)\,w\,\partial_w f)\tfrac{1}{2} + (z\,\partial_z f + (d-1)\,f \\
&- C_2)^2\tfrac{1}{2})\,(\partial_z\partial_w f)^2 - 2\,z\,(w\,(w\,((d-1)\,f + z\,C_1 - C_2)\,\partial_w f + (z\,\partial_z f \\
&+ (d-1)\,f - C_2)^2)\,\partial_w^2 f + (-z^2\,(w\,\partial_w f + z\,\partial_z f + (d-1)\,f - C_2)\,\partial_z^2 f \\
&- w^2\,(d-2)\,(\partial_w f)^2 + 3\,w\,(-z\,(d-\tfrac{4}{3})\,\partial_z f - ((-(d-1)\,f \\
&+ C_2)\,(4-d))\tfrac{1}{3})\,\partial_w f + (-z\,(d-2)\,\partial_z f + 2(d-1)\,f - 2\,C_2)\,(z\,\partial_z f \\
&+ (d-1)\,f - C_2))\,\partial_w f)\,\partial_z\partial_w f - w^2\,(w\,((d-1)\,f + z\,C_1 - C_2)\,\partial_w f \\
&+ (z\,\partial_z f + (d-1)\,f - C_2)^2)\,(\partial_w^2 f)^2 - 2\,w\,(-z^2\,(w\,\partial_w f + z\,\partial_z f \\
&+ (d-1)\,f - C_2)\,\partial_z^2 f - w^2\,(d-1)\,(\partial_w f)^2 + 3\,(z\,(\tfrac{2}{3}-d)\,\partial_z f \\
&- ((-(d-1)\,f + C_2)\,(2-d))\tfrac{1}{3})\,w\,\partial_w f + (z\,\partial_z f + (d-1)\,f \\
&- C_2)\,(-z\,(d-1)\,\partial_z f + (d-1)\,f - C_2))\,\partial_w f\,\partial_w^2 f + 2\,(-z^2\,(w\,\partial_w f \\
&+ z\,\partial_z f + (d-1)\,f - C_2)\,\partial_z^2 f - w^2\,(d-1)\,(\partial_w f)^2 \\
&+ 3\,(z\,(\tfrac{2}{3}-d)\,\partial_z f - ((-(d-1)\,f + C_2)\,(\tfrac{4}{3}-d))\tfrac{1}{2})\,w\,\partial_w f \\
&+ 3\,(z\,(\tfrac{2}{3}-d)\,\partial_z f - ((-(d-1)\,f + C_2)\,(2-d))\tfrac{1}{3})\,(z\,\partial_z f \\
&+ (d-1)\,f - C_2)\tfrac{1}{2})\,(-d)\,(\partial_w f)^2)
\end{aligned}
$$

$$
\partial_z\partial_w^2 f = \frac{1}{2\partial_w f(w\partial_w f + z\partial_z f + (d-1)f - C_2)^2}(2\,z\,(w\,\partial_w f + z\,\partial_z f\,\tfrac{1}{2} \tag{A.3}
$$

$$
\begin{aligned}
&+ (d-1)\,f + z\,\tfrac{C_1}{2} - C_2)\,\partial_w f\,(\partial_z\partial_w f)^2 + ((4\,w^2\,(\partial_w f)^2 \\
&+ 2\,w\,(2\,z\,\partial_z f + 3(d-1)\,f + z\,C_1 - 3\,C_2)\,\partial_w f + 2\,(z\,\partial_z f \\
&+ (d-1)\,f - C_2)^2)\,\partial_w^2 f - 2\,(w\,\partial_w f + (d-1)\,f \\
&- C_2)\,(\partial_w f)^2\,(-d))\,\partial_z\partial_w f + \partial_w f\,(w^2\,(-\partial_z f + C_1)\,(\partial_w^2 f)^2 \\
&- 2\,w\,\partial_w f\,\partial_z f\,d\,\partial_w^2 f - (\partial_w f)^2\,\partial_z f\,d^2))
\end{aligned}
$$

$$
\partial_w^3 f = \frac{1}{2\partial_w f(w\partial_w f + z\partial_z f + (d-1)f - C_2)^2\,w}(3\,(w^2\,(\partial_w f)^2 \tag{A.4}
$$

$$
\begin{aligned}
&- ((-6\,z\,\partial_z f - 5(d-1)\,f + z\,C_1 + 5\,C_2)\,w\,\partial_w f)\tfrac{1}{3} \\
&+ (2\,(z\,\partial_z f + (d-1)\,f - C_2)^2)\tfrac{1}{3})\,w\,(\partial_w^2 f)^2 - 2\,(w\,z^2\,(-\partial_z f
\end{aligned}
$$

$$
\begin{aligned}
&+ C_1)\,\partial_z\partial_w f - w^2\,(d-1)\,(\partial_w f)^2 + 2\,w\,(-z\,(d-1)\,\partial_z f - ((-(d-1)\,f \\
&+ C_2)\,(2-d))\tfrac{1}{2})\,\partial_w f + (z\,\partial_z f + (d-1)\,f - C_2)^2)\,\partial_w f\,\partial_w^2 f \\
&- (z^2\,(w\,\partial_w f + (d-1)\,f + z\,C_1 - C_2)\,(\partial_z\partial_w f)^2 \\
&- 2\,\partial_z f\,\partial_w f\,z^2\,d\,\partial_z\partial_w f - (w\,\partial_w f + 2\,z\,\partial_z f + (d-1)\,f \\
&- C_2)\,d^2\,(\partial_w f)^2)\,\partial_w f).
\end{aligned}
$$

for the third derivatives of the funciton f of the variables

$$
z = \frac{u^3 - u^1}{u^2}, \qquad w = \frac{u^4}{u^2}
$$

realizing

$$
\begin{aligned}
F_1(z,w) &= -\partial_z f(z,w) + C_1 \\
F_2(z,w) &= -z\,\partial_z f(z,w) - w\,\partial_w f(z,w) - (d-1)\,f(z,w) + C_2 \\
F_3(z,w) &= \partial_z f(z,w) \\
F_4(z,w) &= \partial_w f(z,w)
\end{aligned}
$$

for

$$
\eta = (u^2)^{-d} \begin{bmatrix} F_1 & F_2 & 0 & 0 \\ F_2 & 0 & 0 & 0 \\ 0 & 0 & F_3 & F_4 \\ 0 & 0 & F_4 & 0 \end{bmatrix}
$$

where C_1, C_2 are constants. In the case of three blocks of sizes 2, 1 and 1 respectively, the flatness conditions amount to the following system of PDEs

$$
\begin{aligned}
\partial_z^3 f = {} & \frac{1}{2(w\partial_w f + z\partial_z f + (d-1)f - C_2)^2\,(w-z)\partial_w f\,z^2\partial_z f}(-z\partial_z f(2z^2(w \qquad \text{(A.5)} \\
&- z)\,(\partial_z f)^2 + (w-z)\,z\,(w\,\partial_w f + 4(d-1)\,f - 4\,C_2)\,\partial_z f \\
&+ (-(2w-z)\,(1-d)\,f + (C_1\,z - 2\,C_2)\,w + z\,C_2)\,w\,\partial_w f \\
&+ 2\,(w-z)\,(-(d-1)\,f + C_2)^2)\,w\,(\partial_z\partial_w f)^2 - (-3\,((4\,z\,(w-z)\,\partial_z f)\tfrac{1}{3} \\
&+ w\,(w-z)\,\partial_w f - (w - \tfrac{4}{3}z)\,(1-d)\,f + (-\tfrac{C_1}{3}\,z - C_2)\,w \\
&+ \tfrac{4}{3}\,z\,C_2)\,\partial_w f\,z^2\,\partial_z^2 f + (z^2\,(w-z)\,(\partial_z f)^2 - 2\,(w-z)\,z\,(-(d-1)\,f \\
&+ C_2)\,\partial_z f + ((d-1)\,f + C_1\,z - C_2)\,w^2\,\partial_w f \\
&+ (w-z)\,(-(d-1)\,f + C_2)^2)\,w\,\partial_w^2 f + \partial_w f\,(5\,((\tfrac{2}{5} - d)\,w \\
&- (-4\,z\,d)\tfrac{1}{5})\,z^2\,(\partial_z f)^2 + 6\,z\,((w\,(\tfrac{2}{3} - d) \\
&- (-2\,z\,d)\tfrac{1}{3})\,w\,\partial_w f - 2\,(-w\,(d-1) + z\,d)\,(-(d-1)\,f \\
&+ C_2)\tfrac{1}{3})\,\partial_z f + (-w^2\,(d-2)\,(\partial_w f)^2 - (-(d-1)\,f + C_2)\,(-w\,(d-4)
\end{aligned}
$$

$$+ z\,d)\,\partial_w f + 2\,(-(d-1)\,f + C_2)^2)\,w))\,\partial_z f\,w\,\partial_z\partial_w f$$
$$- \partial_w f\,(-(w\,\partial_w f + z\,\partial_z f + (d-1)\,f - C_2)\,(w-z)\,z^2\,(3\,z\,\partial_z f$$
$$+ w\,\partial_w f + (d-1)\,f - C_2)\,(\partial_z^2 f)^2 + z\,(((z-w)\,\partial_z f$$
$$+ (d-1)\,f + w\,C_1 - C_2)\,z\,w^2\,\partial_w^2 f - 4\,z^2\,(d-1)\,(w-z)\,(\partial_z f)^2$$
$$+ 5\,((w\,(\tfrac{8}{5} - d) - (4\,z\,(2-d))\tfrac{1}{5})\,w\,\partial_w f - 4\,(w-z)\,(2$$
$$- d)\,(-(d-1)\,f + C_2)\tfrac{1}{5})\,z\,\partial_z f + 4\,(w - z\,(d+4)\tfrac{1}{4})\,w^2\,(\partial_w f)^2$$
$$- 8\,(w-z)\,(-(d-1)\,f + C_2)\,w\,\partial_w f + 4\,(w-z)\,(-(d-1)\,f$$
$$+ C_2)^2)\,\partial_z f\,\partial_z^2 f + (((2\,w\,z - z^2)\,\partial_z f + w^2\,\partial_w f - (w-z)\,(-(d-1)\,f$$
$$+ C_2))\,w^2\,\partial_w^2 f - 3\,(w-z)\,z^2\,(\tfrac{2}{3} - d)\,(\partial_z f)^2 - 6\,z\,((w\,(\tfrac{2}{3} - d)$$
$$- (2\,z\,(1-d))\tfrac{1}{3})\,w\,\partial_w f - 2\,(w-z)\,(1-d)\,(-(d-1)\,f + C_2)\tfrac{1}{3})\,\partial_z f$$
$$- 2\,(-z + w\,(1-d))\,w^2\,(\partial_w f)^2 + 3\,(-(d-1)\,f + C_2)\,(w\,(\tfrac{4}{3} - d)$$
$$- (2\,z\,(2-d))\tfrac{1}{3})\,w\,\partial_w f - (w-z)\,(2-d)\,(-(d-1)\,f$$
$$+ C_2)^2)\,(-d)\,(\partial_z f)^2))$$

$$\partial_z^2\partial_w f = \frac{1}{2(w\partial_w f + z\partial_z f + (d-1)f - C_2)^2(w-z)\partial_w f\,\partial_z f}(2\,\partial_z f\,(w^2\,(w \tag{A.6}$$
$$- z)\,(\partial_w f)^2 + ((2\,z\,(w-z)\,\partial_z f - (3\,w - 2\,z)\,(1-d)\,f$$
$$+ (C_1\,z - 3\,C_2)\,w + 2\,z\,C_2)\,w\,\partial_w f)\tfrac{1}{2} + ((w-z)\,(z\,\partial_z f$$
$$+ (d-1)\,f - C_2)^2)\tfrac{1}{2})\,(\partial_z\partial_w f)^2 + \partial_w f\,((w^2\,(w-z)\,(\partial_w f)^2$$
$$+ 4\,(w-z)\,(z\,\partial_z f + \tfrac{d-1}{2}\,f - \tfrac{C_2}{2})\,w\,\partial_w f$$
$$+ 2\,z^2\,(w-z)\,(\partial_z f)^2 + z\,(-(4\,w - 3\,z)\,(1-d)\,f + (C_1\,z$$
$$- 4\,C_2)\,w + 3\,z\,C_2)\,\partial_z f + (w-z)\,(-(d-1)\,f + C_2)^2)\,\partial_z^2 f$$
$$+ \partial_z f\,(((z-w)\,\partial_z f + (d-1)\,f + w\,C_1 - C_2)\,w^2\,\partial_w^2 f$$
$$- w^2\,(d-2)\,(\partial_w f)^2 + 2\,(-z\,(d-2)\,\partial_z f + 2(d-1)\,f$$
$$- 2\,C_2)\,w\,\partial_w f - z^2\,(d-2)\,(\partial_z f)^2 + (-w\,d + z\,(d-4))\,((1$$
$$- d)\,f + C_2)\,\partial_z f + 2\,(-(d-1)\,f + C_2)^2))\,\partial_z\partial_w f$$
$$+ ((((w-z)\,\partial_w f + (d-1)\,f + C_1\,z - C_2)\,w^2\,\partial_w^2 f - (w\,(w - 2\,z)\,\partial_w f$$
$$- z^2\,\partial_z f - (w-z)\,(-(d-1)\,f + C_2))\,(-d)\,\partial_w f)\,\partial_z^2 f$$
$$+ (w^2\,(\partial_w f + \partial_z f)\,\partial_w^2 f - (2\,w\,\partial_w f + (w+z)\,\partial_z f + (d-1)\,f$$
$$- C_2)\,(-d)\,\partial_w f)\,\partial_z f\,(-d))\,\partial_z f\,\partial_w f)$$

$$\partial_z\partial_w^2 f = \frac{1}{2\,(w\partial_w f + z\partial_z f + (d-1)f - C_2)^2(w-z)\partial_w f\,\partial_z f}((w^2\,(w \tag{A.7}$$
$$- z)\,(\partial_w f)^2 + 2\,(w-z)\,(z\,\partial_z f + (d-1)\,f - C_2)\,w\,\partial_w f$$
$$+ 2\,z^2\,(w-z)\,(\partial_z f)^2 - z\,(2\,(w - (3\,z)\tfrac{1}{2})\,(1-d)\,f + (C_1\,w - 3\,C_2)\,z$$
$$+ 2\,w\,C_2)\,\partial_z f + (w-z)\,(-(d-1)\,f + C_2)^2)\,\partial_w f\,(\partial_z\partial_w f)^2$$

$$
\begin{aligned}
&+ 2\,((w^2\,(w-z)\,(\partial_w f)^2 - ((-4\,z\,(w-z)\,\partial_z f + (3\,w-4\,z)\,(1-d)\,f \\
&+ (C_1\,w - 4\,C_2)\,z + 3\,w\,C_2)\,w\,\partial_w f)\tfrac{1}{2} + ((w-z)\,(z\,\partial_z f + (d-1)\,f \\
&- C_2)^2)\tfrac{1}{2})\,\partial_w^2 f - \partial_w f\,(((w-z)\,\partial_w f + (d-1)\,f + C_1\,z - C_2)\,z^2\,\partial_z^2 f \\
&+ w^2\,(2-d)\,(\partial_w f)^2 + (2\,w\,z\,(2-d)\,\partial_z f - (-(d-1)\,f + C_2)\,(d\,z \\
&- w\,(d-4)))\,\partial_w f + z^2\,(2-d)\,(\partial_z f)^2 - 4\,(-(d-1)\,f + C_2)\,z\,\partial_z f \\
&+ 2\,(-(d-1)\,f + C_2)^2)\tfrac{1}{2})\,\partial_z f\,\partial_z\partial_w f - \partial_w f\,((z^2\,((z-w)\,\partial_z f \\
&+ (d-1)\,f + w\,C_1 - C_2)\,\partial_z^2 f + ((2\,w\,z - z^2)\,\partial_z f + w^2\,\partial_w f \\
&- (w-z)\,(-(d-1)\,f + C_2))\,(-d)\,\partial_z f)\,\partial_w^2 f - (-z^2\,(\partial_w f + \partial_z f)\,\partial_z^2 f \\
&+ ((w+z)\,\partial_w f + 2\,z\,\partial_z f + (d-1)\,f - C_2)\,(-d)\,\partial_z f)\,\partial_w f\,(-d))\,\partial_z f)
\end{aligned}
$$

$$
\begin{aligned}
\partial_w^3 f = \frac{1}{2(w\partial_w f + z\partial_z f + (d-1)f - C_2)^2(w-z)\partial_w f\,\partial_z f\,w^2}&(-2zw\partial_w f(w^2(w \qquad \text{(A.8)}\\
&- z)\,(\partial_w f)^2 + ((w-z)\,(z\,\partial_z f + 4(d-1)\,f \\
&- 4\,C_2)\,w\,\partial_w f)\tfrac{1}{2} - z\,(-(d-1)\,(w-2\,z)\,f + (C_1\,w - 2\,C_2)\,z \\
&+ w\,C_2)\,\partial_z f\,\tfrac{1}{2} + (w-z)\,(-(d-1)\,f + C_2)^2)\,(\partial_z\partial_w f)^2 \\
&+ 4\,\partial_w f\,z\,((w\,(w-z)\,\partial_w f + (3\,z\,(w-z)\,\partial_z f)\tfrac{1}{4} - (w - \tfrac{3}{4}\,z)\,(1-d)\,f \\
&+ (w\,\tfrac{C_1}{4} + \tfrac{3}{4}C_2)\,z - w\,C_2)\,\partial_z f\,w^2\,\partial_w^2 f - ((w^2\,(w-z)\,(\partial_w f)^2 \\
&- 2\,(w-z)\,(-(d-1)\,f + C_2)\,w\,\partial_w f - ((d-1)\,f + w\,C_1 - C_2)\,z^2\,\partial_z f \\
&+ (w-z)\,(-(d-1)\,f + C_2)^2)\,z\,\partial_z^2 f)\tfrac{1}{4} - ((\tfrac{5\,d-2}{4}\,z \\
&- w\,d)\,w^2\,(\partial_w f)^2 + (z\,(\tfrac{3\,d-2}{2}\,z - w\,d)\,\partial_z f - ((d-1)\,z \\
&- w\,d)\,(-(d-1)\,f + C_2))\,w\,\partial_w f - ((-z^2\,(d-2)\,(\partial_z f)^2 \\
&+ ((d-4)\,z - w\,d)\,(-(d-1)\,f + C_2)\,\partial_z f + 2\,(-(d-1)\,f \\
&+ C_2)^2)\,z)\tfrac{1}{4})\,\partial_z f)\,\partial_z\partial_w f + 3\,((w\,\partial_w f + z\,\partial_z f + (d-1)\,f \\
&- C_2)\,(w-z)\,(z\,\partial_z f\,\tfrac{1}{3} + w\,\partial_w f + \tfrac{d-1}{3}\,f - \tfrac{C_2}{3})\,w^2\,(\partial_w^2 f)^2 \\
&- (4\,\partial_w f\,(-(((w-z)\,\partial_w f + (d-1)\,f + C_1\,z - C_2)\,z^2\,w\,\partial_z^2 f)\tfrac{1}{4} \\
&- w^2\,(d-1)\,(w-z)\,(\partial_w f)^2 + (z\,(\tfrac{5\,d-8}{4}\,z - w\,(d-2))\,\partial_z f \\
&+ (w-z)\,(d-2)\,(-(d-1)\,f + C_2))\,w\,\partial_w f - ((4\,z \\
&- w\,(d-4))\,z^2\,(\partial_z f)^2)\tfrac{1}{4} - 2\,(w-z)\,z\,(-(d-1)\,f + C_2)\,\partial_z f \\
&+ (w-z)\,(-(d-1)\,f + C_2)^2)\,w\,\partial_w^2 f)\tfrac{1}{3} - (\partial_w f)^2\,d\,(-z^2\,(w\,(w \\
&- 2\,z)\,\partial_w f - z^2\,\partial_z f - (w-z)\,(-(d-1)\,f + C_2))\,\partial_z^2 f\,\tfrac{1}{3} \\
&- (d - \tfrac{2}{3})\,(w-z)\,w^2\,(\partial_w f)^2 + 4\,((\tfrac{3\,d-2}{2}\,z - w\,(d-1))\,z\,\partial_z f \\
&+ (w-z)\,(d-1)\,(-(d-1)\,f + C_2))\,w\,\partial_w f\,\tfrac{1}{3} + (2\,z^2\,((d-1)\,z \\
&+ w)\,(\partial_z f)^2)\tfrac{1}{3} - 2\,(\tfrac{3\,d-4}{2}\,z - w\,(d-2))\,(-(d-1)\,f + C_2)\,z\,\partial_z f\,\tfrac{1}{3}
\end{aligned}
$$

$$+\left((w-z)\,(2-d)\,(-(d-1)\,f+C_2)^2)\tfrac{1}{3})\right)\partial_z f).$$

for the third derivatives of the function f of the variables

$$z=\frac{u^3-u^1}{u^2},\qquad w=\frac{u^4-u^1}{u^2}$$

realizing

$$\begin{aligned}
F_1(z,w)&=-\partial_z f(z,w)-\partial_w f(z,w)+C_1\\
F_2(z,w)&=-z\,\partial_z f(z,w)-w\,\partial_w f(z,w)-(d-1)\,f(z,w)+C_2\\
F_3(z,w)&=\partial_z f(z,w)\\
F_4(z,w)&=\partial_w f(z,w)
\end{aligned}$$

for

$$\eta=(u^2)^{-d}\begin{bmatrix}F_1 & F_2 & 0 & 0\\ F_2 & 0 & 0 & 0\\ 0 & 0 & F_3 & 0\\ 0 & 0 & 0 & F_4\end{bmatrix}$$

where C_1, C_2 are constants.

As seen in Chapter 3, when $d=0$ a solution to the system of PDEs amounting to the flatness conditions is provided by (3.64)

$$f(z,w)=a\,z+b\,w+c$$

for some constants a, b and c and for this choice of f the Frobenius metric turns out to be constant in canonical coordinates. In the following example we provide such a metric and the Frobenius potential in the cases when $L=E\circ$ has two Jordan blocks of size 2 and three Jordan blocks of sizes 2, 1 and 1.

Example A.1 *Let the function $f(z,w)$ be of the form* (3.64)*, $d=0$. When $L=E\circ$ has two Jordan blocks of size 2 the metric is given by*

$$\eta=\begin{bmatrix}C_1-a & C_2+c & 0 & 0\\ C_2+c & 0 & 0 & 0\\ 0 & 0 & a & b\\ 0 & 0 & b & 0\end{bmatrix}$$

and up to second-order polynomial terms the Frobenius potential is

$$F(u^1,u^2,u^3,u^4)=\frac{C_1-a}{6}\,(u^1)^3+\frac{C_2+c}{2}\,(u^1)^2\,u^2+\frac{a}{6}\,(u^3)^3+\frac{b}{2}\,(u^3)^2\,u^4.$$

When $L = E\circ$ has three Jordan blocks of sizes 2, 1 and 1 the metric is given by

$$\eta = \begin{bmatrix} C_1 - a - b & C_2 + c & 0 & 0 \\ C_2 + c & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{bmatrix}$$

and up to second-order polynomial terms the Frobenius potential is

$$F(u^1, u^2, u^3, u^4) = \frac{C_1 - a - b}{6} (u^1)^3 + \frac{C_2 + c}{2} (u^1)^2 u^2 + \frac{a}{6} (u^3)^3 + \frac{b}{6} (u^4)^3.$$

We conclude this appendix by providing one last example of non-trivial solution for the case where $L = E\circ$ has two Jordan blocks of size 2 and for the one where $L = E\circ$ has three Jordan blocks of sizes 2, 1 and 1.

Example A.2 *Let $L = E\circ$ have two Jordan blocks of size 2. When looking for a function f of the form*

$$f(z, w) = g(z) + h(w)$$

for some functions $g(z)$ and $h(w)$, the system (A.1)–(A.4) *yields*

$$\begin{aligned} h(w) &= a_1 + a_2\, w^{1-d} \\ g(z) &= a_4\, z^{d-a_3} + a_5\, z^{1-d} - a_1 + \frac{C_2}{d-1} \end{aligned}$$

when $d \neq 1$ and

$$\begin{aligned} h(w) &= a_1 + \frac{a_2}{w} \\ g(z) &= a_4\, z^{a_3} + (C_2 - a_2)\, \ln z + a_5 \end{aligned}$$

when $d = 1$, for some constants a_1, a_2, a_3, a_4, a_5. For instance, when $d = 2$ in the flat coordinates

$$\begin{aligned} x^1(u^1, u^2, u^3, u^4) &= -\frac{1}{u^2} - \frac{a_5}{2\, a_4\, (u^3 - u^1)} \\ x^2(u^1, u^2, u^3, u^4) &= -\frac{1}{u^4} - \frac{a_5}{2\, a_2\, (u^3 - u^1)} \\ x^3(u^1, u^2, u^3, u^4) &= u^1 \\ x^4(u^1, u^2, u^3, u^4) &= u^3 \end{aligned}$$

the metric becomes

$$\widetilde{\eta} = \begin{bmatrix} 0 & 0 & -a_4 & 0 \\ 0 & 0 & 0 & -a_2 \\ -a_4 & 0 & 0 & 0 \\ 0 & -a_2 & 0 & 0 \end{bmatrix}$$

and up to second-order polynomial terms the Frobenius potential is

$$F(x^1,x^2,x^3,x^4) = \frac{a_5}{2}\,(x^3-x^4)\,\ln(x^3-x^4) - \frac{a_4}{2}\,x^1\,(x^3)^2 - \frac{a_2}{2}\,x^2\,(x^4)^2.$$

In flat coordinates the unit and the Euler vector fields are respectively written as

$$e = \tilde{\partial}_3 + \tilde{\partial}_4$$

and

$$E = -x^1\,\tilde{\partial}_1 - x^2\,\tilde{\partial}_2 + x^3\,\tilde{\partial}_3 + x^4\,\tilde{\partial}_4.$$

Example A.3 *Let $L = E\circ$ have three Jordan blocks of sizes 2, 1 and 1. When looking for a function f of the form*

$$f(z,w) = g(z) + h(w)$$

for some functions $g(z)$ and $h(w)$, the system (A.5)–(A.8) *yields*

$$g(z) = a_1 + a_2\,z^{1-d}$$
$$h(w) = a_3 + a_4\,w^{1-d}$$

when $d \neq 1$ and

$$g(z) = a_1 + \ln z$$
$$h(w) = a_3 + \ln w$$

when $d = 1$, for some constants a_1, a_2, a_3, a_4. For instance, when $d = 2$ in the flat coordinates

$$x^1(u^1,u^2,u^3,u^4) = -\frac{1}{u^2} + \frac{a_2}{(C_2-a_1-a_3)\,(u^3-u^1)} + \frac{a_4}{(C_2-a_1-a_3)\,(u^4-u^1)}$$
$$x^2(u^1,u^2,u^3,u^4) = -\ln(u^3-u^1)$$
$$x^3(u^1,u^2,u^3,u^4) = -\ln(u^4-u^1)$$
$$x^4(u^1,u^2,u^3,u^4) = u^1$$

the metric becomes

$$\tilde{\eta} = \begin{bmatrix} 0 & 0 & 0 & C_2-a_1-a_3 \\ 0 & -a_2 & 0 & 0 \\ 0 & 0 & -a_4 & 0 \\ C_2-a_1-a_3 & 0 & 0 & 0 \end{bmatrix}$$

and up to second-order polynomial terms the Frobenius potential is

$$F(x^1,x^2,x^3,x^4) = -\,a_2\,e^{-x^2} - a_4\,e^{-x^3} + \frac{C_2-a_1-a_3}{2}\,x^1\,(x^4)^2$$

$$- \frac{a_2}{2} (x^2)^2 x^4 - \frac{a_4}{2} (x^3)^2 x^4.$$

In flat coordinates the unit and the Euler vector fields are respectively written as

$$e = \tilde{\partial}_4$$

and

$$E = -x^1 \tilde{\partial}_1 - \tilde{\partial}_2 - \tilde{\partial}_3 + x^4 \tilde{\partial}_4.$$

Appendix B

Technical lemmas about regular Lauricella bi-flat F-manifolds

This appendix integrates Chapter 4 by proving technical lemmas which are crucial in the construction of regular Lauricella bi-flat F-manifolds.

Proof of Lemma 4.21

In the wake of the previous result, we consider the following significative cases:

1. α, β, γ are pairwise distinct
2. $\alpha = \gamma \neq \beta$
3. $\alpha = \beta \neq \gamma$
4. $\alpha = \beta = \gamma$.

Case 1: $\alpha \neq \beta \neq \gamma \neq \alpha$**.** Since in this case all the quantities $\Gamma^{k(\gamma)}_{i(\alpha)j(\beta)}$ vanish by (4.61), (4.72) holds trivially for each choice of δ.
Case 2: $\alpha = \gamma \neq \beta$**.** We are going to prove that

$$\frac{\partial\Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}}{\partial u^{l(\delta)}} = \frac{\partial\Gamma^{(k-1)(\alpha)}_{i(\alpha)j(\beta)}}{\partial u^{(l-1)(\delta)}} \tag{B.1}$$

for all $k \in \{2, \dots, m_\alpha\}$, $l \in \{3, \dots, m_\delta\}$ and $\delta \in \{1, \dots, r\}$. This holds trivially for each $\delta \neq \alpha$, as $\Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}$ does not depend on $u^{l(\delta)}$ and $\Gamma^{(k-1)(\alpha)}_{i(\alpha)j(\beta)}$ does not depend on $u^{(l-1)(\delta)}$ for any $l \geq 3$. Moreover, by (4.62) and (4.63), both $\Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}$ and $\Gamma^{(k-1)(\alpha)}_{i(\alpha)j(\beta)}$ vanish if $j \geq 2$ or $k < i$. Therefore we are left to show that

$$\frac{\partial\Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} = \frac{\partial\Gamma^{(k-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{(l-1)(\alpha)}} \tag{B.2}$$

for $k \geq i$. We are going to prove (B.2) by induction over k. If $k = i$ we get

$$\frac{\partial \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} = \frac{\partial \Gamma^{i(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} \overset{(4.63)}{=} \frac{\partial \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} = 0$$

(as $\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}$ does not depend on $u^{l(\alpha)}$ for any $l \geq 3$) and

$$\frac{\partial \Gamma^{(k-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{(l-1)(\alpha)}} = \frac{\partial \Gamma^{(i-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{(l-1)(\alpha)}} = 0$$

(as $\Gamma^{(i-1)(\alpha)}_{i(\alpha)1(\beta)} \overset{(4.63)}{=} 0$), thus (B.2) is verified for $k = i$. Given an integer $h \geq 1$, let us suppose that (B.2) holds whenever $k \leq i + h - 1$. We now want to prove that it holds for $k = i + h$ as well, that is

$$\frac{\partial \Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} = \frac{\partial \Gamma^{(i+h-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{(l-1)(\alpha)}}. \tag{B.3}$$

The left-hand side term reads

$$\begin{aligned}
\frac{\partial \Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} &\overset{(4.63)}{=} \frac{\partial}{\partial u^{l(\alpha)}}\left(-\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h+1} \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)}\, u^{s(\alpha)} \right) \\
&\overset{l\geq 2}{=} -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h+1} \frac{\partial}{\partial u^{l(\alpha)}} \left(\Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)}\, u^{s(\alpha)} \right) \\
&= -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h+1} \frac{\partial \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}}\, u^{s(\alpha)} \\
&\quad - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\, \Gamma^{(i+h-l+1)(\alpha)}_{i(\alpha)1(\beta)} \\
&= -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \frac{\partial \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}}\, u^{s(\alpha)} \\
&\quad - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\, \frac{\partial \Gamma^{i(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}}\, u^{(h+1)(\alpha)} \\
&\quad - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\, \Gamma^{(i+h-l+1)(\alpha)}_{i(\alpha)1(\beta)}
\end{aligned}$$

where

$$\frac{\partial \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} = \frac{\partial \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{(l-1)(\alpha)}}$$

by the inductive hypothesis for each $s \geq 2$ and

$$\frac{\partial \Gamma^{i(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} \overset{(4.63)}{=} \frac{\partial \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} = 0$$

for every $l \geq 3$. Therefore the left-hand side of (B.3) is

$$\frac{\partial \Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{l(\alpha)}} = -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \frac{\partial \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{(l-1)(\alpha)}} u^{s(\alpha)} - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \Gamma^{(i+h-l+1)(\alpha)}_{i(\alpha)1(\beta)}$$

which amounts the right-hand side term

$$\begin{aligned} \frac{\partial \Gamma^{(i+h-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{(l-1)(\alpha)}} &\overset{(4.63)}{=} \frac{\partial}{\partial u^{(l-1)(\alpha)}} \left(-\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)} u^{s(\alpha)} \right) \\ &\overset{l\geq 2}{=} -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \frac{\partial \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{(l-1)(\alpha)}} u^{s(\alpha)} \\ &\quad - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \Gamma^{(i+h-l+1)(\alpha)}_{i(\alpha)1(\beta)}. \end{aligned}$$

This proves (B.2) for $k \geq i$.

Case 3: $\alpha = \beta \neq \gamma$**.** For every $k \in \{2, \dots, m_\gamma\}$, $l \in \{3, \dots, m_\delta\}$ and $\delta \in \{1, \dots, r\}$ we have

$$\begin{aligned} \frac{\partial \Gamma^{k(\gamma)}_{i(\alpha)j(\alpha)}}{\partial u^{l(\delta)}} &\overset{\text{Prop.8.3 (i)}}{=} \frac{\partial \Gamma^{k(\gamma)}_{(i+j-1)(\alpha)1(\alpha)}}{\partial u^{l(\delta)}} \underset{(4.61)}{\overset{(4.79)}{=}} -\frac{\partial \Gamma^{k(\gamma)}_{(i+j-1)(\alpha)1(\gamma)}}{\partial u^{l(\delta)}} \overset{\text{Case 2}}{=} -\frac{\partial \Gamma^{(k-1)(\gamma)}_{(i+j-1)(\alpha)1(\gamma)}}{\partial u^{(l-1)(\delta)}} \\ &\underset{(4.61)}{\overset{(4.79)}{=}} \frac{\partial \Gamma^{(k-1)(\gamma)}_{(i+j-1)(\alpha)1(\alpha)}}{\partial u^{(l-1)(\delta)}} \overset{\text{Prop.8.3 (i)}}{=} \frac{\partial \Gamma^{(k-1)(\gamma)}_{i(\alpha)j(\alpha)}}{\partial u^{(l-1)(\delta)}}. \end{aligned}$$

Case 4: $\alpha = \beta = \gamma$**.** We are going to show that

$$\frac{\partial \Gamma^{k(\alpha)}_{i(\alpha)j(\alpha)}}{\partial u^{l(\delta)}} = \frac{\partial \Gamma^{(k-1)(\alpha)}_{i(\alpha)j(\alpha)}}{\partial u^{(l-1)(\delta)}} \tag{B.4}$$

for all $\delta \in \{1, \dots, r\}$, $k \in \{2, \dots, m_\alpha\}$ and $l \in \{3, \dots, m_\delta\}$. If $i = 1$ (or equivalently $j = 1$) then (B.4) is verified by means of Case 2 and (4.79), as

$$\frac{\partial \Gamma^{k(\alpha)}_{1(\alpha)j(\alpha)}}{\partial u^{l(\delta)}} \overset{(4.79)}{=} -\sum_{\sigma\neq\alpha} \frac{\partial \Gamma^{k(\alpha)}_{1(\sigma)j(\alpha)}}{\partial u^{l(\delta)}} = -\sum_{\sigma\neq\alpha} \frac{\partial \Gamma^{(k-1)(\alpha)}_{1(\sigma)j(\alpha)}}{\partial u^{(l-1)(\delta)}} \overset{(4.79)}{=} \frac{\partial \Gamma^{(k-1)(\alpha)}_{1(\alpha)j(\alpha)}}{\partial u^{(l-1)(\delta)}}$$

for every choice of δ and every $k \in \{2, \dots, m_\alpha\}$, $l \in \{3, \dots, m_\delta\}$. Let us now consider $i, j \geq 2$. Without loss of generality, by (4.69), we can restrict ourselves to the case where $i = j = 2$. If $\delta \neq \alpha$ then

$$\frac{\partial \Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{l(\delta)}} = 0 = \frac{\partial \Gamma^{(k-1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\delta)}}$$

as $\Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)}$ only contains the terms $\{u^{s(\alpha)}\,|\,2 \le s \le k\}$ and $\{u^{1(\sigma)}\,|\,1,\dots,r\}$, where $l(\delta)$ is not included ($l \ge 3$). It only remains to prove (B.4) for $i = j = 2$ and $\delta = \alpha$, that is

$$\frac{\partial\Gamma^{k(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{l(\alpha)}} = \frac{\partial\Gamma^{(k-1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} \tag{B.5}$$

for all $k \in \{2,\dots,m_\alpha\}$ and $l \in \{3,\dots,m_\alpha\}$. We will proceed by induction over k. For $k = 2$ both the left and the right-hand sides vanish, as $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} = -\frac{m_\alpha\varepsilon_\alpha}{u^{2(\alpha)}}$ does not depend on any of the terms $\{u^{l(\alpha)}\,|\,l \ge 3\}$ and $\Gamma^{1(\alpha)}_{2(\alpha)2(\alpha)} = 0$. Let us suppose that (B.5) holds for all $k \in \{2,\dots,h\}$ (given an integer $2 \le h \le m_\alpha - 1$) and $l \in \{3,\dots,m_\alpha\}$. We must prove that

$$\frac{\partial\Gamma^{(h+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{l(\alpha)}} = \frac{\partial\Gamma^{h(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} \tag{B.6}$$

for all $l \in \{3,\dots,m_\alpha\}$. The left-hand side term reads

$$\begin{aligned}
\frac{\partial\Gamma^{(h+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{l(\alpha)}} &\overset{(4.68)}{=} \frac{\partial}{\partial u^{l(\alpha)}}\Bigg(\Gamma^{(h-1)(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{(h+1)(\alpha)}}{u^{2(\alpha)}} \\
&\quad - \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-2}\big(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{(h-s+1)(\alpha)}\Bigg) \\
&\overset{l\ge 2}{=} \frac{\partial\Gamma^{(h-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{l(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h+1}_l \\
&\quad - \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-2}\Bigg(\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{l(\alpha)}} - \frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{l(\alpha)}}\Bigg)u^{(h-s+1)(\alpha)} \\
&\quad - \frac{1}{u^{2(\alpha)}}\big(\Gamma^{(h-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\big)
\end{aligned}$$

where by what we said above ($i = 1$) we have

$$\begin{aligned}
\frac{\partial\Gamma^{(h-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{l(\alpha)}} &= \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}, \\
\frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{l(\alpha)}} &= \frac{\partial\Gamma^{(s-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}, \qquad 1 \le s \le h-2,
\end{aligned}$$

and by the inductive hypothesis

$$\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{l(\alpha)}} = \frac{\partial\Gamma^{(s+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}}, \qquad 1 \le s \le h-2.$$

It follows that the left-hand side of (B.6) is

$$\frac{\partial\Gamma^{(h+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{l(\alpha)}} = \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h+1}_l$$

$$
\begin{aligned}
&- \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-2}\left(\frac{\partial\Gamma^{(s+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\partial\Gamma^{(s-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}\right)u^{(h-s+1)(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\left(\Gamma^{(h-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\right)\\
&= \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h+1}_{l}\\
&- \frac{1}{u^{2(\alpha)}}\frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}}u^{h(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\sum_{s=2}^{h-2}\left(\frac{\partial\Gamma^{(s+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\partial\Gamma^{(s-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}\right)u^{(h-s+1)(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\left(\Gamma^{(h-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\right).
\end{aligned}
$$

When $l \geq 4$, since the right-hand side reads

$$
\begin{aligned}
\frac{\partial\Gamma^{h(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} &\overset{(4.68)}{=} \frac{\partial}{\partial u^{(l-1)(\alpha)}}\Bigg(\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{h(\alpha)}}{u^{2(\alpha)}}\\
&- \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-3}\left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(h-s)(\alpha)}\Bigg)\\
&\overset{l-1>2}{=} \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h}_{l-1} - \frac{u^{h(\alpha)}}{u^{2(\alpha)}}\frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}}\\
&- \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-3}\left(\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}\right)u^{(h-s)(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\left(\Gamma^{(h-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\right)
\end{aligned}
$$

we get that the difference between the left and the right-hand side terms is

$$
\begin{aligned}
\frac{\partial\Gamma^{(h+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{l(\alpha)}} - \frac{\partial\Gamma^{h(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} &= \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h+1}_{l}\\
&- \frac{1}{u^{2(\alpha)}}\frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}}u^{h(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\sum_{s=2}^{h-2}\left(\frac{\partial\Gamma^{(s+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\partial\Gamma^{(s-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}\right)u^{(h-s+1)(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\left(\Gamma^{(h-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\right)\\
&- \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}} + \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h}_{l-1} + \frac{u^{h(\alpha)}}{u^{2(\alpha)}}\frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}}\\
&+ \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-3}\left(\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}\right)u^{(h-s)(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{u^{2(\alpha)}}\big(\Gamma^{(h-l+3)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-l+1)(\alpha)}_{1(\alpha)1(\alpha)}\big)\\
&= -\frac{1}{u^{2(\alpha)}}\sum_{s=2}^{h-2}\left(\frac{\partial\Gamma^{(s+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\partial\Gamma^{(s-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}\right)u^{(h-s+1)(\alpha)}\\
&+ \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-3}\left(\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{(l-1)(\alpha)}} - \frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{(l-1)(\alpha)}}\right)u^{(h-s)(\alpha)} = 0
\end{aligned}
$$

by changing the variable in one of the summations. When $l = 3$, the left-hand side term reads

$$
\begin{aligned}
\frac{\partial\Gamma^{(h+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{3(\alpha)}} &= \frac{\partial\Gamma^{(h-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{3(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h+1}_{3}\\
&- \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-2}\left(\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{3(\alpha)}} - \frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{3(\alpha)}}\right)u^{(h-s+1)(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\left(\Gamma^{h(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}\right)\\
&= \frac{\partial\Gamma^{(h-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{3(\alpha)}} - \frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h+1}_{3} - \frac{1}{u^{2(\alpha)}}\left(\frac{\partial\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{3(\alpha)}} - \frac{\partial\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{3(\alpha)}}\right)u^{h(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\sum_{s=2}^{h-2}\left(\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{3(\alpha)}} - \frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{3(\alpha)}}\right)u^{(h-s+1)(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\left(\Gamma^{h(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}\right)
\end{aligned}
$$

and the right-hand side one reads

$$
\begin{aligned}
\frac{\partial\Gamma^{h(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}} &= \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{2(\alpha)}} - \frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}}\frac{u^{h(\alpha)}}{u^{2(\alpha)}} + \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{(u^{2(\alpha)})^2}u^{h(\alpha)} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h}_{2}\\
&+ \frac{1}{(u^{2(\alpha)})^2}\sum_{s=1}^{h-3}\left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(h-s)(\alpha)}\\
&- \frac{1}{u^{2(\alpha)}}\sum_{s=1}^{h-3}\left(\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}} - \frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{2(\alpha)}}\right)u^{(h-s)(\alpha)}
\end{aligned}
$$

where, by means of the inductive hypotesis, we have

$$
\begin{aligned}
&\frac{\partial\Gamma^{(h-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{3(\alpha)}} = \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{2(\alpha)}}\\
&\frac{\partial\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{3(\alpha)}} - \frac{\partial\Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{3(\alpha)}} = \frac{\partial\Gamma^{(s+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}} - \frac{\partial\Gamma^{(s-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{2(\alpha)}} \qquad \text{for each } \ s \leq h-2
\end{aligned}
$$

thus (by changing the variable in the last summation of the right-hand side term) their difference is

$$
\begin{aligned}
&\frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{2(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^{h+1}_3 - \frac{1}{u^{2(\alpha)}}\frac{\partial}{\partial u^{3(\alpha)}}\big(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\big)u^{h(\alpha)} \\
&- \frac{1}{u^{2(\alpha)}}\sum_{s=2}^{h-2}\left(\frac{\partial\Gamma^{(s+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}} - \frac{\partial\Gamma^{(s-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{2(\alpha)}}\right)u^{(h-s+1)(\alpha)} \\
&- \frac{1}{u^{2(\alpha)}}\left(\Gamma^{h(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}\right) \\
&- \frac{\partial\Gamma^{(h-2)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{2(\alpha)}} + \frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}}\frac{u^{h(\alpha)}}{u^{2(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{(u^{2(\alpha)})^2}u^{h(\alpha)} + \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}}\delta^h_2 \\
&- \frac{1}{(u^{2(\alpha)})^2}\sum_{s=1}^{h-3}\left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(h-s)(\alpha)} \\
&+ \frac{1}{u^{2(\alpha)}}\sum_{t=2}^{h-2}\left(\frac{\partial\Gamma^{(t+1)(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}} - \frac{\partial\Gamma^{(t-1)(\alpha)}_{1(\alpha)1(\alpha)}}{\partial u^{2(\alpha)}}\right)u^{(h-t+1)(\alpha)} \\
&\overset{(4.68)}{=} -\frac{1}{u^{2(\alpha)}}\frac{\partial}{\partial u^{3(\alpha)}}\left(-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\right)u^{h(\alpha)} \\
&+ \frac{1}{(u^{2(\alpha)})^2}\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}u^{h(\alpha)} + \frac{1}{(u^{2(\alpha)})^2}\sum_{l=1}^{h-3}\left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(h-l)(\alpha)} \\
&+ \frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}}\frac{u^{h(\alpha)}}{u^{2(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{(u^{2(\alpha)})^2}u^{h(\alpha)} \\
&- \frac{1}{(u^{2(\alpha)})^2}\sum_{s=1}^{h-3}\left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(h-s)(\alpha)} \\
&= \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{(u^{2(\alpha)})^2}u^{h(\alpha)} + \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{(u^{2(\alpha)})^2}u^{h(\alpha)} + \frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}}\frac{u^{h(\alpha)}}{u^{2(\alpha)}} - \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{(u^{2(\alpha)})^2}u^{h(\alpha)} \\
&= \frac{u^{h(\alpha)}}{u^{2(\alpha)}}\left(\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{2(\alpha)}} + \frac{\partial\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{\partial u^{2(\alpha)}}\right) \\
&\overset{(4.66)}{=} \frac{u^{h(\alpha)}}{u^{2(\alpha)}}\left(-\frac{m_\alpha\varepsilon_\alpha}{(u^{2(\alpha)})^2} + \frac{m_\alpha\varepsilon_\alpha}{(u^{2(\alpha)})^2}\right) = 0.
\end{aligned}
$$

This proves (B.6), thus (B.4) has been proved for all $\delta \in \{1, \dots, r\}$, $k \in \{2, \dots, m_\alpha\}$ and $l \in \{3, \dots, m_\delta\}$.

Let us now prove that (4.72) holds for $l = 2$ as well when $\beta \neq \alpha = \gamma = \delta$, that is

$$\frac{\partial\Gamma^{k(\alpha)}_{i(\alpha)j(\beta)}}{\partial u^{2(\alpha)}} = \frac{\partial\Gamma^{(k-1)(\alpha)}_{i(\alpha)j(\beta)}}{\partial u^{1(\alpha)}}$$

for all $k \in \{2, \dots, m_\gamma\}$. As in the proof of Case 2, this is trivially true when $j \geq 2$ or

$k < i$. We are going to prove (by induction over k) that

$$\frac{\partial \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} = \frac{\partial \Gamma^{(k-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}} \tag{B.7}$$

for $k \geq i$. If $k = i$ we get

$$\frac{\partial \Gamma^{k(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} = \frac{\partial \Gamma^{i(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} \overset{(4.63)}{=} \frac{\partial \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} = 0$$

(as $\Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}$ does not depend on $u^{2(\alpha)}$) and

$$\frac{\partial \Gamma^{(k-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}} = \frac{\partial \Gamma^{(i-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}} = 0$$

(as $\Gamma^{(i-1)(\alpha)}_{i(\alpha)1(\beta)} \overset{(4.63)}{=} 0$). This proves (B.7) for $k = i$. Let us suppose that (B.7) holds whenever $k \leq i + h - 1$, for a given integer $h \geq 1$. Let us show that it holds when $k = i + h$ as well, that is

$$\frac{\partial \Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} = \frac{\partial \Gamma^{(i+h-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}}. \tag{B.8}$$

The left-hand side of (B.8) reads

$$\begin{aligned}
\frac{\partial \Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} &\overset{(4.63)}{=} \frac{\partial}{\partial u^{2(\alpha)}}\left(- \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h+1} \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)}\, u^{s(\alpha)} \right) \\
&= -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h+1} \frac{\partial \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}}\, u^{s(\alpha)} \\
&\quad - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\, \Gamma^{(i+h-1)(\alpha)}_{i(\alpha)1(\beta)}
\end{aligned}$$

where in the first summation only the terms for $s \leq h$ survive, as for $s = h + 1$ we get

$$\frac{\partial \Gamma^{i(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} = \frac{\partial \Gamma^{1(\alpha)}_{1(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} = 0,$$

and (by the inductive hypothesis)

$$\frac{\partial \Gamma^{(i+h-s+1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} = \frac{\partial \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}}$$

for each $2 \leq s \leq h$. Then

$$\frac{\partial \Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} = -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \frac{\partial \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}}\, u^{s(\alpha)}$$

$$- \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \Gamma^{(i+h-1)(\alpha)}_{i(\alpha)1(\beta)}.$$

The right-hand side of (B.8) reads

$$\begin{aligned}
\frac{\partial \Gamma^{(i+h-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}} &\overset{(4.63)}{=} \frac{\partial}{\partial u^{1(\alpha)}} \left(- \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)} u^{s(\alpha)} \right) \\
&= \frac{1}{(u^{1(\alpha)} - u^{1(\beta)})^2} \sum_{s=2}^{h} \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)} u^{s(\alpha)} \\
&\quad - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \frac{\partial \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}} u^{s(\alpha)}
\end{aligned}$$

thus their difference is

$$\begin{aligned}
\frac{\partial \Gamma^{(i+h)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{2(\alpha)}} - \frac{\partial \Gamma^{(i+h-1)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}} &= - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \frac{\partial \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}} u^{s(\alpha)} \\
&\quad - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \Gamma^{(i+h-1)(\alpha)}_{i(\alpha)1(\beta)} \\
&\quad - \frac{1}{(u^{1(\alpha)} - u^{1(\beta)})^2} \sum_{s=2}^{h} \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)} u^{s(\alpha)} \\
&\quad + \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h} \frac{\partial \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)}}{\partial u^{1(\alpha)}} u^{s(\alpha)} \\
&\overset{(4.63)}{=} \frac{1}{(u^{1(\alpha)} - u^{1(\beta)})^2} \sum_{s=2}^{h} \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)} u^{s(\alpha)} \\
&\quad - \frac{1}{(u^{1(\alpha)} - u^{1(\beta)})^2} \sum_{s=2}^{h} \Gamma^{(i+h-s)(\alpha)}_{i(\alpha)1(\beta)} u^{s(\alpha)} = 0.
\end{aligned}$$

■

Proof of Lemma 4.22

Let us first consider $\alpha \neq \beta$ and prove $\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\beta)k} u^k = \delta^i_1 \delta^1_j m_\beta \varepsilon_\beta$. We have

$$\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\beta)k} u^k = \sum_{\gamma=1}^{r} \sum_{k=1}^{m_\gamma} \Gamma^{i(\alpha)}_{j(\beta)k(\gamma)} u^{k(\gamma)} \overset{(4.61)}{=} \sum_{k=1}^{m_\alpha} \Gamma^{i(\alpha)}_{j(\beta)k(\alpha)} u^{k(\alpha)} + \sum_{k=1}^{m_\beta} \Gamma^{i(\alpha)}_{j(\beta)k(\beta)} u^{k(\beta)}$$

which vanishes automatically when $j \geq 2$, by (4.62). Let us then fix $j = 1$, thus

$$\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\beta)k} u^k = \sum_{k=1}^{n} \Gamma^{i(\alpha)}_{1(\beta)k} u^k = \sum_{k=1}^{m_\alpha} \Gamma^{i(\alpha)}_{1(\beta)k(\alpha)} u^{k(\alpha)} + \sum_{k=1}^{m_\beta} \Gamma^{i(\alpha)}_{1(\beta)k(\beta)} u^{k(\beta)}$$

$$\overset{(4.63)}{\underset{(4.64)}{=}} \sum_{k=1}^{i} \Gamma^{i(\alpha)}_{1(\beta)k(\alpha)} u^{k(\alpha)} - \sum_{k=1}^{m_\beta} \Gamma^{i(\alpha)}_{1(\alpha)k(\beta)} \delta^1_k u^{k(\beta)}$$

$$= \sum_{k=1}^{i} \Gamma^{i(\alpha)}_{1(\beta)k(\alpha)} u^{k(\alpha)} - \Gamma^{i(\alpha)}_{1(\alpha)1(\beta)} u^{1(\beta)}$$

$$= \sum_{k=2}^{i} \Gamma^{i(\alpha)}_{1(\beta)k(\alpha)} u^{k(\alpha)} + \Gamma^{i(\alpha)}_{1(\beta)1(\alpha)}(u^{1(\alpha)} - u^{1(\beta)})$$

$$\overset{(4.63)}{=} \begin{cases} \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}(u^{1(\alpha)} - u^{1(\beta)}) & \text{if } i = 1 \\ \sum\limits_{k=2}^{i} \Gamma^{(i-k+1)(\alpha)}_{1(\beta)1(\alpha)} u^{k(\alpha)} - \sum\limits_{s=2}^{i} \Gamma^{(i-s+1)(\alpha)}_{1(\beta)1(\alpha)} u^{s(\alpha)} & \text{if } i \geq 2 \end{cases}$$

$$\overset{(4.63)}{=} \begin{cases} m_\beta \varepsilon_\beta & \text{if } i = 1, \\ 0 & \text{if } i \geq 2, \end{cases} = m_\beta \varepsilon_\beta \, \delta^i_1.$$

In order to complete the proof we must show that

$$\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\alpha)k} u^k = \begin{cases} 0 & \text{if } i \neq j, \\ -\sum\limits_{\sigma \neq \alpha} m_\sigma \varepsilon_\sigma & \text{if } i = j = 1, \\ -\sum\limits_{\tau=1}^{r} m_\tau \varepsilon_\tau & \text{if } i = j \neq 1. \end{cases}$$

Let us first consider the case where $i \neq j$. Without loss of generality we assume $i > j$, as $\Gamma^{i(\alpha)}_{j(\alpha)k} = 0$ trivially whenever $i < j$, by (4.63) and (4.69). We have

$$\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\alpha)k} u^k \overset{(4.62)}{=} \sum_{k=1}^{m_\alpha} \Gamma^{i(\alpha)}_{j(\alpha)k(\alpha)} u^{k(\alpha)} + \sum_{\sigma \neq \alpha} \sum_{k=1}^{m_\sigma} \Gamma^{i(\alpha)}_{j(\alpha)k(\sigma)} u^{k(\sigma)} \delta^1_k$$

$$= \Gamma^{i(\alpha)}_{j(\alpha)1(\alpha)} u^{1(\alpha)} + \sum_{k=2}^{m_\alpha} \Gamma^{i(\alpha)}_{j(\alpha)k(\alpha)} u^{k(\alpha)} + \sum_{\sigma \neq \alpha} \Gamma^{i(\alpha)}_{j(\alpha)1(\sigma)} u^{1(\sigma)}$$

$$\overset{(4.79)}{=} -\sum_{\sigma \neq \alpha} \Gamma^{i(\alpha)}_{j(\alpha)1(\sigma)} u^{1(\alpha)} + \sum_{k=2}^{m_\alpha} \Gamma^{i(\alpha)}_{j(\alpha)k(\alpha)} u^{k(\alpha)} + \sum_{\sigma \neq \alpha} \Gamma^{i(\alpha)}_{j(\alpha)1(\sigma)} u^{1(\sigma)}$$

$$\overset{(4.69)}{=} -\sum_{\sigma \neq \alpha} \Gamma^{i(\alpha)}_{j(\alpha)1(\sigma)} \left(u^{1(\alpha)} - u^{1(\sigma)}\right) + \sum_{k=2}^{i-j+2} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} u^{k(\alpha)}$$

where

$$\sum_{k=2}^{i-j+2} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} u^{k(\alpha)} = \Gamma^{(i-j+2)(\alpha)}_{2(\alpha)2(\alpha)} u^{2(\alpha)} + \sum_{k=3}^{i-j+2} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} u^{k(\alpha)}$$

$$\overset{(4.68)}{=} \Bigg(\Gamma^{(i-j)(\alpha)}_{1(\alpha)1(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{(i-j+2)(\alpha)}}{u^{2(\alpha)}}$$

$$
\begin{aligned}
&- \frac{1}{u^{2(\alpha)}} \sum_{l=1}^{i-j-1} \Big(\big(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\big) u^{(i-j+2-l)(\alpha)} \Big) u^{2(\alpha)} \\
&+ \sum_{k=3}^{i-j+2} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} \, u^{k(\alpha)} \\
&= \Gamma^{(i-j)(\alpha)}_{1(\alpha)1(\alpha)} \, u^{2(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \, u^{(i-j+2)(\alpha)} \\
&- \sum_{l=1}^{i-j-1} \Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} \, u^{(i-j+2-l)(\alpha)} + \sum_{l=1}^{i-j-1} \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)} \, u^{(i-j+2-l)(\alpha)} \\
&+ \sum_{k=3}^{i-j+2} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} \, u^{k(\alpha)} \\
&= \Gamma^{(i-j)(\alpha)}_{1(\alpha)1(\alpha)} \, u^{2(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \, u^{(i-j+2)(\alpha)} \\
&- \sum_{k=3}^{i-j+1} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} \, u^{k(\alpha)} + \sum_{k=3}^{i-j+1} \Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\alpha)} \, u^{k(\alpha)} \\
&+ \sum_{k=3}^{i-j+2} \Gamma^{(i-j-k+4)(\alpha)}_{2(\alpha)2(\alpha)} \, u^{k(\alpha)} \\
&= -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \, u^{(i-j+2)(\alpha)} + \sum_{k=2}^{i-j+1} \Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\alpha)} \, u^{k(\alpha)} \\
&+ \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \, u^{(i-j+2)(\alpha)} = \sum_{k=2}^{i-j+1} \Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\alpha)} \, u^{k(\alpha)}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\alpha)k} \, u^k &= -\sum_{\sigma \neq \alpha} \Gamma^{i(\alpha)}_{j(\alpha)1(\sigma)} \left(u^{1(\alpha)} - u^{1(\sigma)}\right) + \sum_{k=2}^{i-j+1} \Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\alpha)} \, u^{k(\alpha)} \\
&\overset{(4.79)}{\underset{(4.63)}{=}} \sum_{\sigma \neq \alpha} \sum_{s=2}^{i-j+1} \Gamma^{(i-j-s+2)(\alpha)}_{1(\alpha)1(\sigma)} \, u^{s(\alpha)} - \sum_{\sigma \neq \alpha} \sum_{k=2}^{i-j+1} \Gamma^{(i-j-k+2)(\alpha)}_{1(\alpha)1(\sigma)} \, u^{k(\alpha)} = 0.
\end{aligned}
$$

Let us now consider the case where $i = j$. We have

$$
\begin{aligned}
\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{i(\alpha)k} \, u^k &\overset{(4.62)}{=} \sum_{k=1}^{m_\alpha} \Gamma^{i(\alpha)}_{i(\alpha)k(\alpha)} \, u^{k(\alpha)} + \sum_{\sigma \neq \alpha} \sum_{k=1}^{m_\sigma} \Gamma^{i(\alpha)}_{i(\alpha)k(\sigma)} \, u^{k(\sigma)} \delta^1_k \\
&= \Gamma^{i(\alpha)}_{i(\alpha)1(\alpha)} \, u^{1(\alpha)} + \sum_{k=2}^{m_\alpha} \Gamma^{i(\alpha)}_{i(\alpha)k(\alpha)} \, u^{k(\alpha)} + \sum_{\sigma \neq \alpha} \Gamma^{i(\alpha)}_{i(\alpha)1(\sigma)} \, u^{1(\sigma)} \\
&\overset{(4.79)}{=} -\sum_{\sigma \neq \alpha} \Gamma^{i(\alpha)}_{i(\alpha)1(\sigma)} \big(u^{1(\alpha)} - u^{1(\sigma)}\big) + \sum_{k=2}^{m_\alpha} \Gamma^{i(\alpha)}_{i(\alpha)k(\alpha)} \, u^{k(\alpha)} \\
&\overset{(4.63)}{=} -\sum_{\sigma \neq \alpha} \Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)} \big(u^{1(\alpha)} - u^{1(\sigma)}\big) + \sum_{k=2}^{m_\alpha} \Gamma^{i(\alpha)}_{i(\alpha)k(\alpha)} \, u^{k(\alpha)}.
\end{aligned}
$$

If $i = 1$ then

$$\sum_{k=1}^{n} \Gamma^{1(\alpha)}_{1(\alpha)k}\, u^k = -\sum_{\sigma\neq\alpha} \Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\big(u^{1(\alpha)} - u^{1(\sigma)}\big) + \sum_{k=2}^{m_\alpha} \Gamma^{1(\alpha)}_{1(\alpha)k(\alpha)}\, u^{k(\alpha)}$$

where

$$\Gamma^{1(\alpha)}_{1(\alpha)k(\alpha)} \overset{(4.79)}{=} -\sum_{\sigma\neq\alpha} \Gamma^{1(\alpha)}_{1(\sigma)k(\alpha)} \overset{(4.63)}{=} 0$$

for every $k \geq 2$. It follows that

$$\sum_{k=1}^{n} \Gamma^{1(\alpha)}_{1(\alpha)k}\, u^k = -\sum_{\sigma\neq\alpha} \Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\big(u^{1(\alpha)} - u^{1(\sigma)}\big) \overset{(4.63)}{=} -\sum_{\sigma\neq\alpha} m_\sigma \varepsilon_\sigma.$$

If $i \neq 1$ then

$$\begin{aligned}
\sum_{k=1}^{n} \Gamma^{i(\alpha)}_{i(\alpha)k}\, u^k &\overset{(4.69)}{=} -\sum_{\sigma\neq\alpha} \Gamma^{1(\alpha)}_{1(\alpha)1(\sigma)}\big(u^{1(\alpha)} - u^{1(\sigma)}\big) + \sum_{k=2}^{m_\alpha} \Gamma^{(4-k)(\alpha)}_{2(\alpha)2(\alpha)}\, \delta^2_k\, u^{k(\alpha)} \\
&\overset{(4.63)}{=} -\sum_{\sigma\neq\alpha} m_\sigma \varepsilon_\sigma + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\, u^{2(\alpha)} \\
&\overset{(4.66)}{=} -\sum_{\sigma\neq\alpha} m_\sigma \varepsilon_\sigma - m_\alpha \varepsilon_\alpha = -\sum_{\tau=1}^{r} m_\tau \varepsilon_\tau.
\end{aligned}$$

■

Proof of Lemma 4.23

Let us recall that

$$\nabla_{j(\beta)} E^{i(\alpha)} = \partial_{j(\beta)} E^{i(\alpha)} + \sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\beta)k}\, E^k = \delta^{i(\alpha)}_{j(\beta)} + \sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\beta)k}\, u^k.$$

If $\alpha \neq \beta$ then

$$\nabla_{j(\beta)} E^{i(\alpha)} = \delta^i_1 \delta^1_j\, m_\beta \varepsilon_\beta$$

by (4.73). Let us now consider the case where $\alpha = \beta$. Here we have

$$\nabla_{j(\alpha)} E^{i(\alpha)} = \delta^i_j + \sum_{k=1}^{n} \Gamma^{i(\alpha)}_{j(\alpha)k}\, u^k$$

which vanishes if $i \neq j$ by (4.73). Let now consider $i = j$. By virtue of (4.73), we get

$$\begin{aligned}\nabla_{i(\alpha)}E^{i(\alpha)} &= 1 + \sum_{k=1}^{n} \Gamma^{i(\alpha)}_{i(\alpha)k}\, u^k \\ &= \begin{cases} 1 - \sum\limits_{\sigma\neq\alpha} m_\sigma \varepsilon_\sigma & \text{if } i = 1, \\ 1 - \sum\limits_{\tau=1}^{r} m_\tau \varepsilon_\tau & \text{if } i \neq 1. \end{cases}\end{aligned}$$

■

Proof of Lemma 4.24

We will proceed by induction over l. For $l = 3$ we get

$$A^{3(\alpha)} = \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\, u^{3(\alpha)} - u^{4(\alpha)} \right) - \left(\Gamma^{4(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{2(\alpha)}$$

where

$$\begin{aligned}\Gamma^{4(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{2(\alpha)}_{1(\alpha)1(\alpha)} &\overset{(4.68)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{4(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}} \left(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{3(\alpha)} \\ &\overset{(4.68)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{4(\alpha)}}{u^{2(\alpha)}} + \frac{u^{3(\alpha)}}{u^{2(\alpha)}}\, \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{3(\alpha)}}{u^{2(\alpha)}}.\end{aligned}$$

It follows that

$$\begin{aligned}A^{3(\alpha)} = \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\, u^{3(\alpha)} - u^{4(\alpha)} \right) - \Bigg(&- \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\, u^{4(\alpha)} \\ &+ \frac{u^{3(\alpha)}}{u^{2(\alpha)}}\, \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\, u^{3(\alpha)} \Bigg) = 0.\end{aligned}$$

Given an integer $h \geq 1$, $h \leq m_\alpha - 2$, let us suppose that

$$A^{l(\alpha)} = 0, \qquad 3 \leq l \leq h, \tag{B.9}$$

and show $A^{(h+1)(\alpha)} = 0$. We have

$$\begin{aligned}A^{(h+1)(\alpha)} &= \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\, u^{(h+1)(\alpha)} - u^{(h+2)(\alpha)} \right) - \sum_{s=2}^{h} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(h-s+2)(\alpha)} \\ &= \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\, u^{(h+1)(\alpha)} - u^{(h+2)(\alpha)} \right) - \sum_{s=2}^{h-1} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{(h-s+2)(\alpha)} \\ &\quad - \left(\Gamma^{(h+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{h(\alpha)}_{1(\alpha)1(\alpha)} \right) u^{2(\alpha)}\end{aligned}$$

where

$$\Gamma^{(h+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{h(\alpha)}_{1(\alpha)1(\alpha)} \overset{(4.68)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{(h+2)(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}} \sum_{l=1}^{h-1} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(h-l+2)(\alpha)}.$$

It follows that

$$\begin{aligned}
A^{(h+1)(\alpha)} &= \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{(h+1)(\alpha)} - u^{(h+2)(\alpha)}\right) - \sum_{s=2}^{h-1} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(h-s+2)(\alpha)} \\
&+ \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} u^{(h+2)(\alpha)} + \sum_{l=1}^{h-1} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(h-l+2)(\alpha)} \\
&= \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{(h+1)(\alpha)} - \sum_{s=2}^{h-1} \left(\Gamma^{(s+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{s(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(h-s+2)(\alpha)} \\
&+ \left(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(h+1)(\alpha)} + \sum_{l=2}^{h-1} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(h-l+2)(\alpha)} \\
&\overset{(4.68)}{=} \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{(h+1)(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{3(\alpha)}}{u^{2(\alpha)}} u^{(h+1)(\alpha)} = 0.
\end{aligned}$$

■

Proof of Lemma 4.25

We will proceed by induction over l. For $l = 1$ we get

$$\partial_{1(\sigma)}\left(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\right) \overset{(4.68)}{=} \partial_{1(\sigma)}\left(-\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{3(\alpha)}}{u^{2(\alpha)}}\right) = 0$$

as $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}$ does not depend on $u^{1(\sigma)}$ for any choice of σ. Given an integer $h \geq 2$, $h \leq m_\alpha - 2$, let us suppose

$$\partial_{1(\sigma)}\left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) = 0, \qquad l \leq h-1, \tag{B.10}$$

and show $\partial_{1(\sigma)}\left(\Gamma^{(h+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{h(\alpha)}_{1(\alpha)1(\alpha)}\right) = 0$. We have

$$\Gamma^{(h+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{h(\alpha)}_{1(\alpha)1(\alpha)} \overset{(4.68)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{(h+2)(\alpha)}}{u^{2(\alpha)}} - \frac{1}{u^{2(\alpha)}} \sum_{l=1}^{h-1} \left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(h-l+2)(\alpha)}$$

where neither $\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)} \frac{u^{(h+2)(\alpha)}}{u^{2(\alpha)}}$ nor

$$\left(\Gamma^{(l+2)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{l(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(h-l+2)(\alpha)}, \qquad 1 \leq l \leq h-1,$$

(by means of (B.10) and of the requirement $\sigma \neq \alpha$) depend on $u^{1(\sigma)}$. ■

Proof of Lemma 4.26

We will proceed by induction over s. For $s = 1$ we get

$$\begin{aligned}
B^{1(\alpha)}_{\beta\epsilon} &= -\sum_{t=1}^{2} \Gamma^{(3-t)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)} \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} + \Gamma^{2(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} \\
&= -\Gamma^{2(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)} \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} + \Gamma^{2(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} \\
&= \Gamma^{2(\alpha)}_{1(\epsilon)1(\alpha)}\left(\Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} - \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\right) + \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)}\left(\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}\right) \\
&\overset{(4.63)}{=} \Gamma^{2(\alpha)}_{1(\epsilon)1(\alpha)} m_\beta \varepsilon_\beta \left(\frac{1}{u^{1(\epsilon)} - u^{1(\beta)}} - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\right) \\
&+ \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)} m_\epsilon \varepsilon_\epsilon \left(\frac{1}{u^{1(\beta)} - u^{1(\epsilon)}} - \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}}\right) \\
&\overset{(4.63)}{=} -\frac{\Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}}{u^{1(\alpha)} - u^{1(\epsilon)}} u^{2(\alpha)} m_\beta \varepsilon_\beta \frac{-(u^{1(\alpha)} - u^{1(\epsilon)})}{(u^{1(\beta)} - u^{1(\epsilon)})(u^{1(\alpha)} - u^{1(\beta)})} \\
&- \frac{\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}}{u^{1(\alpha)} - u^{1(\beta)}} u^{2(\alpha)} m_\epsilon \varepsilon_\epsilon \frac{u^{1(\alpha)} - u^{1(\beta)}}{(u^{1(\beta)} - u^{1(\epsilon)})(u^{1(\alpha)} - u^{1(\epsilon)})} \\
&\overset{(4.63)}{=} \frac{m_\epsilon \varepsilon_\epsilon}{u^{1(\alpha)} - u^{1(\epsilon)}} u^{2(\alpha)} \frac{m_\beta \varepsilon_\beta}{(u^{1(\beta)} - u^{1(\epsilon)})(u^{1(\alpha)} - u^{1(\beta)})} \\
&- \frac{m_\beta \varepsilon_\beta}{u^{1(\alpha)} - u^{1(\beta)}} u^{2(\alpha)} \frac{m_\epsilon \varepsilon_\epsilon}{(u^{1(\beta)} - u^{1(\epsilon)})(u^{1(\alpha)} - u^{1(\epsilon)})} = 0.
\end{aligned}$$

Given an integer $h \geq 2$, $h \leq m_\alpha - 1$, let us suppose that

$$B^{s(\alpha)}_{\beta\epsilon} = 0, \qquad s \leq h-1, \tag{B.11}$$

and show $B^{h(\alpha)}_{\beta\epsilon} = 0$. We have

$$\begin{aligned}
B^{h(\alpha)}_{\beta\epsilon} &= -\sum_{t=1}^{h+1} \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{(h+1)(\alpha)}_{1(\beta)1(\alpha)} \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} + \Gamma^{(h+1)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} \\
&= \Gamma^{(h+1)(\alpha)}_{1(\beta)1(\alpha)} \left(\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}\right) + \Gamma^{(h+1)(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} - \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\right) \\
&- \sum_{t=2}^{h} \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} \\
&\overset{(4.63)}{=} -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{s=2}^{h+1} \Gamma^{(h-s+2)(\alpha)}_{1(\beta)1(\alpha)} u^{s(\alpha)} \left(\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{s=2}^{h+1} \Gamma^{(h-s+2)(\alpha)}_{1(\epsilon)1(\alpha)}\, u^{s(\alpha)} \left(\Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} - \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\right) \\
&- \sum_{t=2}^{h} \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)}\, \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)}
\end{aligned}
$$

where in the first two summations only the terms corresponding to $s \leq h$ survive, as the sum of their two $(s = h+1)$-th terms is

$$
\begin{aligned}
&- \frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\, \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\, u^{(h+1)(\alpha)} \left(\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}\right) \\
&- \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}}\, \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}\, u^{(h+1)(\alpha)} \left(\Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} - \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\right) \\
&\overset{(4.63)}{=} - \frac{m_\beta \varepsilon_\beta\, u^{(h+1)(\alpha)}}{(u^{1(\alpha)} - u^{1(\beta)})^2} \left(\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}\right) \\
&- \frac{m_\epsilon \varepsilon_\epsilon\, u^{(h+1)(\alpha)}}{(u^{1(\alpha)} - u^{1(\epsilon)})^2} \left(\Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} - \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\right) \\
&\overset{(4.63)}{=} - \frac{m_\beta \varepsilon_\beta\, u^{(h+1)(\alpha)}}{(u^{1(\alpha)} - u^{1(\beta)})^2}\, m_\epsilon \varepsilon_\epsilon \left(\frac{1}{u^{1(\beta)} - u^{1(\epsilon)}} - \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}}\right) \\
&- \frac{m_\epsilon \varepsilon_\epsilon\, u^{(h+1)(\alpha)}}{(u^{1(\alpha)} - u^{1(\epsilon)})^2}\, m_\beta \varepsilon_\beta \left(\frac{1}{u^{1(\epsilon)} - u^{1(\beta)}} - \frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\right) \\
&= - \frac{m_\beta \varepsilon_\beta\, u^{(h+1)(\alpha)}}{(u^{1(\alpha)} - u^{1(\beta)})^2}\, m_\epsilon \varepsilon_\epsilon\, \frac{u^{1(\alpha)} - u^{1(\beta)}}{(u^{1(\beta)} - u^{1(\epsilon)})(u^{1(\alpha)} - u^{1(\epsilon)})} \\
&- \frac{m_\epsilon \varepsilon_\epsilon\, u^{(h+1)(\alpha)}}{(u^{1(\alpha)} - u^{1(\epsilon)})^2}\, m_\beta \varepsilon_\beta\, \frac{-(u^{1(\alpha)} - u^{1(\epsilon)})}{(u^{1(\beta)} - u^{1(\epsilon)})(u^{1(\alpha)} - u^{1(\beta)})} = 0.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
B^{h(\alpha)}_{\beta\epsilon} = &- \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{t=2}^{h} \Gamma^{(h-t+2)(\alpha)}_{1(\beta)1(\alpha)}\, u^{t(\alpha)} \left(\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)}\right) \\
&- \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{t=2}^{h} \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)}\, u^{t(\alpha)} \left(\Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} - \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\right) \\
&- \sum_{t=2}^{h} \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)}\, \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)}
\end{aligned}
$$

where

$$
\begin{aligned}
-\sum_{t=2}^{h} \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)}\, \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} &\overset{(4.63)}{=} -\sum_{t=2}^{h} \left(- \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{l=2}^{h-t+2} \Gamma^{(h-t-l+3)(\alpha)}_{1(\epsilon)1(\alpha)}\, u^{l(\alpha)} \right) \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} \\
&= \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{t=2}^{h} \sum_{l=2}^{h-t+2} \Gamma^{(h-t-l+3)(\alpha)}_{1(\epsilon)1(\alpha)}\, u^{l(\alpha)}\, \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)}
\end{aligned}
$$

$$
\begin{aligned}
&\overset{s=h-l+1}{=} \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{t=2}^{h} \sum_{s=t-1}^{h-1} \Gamma^{(s-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \, u^{(h-s+1)(\alpha)} \, \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} \\
&= \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{s=1}^{h-1} \sum_{t=2}^{s+1} \Gamma^{(s-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \, u^{(h-s+1)(\alpha)} \, \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} \\
&= \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{s=1}^{h-1} \left(\sum_{t=1}^{s+1} \Gamma^{(s-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \, \Gamma^{t(\alpha)}_{1(\beta)1(\alpha)} \right. \\
&\left. - \Gamma^{(s+1)(\alpha)}_{1(\epsilon)1(\alpha)} \, \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} \right) u^{(h-s+1)(\alpha)} \\
&\overset{(B.11)}{=} \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{s=1}^{h-1} \left(\Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)} \, \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} + \Gamma^{(s+1)(\alpha)}_{1(\epsilon)1(\alpha)} \, \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} \right. \\
&\left. - \Gamma^{(s+1)(\alpha)}_{1(\epsilon)1(\alpha)} \, \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} \right) u^{(h-s+1)(\alpha)} \\
&\overset{t=h-s+1}{=} \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \sum_{t=2}^{h} \left(\Gamma^{(h-t+2)(\alpha)}_{1(\beta)1(\alpha)} \, \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} \right. \\
&\left. + \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \, \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} - \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \, \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} \right) u^{t(\alpha)}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
B^{h(\alpha)}_{\beta\epsilon} &= \sum_{t=2}^{h} \left(- \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \, \Gamma^{(h-t+2)(\alpha)}_{1(\beta)1(\alpha)} \left(\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)} \right) \right. \\
&- \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \, \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \left(\Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} - \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} \right) \\
&+ \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \, \Gamma^{(h-t+2)(\alpha)}_{1(\beta)1(\alpha)} \, \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} + \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \, \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \, \Gamma^{1(\epsilon)}_{1(\beta)1(\epsilon)} \\
&\left. - \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \, \Gamma^{(h-t+2)(\alpha)}_{1(\epsilon)1(\alpha)} \, \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)} \right) u^{t(\alpha)} \\
&= \sum_{t=2}^{h} \Gamma^{(h-t+2)(\alpha)}_{1(\beta)1(\alpha)} \left(- \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \left(\Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} - \Gamma^{1(\alpha)}_{1(\epsilon)1(\alpha)} \right) \right. \\
&\left. + \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \, \Gamma^{1(\beta)}_{1(\epsilon)1(\beta)} \right) u^{t(\alpha)} \\
&\overset{(4.63)}{=} \sum_{t=2}^{h} \Gamma^{(h-t+2)(\alpha)}_{1(\beta)1(\alpha)} \left(- \frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \, \frac{m_\epsilon \varepsilon_\epsilon \left(u^{1(\alpha)} - u^{1(\beta)} \right)}{\left(u^{1(\beta)} - u^{1(\epsilon)} \right) \left(u^{1(\alpha)} - u^{1(\epsilon)} \right)} \right. \\
&\left. + \frac{1}{u^{1(\alpha)} - u^{1(\epsilon)}} \, \frac{m_\epsilon \varepsilon_\epsilon}{u^{1(\beta)} - u^{1(\epsilon)}} \right) u^{t(\alpha)} = 0.
\end{aligned}
$$

■

Proof of Lemma 4.27

For $s = 0$ we get

$$\begin{aligned}
C_\beta^{0(\alpha)} &= \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\Gamma^{2(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\,\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&\overset{(4.63)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\frac{\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}}{u^{1(\alpha)} - u^{1(\beta)}}\,u^{2(\alpha)} + \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\,\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&= \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\left(- \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\,u^{2(\alpha)} + \Gamma^{1(\beta)}_{1(\alpha)1(\beta)}\right) \\
&\underset{(4.66)}{\overset{(4.63)}{=}} \Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}\left(\frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}} - \frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}}\right) = 0.
\end{aligned}$$

For $s = 1$ we get

$$\begin{aligned}
C_\beta^{1(\alpha)} &= \left(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\right)\Gamma^{2(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\Gamma^{3(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)}\,\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&\underset{(4.68)}{\overset{(4.63)}{=}} \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\,\frac{\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}}{u^{1(\alpha)} - u^{1(\beta)}}\,u^{2(\alpha)} - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\frac{\Gamma^{2(\alpha)}_{1(\beta)1(\alpha)}}{u^{1(\alpha)} - u^{1(\beta)}}\,u^{2(\alpha)} \\
&\quad - \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\frac{\Gamma^{1(\alpha)}_{1(\beta)1(\alpha)}}{u^{1(\alpha)} - u^{1(\beta)}}\,u^{3(\alpha)} + \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)}\,\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&= \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)}\left(- \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\frac{1}{u^{1(\alpha)} - u^{1(\beta)}}\,u^{2(\alpha)} + \Gamma^{1(\beta)}_{1(\alpha)1(\beta)}\right) \\
&\underset{(4.66)}{\overset{(4.63)}{=}} \Gamma^{2(\alpha)}_{1(\beta)1(\alpha)}\left(\frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}} - \frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}}\right) = 0.
\end{aligned}$$

Let us now consider an integer $s \in \{1, \dots, m_\alpha - 2\}$. Since

$$\begin{aligned}
\Gamma^{l(\alpha)}_{1(\beta)1(\alpha)} &\overset{(4.63)}{=} -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{t=2}^{l} \Gamma^{(l-t+1)(\alpha)}_{1(\beta)1(\alpha)}\,u^{t(\alpha)} \\
&\overset{k=l-t+1}{=} -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{k=1}^{l-1} \Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}\,u^{(l-k+1)(\alpha)}
\end{aligned}$$

for each $l \in \{2, \dots, s+1\}$, we have

$$\begin{aligned}
C_\beta^{s(\alpha)} &= -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{l=2}^{s+1} \left(\Gamma^{(s-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(s-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right) \sum_{k=1}^{l-1} \Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}\,u^{(l-k+1)(\alpha)} \\
&\quad + \Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\Gamma^{(s+2)(\alpha)}_{1(\beta)1(\alpha)} + \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)}\,\Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&= -\frac{1}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{k=1}^{s} \Gamma^{k(\alpha)}_{1(\beta)1(\alpha)} \sum_{l=k+1}^{s+1} \left(\Gamma^{(s-l+4)(\alpha)}_{2(\alpha)2(\alpha)} - \Gamma^{(s-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right) u^{(l-k+1)(\alpha)}
\end{aligned}$$

$$+\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\Gamma^{(s+2)(\alpha)}_{1(\beta)1(\alpha)}+\Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)}\,\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}$$

where (by taking $\widetilde{s} := s-l+2$ and $\widetilde{l} := s-k+2$)

$$\begin{aligned}
\sum_{l=k+1}^{s+1}\left(\Gamma^{(s-l+4)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{(s-l+2)(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(l-k+1)(\alpha)} &= \sum_{\widetilde{s}=1}^{\widetilde{l}-1}\left(\Gamma^{(\widetilde{s}+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{\widetilde{s}(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(\widetilde{l}-\widetilde{s}+1)(\alpha)}\\
&= \left(\Gamma^{3(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{1(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{\widetilde{l}(\alpha)}\\
&+\sum_{\widetilde{s}=2}^{\widetilde{l}-1}\left(\Gamma^{(\widetilde{s}+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{\widetilde{s}(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(\widetilde{l}-\widetilde{s}+1)(\alpha)}\\
&\overset{(4.68)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\,u^{\widetilde{l}(\alpha)}\\
&+\sum_{\widetilde{s}=2}^{\widetilde{l}-1}\left(\Gamma^{(\widetilde{s}+2)(\alpha)}_{2(\alpha)2(\alpha)}-\Gamma^{\widetilde{s}(\alpha)}_{1(\alpha)1(\alpha)}\right)u^{(\widetilde{l}-\widetilde{s}+1)(\alpha)}\\
&\overset{(4.75)}{=} -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\,u^{\widetilde{l}(\alpha)}\\
&+\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\left(\frac{u^{3(\alpha)}}{u^{2(\alpha)}}\,u^{\widetilde{l}(\alpha)}-u^{(\widetilde{l}+1)(\alpha)}\right)\\
&= -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,u^{(\widetilde{l}+1)(\alpha)}\\
&= -\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,u^{(s-k+3)(\alpha)}.
\end{aligned}$$

It follows that

$$\begin{aligned}
C^{s(\alpha)}_{\beta} &= \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\beta)}}\sum_{k=1}^{s}\Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}\,u^{(s-k+3)(\alpha)}\\
&\quad+\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}\,\Gamma^{(s+2)(\alpha)}_{1(\beta)1(\alpha)}+\Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)}\,\Gamma^{1(\beta)}_{1(\alpha)1(\beta)}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma^{(s+2)(\alpha)}_{1(\beta)1(\alpha)} &\overset{(4.63)}{=} -\frac{1}{u^{1(\alpha)}-u^{1(\beta)}}\sum_{t=2}^{s+2}\Gamma^{(s-t+3)(\alpha)}_{1(\beta)1(\alpha)}\,u^{t(\alpha)}\\
&\overset{k=s-t+3}{=} -\frac{1}{u^{1(\alpha)}-u^{1(\beta)}}\sum_{k=1}^{s+1}\Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}\,u^{(s-k+3)(\alpha)}
\end{aligned}$$

thus

$$C^{s(\alpha)}_{\beta} = \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)}-u^{1(\beta)}}\sum_{k=1}^{s}\Gamma^{k(\alpha)}_{1(\beta)1(\alpha)}\,u^{(s-k+3)(\alpha)}$$

$$
\begin{aligned}
&- \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\beta)}} \sum_{k=1}^{s+1} \Gamma^{k(\alpha)}_{1(\beta)1(\alpha)} \, u^{(s-k+3)(\alpha)} \\
&+ \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)} \, \Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&= -\frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\beta)}} \, \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)} \, u^{2(\alpha)} + \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)} \, \Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \\
&= \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)} \left(- \frac{\Gamma^{2(\alpha)}_{2(\alpha)2(\alpha)}}{u^{1(\alpha)} - u^{1(\beta)}} \, u^{2(\alpha)} + \Gamma^{1(\beta)}_{1(\alpha)1(\beta)} \right) \\
&\underset{(4.66)}{\overset{(4.63)}{=}} \Gamma^{(s+1)(\alpha)}_{1(\beta)1(\alpha)} \left(\frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}} - \frac{m_\alpha \varepsilon_\alpha}{u^{1(\alpha)} - u^{1(\beta)}} \right) = 0.
\end{aligned}
$$

■